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On Local Systems over Complements to Arrangements of Hyperplanes Associated to Grassmann Strata (*).

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Summary. – *In the first two parts we recall the construction of generalized hypergeometric functions and of the cellular complex homotopy equivalent to the complement of a family of hyperplanes in* $\mathbb{C}^N$. *In the third part we find a generalization of some results in* [2] *about the homology of local systems on an affine space less some hyperplanes. Our method is based on* [7] *and it gives also informations about the cellular complex there constructed. In the last part explicit bases for the only non-vanishing homology group are described in terms of the cells of the above mentioned complex. The configurations of hyperplanes which we examine are those giving fundamental strata in the grassmanian* ([2], [3]) *and strata in* $G_{3,n}$ *allowing also triple points.*

Introduction.

In [1] and a sequence of following papers a theory of the generalized hypergeometric function was developed. Such function satisfies a second order holonomic system of differential equations and it can be defined by Eulerian integrals over cycles which are contained in a certain covering of an affine space less some hyperplanes ([2]). The configuration of these hyperplanes individuates a stratum (see [3]) of a suitable Grassmann variety, and to each stratum a hypergeometric function is associated. Thus this theory is based on a strong connection between combinatorial and analytic methods.

One of the main tools of the theory is the study of rank 1-local systems over the complement of hyperplanes and the explicit description of bases for the homology with coefficients in such local systems (see [2]). Since in [7] one of the authors produced an explicit cellular-complex homotopy equivalent to the complement of the hyperplanes (when these hyperplanes are real) it is natural to try to study local systems through such complex. Calling Y the complement to the hyperplanes in $\mathbb{C}^N$ and $\mathcal{L}$ a local system on Y, it was proved in [2] that when the hyperplanes have normal crossings (in this case their configuration determines a so called fundamental stratum) and for generic $\mathcal{L}$ then $H_i(Y, \mathcal{L}) = 0$ for $i \neq N$. In the third part we obtain analog results under less restrictive genericity conditions for $\mathcal{L}$. The method here used is

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quite different from that of [2], since it uses the above mentioned complex X. To obtain the result we just split X into two parts and apply the Mayer-Vietoris sequence and induction on the number of hyperplanes.

In the fourth part we describe for $N=2$ an explicit basis of $H_2(Y, \mathcal{L})$ in terms of the cells of X, in the two following cases: when the configuration of lines has normal crossings, and when it admits also triple points as singularities. The methods we used can be easily generalized to higher dimension: in particular, bases for the so called fundamental strata of any dimension can be found similar to those of the first case above. In particular the so called «double knots» of [2] are given here as a linear combination of the cells of X.

At last we report about our computer program (based on the computations in [8]), which given any arrangement of lines returns the dimensions of the homology groups and their bases.

We thought useful for the reader to dedicate the first two parts to resume the essential parts of the theories: the first part is devoted to recall the definition of hypergeometric functions and its main properties, and it was put here expecially to stress the role of local systems in the theory of hypergeometric functions; in the second one we recall the construction of [7] stressing its geometrical description.

It is a pleasure to thank Prof. I. M. GELFAND for having exposed his theory of hypergeometric functions in Pisa and for having pushed us to learn it.

1. – Recall of the definition of generalized hypergeometric function.

Consider the Grassman manifold $G_k(\mathbb{CP}^n) \cong G_{k+1}(\mathbb{C}^{n+1})$, with coordinates $(x_0 : \ldots : x_n)$. Indicate by $\bar{\gamma}$ a $(k+1)$-dimensional space in $\mathbb{C}^{n+1}$ and with γ its projective image, and let $\sigma_j = \{x_j = 0\}$, $\sigma_J = \{x_{j_1} = \ldots = x_{j_r} = 0\}$, where $J = \{j_1, \ldots, j_r\} \subset \{0, \ldots, n\}$.

DEFINITION. – One says that γ, $\gamma' \in G_k(\mathbb{P}^n)$ are equivalent iff $\dim(\gamma \cap \sigma_J) = \dim(\gamma' \cap \sigma_J)$, $\forall J \subset \{0, \ldots, n\}$.

