Existence of Global Solutions with Prescribed Asymptotic Behavior for Nonlinear Ordinary Differential Equations (*).

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Summary. – *Conditions are given for the nonlinear differential equation* (1) $L_n y + f(t, y, \ldots, \ldots, y^{(n-1)}) = 0$ *to have solutions which exist on a given interval* $[t_0, \infty)$ *and behave in some sense like specified solutions of the linear equation* (2) $L_n z = 0$ *as* $t \to \infty$. *The global nature of these results is unusual as compared to most theorems of this kind, which guarantee the existence of solutions of* (1) *only for sufficiently large t. The main theorem requires no assumptions regarding oscillation or nonoscillation of solutions of* (2). *A second theorem is specifically applicable to the situation where* (2) *is disconjugate on* $[t_0, \infty)$, *and a corollary of the latter applies to the case where* $Lz = z^{(n)}$.

1. – Introduction.

We consider the nonlinear differential equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y + f(t, y, \ldots, y^{(n-1)}) = 0\,, \qquad t > t_0 \tag{1}$$

as a perturbation of the linear equation

$$z^{(n)} + a_1(t)z^{(n-1)} + \ldots + a_n(t)z = 0\,, \qquad t > t_0\,. \tag{2}$$

It is assumed throughout that $a_i \in C[t_0, \infty)$, $1 \leqslant i \leqslant n$. We give conditions which imply that (1) has a solution $\hat{y}$ which is defined on $[t_0, \infty)$ and behaves as $t \to \infty$ in some sense like $c\hat{z}$, where $\hat{z}$ is a given solution of (2) and c is a constant. Although much has been written on the existence of solutions with prescribed asymptotic behavior for nonlinear equations, almost all such results are «local» near infinity, in that the desired solutions are shown to exist only for t sufficiently large. Global conditions, i.e., conditions which imply the existence of solutions on the given interval $[t_0, \infty)$ are relatively rare (see, e.g., [3], [5], and [6]), and—as far as we

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know—confined to equations of the form

$$y^{(n)} + f(t, y) = 0 .$$

Here we give a global existence theorem for the general equation (1), which requires no assumptions concerning oscillation or nonoscillation of the solutions of (2). We also obtain from this theorem a result which applies specifically to the case where (2) is disconjugate on $[t_0, \infty)$.

2. – The main results.

We impose the following standing assumption on the nonlinear term in (1).

ASSUMPTION A. – The function $f\colon [t_0, \infty) \times \mathcal{R}^n \to \mathcal{R}$ is continuous and satisfies the inequality

$$|f(t, u_0, \ldots, u_{n-1})| \leqslant F(t, |u_0|, \ldots, |u_{n-1}|) , \tag{3}$$

where $F\colon [t_0, \infty) \times \mathcal{R}_+^n \to \mathcal{R}_+$ is continuous and $F(t, v_0, \ldots, v_{n-1})$ is nondecreasing in each v_r, $0 \leqslant r \leqslant n-1$, and satisfies one of the following hypotheses:

(H_1) For fixed $(t, v_0, \ldots, v_{n-1})$, $\lambda^{-1}F(t, \lambda v_0, \ldots, \lambda v_{n-1})$ is nondecreasing in λ for $\lambda > 0$, and

$$\lim_{\lambda \to +0} \lambda^{-1} F(t, \lambda v_0, \ldots, \lambda v_{n-1}) = 0 ; \tag{4}$$

or

(H_2) For fixed $(t, v_0, \ldots, v_{n-1})$, $\lambda^{-1}F(t, \lambda v_0, \ldots, \lambda v_{n-1})$ is nonincreasing in λ for $\lambda > 0$, and

$$\lim_{\lambda \to \infty} \lambda^{-1} F(t, \lambda v_0, \ldots, \lambda v_{n-1}) = 0 .$$

Hypotheses (H_1) and (H_2) were employed in [4] for the study of second order semilinear elliptic equations.

It will be convenient below to abbreviate

$$f\big(t, y(t), \ldots, y^{(n-1)}(t)\big) = (fy)(t) . \tag{5}$$

It is to be understood that all equations and inequalities involving t hold for $t \geqslant t_0$ unless otherwise specified, and that «o» and «O» have their standard meanings as $t \to \infty$.

