Hilbert Functions and Betti Numbers in a Flat Family (*).

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Summary. – *This paper is dedicated to the study of Hilbert functions and Betti numbers of the projective varieties in a flat family. We prove that the Hilbert function $H(X_y, n)$, $y \in Y$—a parameter scheme—is lower semicontinuous for any fixed n. In case Y is integral and noetherian we obtain the well-known fact that the set $V \subset Y$ where $H(X_y, n)$ is maximal for all n's is open and nonempty. We show also that $b_i(X_y)$—the i-th Betti number of X_y—is upper semicontinuous for $y \in V$. The paper contains also a number of results concerning the relations among the various Betti numbers.*

Introduction.

Let $f\colon X \to Y$ be a flat family of closed subschemes of a fixed projective space. The aim of this paper is to study how the Hilbert functions and the Betti numbers of the fibers X_y vary with respect to $y \in Y$.

In § 1 we show that the Hilbert function $H(X_y, n)$ is lower semicontinuous on Y for any fixed n (Prop. 1.5). When Y is integral and noetherian this implies the well known fact that subset V of Y where $H(X_y, n)$ is maximal for all n's is open and non-empty (Prop. 1.7) This set V is particularly important, because over it the ideals of the fibers are « well behaved » (Th. 1.10), in a sense which is essential later. All these facts are easy consequences of the semicontinuity theorem for cohomology in a flat family.

In § 2, which is, but for the geometric applications, independent of § 1, we deal with a finitely generated, graded $R[X_0, ..., X_r]$—module E, flat over R, and we study the Betti numbers of the fibers $E \otimes k(\mathfrak{p})$, as $\mathfrak{p}$ varies in spec (R). We show first that these numbers do not decrease under specialization (Cor. 2.6), and that they are upper semicontinuous (Th. 2.12). These facts, applied to homogeneous ideals of $R[X_0, ..., X_r]$, in connection with the results of § 1, show that the Betti numbers $b_i(X_y)$ of X_y are upper semicontinuous on V—a not too unexpected fact (Prop. 2.15). This is obviously false on the whole of Y (see 2.16).

A number of other results are proved, mainly concerned with the relations among the various Betti numbers. For example, it is shown that if $b_1(X_y)$ is minimum

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so is $b_0(X_y)$ (always $y \in V$), while an example due to A. Geramita shows that $b_2(X_y)$ minimal does not necessarily imply that b_0 is minimal (see 2.17).

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Foreward.

In this paper all rings are commutative and noetherian, and all schemes are locally noetherian. Basic facts and definitions are recalled at the beginning of each section. For general reference see [H] and [M].

1. – Hilbert functions.

In this section we show that the Hilbert function in a flat family of projective schemes is lower semicontinuous (Prop. 1.5 and 1.7), and that the ideals of the family are well behaved in the set where the Hilbert function is maximal (Th. 1.10). The section is divided into three subsections: the first one contains preparatory material, the second deals with Hilbert functions and the third with ideals.

A. Preliminaries and notation.

1.1 Let k be any field and let $Z \subset \boldsymbol{P}^r_k = \boldsymbol{P}^r$ be a closed subscheme of projective r-dimensional space over k. Let $S = k[X_0, \dots, X_r]$ and let $J \subset S$ be the (saturated) graded ideal of Z. The *Hilbert function* of Z (with respect to the given embedding) is defined for all integers $n \geqq 0$, as

$$H(Z, n) = \dim_k S_n/J_n .$$

Now from the exact sequence of sheaves:

$$0 \to \tilde{J} \to \mathcal{O}_{\boldsymbol{P}^r} \to \mathcal{O}_Z \to 0 ,$$

twisting by $n \geqq 0$ and taking cohomology one gets the exact sequence:

$$0 \to H^0(\boldsymbol{P}^r, \tilde{J}(n)) \to H^0(\boldsymbol{P}^r, \mathcal{O}_{\boldsymbol{P}^r}(n)) \to H^0(Z, \mathcal{O}_Z(n)) \to H^1(\boldsymbol{P}^r, \tilde{J}(n)) \to 0 ;$$

and since $H^0(\boldsymbol{P}^r, \tilde{J}(n)) = J_n$, $H^0(\boldsymbol{P}^r, \mathcal{O}_{\boldsymbol{P}^r}(n)) = S_n$ one gets:

$$H(Z, n) = h^0(\mathcal{O}_{\boldsymbol{P}^r}(n)) - h^0(\tilde{J}(n), H(Z, n)) = h^0(\mathcal{O}_Z(n)) - h^1(\tilde{J}(n)) .$$

1.2 By a *family of projective schemes* $f: X \to Y$ we mean a morphism f of (locally noetherian) schemes which factors through a closed embedding $X \subset \boldsymbol{P}_Y^r = \boldsymbol{P}$. *Such an embedding will be fixed once and for all,* and its ideal sheaf will be denoted by $\mathcal{J}$. We denote by $g: \boldsymbol{P} \to Y$ the canonical projection.

For each $y \in Y$ the embedding $X \subset \boldsymbol{P}$ induces a closed immersion of the fiber X_y into the fiber $\boldsymbol{P}_y$, which is naturally isomorphic to the projective case $\boldsymbol{P}^r_{k(y)}$.

We denote by $I(X_y)$ the homogeneous ideal of this embedding, and by $H(X_y, n)$ the Hilbert function of X_y. Note that both $I(X_y)$ and $H(X_y, n)$ depend on the fixed embedding $X \subset \boldsymbol{P}$.