The preceding equivalence produces a *stratification* of the Grassman manifold (see [3] for many other equivalent definitions). Let now $\gamma \in G_k(\mathbb{P}^n)$, and let $D(\gamma)$ be the set of forms on $\bar{\gamma}$ of the shape $\omega = \sum_{i=0}^{n} t_i \, dt_0 \wedge \ldots \wedge \widehat{dt_i} \wedge \ldots \wedge dt_n$, where t_i are linear coordinates on $\bar{\gamma}$. The set of pairs (γ, ω) determines a rank 1 bundle $\widetilde{G}_{k,n}$ on $G_k(\mathbb{P}^n)$ (a determinant bundle). Note that the forms ω have homogeneity degree $n+1$.

Consider also the function $\pi_\alpha(x) = \prod_{j=0}^{n} (x_j)^{\alpha_j - 1}$ where $\alpha = (\alpha_0, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{C}$, and it holds $\sum \alpha_j = n - k$. π_α is a multivalued function ramified over $\sigma = \bigcup \sigma_j$. Let us construct the covering P_γ of $\gamma \setminus \sigma$ on which $\pi_\alpha \omega$ is univalued.

First let $\bar{P}$ be the covering of $\mathbb{C}^{n+1} \setminus \bar{\sigma}$ obtained from the universal covering $\mathbb{C}^{n+1}$ dividing by the subgroup generated by $2\pi i(1, \ldots, 1)$. Here π_α lifts to a univalued function by the formula

$$\pi_\alpha(a_0, \ldots, a_n) = \exp\left(\sum a_j(\alpha_j - 1)\right).$$

Let us denote again by ω the lifting of ω to $\overline{P}|\overline{\gamma}\setminus\overline{\sigma}$ and with $\pi_\alpha\omega$ the associated k form. Now let us take the projective image γ of $\overline{\gamma}$: let $P_\gamma = (\overline{P}|\overline{\gamma}\setminus\overline{\sigma})/\mathbb{C}^*$, where one lifts the action of $\mathbb{C}^*$ to $\overline{P}|\overline{\gamma}\setminus\overline{\sigma}$, endowed with the natural projection onto $\gamma\setminus\sigma$. One can see that P_γ corresponds to the commutator subgroup of $\pi_1(\gamma\setminus\sigma)$, so its automorphism group is $H_1(\gamma\setminus\sigma)$. Note that this last group is easily described, since it is isomorphic to $\mathbb{Z}^n$ and it is generated by elementary loops around the hyperplanes subjected to the only relation that their sum is zero.

Continue to indicate by $\pi_\alpha\omega$ the lifting on P_γ: the condition $\sum_{j=0}^{n} \alpha_j = n-k$ guarantees that $\pi_\alpha\omega$ is homogeneous of degree 0 so it can be lifted to P_γ.

The hypergeometric functions will be defined as integrals on cycles of P_γ. So first let us study the structure of these cycles.

Let $C_*^{rf}(P_\gamma)$ be the complex of those locally finite chains on P_γ which project onto locally finite chains in $\gamma\setminus\sigma$, and let $H_*^{rf}(P_\gamma)$ the homology of this complex (rf: relatively finite). Let α be as above and set $\tau_j = \exp(2\pi i\alpha_j)$ $\left(\text{so } \prod_{j=0}^{n} \tau_j = 1\right)$. Let $B_*(\gamma, \alpha) \subset C_*^{rf}(P_\gamma)$ be the subcomplex generated by the chains of the form

$$s\Delta - \left(\prod \tau_j^{s_j}\right)\Delta,$$

where $\Delta \in C_*^{rf}(P_\gamma)$, $s \in H_1(\gamma\setminus\sigma)$, s_j is the linking number of s with σ_j, $s\Delta$ is the chain obtained from Δ by parallel transport along s. Indicate by $C_*(\gamma, \alpha) = C_*^{rf}(P_\gamma)/B_*(\gamma, \alpha)$, and by $H_*(\gamma, \alpha)$ its homology.

Let $Y_\gamma = \gamma\setminus\sigma_0$, which is an affine k-dimensional space such that $Y_\gamma \setminus \bigcup_{j=1}^{n} M_j$ is still covered by P_γ (we set M_j as the affine hyperplane $Y_\gamma \cap \sigma_j$).

DEF. – A local system L_τ on Y_γ with monodromy indices τ_i, $i = 1, \ldots, n$, is a rank 1 bundle on Y_γ endowed with a flat connexion, such that a loop around M_i produces on the fiber the multiplication by τ_i.