Let $z_1, \ldots, z_n$ form a fundamental system for (2), and denote

$$w_i = \frac{W(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)}{W(z_1, \ldots, z_n)},$$

where $W(\varphi_1, \ldots, \varphi_m)$ is the Wronskian of $\varphi_1, \ldots, \varphi_m$. The following well known identities will be useful below:

$$\sum_{i=1}^{n} (-1)^{n-i} z_i^{(r)}(t) w_i(t) = \delta_{r,n-1}, \quad 0 \leqslant r \leqslant n-1 . \tag{6}$$

THEOREM 1. – *Let $\hat{z}$ be a given solution of* (2), *and suppose that there are positive continuous functions $\varrho_0, \ldots, \varrho_{n-1}$ on $[t_0, \infty)$ and an integer k, $1 \leqslant k \leqslant n$, such that*

$$|\hat{z}^{(r)}(t)| \leqslant \varrho_r(t), \quad 0 \leqslant r \leqslant n-1, \tag{7}$$

$$|z_i^{(r)}(t)| \int_{t_0}^{t} |w_i(s)| F(s, \lambda\varrho_0(s), \ldots, \lambda\varrho_{n-1}(s))\, ds = o(\varrho_r(t)), \quad 1 \leqslant i \leqslant k-1,\ 0 \leqslant r \leqslant n-1, \tag{8}$$

and

$$|z_i^{(r)}(t)| \int_{t}^{\infty} |w_i(s)| F(s, \lambda\varrho_0(s), \ldots, \lambda\varrho_{n-1}(s))\, ds = o(\varrho_r(t)), \quad k \leqslant i \leqslant n,\ 0 \leqslant r \leqslant n-1, \tag{9}$$

for $\lambda > 0$. Let θ be an arbitrary positive number, and suppose that c is a given constant. Then (1) *has a solution $\hat{y}$ on $[t_0, \infty)$ such that*

$$|\hat{y}^{(r)}(t) - c\hat{z}^{(r)}(t)| \leqslant \theta |c| \varrho_r(t), \quad 0 \leqslant r \leqslant n-1, \tag{10}$$

provided that $|c|$ is sufficiently small if (H_1) *holds, or sufficiently large if* (H_2) *holds. Moreover,*

$$\hat{y}^{(r)}(t) = c\hat{z}^{(r)}(t) + o(\varrho_r(t)), \quad 0 \leqslant r \leqslant n-1 . \tag{11}$$

PROOF. – It is convenient to define

$$\varphi_i(t, \lambda) = \int_{t_0}^{t} |w_i(s)| F(s, \lambda\varrho_0(s), \ldots, \lambda\varrho_{n-1}(s))\, ds, \quad 1 \leqslant i \leqslant k-1, \tag{12}$$

$$\varphi_i(t, \lambda) = \int_{t}^{\infty} |w_i(s)| F(s, \lambda\varrho_0(s), \ldots, \lambda\varrho_{n-1}(s))\, ds, \quad k \leqslant i \leqslant n, \tag{13}$$

and

$$\Phi_r(t, \lambda) = \sum_{i=1}^{n} |z_i^{(r)}(t)| \varphi_i(t, \lambda), \quad 0 \leqslant r \leqslant n-1 . \tag{14}$$

Then (8) and (9) imply that

$$\Phi_r(t,\lambda)=o(\varrho_r(t)), \quad 0\leqslant r\leqslant n-1. \tag{15}$$

Now let γ be an arbitrary positive number. Then

$$\varphi_r(t,\lambda)\leqslant\gamma\lambda\varrho_r(t), \quad 0\leqslant r\leqslant n-1, \tag{16}$$

for λ sufficiently small if (H_1) holds. To see this, let λ_0 be an arbitrary positive number, and choose $T\geqslant t_0$ such that

$$\Phi_r(t,\lambda_0)\leqslant\gamma\lambda_0\varrho_r(t), \quad t\geqslant T,\ 0\leqslant r\leqslant n-1. \tag{17}$$

(This is possible because of (15).) Since (H_1) implies that $\lambda^{-1}\Phi_r(t,\lambda)$ is nondecreasing in λ, (17) implies (16) for $t\geqslant T$ and $0<\lambda<\lambda_0$. If $t_0\leqslant t\leqslant T$, then (12), (13), and (14) imply that

$$\Phi_r(t,\lambda)\leqslant\sum_{i=1}^{k-1}|z_i^{(r)}(t)|\varphi_i(T,\lambda)+\sum_{i=k}^{n}|z_i^{(r)}(t)|\varphi_i(t_0,\lambda), \quad 0\leqslant r\leqslant n-1. \tag{18}$$