1.3 When Y is affine we always put $Y = \operatorname{spec}(R)$ and $\boldsymbol{P} = \operatorname{Proj}(A)$ where $A = R[X_0, \ldots, X_r]$ (graded with respect to the variables). Thus $X = \operatorname{Proj}(A/I)$ where I is a graded saturated ideal of A. If $y \in Y$ we then have $\boldsymbol{P}_y = \operatorname{Proj}(A \otimes k(y))$, $X_y = \operatorname{Proj}((A/I) \otimes k(y))$ where all the tensor products are over R.

Note that in general $I \otimes k(y)$ *is not* equal to $I(X_y)$; however the image of the canonical map $I \otimes k(y) \to A \otimes k(y)$ is a graded ideal corresponding to the embedding $X_y \subset \boldsymbol{P}_y$.

1.4 We recall that if $f: X \to Y$ is a projective morphism of noetherian schemes and $\mathcal{F}$ is a coherent sheaf over X which is also f-flat, then the functions $y \mapsto \dim_{k(y)}(H^i(X_y, F_y))$ are upper semicontinuous on Y (see [H], III, 12.8).

In particular the sets $\{y \in Y: \dim_{k(y)} H^i(X_y, \mathcal{F}_y) \text{ is minimal}\}$ are open in Y; and if moreover Y is irreducible with generic point y_0 then

$$\dim_{k(y)} H^i(X_y, \mathcal{F}_y) \geq \dim_{k(y_0)} H^i(X_{y_i}, \mathcal{F}_{y_0}) \quad \text{for all } y \in Y .$$

B. Semicontinuity of the Hilbert functions.

In this subsection we show that given a flat family of projective schemes $X \to Y$, the function $H(X_y, n)$ is lower semicontinuous on Y for each n, and we characterize, under suitable assumptions, the set $V \subset Y$ where it reaches the maximum for all n's.

1.5 PROPOSITION. – Let $f: X \to Y$ be a flat family of projective schemes (see 1.2), and let n be a fixed integer. Then the function from Y to $\boldsymbol{N}$ defined by: $y \mapsto H(X_y, n)$ is lower semicontinuous.

PROOF. – Let $y \in Y$, and let $\mathcal{J}$ and g be as in 1.2. From the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_{\boldsymbol{P}} \to \mathcal{O}_X \to 0 \tag{1}$$

tensoring with $k(y)$ and using the g-flatness of $\mathcal{O}_X$ we get the exact sequence

$$0 \to \mathcal{J}_y \to \mathcal{O}_{\boldsymbol{P}_y} \to \mathcal{O}_{X_y} \to 0$$

which shows that $\mathcal{J}_y = \mathcal{J} \otimes k(y)$ is the ideal sheaf of the embedding $X_y \subset \boldsymbol{P}_y$. Hence by 1.1 we have:

$$H(X_y, n) = h^0(\mathcal{O}_{\boldsymbol{P}_y}(n)) - h^0(\mathcal{J}_y(n)) .$$

Now by (1) and the g-flatness of $\mathcal{O}_X$ we have that $\mathcal{J}(n)$ is g-flat; hence we can apply the semicontinuity theorem (see 1.4) to show that the function $h^0(\mathcal{J}_y(n))$ is upper semicontinuous on Y. The conclusion follows from the fact that $h^0(\mathcal{O}_{\boldsymbol{P}_y}(n))$ is constant on Y.

1.6 COROLLARY. - Let the assumptions be as in 1.5. Then:

(*a*) the set $V_n = \{y \in Y: H(X_y, n) \text{ is maximum}\}$ is open in Y;

(*b*) if $y, y_0 \in Y$ and y is a specialization of y_0, then $H(X_y, n) \leq H(X_{y_0}, n)$;

(*c*) if Y is irreducible with generic point y_0 then $H(X_y, n) \leq H(X_{y_0}, n)$ for all $y \in Y$.

PROOF. - It follows from 1.5 and 1.4.

1.7 PROPOSITION. - Let the assumptions be as in 1.5, and assume further that Y is integral and noetherian. Then the set $V = \{y \in Y: H(X_y, n) \text{ is maximum for all } n\}$ is open and non-empty.

PROOF. - For each n let $V_n = \{y \in Y: H(X_y, n) \text{ is maximum}\}$. Since V_n is open by 1.6 it is sufficient to show that $V_n = Y$ for $n \gg 0$.

Now for each n the sheaves $\mathcal{O}_X(n)$ are g-flat and hence by 1.4 and quasi-compactness we get that $H^i(X_y, \mathcal{O}_{X_y}(n)) = 0$ for all $i > 0$, all $y \in Y$ and $n \gg 0$. Since $\mathcal{J}(n)$ is g-flat we can apply the same kind of argument to $H^1(X_y, \mathcal{J}_y(n))$ and in view of the remark made in 1.1 we obtain $H(X_y, n) = h^0(X_y, \mathcal{O}_{X_y}(n)) = \chi(\mathcal{O}_{X_y}(n))$ for all $y \in Y$ and $n \gg 0$. But since f is flat, $\chi(\mathcal{O}_{X_y}(n))$ is constant on Y (see [H], III, 9.9.) and the conclusion follows.