So the τ_j induce a local system $\mathcal{L}_\tau$ on $Y_\gamma\setminus\mathcal{M}$ $\left(\text{set } \mathcal{M} = \bigcup M_j\right)$. It is not difficult to see that ([2]):

i) *The projection induces an isomorphism*

$$H_*(\gamma, \alpha) \cong H_*^{lf}(Y_\gamma\setminus\mathcal{M}, \mathcal{L}_\tau);$$

ii) *The projection* $H_k^{rf}(P_\gamma) \to H_k(\gamma, \alpha)$ *is surjective.*

Let now $h \in H_k^{rf}(P_\gamma)$ and consider $\int_h \pi_\alpha\omega$. When h is not compact such integral can diverge, but with a method of subdivision and analytic continuation it is possible to define it correctly (see [1]). By definition one has:

iii) $\int_h \pi_\alpha\omega$ *depends only on the class of the image of* h *in* $H_k(\gamma, \alpha)$.

Fix now a stratum $\Gamma \subset G_k(\mathbb{P}^n)$ and define the generalized hypergeometric function on Γ as follows. Let $\gamma \in \Gamma$, $h \in H_k^{lf}(P_\gamma)$, let U be a simply connected neighborhood of γ in Γ. Then $\forall \xi \in U$, $\omega \in D(\gamma)$ define

$$\phi_h(\alpha; \xi, \omega) = \int_{h(\xi)} \pi_\alpha \omega ,$$

where $h(\xi)$ is obtained from h by parallel transport through the Gauss-Manin connection. Here the Gauss-Manin connection lives in the fiber bundle on Γ with fiber $H_k^{lf}(P_\gamma)$.

The main properties of ϕ are the following ones:

— For fixed γ, ω, ϕ_h is meromorphic in α.

— For fixed α, ϕ_h is analytic in (γ, ω) for $\gamma \in \Gamma$.

— Let T be the torus $\{\delta = \mathrm{diag}(\delta_0, \dots, \delta_n), \delta_j \in \mathbb{C}^*\}$, acting on $G_k(\mathbb{P}^n)$ and on $\tilde{G}_{k,n}$: then

$$\phi_h(\alpha; \delta(\xi, \omega)) = \pi_\alpha(\delta)\, \phi_h(\alpha; \xi, \omega) .$$

So ϕ_h is determined by its value at one point ξ of Γ, and ϕ_h will be thought of as a function on the space Γ/H.

— Let h_0 be the image of h in $H_k(\gamma, \alpha)$: then ϕ_h depends only on h_0 and it can be indicated by $\phi_{h0}(\alpha; \xi, \omega)$. So the hypergeometric functions with fixed α are parametrized by $H_k^{lf}(Y_\gamma \setminus \mathcal{M}, \mathcal{L}_\tau)$.

Call a stratum Γ a *fundamental* stratum if for any $\gamma \in \Gamma$ there is an index $i \in \{0, \dots, n\}$ such that the arrangement of hyperplanes $\gamma \cap \sigma_j$, $j = 1, \dots, \hat{\imath}, \dots, n$, in the affine space $\gamma \setminus \sigma_i$ has only normal crossing singularities. Recall from [2]:

— Let γ be a real point of a fundamental stratum Γ, and assume $\alpha_j \notin \mathbb{Z}$, $\forall j$. Then if $h_1, \dots, h_N \in H_k^{lf}(Y_\gamma \setminus \mathcal{M}, \mathcal{L}_\tau)$ are linearly independent, the associated hypergeometric functions are independent.

2. – Recall of the cellular complex associated to an arrangement of hyperplanes.

Let $\mathcal{M} = \bigcup_i M_i$ be a family of affine hyperplanes in $\mathbb{C}^n$ which have real equations. We shall assume that the family is locally-finite and set $Y = \mathbb{C}^N \setminus \bigcup_i M_i$ as its complement. We recall from [7] the construction of a cellular complex $X \subset Y$ which is homotopy equivalent to Y (see also [6; ch. 8]). It is convenient to give a geometrical picture in the two dimensional case, for which X is a 2-dimensional cellular complex. So, assume that the arrangement is the one given in picture 1: it determines a cellular decomposition of $\mathbb{R}^2$ into facets (*chambers* are the interiors of the 2-dimensional cells, *faces* are the interiors of the 1-dimensional cells, and so on).