From (4) and Lebesgue's bounded convergence theorem,

$$\lim_{\lambda\to+0}\lambda^{-1}\varphi_i(\tau,\lambda)=0, \quad 1\leqslant i\leqslant k, \tag{19}$$

for any fixed $\tau\geqslant t_0$. Since the functions $\varrho_r^{-1}|z_i^{(r)}|$ $(0\leqslant r\leqslant n-1,\ 1\leqslant i\leqslant k)$ are all bounded on $[t_0,T]$, (18) and (19) now imply that if λ is sufficiently small, then (16) also holds on $[t_0,T]$.

A similar argument shows that (H_2) implies (16) for sufficiently large λ.

We will now use the Schauder-Tychonoff theorem to obtain $\hat{y}$ as a fixed point (function) of the transformation $\mathcal{C}$ defined by

$$\begin{aligned}(\mathcal{C}y)(t)=c\hat{z}(t)&-\sum_{i=1}^{k-1}(-1)^{n-i}z_i(t)\int_{t_0}^{t}w_i(s)(fy)(s)\,ds\\&+\sum_{i=k}^{n}(-1)^{n-i}z_i(t)\int_{t}^{\infty}w_i(s)(fy)(s)\,ds\end{aligned}$$

(recall (5)) on a suitable subset of $C^{(n-1)}[t_0,\infty)$. Let $C^{(n-1)}[t_0,\infty)$ be given the topology of uniform convergence on finite intervals; i.e., $y_n\to y$ means that $\lim_{n\to\infty}y_n^{(r)}(t)=y^{(r)}(t)$, $0\leqslant r\leqslant n-1$, where the convergence is uniform on $[t_0,T]$ for every $T\geqslant t_0$. For a given constant c, let V be the closed convex subset of $C^{(n-1)}[t_0,\infty)$ defined by

$$V=\{y\in C^{(n-1)}[t_0,\infty):\ |y^{(r)}(t)-c\hat{z}^{(r)}(t)|\leqslant\theta|c|\varrho_r(t),\ 0\leqslant r\leqslant n-1\}. \tag{20}$$

Because of (3) and the convergence of the integrals in (9) for all λ, $\mathcal{T}y$ is defined on $[t_0, \infty)$ if $y \in V$; moreover, the identities (6) and the fact that $z_1, \ldots, z_n$ are solutions of (2) imply that

$$(\mathcal{T}y)^{(r)}(t) = c\hat{z}^{(r)}(t) - \sum_{i=1}^{k-1} (-1)^{n-i} z_i^{(r)}(t) \int_{t_0}^{t} w_i(s)(fy)(s)\, ds + \sum_{i=k}^{n} (-1)^{n-i} z_i^{(r)}(t) \int_{t}^{\infty} w_i(s)(fy)(s)\, ds\,, \quad 0 \leqslant r \leqslant n-1\,, \tag{21}$$

and that

$$(\mathcal{T}y)^{(n)}(t) = -\sum_{j=1}^{n} a_j(t) y^{(n-j)}(t) - f\big(t, y(t), \ldots, y^{(n-1)}(t)\big)\,. \tag{22}$$

From (14) with $\lambda = |c|(1+\theta)$ and (21),

$$|(\mathcal{T}y)^{(r)}(t) - c\hat{z}^{(r)}(t)| \leqslant \varphi_r\big(t, |c|(1+\theta)\big)\,, \quad 0 \leqslant r \leqslant n-1,\ y \in V\,. \tag{23}$$

Therefore, to guarantee that $\mathcal{T}y \in V$ whenever $y \in V$ (see (20)), we have only to choose c so that

$$\Phi_r\big(t, |c|(1+\theta)\big) \leqslant |c|\theta\varrho_r(t)\,, \quad 0 \leqslant r \leqslant n-1\,,$$

which is possible for $|c|$ sufficiently small if (H_1) holds, or for $|c|$ sufficiently large if (H_2) holds. (See (16) with $\gamma = \theta/(1+\theta)$ and $\lambda = |c|(1+\theta)$.)