Now we want to give a better description of the open set V, under suitable assumptions on the fibers of f. For this consider the exact sequence of graded $\mathcal{O}_Y$-modules

$$0 \to \bigoplus_n S^n(\mathcal{O}_Y^{r+1})/g_*(\mathcal{J}(n)) \xrightarrow{\varphi} \bigoplus_n f_*(\mathcal{O}_X(n)) \to \bigoplus_n \mathcal{K}_n \to 0 .$$

$S^n(\mathcal{O}_Y^{r+1})$ denotes the n-th symmetric power of $\mathcal{O}_Y^{r+1}$, φ is the canonical map and $\mathcal{K} = \bigoplus \mathcal{K}_n$ is its cokernel. Notice that if $Y = \operatorname{spec} R$, the above exact sequence corresponds in each degree n to the exact sequence of R-modules (notation as in 1.3):

$$0 \to A_n/I_n \to H^0(X, \mathcal{O}_X(n)) \to K_n \to 0 \tag{2}$$

were $K_n = H^1(\boldsymbol{P}, \tilde{I}(n))$.

1.8 PROPOSITION. – Let $f: X \to Y$ be a flat family of projective schemes with Y integral and noetherian. Assume further that $H^1(X_y, \mathcal{O}_{X_y}(n)) = 0$ for all $y \in Y$ and all $n \geq 0$. Then $V = \{y \in Y: \mathcal{K}_y \text{ is } \mathcal{O}_{y,Y}\text{-free}\}$.

PROOF. – Since everything in local on Y we may assume that $Y = \operatorname{spec}(R)$ and use the notation as in 1.3. The assumption on the fibers implies (see [MF]) that $f_*(\mathcal{O}_X(n))$ is locally free over R (of constant rank r_n), and $H^0(X_y, \mathcal{O}_{X_y}(n)) = H^0(X, \mathcal{O}_X(n)) \otimes k(y)$ for all $n \geq 0$, and for all $y \in Y$. Tensoring (2) by $k(y)$ we get the exact sequence of $k(y)$-vector spaces:

$$A_n/I_n \otimes k(y) \to H^0(X, \mathcal{O}_X(n)) \otimes k(y) \to K_n \otimes k(y) \to 0 .$$

But $H^0(X, \mathcal{O}_X(n)) \otimes k(y) = H^0(X_y, \mathcal{O}_{X_y}(n))$, and the image of $A/I \otimes k(y)$ in $\bigoplus_n H^0(X_y, \mathcal{O}_{X_y}(n))$ is a coordinate ring of $X_y \subset \boldsymbol{P}_y$ (see 1.3); hence by 1.1 we have:

$$H(X_y, n) = \dim_{k(y)} H^0(X, \mathcal{O}_X(n)) \otimes k(y) - \dim_{k(y)} K_n \otimes k(y) = r_n - \dim_{k(y)} K_n \otimes k(y) .$$

In particular if y_0 is the generic point of Y, we have $H(X_{y_0}, n) = r_n - rk(K_n)$, and hence $H(X_y, n) = H(X_{y_0}, n)$ if and only if $rk(K_n) = \dim_{k(y)} K_n \otimes k(y)$, that is if and only if $(K_n)_y$ is free over R_y (see e.g. [H], II.8.9). The conclusion follows.

1.9 REMARKS. – (i) Proposition 1.7 remains true, with V possibly empty, also without the assumption « Y integral »; indeed it is easy to reduce the problem to the integral case by working on each irreducible component.

(ii) Proposition 1.8 can be applied for example, to families of zero dimensional schemes. We do not know whether a similar description of V can be given for more general families.

C. Ideals of the fibers.

In this subsection we show that the ideals of the fibers of a flat family have a good behaviour over the set V described in 1.7.

We use the notation stated in 1.2.

1.10 THEOREM. – Let $f: X \to Y$ be a flat family as in 1.2, with Y integral and noetherian and let $V \subset Y$ be as in 1.7. Then:

(*a*) $g_* \mathcal{I}(n)$ is locally free at all $y \in V$ and for all $n \geq 0$;

(*b*) $g_* \mathcal{I}(n) \otimes k(y) = I(X_y)_n$ for all $y \in V$ and all $n \geq 0$.

PROOF. – From the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_{\boldsymbol{P}} \to \mathcal{O}_X \to 0$ and the flatness of f we deduce that $\mathcal{I}(n)$ is g-flat for all $n \geq 0$. Moreover $H(X_y, n)$ is constant on V for all $n \geq 0$ (see 1.7), and hence the same is true for $\dim_{k(y)} H^0(\boldsymbol{P}_y, \mathfrak{J}_y(n))$, because $\mathfrak{J}_y$ is the ideal sheaf of the embedding $X_y \subset \boldsymbol{P}_y$ (see proof of 1.5).

Hence we can apply a theorem of Grauert ([H], III.12.9) which gives (a) and

$$(b') \quad g_* \mathfrak{J}(n) \otimes k(y) \to H^0(\boldsymbol{P}_y, \mathfrak{J}_y(n)) \text{ is an isomorphism for all } y \in V.$$

Since $\mathfrak{J}_y$ is the ideal sheaf of $X_y \subset \boldsymbol{P}_y$ it follows that $H^0(\boldsymbol{P}_y, \mathfrak{J}_y(n))$ is the n-th homogeneous component of the saturated graded ideal of the same embeddnng, namely $H^0(\boldsymbol{P}_y, \mathfrak{J}_y(n)) = I(X_y)_n$.

Thus (b) follows from (b') and the proof is complete.

1.11 REMARKS. – (i) It is not clear to us which is the subset of Y where $I(X_y) = \bigoplus_n g_*(\mathfrak{J}(n)) \otimes k(y)$. It is not even clear to us if this condition is open on Y.