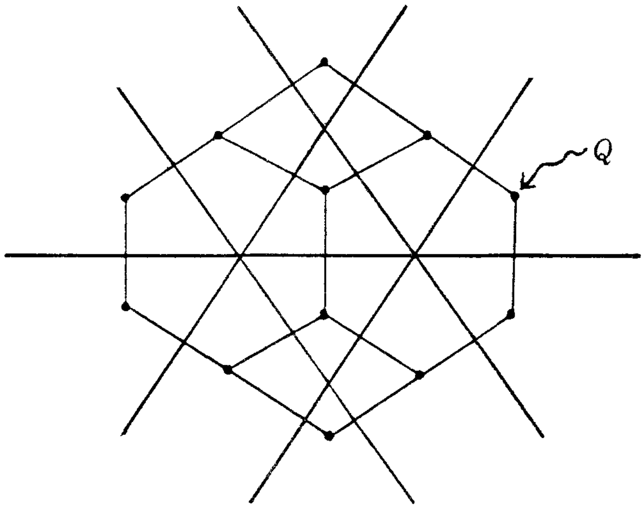

Fig. 1.

Let Q be the cellular complex in $\mathbb{R}^2$ which is *dual* to this decomposition (Q is obtained by a baricentrical subdivision). Each vertex of Q is dual to a chamber, each edge is dual to a face, each 2-cell is dual to a 0-cell (that is, a singularity) of the arrangement. First we describe the complex X above and a dimension-preserving cellular map ψ: $X \to Q$ in an abstract manner, without taking into account its realization inside Y. The 0-skeleton X_0 is isomorphic to Q_0 through ψ. The 1-skeleton X_1 is obtained from Q_1 by «doubling» each edge : that is, substitute to each edge of Q a pair of edges with same ends. Such edges are oriented (so that their union represents a loop around a hyperplane). The map ψ is defined naturally as taking each 1-cell of X into the 1-cell of Q with same ends (see fig. 2).

Let e^2 be a 2-cell of Q. There are in X as many 2-cells going to e^2 through ψ as the vertices of e^2. If $v \in e^2$ is a vertex, let us indicate by $e^2(v) \in X$ the 2-cell whose bound-

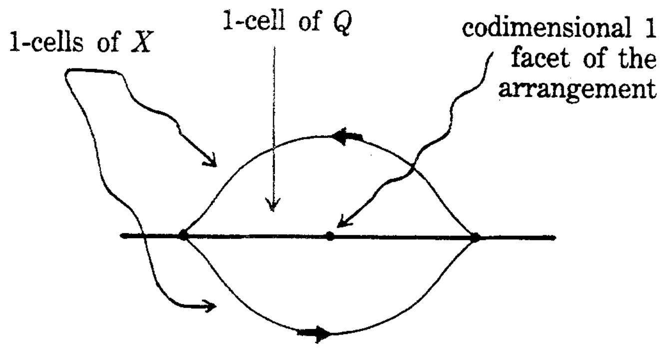

Fig. 2.

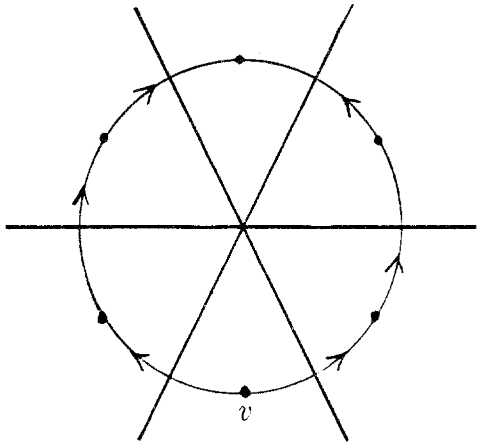

Fig. 3.

ary is given by the two positive paths of X going from v to the vertex opposite to v in e^2 (see fig. 3).

For higher dimension the construction is similar.