Having chosen c in this way, we have $\mathcal{T}(V) \subset V$. We now show that $\mathcal{T}$ is continuous on V. To this end, suppose that $\{y_n\}$ is a sequence in V such that $y_n \to y$. We must show that

$$\mathcal{T}y_n \to \mathcal{T}y\,. \tag{24}$$

From (21), if $T \geqslant t_0$, then

$$|(\mathcal{T}y_n)^{(r)}(t) - (\mathcal{T}y)^{(r)}(t)| \leqslant \sum_{i=1}^{k-1} |z_i^{(r)}(t)| \int_{t_0}^{t} |w_i(s)||(fy_n)(s) - (fy)(s)|\, ds + \sum_{i=k}^{n} |z_i^{(r)}(t)| \int_{t_0}^{\infty} |w_i(s)||(fy_n)(s) - (fy)(s)|\, ds\,, \quad t_0 \leqslant t \leqslant T,\ 0 \leqslant r \leqslant n-1\,. \tag{25}$$

The integrands on the right of (25) converge pointwise to zero as $t \to \infty$, and they are respectively dominated by

$$2|w_i(s)|F\big(s, |c|(1+\theta)\varrho_0(s), \ldots, |c|(1+\theta)\varrho_{n-1}(s)\big)\,, \quad 1 \leqslant i \leqslant n\,.$$

Our integrability conditions on F and Lebesgue's dominated convergence theorem imply that the integrals in (25) converge to zero as $n \to \infty$. This implies (24).

From (23), the families

$$\{(\mathcal{T}y)^{(r)}: y \in V\}, \quad 0 \leqslant r \leqslant n-1, \tag{26}$$

are all uniformly bounded on finite subintervals of $[t_0, \infty)$. Moreover, (3), (20), and (22) imply that

$$|(\mathcal{T}y)^{(n)}(t)| \leqslant |c|(1+\theta) \sum_{j=1}^{n} |a_j(t)|\varrho_{n-j}(t) + F(t, |c|(1+\theta)\varrho_0(t), \ldots, |c|(1+\theta)\varrho_{n-1}(t)), \qquad y \in V,$$

which, together with (23), implies that the families (26) are also equicontinuous on finite subintervals of $[t_0, \infty)$. This and the Ascoli-Arzela theorem imply that $\mathcal{T}(V)$ has compact closure, which completes the verification of the hypotheses of the Schauder-Tychonoff theorem. Therefore, $\mathcal{T}\hat{y} = \hat{y}$ for some $\hat{y}$ in V. That $\hat{y}$ satisfies (1), (10), and (11) can be seen from (22) and (23) with $y = \mathcal{T}y = \hat{y}$, and (15). This completes the proof.

REMARK 1. – As will be seen in Example 1, (23) may yield estimates of $\hat{y}^{(r)}(t) - c\hat{z}^{(r)}(t)$ as $t \to \infty$ which are sharper than (11).

Theorem 1 implies and extends Theorem 1 of [3].

We now apply Theorem 1 to the case where (2) is disconjugate; i.e., none of its nontrivial solutions has more than $n-1$ zeros, counting multiplicities, on $[t_0, \infty)$. Then it is possible to choose a fundamental system $z_1, \ldots, z_n$ with the properties assumed in the following theorem. (For a convenient reference for this statement, see [7, Lemma 1]; however, it is clearly implicit in the earlier papers of HARTMAN [1] and WILLETT [8].)

THEOREM 2. – *Suppose that the fundamental system $z_1, \ldots, z_n$ for (2) is such that*

$$z_i > 0, \quad w_i > 0, \quad 1 \leqslant i \leqslant n, \tag{27}$$

$$\left(\frac{w_i}{w_j}\right)' > 0, \quad 1 \leqslant i < j \leqslant n, \tag{28}$$

and

$$\lim_{t\to\infty} \frac{w_j(t)}{w_i(t)} = \lim_{t\to\infty} \frac{z_i(t)}{z_j(t)} = 0, \quad 1 \leqslant i < j \leqslant n. \tag{29}$$

Let k be an integer, $1 \leqslant k \leqslant n$, define

$$v_{rk} = w_k^{-1} \sum_{i=1}^{n} w_i |z_i^{(r)}|, \quad 0 \leqslant r \leqslant n-1, \tag{30}$$

and suppose that

$$\int_{t_0}^{\infty} w_k(s)F\big(s, \lambda v_{0k}(s), \dots, \lambda v_{n-1,k}(s)\big)\, ds < \infty , \qquad \lambda > 0 . \tag{31}$$