(ii) Under the assumptions of Proposition 1.8 it is easy to show that the given family is *very flat* over V; this means, when Y is affine and the notation is as in 1.3, that A_n/I_n is flat at all points of V *for all* n (see [H], p. 266, 9.5).

It might be interesting to know whether this is true more generally, and which are the relations among being very flat, maximality of the Hilbert function, and good behaviour of the ideals of the fibers (see 1.10).

2. – Betti numbers.

In this section we show that given a flat irreducible family $f: X \to Y$ of projective schemes, the Betti numbers $b_i(X_y)$ (2.2) are upper semicontinuous as y varies in the set where th Hilbert function is constant (see 1.7). Our treatment is purely algebraic and independent of § 1, but for the geometric applications.

For a better exposition we divide this section into four subsections. After giving some definitions and notation in subsection A, we show that the Betti numbers cannot decrease under specialization (subsection B), and from this we deduce the semicontinuity theorem in subsection C. Geometric applications are given at the end.

A. Notation and definitions.

2.1 Let k be a field and let S be a finitely generated graded k-algebra. If M is a finitely generated graded S-module, the *Betti numbers* of M are defined as:

$$b_i(M) = \dim_k \operatorname{Tor}_i^S(k, M) \quad \text{ for } i = 0, 1, \ldots$$

where k is viewed as an S-module in the obvious way.

Clearly if S is a polynomial ring in $r + 1$ variables over k one has $b_i(M) = 0$ for $i \geq r + 2$.

One can also show, by using the graded Nakayama lemma, that M has a *minimal free resolution*

$$F^{\cdot}: \ldots \to F_i \to F_{i-1} \to \ldots \to F_0 \to M \to 0$$

i.e. a resolution where F_i is free graded and the maps $F_i \to F_{i-1}$ are zero modulo the irrelevant maximal ideal of S; this easily implies that $b_i(M) = rk(F_i)$ for all $i \geq 0$.

In particular the ranks of the F_i's are independent of the minimal resolution chosen.

Observe also that $b_0(M)$ is equal to the minimum number of homogeneous generators of M.

2.2 If Z is a closed subscheme of I and $J \subset k[X_0, \ldots, X_r]$ is the ideal of Z we put $b_i(Z) = b_i(J)$ (considering J as a graded $k[X_0, \ldots, X_r]$-module).

2.3 NOTATION. – (i) In the rest of this section R is a (noetherian commutative) ground ring, and all tensor products will be taken over R, unless differently written.

(ii) If R is a domain we always denote by K its field of fractions; when R is local we denote by k its residue field.

(iii) We shall denote by $A = R[X_0, \ldots, X_n]$ the ring of polynomials in $n+1$ variables, and by E a fixed finitely generated graded A-module.

Note that if $\mathfrak{p} \in \operatorname{spec}(R)$, $E \otimes k(\mathfrak{p})$ has a natural structure of a graded $k(\mathfrak{p})[X_0, \ldots, \ldots, X_n]$-module. We shall always consider this structure, and the Betti numbers $b_i(E \otimes k(\mathfrak{p}))$ are thus well defined.

(iv) Whenever M is a module over a graded ring we denote by $\mu(M)$ the minimal number of homogeneous generators of M (thus $\mu(M) = b_0(M)$ whenever $b_0(M)$ is defined).

B. Behaviour of the Betti numbers under specialization.

We want to study how $b_i(E \otimes k(\mathfrak{p}))$ changes when specializing $\mathfrak{p}$, and give conditions for it to remain constant. The key results are 2.5 and 2.10, from which a number of corollaries are deduced. Clearly, it is sufficient to deal with a local ground ring R; so in this subsection R is always assumed to be local, unless the contrary is stated.

2.4 LEMMA. – Let E be a finitely generated graded A-module, and assume R is a local ring. Then

$$\mu(E) = \mu(E \otimes k) = \dim_k (E \otimes_A k) .$$

PROOF. – By the graded Nakayama lemma and the local Nakayama lemma over R one has:

$$\mu(E) = \mu(E \otimes_A R) = \dim_k (E \otimes_A k) = \mu(E \otimes k) .$$

2.5 PROPOSITION. – Assume R is a local domain, and E is R-flat. For each i put

$$d_i = d_i(E) = b_i(E \otimes k) - b_i(E \otimes K) .$$

Then:

$$\sum_{i=0}^{r} (-1)^{r+i} d_i \geq 0 \quad \text{for all } r \geq 0 .$$

PROOF. – We use induction on r. If $r = 0$ we have to prove that $b_0(E \otimes k) \geq$ $\geq b_0(E \otimes K)$; but by 2.4 we have:

$$b_0(E \otimes k) = \mu(E) \geq \mu(E \otimes K) = b_0(E \otimes K)$$

and we are done in this case.