The realization of X into Y is obtained by using some formulas, as follows. First, give to the set of all facets of the arrangement the following partial ordering:

$$F^i > F^j \Leftrightarrow \text{closure } (F^i) \supset \text{closure } (F^j)$$

(the exponent i of F^i means *codimension*). Moreover, given a vertex v of Q and a facet F^i of the arrangement, let $w_{F^i,v}$ be the vertex of the i-cell of Q dual to F^i which is closest to v; here distance between two vertices is understood as the minimum number of edges of a path in Q_1 connecting the two vertices. The $w_{F^i,v}$ belongs to the dual cell to F^i iff it is dual to a chamber which has F^i in its closure; it is convenient to indicate by $\mathcal{V}(F^i)$ the set of vertices which satisfy this property. So a realization of Q in $\mathbb{R}^N$ is given by

$$Q = \bigcup s(F^h > \ldots > F^k),$$

where:

the union above is taken over all chains $F^h > \ldots > F^k$;

if $w(F^j)$ is the point chosen in F^j for the baricentrical subdivision which defines Q, then $s(F^h > \ldots > F^k)$ is the simplex associated to the chain $F^h > \ldots > F^k$ which is given by

$$s(F^h > \ldots > F^k) = \left\{ \sum_{j=h}^{k} \lambda_j w(F^j) \colon 0 \leqslant \lambda_j \leqslant 1, \sum \lambda_j = 1 \right\}.$$

See an example in fig. 4.

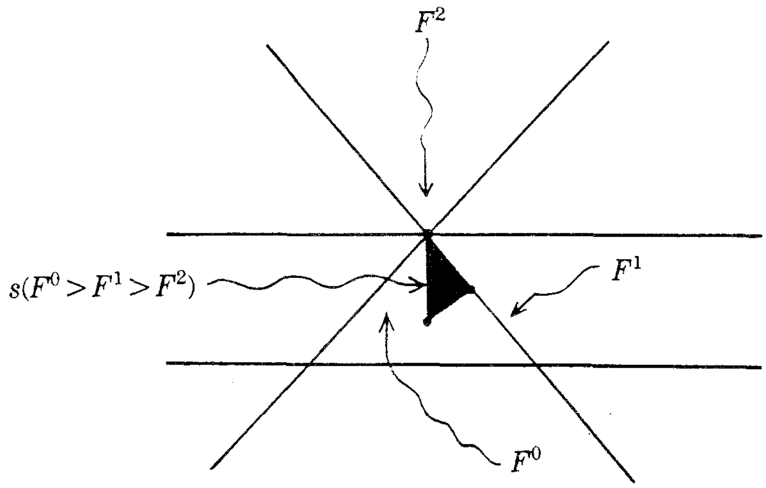

Fig. 4.

Fixing F^k and varying the chain which contains F^k as its last element we obtain the k-cell $e^k \subset \mathbb{R}^N$ of Q which is dual to F^k. The complex X is realized in $Y \subset \mathbb{C}^N$ as

$$X = \bigcup s(F^h > \ldots > F^k; v),$$

where:

the union is taken over all chains $F^h > \ldots > F^k$ and all vertices v of Q which are dual to a chamber C such that closure $(C) \supset F^k$;

$s(F^h > \ldots > F^k; v)$ is the simplex given by

$$s(F^h > \ldots > F^k;\ v) =$$
$$= \left\{ \sum_{j=h}^{k} \lambda_j w(F^j) + \sqrt{-1} \left[\sum_{j=k}^{k} \lambda_j (w_{F^j,v} - w(F^j)) \right] \middle| \ 0 \leq \lambda_j \leq 1,\ \sum \lambda_j = 1 \right\}.$$

Here $w_{F^j,v}$ is defined as above.

Fixing F^k and v as above, the union of all simplexes $s(F^h > \ldots > F^k; v)$ varying the chain which contain F^k as last element is a k-cell $e^k(v)$ of X which is cellularly homeomorphic (through ψ) to the k-cell e^k which is dual to the facet F^k. So any cell of X has a natural preferred point (v for $e^k(v)$). Note that when X is realized in Y as above, ψ is just the real projection $\mathcal{R}$: $\mathbb{C}^N \to \mathbb{R}^N$.

3. - Computations with local systems.

Let Y be as in the preceding part. The following results hold for any local system L_τ, and are proved by using Poincaré duality and conjugation in $\mathbb{C}^N$:

(a) $$H_i(Y, L_\tau) \cong (H^{lf}_{2N-i}(Y, L^\vee_\tau))^*;$$

$$(b) \qquad H_N(Y, L_\tau) \cong H_N(Y, L_\tau^\vee);$$

$$(c) \qquad H_i(Y, L_\tau) = 0 \quad \text{for} \quad i > N, \qquad H_i^{lf}(Y, L_\tau) = 0 \quad \text{for} \quad i < N.$$

Here $L_\tau^\vee$ is the local system with monodromy indices τ_i^{-1}.