Let θ be an arbitrary positive number. Then (1) *has a solution y_k on $[t_0, \infty)$ such that*

$$|y_k^{(r)}(t) - cz_k^{(r)}(t)| \leqslant \theta |c| v_{rk}(t) , \qquad 0 \leqslant r \leqslant n-1 ,$$

and

$$y_k^{(r)}(t) = cz_k^{(r)}(t) + o\big(v_{rk}(t)\big) , \qquad 0 \leqslant r \leqslant n-1 ,$$

provided that $|c|$ is sufficiently small if (H_1) *holds, or sufficiently large if* (H_2) *holds.*

PROOF. - We apply Theorem 1 with $\hat{z} = z_k$ and $\varrho_r = v_{rk}$. Then (30) obviously implies (7). Because of (27), (28), and (29), it is easily inferred from (31) that

$$\int_{t_0}^{t} w_i(s)F\big(s, \lambda v_{0k}(s), \dots, \lambda v_{n-1,k}(s)\big)\, ds = o\big(w_i(t)/w_k(t)\big) , \qquad 1 \leqslant i \leqslant k-1 ,$$

and

$$\int_{t}^{\infty} w_i(s)F\big(s, \lambda v_{0k}(s), \dots, \lambda v_{n-1,k}(s)\big)\, ds = o\big(w_i(t)/w_k(t)\big) , \qquad k \leqslant i \leqslant n .$$

The last two equations imply (8) and (9) with $\varrho_r = v_{rk}$; hence, Theorem 1 implies the conclusion.

COROLLARY 1. - *Suppose that k is an integer, $1 \leqslant k \leqslant n$, such that*

$$\int_{t_0}^{\infty} t^{n-k} F(t, Mt^{k-1}, Mt^{k-2}, \dots, M, Mt^{-1}, \dots, Mt^{-n+k})\, dt < \infty , \qquad M > 0 . \tag{32}$$

Let μ be an arbitrary positive number. Then the equation

$$y^{(n)} + f(t, y, \dots, y^{(n-1)}) = 0 , \qquad t > t_0 > 0 ,$$

has a solution y_k on $[t_0, \infty)$ such that

$$|y_k^{(r)}(t) - ct^{k-r-1}/(k-r-1)!| < \mu |c| t^{k-r-1} , \qquad 0 \leqslant r \leqslant k-1 , \tag{33}$$

$$|y_k^{(r)}(t)| \leqslant \mu |c| t^{k-r-1} , \qquad k \leqslant r \leqslant n-1 , \tag{34}$$

and

$$y_k^{(r)}(t) = \begin{cases} (c + o(1))t^{k-r-1}/(k-r-1)!\,, & 0 \leqslant r \leqslant k-1\,, \\ o(t^{k-r-1})\,, & k \leqslant r \leqslant n-1\,, \end{cases} \tag{35}$$

provided that $|c|$ *is sufficiently small if* (H_1) *holds, or sufficiently large if* (H_2) *holds.*

PROOF. – In this case we can take $z_i(t) = t^{i-1}/(i-1)!$ and $w_i(t) = t^{n-i}/(n-i)!$. Then $v_{rk}(t) = c_{rk}t^{k-r-1}$, where

$$c_{rk} = (n-k)! \sum_{i=r+1}^{n} \frac{1}{(n-i)!(i-r-1)!}\,.$$

Now choose θ so that $c_{rk}\theta < \mu$ $(0 \leqslant r \leqslant n-1)$, and apply Theorem 2.

Corollary 1 extends Theorem 1 of [3].

REMARK 2. – Trivial modifications of the proofs show that Theorems 1 and 2 and Corollary 1 still hold under (H_1) if the integrability conditions on F (i.e., (8) and (9) for Theorem 1, (31) for Theorem 2, and (32) for Corollary 1) are assumed only for sufficiently small λ (or M in (32)).

3. – Examples.

In this section we apply our results to equations of the form (1) with

$$f(t, u_0, \ldots, u_{n-1}) = \sum_{r=0}^{n-1} p_r(t)(u_r)^{\gamma_r}\,, \tag{36}$$

where $p_0, \ldots, p_{n-1} \in C[t_0, \infty)$ and $\gamma_0, \ldots, \gamma_{n-1}$ are positive rationals with odd denominators, so that f is real-valud for all $(t, u_0, \ldots, u_{n-1})$ with $t \geqslant t_0$. (We depart slightly from these conventions in Example 2.) Clearly (36) implies (3) with

$$F(t, v_0, \ldots, v_{n-1}) = \sum_{r=0}^{n-1} |p_r(t)| v_r^{\gamma_r}\,,$$

and (H_1) holds if $\gamma_r > 1$ $(0 \leqslant r \leqslant n-1)$, while (H_2) holds if $0 < \gamma_r < 1$ $(0 \leqslant r \leqslant n-1)$.