In order to be able to make the induction step we need some preliminaries. So let $0 \to N \to F \to E \to 0$ be an exact sequence of finitely generated graded A-modules, with F free. Since E is flat over R, we get an exact sequence of graded $k[\mathbf{X}]$-modules

$$0 \to N \otimes k \to F \otimes k \to E \otimes k \to 0 . \tag{3}$$

Tensoring (3) with k over $k[\mathbf{X}]$ and using the Tor-sequence we get an exact sequence of k-vector spaces

$$0 \to \operatorname{Tor}_1^{k[\mathbf{X}]}(k, E \otimes k) \to \overline{N \otimes k} \to \overline{F \otimes k} \to \overline{E \otimes k} \to 0$$

where the upper bars mean reduction modulo $(\mathbf{X})$. Likewise we get an exact sequence of K-vector spaces:

$$0 \to \operatorname{Tor}_1^{K[\mathbf{X}]}(K, E \otimes K) \to \overline{N \otimes K} \to \overline{F \otimes K} \to \overline{E \otimes K} \to 0 . \tag{5}$$

Taking the alternating sum of dimensions in (4) and (5) and using the obvious equality $\dim_k \overline{F \otimes k} = \dim_K \overline{F \otimes K}$ we get

$$d_1(E) - d_0(E) = d_0(N) . \tag{6}$$

Moreover by again using (3) and the Tor-sequence we have, for $i \geq 2$;

$$\operatorname{Tor}_i^{k[\mathbf{X}]}(k, E \otimes k) = \operatorname{Tor}_{i-1}^{k[\mathbf{X}]}(k, N \otimes k)$$

and similarly with k replaced by K. From this it follows:

$$d_i(E) = d_{i-1}(N) \quad \text{for } i \geq 2 . \tag{7}$$

Assume now $r > 0$. Since N is flat, we have, by induction,

$$\sum_{i=0}^{r-1} (-1)^{i+r-1} d_i(N) \geq 0$$

and the conclusion follows by using (6) and (7).

2.6 COROLLARY. – Let the assumptions be as in 2.5. Then we have:

(*a*) $d_s \geq \sum_{i=0}^{s-1} (-1)^{i+s-1} d_i \geq 0$ for all $s \geq 0$;

(*b*) $b_s(E \otimes k) \geq b_s(E \otimes K)$ for all $s \geq 0$;

(*c*) $\sum_{i=1}^{\infty} (-1)^i d_i = 0$.

PROOF. – (*a*) By 2.5, with $r = s$, we get the first inequality, while the second one follows again by 2.5 applied with $r = s - 1$.

(*b*) Follows from (*a*) and the definition of d_i.

(*c*) We have $b_i(E \otimes K) = 0 = b_i(E \otimes k)$ for $i \geq n + 2$; hence (*c*) follows from (*a*) applied with $s = n + 2$.

2.7 COROLLARY. – Let s be a non-negative integer and let the assumptions be as in 2.5. Then the following are equivalent:

(*a*) $d_s = 0$;

(*b*) $\sum_{i<s} (-1)^i d_i = 0 = \sum_{i>v} (-1)^i d_i$.

PROOF. – Assume (*a*). Then $\sum_{i<s} (-1)^i d_i = 0$ by 2.6 (*a*); from this and 2.6 (*c*) we have also $\sum_{i>v} (-1)^i d_i = 0$. The converse follows immediately from 2.6 (*c*).

2.8 COROLLARY. – Let the assumptions be as in 2.5. Then we have:

(*a*) If $b_1(E \otimes K) = b_1(E \otimes k)$, then $b_0(E \otimes K) = b_0(E \otimes k)$;

(*b*) If $b_h(E \otimes K) = b_h(E \otimes k)$ and $b_i(E \otimes k) = 0$ for $i \geq h + 2$ then $b_{h+1}(E \otimes K) = b_{h+1}(E \otimes k)$.

PROOF. – To prove (*a*) apply 2.7 with $s = 1$. To prove (*b*) observe that $b_i(E \otimes K) = 0$ for $i \geq h + 2$ by 2.6 (*b*), whence $d_i = 0$ for $i \geq h + 2$; the conclusion follows from 2.7 applied with $s = h$.

We have also the following well known:

2.9 COROLLARY. – Let the assumptions be as in 2.5. Then $rk(E\otimes k) = rk(E\otimes K)$.

PROOF. – By using minimal resolutions (see 2.1) one has:

$$rk(E\otimes k) = \sum_{i\geqslant 0} (-1)^i b_i(E\otimes k)$$

and similarly for $E\otimes K$. The conclusion then follows from 2.6 (c).

The next proposition gives a condition which is equivalent to the minimality of the first s Betti numbers, and should be compared with 2.7.

2.10. PROPOSITION. – Let the assumptions be as in 2.5. Then for any fixed integer s the following conditions are equivalent:

(a) $d_i = 0$ $\big($i.e. $b_i(E\otimes K) = b_i(E\otimes k)$ for $i \leq s\big)$.

(b) $\operatorname{Tor}_i^A(E, R)$ is R-free, for $i \leq s$.

(c) There is a free graded resolution of length $s+1$:

$$F'\colon F_{s+1} \to F_s \to \dots \to F_0 \to E \to 0$$

which is minimal with respect to $(\mathbf{X})$ $\big($i.e. the maps $F_i \otimes_A R \to F_{i-1}\otimes_A R$ are zero for $1 \leq i \leq s+1\big)$.

PROOF. – $(a) \Rightarrow (b)$. Since $d_0 = 0$, by 2.4 and the graded Nakayama's Lemma we have:

$$\dim_k (E\otimes_A R)\otimes_R k = \dim_K (E\otimes_A R)\otimes_R K\,.$$

and hence $E\otimes_A R$ is R-free for $i = 0$ and any s; thus if $s = 0$ there is nothing else to be proved.