Let us suppose that $\bigcup_{i \in \mathfrak{J}} M_i$ has only normal singularities; so, when $\mathfrak{J}$ is finite, the given arrangement represents a fundamental stratum in $G_{N+1}(\mathcal{R}^{|\mathfrak{J}|})$ (see part 1).

Let $X \subset Y$ be the associated cellular N-complex, according to [7]. (Recall ([7; Lemma 13]) that in the considered case X can be obtained by its real projection by glueing the cells $D^j_{F^j}$, $D^j_{F'^j}$ if and only if $|F^j| = |F'^j|$. However, here we prefer to use another method).

Let M_i be a hyperplane of $\mathcal{M}$, $M_i = \{{}^t a_i \cdot x + b_i = 0\}$, $a_i \in \mathbb{R}^N$, $b_i \in \mathbb{R}$, and indicate by $M_i^+ = \{{}^t a_i \cdot x + b_i \geq 0\}$, $M_i^- = \{{}^t a_i \cdot x + b_i \leq 0\}$ as the two half-spaces delimited by M_i. Set also $\mathcal{M}_i' = \mathcal{M} \setminus \{M_i\}$ and let $\mathcal{M}_i = \{M_i \cap M_j\}_{j \in J, j \neq i}$ be the configuration determined on M_i by the remaining hyperplanes. Clearly both the arrangements $\mathcal{M}_i'$, $\mathcal{M}_i$ have normal crossing singularities. Correspondingly to $\mathcal{M}_i'$ and $\mathcal{M}_i$ one constructs cellular complexes $X(\mathcal{M}_i)$, $X(\mathcal{M}_i')$ in the same manner as X ($X(\mathcal{M}_i')$ in general will be still an N-complex while $X(\mathcal{M}_i)$ is an $(N-1)$-complex).

Set $K_i' = \{s(F^h > \ldots > F^k; v) \in X \mid$ either F^k is not contained into M_i and v varies through $\mathcal{V}(F^k)$, or $F^k \subset M_i$ and $v \in \mathcal{V}(F^k) \cap M_i^+\}$;

$$X_i^+ = K_i' \cup \{s(F^h > \ldots > F^k; v) \mid F^k \subset M_i,\ \ F^h \subset M_i^+, v \in \mathcal{V}(F^k) \cap M_i^-\};$$

$$X_i^- = K_i' \cup \{s(F^h > \ldots > F^k; v) \mid F^k \subset M_i,\ \ F^h \subset M_i^-, v \in \mathcal{V}(F^k) \cap M_i^-\}.$$

The complexes above are shown in picture 5, in the case when the arrangement is 1-dimensional (a set of points in $\mathbb{R}$).

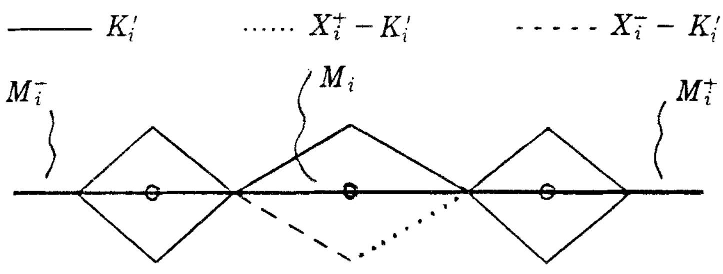

Fig. 5.

THEOREM 1. – One has:

(1) $X_i^+ \cup X_i^- = X$;

(2) $X_i^+ \cap X_i^-$ has two connected components: K_i' and another one (say K_i); K_i' is a deformation retract of $X(\mathcal{M}_i')$ and K_i is isomorphic to $X(\mathcal{M}_i)$;

(3) K_i' is a deformation retract of both X_i^+ and X_i^-.

PROOF. – (1) is clear since each simplex of X is in X_i^+ or in X_i^-.