EXAMPLE 1. – The equation

$$z''' - z'' + z' - z = 0$$

has the fundamental system

$$z_1(t) = \cos t\,, \quad z_2(t) = \sin t\,, \quad z_3(t) = e^t\,, \tag{37}$$

and it is easily verified that

$$w_1(t) = \frac{\cos t - \sin t}{2}, \qquad w_2(t) = \frac{\cos t + \sin t}{2}, \qquad w_3(t) = -\frac{1}{2}e^{-t}.$$

Applying Theorem 1 to the perturbed equation

$$y''' - y'' + y' - y + p_0(t)y^{\gamma_0} + p_1(t)(y')^{\gamma_1} + p_2(t)(y'')^{\gamma_2} = 0 \tag{38}$$

yields the following results.

(*a*) If p_0, p_1, and p_2 are either bounded or absolutely integrable on $[t_0, \infty)$ and

$$\max(\gamma_0, \gamma_1, \gamma_2) = \gamma < 1, \tag{39}$$

then (38) has a solution $\hat{y}$ on $[t_0, \infty)$ such that

$$\hat{y}^{(r)}(t) = ce^t + O(e^{\gamma t}), \qquad r = 0, 1, 2,$$

provided that $|c|$ is sufficiently large.

(*b*) If p_0, p_1, and p_2 are absolutely integrable on $[t_0, \infty)$ and δ is a fixed real number, then (38) has a solution $\hat{y}$ on $[t_0, \infty)$ such that

$$\hat{y}^{(r)}(t) = \begin{cases} c\sin(t+\delta) + o(1), & r = 0, \\ c\cos(t+\delta) + o(1), & r = 1, \\ -c\sin(t+\delta) + o(1), & r = 2, \end{cases}$$

provided that (39) holds and $|c|$ is sufficiently large, or that

$$\min(\gamma_0, \gamma_1, \gamma_2) > 1$$

and $|c|$ is sufficiently small.

PROOF. – (*a*) Let $k = 3$ and $\hat{z}(t) = \varrho_0(t) = \varrho_1(t) = \varrho_2(t) = e^t$. Then our assumptions imply that the integrals in (8) are $O(e^{\gamma t})$ for $i = 1, 2$, and that the integral in (9) is $O(e^{(\gamma-1)t})$ for $k = 3$. From this and (37),

$$\Phi_r(t, \lambda) = O(e^{\gamma t}) = o(e^t), \qquad r = 0, 1, 2$$

(see (15)); hence, Theorem 1 and Remark 1 imply the conclusion.

(*b*) Let $k = 1$, $\hat{z}(t) = \sin(t + \delta)$, and $\varrho_r(t) = 1$, $r = 0, 1, 2$. Then our assumptions imply (9) for $i = 1, 2, 3$, and Theorem 1 implies the conclusion.

EXAMPLE 2. – Consider the elliptic equation

$$\Delta u + m^2 u + \varphi(|x|)|u|^\alpha + \psi(|x|)|\nabla u|^\beta = 0 \tag{40}$$

in the exterior domain

$$\Omega_R = \{x \in \mathcal{R}^3 \colon |x| > R\}\,, \qquad R > 0\,, \tag{41}$$

where m, α, and β are positive constants and $\varphi, \psi \colon [R, \infty) \to \mathcal{R}$ are continuous functions. A radially symmetric function $u = u(|x|)$ is a solution of (40) if and only if the function $y(t) = tu(t)$ is a solution of the ordinary differential equation

$$y'' + m^2 y + t^{1-\alpha}\varphi(t)|y|^\alpha + t^{1-2\beta}\psi(t)|y - ty'|^\beta = 0\,, \qquad t > R\,, \tag{42}$$

which can be regarded as a perturbation of the linear equation

$$z'' + m^2 z = 0\,, \tag{43}$$

with

$$f(t, u_0, u_1) = t^{1-\alpha}\varphi(t)|u_0|^\alpha + t^{1-2\beta}\psi(t)|u_0 - tu_1|^\beta\,.$$