Assume then $s > 0$, and let us proceed by induction on s. For this let $0 \to N \to F \to E \to 0$ be an exact sequence of graded A-modules, with F free and $\operatorname{rk}(F) = \mu(E)$, the minimal number of homogeneous generators of E. By flatness we have exact sequences:

$$0 \to N\otimes K \to F\otimes K \to E\otimes K \to 0 \tag{8}$$

$$0 \to N\otimes k \to F\otimes k \to E\otimes k \to 0\,. \tag{9}$$

Moreover by assumption and by 2.4 we have:

$$b_0(E\otimes K) = \operatorname{rk}(F\otimes K) \quad \text{and} \quad b_0(E\otimes k) = \operatorname{rk}(F\otimes k)\,.$$

Then the maps $F \otimes K \to E \otimes K \to 0$ and $F \otimes k \to E \otimes k \to 0$ give the beginning of a minimal resolution of $E \otimes K$ and of $E \otimes k$ respectively. Hence by (8), (9) and the assumption we have, for $i \leq s-1$:

$$b_i(N \otimes k) = b_{i+1}(E \otimes k) = b_{i+1}(E \otimes K) = b_i(N \otimes K) .$$

Now N is R-flat, and hence by the induction hypothesis $\mathrm{Tor}_i^A(N, R)$ is R-free for $0 \leq i \leq s-1$, and from Tor-sequence it follows that $\mathrm{Tor}_i^A(E, R)$ is R-free for $i \geq 2$.

Moreover we have the exact sequence

$$0 \to \mathrm{Tor}_1^A(E, R) \to N \otimes_A R \to F \otimes_A R \to E \otimes_A R \to 0$$

and since the last three R-modules are free we have that $\mathrm{Tor}_1^A(E, R)$ is free too.

(*b*) $\Rightarrow$ (*c*). Induction on s. If $s = 0$ we have, by assumption, that $E \otimes_A R$ is R-free of rank r, say. Then by 2.4, we have $\mu(E) = r$, and hence there is a free resolution $F_1 \to F_0 \to E \to 0$ of graded A-modules, where $\mathrm{rk}(F_0) = r$. Tensoring with R over A this gives an exact sequence $F_1 \otimes_A R \to F_0 \otimes_A R \to E \otimes_A R \to 0$. But $F_0 \otimes_A R$ and $E \otimes_A R$ are free of the same rank, and hence $F_0 \otimes_A R \to E \otimes_A R$ is an isomorphism, which implies that $F_1 \otimes R \to F_0 \otimes R$ is the zero map. Thus our claim is proved for $s = 0$.

To make the induction step consider the exact sequence $0 \to N \to F_0 \to E \to 0$ where $F_0 \to E$ is as before, and N is the kernel. Then N is flat over R and from the Tor-sequence and the assumption we have that $\mathrm{Tor}_i^A(N, R)$ is free for $1 \leq i \leq s-1$. Moreover in the exact sequence

$$0 \to \mathrm{Tor}_1^A(E, R) \to N \otimes_A R \to F_0 \otimes_A R \to E \otimes_A R \to 0$$

the first, third and fourth R-module are free.

Hence $N \otimes_A R$ in free as well, and we can apply the induction hypothesis to N. Then we then have a free resolution $F_{z+1} \to F_z \to \ldots \to F_1 \to N \to 0$ with all the required properties, and the conclusion follows.

(*c*) $\Rightarrow$ (*a*). By assumption and flatness of E it follows that $F \otimes K$ and $F \otimes k$ are minimal resolutions of $E \otimes K$ and $E \otimes k$ respectively; hence the conclusion.

2.11 REMARK. – The following fact holds: let the assumptions be as in 2.10, and assume further that $\mathrm{Tor}_i^A(E, R)$ is R-free for $i \leq s$.

Then $\mathrm{Tor}_i^{k[X]}(E \otimes k, k) = \mathrm{Tor}_i^A(E, R) \otimes k$ for $i \leq s$ (i.e. Tor_i « commutes with taking fibers »). This can be proved by using 2.10. We leave the details to the reader.

C. Semicontinuity of the Betti numbers.

In this subsection we show that the Betti numbers are upper semicontinuous

2.12 THEOREM. – Let the notation be as in 2.3 and assume E is R-flat. Let i, a be two integers. Then the sets

$$B_i^a = \{\mathfrak{p} \in \operatorname{spec}(R) | b_i(E \otimes k(\mathfrak{p})) \leq a\}$$

are open in spec (R).

PROOF. – By a well known lemma of Nagata it is sufficient to prove (see [M], p. 156, (22.*B*)):

(*a*) If $\mathfrak{p} \in B_i^a$ and $\mathfrak{q} \subset \mathfrak{p}$, then $\mathfrak{q} \in B_i^a$.

(*b*) If $\mathfrak{p} \in B_i^a$ then $B_i^a \cap \operatorname{spec}(R/\mathfrak{p})$ contains a non empty open subset of spec $(R/\mathfrak{p})$.

Now (*a*) is an easy consequence of 2.6 (*b*), because the fibers do not change after the base change $R \to R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}$.

In order to prove (*b*) we may assume $\mathfrak{p} = 0$ and it is sufficient to show that for all $s \geq 0$ the sets $U_s = \{\mathfrak{q} \in \operatorname{spec}(R) | b_i(E \otimes k(\mathfrak{q})) = b_i(E \otimes K)\}$ are open (here, as usual, K is the quotient field of R). But for all $\mathfrak{q} \in \operatorname{spec}(R)$ and all $i \geq 0$ there are somorphism $\operatorname{Tor}_i^A(E, R) \otimes R_\mathfrak{q} = \operatorname{Tor}_i^{A \otimes R_\mathfrak{q}}(E \otimes R_\mathfrak{q}, R_\mathfrak{q})$, and hence by 2.10 we have

$$U_s = \{\mathfrak{q} \in \operatorname{spec}(R) | \operatorname{Tor}_i^A(E, R) \otimes R_\mathfrak{q} \text{ is } R_\mathfrak{q}\text{-free for } i \leq s\}.$$

The conclusion follows because $\operatorname{Tor}_i^A(E, R)$ is a finitely generated R-module for all i's.