(2) Clearly $K_i' \subset X_i^+ \cap X_i^-$ and K_i' is connected. A simplex $s = s(F^h > \ldots > F^k; v) \subset X_i^+$, which is not contained in K_i', is also a simplex of X_i^- if and only if $F^h \subset M_i$ and in that case $s \cap K_i' = \emptyset$. Then $X_i^+ \cap X_i^- = K_i' \coprod K_i$, where

$$K_i = \{s(F^h > \ldots > F^k;\ v) | F^h \subset M_i,\ \ v \in \mathcal{V}(F^k) \cap M_i^-\}$$

(and this is clearly connected).

Note that, because of the normality conditions on the singularities, the chambers $F^0 \subset M_i^-$ having M_i as a wall bijectively correspond to the codimensional 1 facets $F^1 \subset M_i$. From here it easily follows that K_i is isomorphic to $X(\mathcal{M}_i)$: one can construct an isotopy in $\mathbb{C}^N$ taking K_i into $X(\mathcal{M}_i)$ by moving each vertex $w(F^0)$ belonging to a chamber F^0 having M_i as a wall into the corresponding $w(F^1) \in M_i$.

For K_i' one obtains the result by crushing each simplex $s(F^h > \ldots > F^k; v), F^k \subset M_i$, into M_i, through an homotopy which fixes the vertices $w(F^j) \in M_i$ and takes $w(F^j)$ into $w(F^{j+1})$ when $F^j > F^{j+1} \subset M_i, F^j \notin M_i$ (we use again the normality condition). The obtained complex will coincide with $X(\mathcal{M}_i')$.

(3) is obtained in a similar way as (2), by contracting onto K_i' those simplexes of $X_i^+ (X_i^-)$ which are not contained into K_i': the homotopy is similar to that exploited in (2) (see pic. 5). Q.E.D.

DEF. – A *vertex* of an arrangement is a facet of dimension 0, and an arrangement containing vertices will be called of *maximal rank*.

For brevity, we shall set $H_i(\mathcal{M})$ for $H_i(X(\mathcal{M}); \mathcal{L}_\tau)$ and similarly for $\mathcal{M}_i, \mathcal{M}_i'$ where the local system on $X(\mathcal{M}_i)$ is the restriction of $\mathcal{L}_\tau$ and that on $X(\mathcal{M}_i')$ is the one with coefficients $\{\tau_j\}_{j \neq i}$. Moreover, $c^{bd}(\mathcal{M})$ will indicate the number of bounded chambers of $\mathcal{M}$.

THEOREM 2. – Assume that the arrangement $\mathcal{M}$ has only normal crossing singularities and is finite ($\# \mathcal{M} = r$) and of maximal rank. Let $\mathcal{L}_\tau$ be a local system on Y and assume that there are hyperplanes $M_{i_1}, \ldots, M_{i_s}$ such that $\bigcap_{j=1}^{s} M_{i_j} \neq \emptyset$ and $\bigcup_{j=1}^{s} M_{i_j}$ crosses all the intersections of hyperplanes of dimension ≥ 1; moreover $\tau_{i_j} \neq 1$ for $j = 1, \ldots, s$. Then:

$$H_i(\mathcal{M}) = 0 \quad \text{ for } i < N.$$

PROOF. – If the arrangement is in general position then the shortest way is to use a homotopy operator as in [4, prop. 3.4]. For a generic arrangement verifying the above hypotheses, we proceed by induction on N and $r - s$.

For $r = s$ $(= N)$ the arrangement is in general position, as it is for $N = 1$, and the above applies. Let us assume the thesis for all pairs N', $r' - s'$ such that either $N' \leq N - 1$, whatever $r' - s'$, or $N' = N$ and $r' - s' < r - s$. For $r > s$, let $M_i \in \mathcal{M}$ such that $\bigcap_{j=1}^{s} M_{i_j}$ contains a vertex which is not in M_i $\left(\text{if } \bigcap_{j=1}^{s} M_{i_j}\right.$ contains two vertices such M_i always exists; if it contains only one vertex and all the hyperplanes pass through it, then we return to the general position case $\Big)$. Then $\mathcal{M}_j'$ verifies the hypotheses of Theorem 2 with $r-1$ hyperplanes and the same N and s; $\mathcal{M}_i$ verifies the hypotheses with $N-1$. By Theorem 1 there is a Mayer-Vietoris exact sequence

$$\to H_j(\mathcal{M}_i') \oplus H_j(\mathcal{M}_i') \to H_j(\mathcal{M}) \to H_{j-1}(\mathcal{M}_i') \oplus H_{j-1}(\mathcal{M}_i) \to$$

and by induction $H_j(\mathcal{M}_i') = 0$ for $j \leqslant N-1$, $H_j(\mathcal{M}_i) = 0$ for $j \leqslant N-2$ from which theorem follows. Q.E.D.