Here (3) holds, with

$$F(t, v_0, v_1) = t^{1-\alpha}|\varphi(t)|v_0^\alpha + t^{1-2\beta}|\psi(t)||v_0 + tv_1|^\beta\,,$$

which satisfies (H_1) if $\alpha, \beta > 1$, or (H_2) if $\alpha, \beta < 1$. For (43), we take $z_1(t) = mw_2(t) = \cos mt$ and $z_2(t) = mw_1(t) = \sin mt$. If δ is a fixed real number, then applying Theorem 1 with $k = 1$, $z(t) = \cos(mt + \delta)$, $\varrho_0(t) = 1$, and $\varrho_1(t) = m$ shows that if

$$\int_R^\infty t^{1-\alpha}|\varphi(t)|\,dt < \infty \quad \text{and} \quad \int_R^\infty t^{1-\beta}|\psi(t)|\,dt < \infty\,,$$

then (42) has a solution $\hat{y}$ on $[R, \infty)$ such that

$$\hat{y}(t) = c\cos(mt + \delta) + o(1)\,,$$

provided that $|c|$ is sufficiently small if $\alpha, \beta > 1$, or sufficiently large if $\alpha, \beta < 1$. This implies that (40) has a solution $\hat{u}$ on Ω_R such that

$$\lim_{|x|\to\infty}\Big(|x|\hat{u}(|x|) - c\cos(m|x| + \delta)\Big) = 0\,.$$

Note that $\hat{u}$ is oscillatory with respect to $|x|$. It would be of interest to develop a theory which establishes the existence of oscillatory solutions for second order elliptic equations on unbounded domains.

EXAMPLE 3. – Consider the equation

$$y^{(n)} + \sum_{r=0}^{n-1} p_r(t)(y^{(r)})^{\gamma_r} = 0 \,, \tag{44}$$

where $p_r \in C[t_0, \infty)$, $0 \leqslant r \leqslant n-1$, and

$$\int_{t_0}^{\infty} t^{n-k+\gamma_r(k-r-1)} |p_r(t)|\, dt < \infty \,, \qquad 0 \leqslant r \leqslant n-1 \,,$$

for some integer k, $1 \leqslant k \leqslant n$. Suppose that $\mu > 0$. Then Corollary 1 implies that (44) has a solution y_k on $[t_0, \infty)$ which satisfies (33), (34), and (35) if

$$\min(\gamma_0, \ldots, \gamma_{n-1}) > 1$$

and $|c|$ is sufficiently small, or if

$$\max(\gamma_0, \ldots, \gamma_{n-1}) < 1$$

and $|c|$ is sufficiently large.

EXAMPLE 4. – Consider the $2m$-th order elliptic equation

$$\Delta^m u + \sum_{l=0}^{m-1} \varphi_l(|x|)(\Delta^l u)^{\gamma_l} = 0 \tag{45}$$

in the exterior domain (41), where $\varphi_l \colon [R, \infty) \to \mathfrak{R}$, $0 \leqslant l \leqslant m-1$, are continuous. It is easy to see that a radially symmetric function $u = u(|x|)$ is a solution of (45) in Ω_R if and only if

$$t^{-1}(tu(t))^{(2m)} + \sum_{l=0}^{m-1} \varphi_l(t)[t^{-1}(tu(t))^{(2l)}]^{\gamma_l} = 0 \,, \qquad t > R \,,$$

or, equivalently, if and only if $y(t) = tu(t)$ is a solution of

$$y^{(2m)} + \sum_{l=0}^{m-1} t^{1-\gamma_l}\varphi_l(t)(y^{(2l)})^{\gamma_l} = 0 \,, \qquad t > R \,. \tag{46}$$

Corollary 1 implies that if k is an integer, $1 \leqslant k \leqslant 2m$, and

$$\int_{R}^{\infty} t^{2m-k+\gamma_l(k-2l-1)} |\varphi_l(t)|\, dt < \infty \,, \qquad 0 \leqslant l \leqslant m-1 \,,$$

then (44) has a solution y_k on $[R, \infty)$ which satisfies (35) (with $n = 2m$), provided that $|c|$ is sufficiently small if $\gamma_l > 1$ $(0 \leqslant l \leqslant m-1)$, or sufficiently large if $\gamma_l < 1$ $(0 \leqslant l \leqslant m-1)$. This implies that (45) has a solution u_k on Ω_R such that

$$\lim_{|x|\to\infty} |x|^{-k+2} u_k(|x|) = C\,,$$

under the same conditions on C.

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