2.13 COROLLARY. – Let the assumptions be as in 2.12. Then the sets

$$B_i = \{\mathfrak{p} \in \operatorname{spec}(R) | b_i(E \otimes k(\mathfrak{p})) \text{ is minimum}\}$$

are open and non-empty in spec (R).

2.14 REMARK. – The assumption « E flat » in 2.6 (*b*) and 2.12 can be omitted, see Appendix. However we do not know any application of these more general statements.

D. Betti numbers in a flat family.

Now we show how the previous results on Betti numbers can be applied to a flat family $X \to Y$ of projective schemes, such as we considered in § 1.

Since the properties we are interested in are local on Y we may assume $Y = \operatorname{Spec}(R)$ and $X = \operatorname{Proj}(A/I)$, where $A = R[X_0, \ldots, X_n]$ and I is a homogeneous saturated ideal of A. Recall that if $y \in Y$ we have, by definition, $b_i(X_y) = b_i(I(X_y))$, where $I(X_y) \subset A \otimes k(y) = k(y)[X_0, \ldots, X_n]$ is the homogeneous (saturated) ideal corresponding to the embedding $X_y \subset \boldsymbol{P}_y = \operatorname{Proj}(A \otimes k(y))$, see 2.2, 1.2, 1.3.

The results of subsections B and C can be applied to the A-module I, provided we know that:

$$I \quad \text{is } R\text{-flat}. \tag{10}$$

In this case we get informations about $b_i(I \otimes k(y))$ as y varies in Y. In order to also get informations about $b_i(X_y)$ we also have to know when

$$I(X_y) = I \otimes k(y). \tag{11}$$

Now conditions (10) and (11) have been discussed in 1.10, where it was shown that they hold, if Y is integral and noetherian, at all points of the open set V where the Hilbert function $H(X_y, n)$ is maximal for all n's. It is then clear how to give geometric applications. As an example we give the following:

2.15 PROPOSITION. – Let $f: X \to Y$ be a flat family of projective schemes (see 1.2), and assume Y is integral and noetherian with generic point y_0. Let $V \subset Y$ be the open set where the Hilbert function is maximal (see 1.7). Then:

(*a*) $b_i(X_y) \geq b_i(X_{y_0})$ for all $y \in V$;

(*b*) $B_i = \{y \in V | b_i(X_y) = b_i(X_{y_0})\}$ is open and non-empty;

(*c*) $B_0 \supset B_1$.

PROOF. – It follows from 1.10, 2.6, 2.12, 2.8 (*b*) and the above discussion (we leave the details to the reader).

2.16 REMARKS. – (i) Proposition 2.15 (*a*) is false if we do not restrict to V. Indeed let d be any integer and let $f: X \to Y$ be the flat family of all subschemes of $\boldsymbol{P}_k^n$ consisting of d distinct points (here Y is a suitable open set of the symmetric product of d copies of $\boldsymbol{P}^n$, or, equivalently a suitable open set of he Hilbert scheme which parametrizes all zero dimensional subschemes of $\boldsymbol{P}^n$ having degree d).

Now if $n = 2$ among the X_y's there are complete intersections of two curves of degrees a and b, with $ab = d$; and for any such, one has $b_0(X_y) = 2$. But if $d \geq 3$ and X_y is general one has $b_0(X_y) > 2$: for example if $d = 3$ and X_y consists of 3 non-collinear points, then $b_0(X_y) = 3$. Notice also that complete intersections, for $d \geq 3$, do not have maximal Hilbert function, in agreement with 2.15. For references on Hilbert functions of finite subschemes of $\boldsymbol{P}^n$ see for example the bibliography of [GGR].

(ii) Let $X \to Y$ be as in (i). The fact that $b_0(X_y)$ is constant on a non-empty open set (see 2.15 (*b*)) was proved by GERAMITA and MAROSCIA [GM], with different methods. The problem of computing this value is not yet settled; partial results in this direction can be found in [GGR], where a conjecture concerning the above value of $b_0(X_y)$ stated in [GO] is proved to be true in many cases.

(iii) If $X \to Y$ is as above, then $b_n(X_y)$, the last non-zero Betti number, coincides with the CM type of the graded ring of the embedding $X_y \subset \boldsymbol{P}_y$. A proof that this number is constant on a non-empty open set was given by L. ROBERTS [R], who also conjectured a value for such a constant. This conjecture turns out to be true in many cases, see [GGR]. The general problem is still unsettled.

(iv) We do not know whether the set $B_i' = \{y \in Y | b_i(X_y) = b_i(X_{y_0})\}$ (notation as in 2.15) has some nice topological property. It might be possible that repeated applications of 2.15 allow one to prove that B_i' is constructible for all i's.

(v) With the notation as in 2.15 it is false, in general, that $B_2 \subset B_0$, not even if we assume that X_y is finite for all $y \in Y$. We have indeed the following example, kindly communicated to us by A. GERAMITA.

2.17 EXAMPLE. – Let $X \to Y$ be the family of all subschemes of $\boldsymbol{P}_k^3$ consisting of 15 distinct points, and let $V \subset Y$ be as usual. If $y \in V$ then the points of X_y are said to be « in generic position », and conversely. For 15 points in generic position one can prove the « C-M conjecture », which implies that the minimum value of $b_2(X_y)$, $y \in V$ is 5, see [GGR], 4.6. Moreover one can show directly that the minimum value for $b_0(X_y)$, $y \in V$, is 5 (see [GGR]). On the other hand one can construct a set of 15 points in generic position having $b_2 = 5$, $b_1 = 11$, $b_0 = 7$, as follows. Start with the ideal

$$I = \langle X_1^3, X_2^3, X_3^3, X_1X_2^2, X_1X_2X_3, X_3^2X_2^2, X_1^2X_2^2 \rangle \subset k[X_1, X_2, X_3] = B .$$

It is easy to check that

(i) The Hilbert function of B/I is: 1 3 6 5 0 $\to$.

(ii) If $\mathfrak{m}$ is the maximal homogeneous ideal of B/I then $\operatorname{Ann}(\mathfrak{m}) = (B/I)_3$ and so $\dim_k(\operatorname{Ann}(\mathfrak{m})) = 5$; hence the Cohen-Macaulay type of B/I is 5.

Since I is a monomial ideal, by [H1] (see [GGR] for details) there are 15 distinct points in $\boldsymbol{P}^3$, whose ideal $J \subset k[X_0, X_1, X_2, X_3] = C$ has the following properties:

(iii) X_0 is not a zero divisor on C/J;

(iv) $(J, X_0)/(X_0) = I \subset B = C/X_0C$.

It follows that the Hilbert function of C/J is: 1 4 10 15 $\to$, which means that the points corresponding to J are in generic position. Moreover I is minimally generated by 7 elements (e.g. the ones given above), and hence by (iii) and (iv) the same holds for J, i.e. $b_0(J) = 7$. Finally, by (ii), (iii) and (iv) the C-M type of C/J is 5, and by general facts this means that $b_2(J) = 5$, which concludes our proof. Notice that from the above it follows that $b_1(J) = 11$ (e.g. by 2.6).

Appendix.

We generalize two results of section 2, by showing that 2.6 (b) and 2.12 do hold without flatness assumptions.

A1 PROPOSITION. – Let the notation be as in 2.3 and assume that R is a local domain. Then $b_i(E\otimes k) \geq b_i(E\otimes K)$.

PROOF. – We assume first that R is a DVR. Then the R-torsion submodule T of E is a graded A-submodule of E, and hence it is easy to see that $E = F\oplus T$ where $F = E/T$ is A-graded and R-flat. Then by 2.6 (ii) and the equality $F\otimes K = = E\otimes K$ it follows:

$$\begin{aligned} b_i(E\otimes k) &= b_i(F\otimes k) + b_i(T\otimes k) \geq \\ &\geq b_i(F\otimes K) = \\ &= b_i(E\otimes K) \end{aligned}$$

whence the conclusion in this case.

To prove the general case recall that there is a DVR R', dominating R, and with quotient field K. If k' is the residue field of R', by the previous step we have:

$$\begin{aligned} b_i(E\otimes_R k) &= b_i[(E\otimes_R k)\otimes_k k'] = \\ &= b_i[(E\otimes_R R')\otimes_{R'} k'] \geq \\ &\geq b_i[(E\otimes_R R')\otimes_{R'} K] = \\ &= b_i(E\otimes_R K) \end{aligned}$$

and the proof is complete.

A2 THEOREM. – Let the notation be as in 2.3 (with R any noetherian ring and E not necessarily R-flat). Then the sets

$$B_i^a = \{\mathfrak{p} \in \operatorname{spec}(R) | b_i(E\otimes k(\mathfrak{p})) \leq a\} .$$

are open. If moreover R is a domain with quotient field K the sets

$$B_i = \{\mathfrak{p} \in \operatorname{spec}(R) | b_i(E\otimes k(\mathfrak{p})) = b_i(E\otimes K \}$$

are open (and non empty).

PROOF. – We have to prove (a) and (b) as in the proof of 2.12. Now (a) follows immediately from A1. In order to prove (b) we may assume $\mathfrak{p} = 0$. Now by the theorem of generic flatness (see [M], p. 156, 22.A), there is a non-zero $f \in R$ such that $E \otimes R_f$ is R_f-flat. The conclusion follows then by 2.10.

BIBLIOGRAPHY

[GGR] A. V. GERAMITA - D. GREGORY - L. ROBERTS, *Monomial ideals and points in Projective Space*, Queen's University Mathematical Preprint No. 1983-23.

[GM] A. GERAMITA - P. MAROSCIA, *The ideal of forms vanishing at a finite set of points in R^n*, J. of Alg. (to appear).

[GO] A. GERAMITA - F. ORECCHIA: *Minimally generating ideals defining tangent cones*, J. of Alg., **78** (1982), pp. 36-57.

[H] R. HARTSHORNE, *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977.

[H1] R. HARTSHORNE, *Connectedness of the Hilbert scheme*, Publ. Math. I.H.E.S. No., **29** (1966), pp. 261-304.

[M] H. MATSUMURA, *Commutative Algebra*, Benjamin/Cummings Publ. Co., 1980.

[MF] D. MUMFORD - J. FOGARTY, *Geometric Invariant theory*, Erg. der Math. No., **34**, Springer-Verlag, 1982.

[R] L. G. ROBERTS, *A conjecture on Cohen-Macaulay type*, C.R. Math. Rep. Acad. Sci. Canada, **3** (1981), pp. 63-68.