COROLLARY. – In the hypotheses of Theorem 2, the only non trivial homology group is the one in dimension N, and

$$\dim H_N(\mathcal{M}) = c^{bd}(\mathcal{M}).$$

PROOF. – Since $H_i(\mathcal{M}) = 0$ for $i > N$ the only non trivial homology group is the one in dimension N, so $\dim H_N(\mathcal{M})$ is the Euler characteristic of Y. But it is known (see for instance [6, Corollary 6.10]) that $\chi(Y) = c^{bd}(\mathcal{M})$, so the corollary follows. Q.E.D.

The preceding corollary could be proved using again a Mayer-Vietoris sequence and [11, Corollary 7.2].

4. – Construction of special bases for the homology.

We give now an explicit description of a base of $H_N(\mathcal{M})$ in two cases: when $\mathcal{M}$ lies in a fundamental stratum of $G_{3,n}$, that is $\mathcal{M}$ is a union of lines in $\mathbb{C}^2$ with normal crossing singularities, and when $\mathcal{M}$ is a union of lines with at most triple points (these last strata are the next interesting case after the fundamental ones). The basis which we give in the first case is easily generalizable to fundamental strata in $G_{k,n}$.

THEOREM 3. – Let $\mathcal{M}$ be a union of lines with real equations with only double points as singularities (such arrangement determines a fundamental stratum in $G_{3,n}$). Let $\mathcal{L}_\tau$ be a local system on the complement Y of $\mathcal{M}$. Then to each bounded chamber C in $\mathbb{R}^2$ a 2-cycle $\gamma(C)$ in $X(\mathcal{M})$ is associated, with coefficients in L_τ, where

$$(*) \qquad \gamma(C) = \sum_{F^2 \subset \text{closure } (C)} \left(\prod_{C > M_i,\, F^2 \notin M_i} (1-\tau_i) \right) \gamma(C, F^2).$$

Here $C > M_i$ means that the line M_i intersects the closure of C in a 1-cell (a

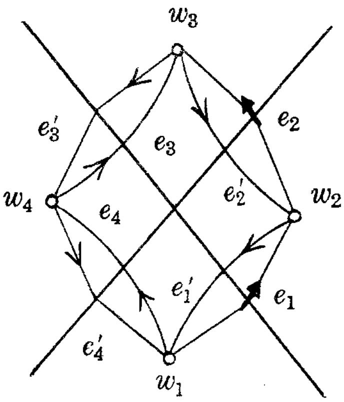

Fig. 6.

non-degenerate segment), while $\gamma(C, F^2)$ is the alternate sum of the 2-cells of X whose real projection is the 2-cell of Q dual to F^2 (see picture 6).

PROOF. – Since $X(\mathcal{M})$ is decomposed into cells $e^2_{F^2}(v)$ (where F^2 is here a vertex of $\mathcal{M}$ and v is a vertex of $X(\mathcal{M})$ belonging to $\mathcal{V}(F^2)$; see part 2) every 2-chain with coefficients in the local system can be given the form

$$\sum_{F^2, v} \sigma(F^2, v)\, e^2_{F^2}(v),$$

where $\sigma(F^2, v)$ is a section of L_τ over v (for definition and calculation of local system over cellular complexes a good reference is [10]). Similar expressions hold for the 1-chains and the 0-chains. The boundary $\partial(\sigma e^2_{F^2}(v))$ is computed as in singular homology, but giving as coefficients to the boundary cells those obtained by trasporting σ onto their preferred point.

Let w be the 0-cell of X contained into the bounded chamber C. If F^2 is a vertex of C there are four cells $E_i = e^2_{F^2}(w_i)$, $i = 1, \ldots, 4$ $(w_1 = w)$, around F^2 which are attached to eight edges $e_1, e_1', \ldots, e_4, e_4'$ ([8]; see pic. 6).

In the following the symbol $\sim$ over a cell $e^i(v)$ means that such cell is endowed with the section of L_τ obtained by transporting the trivial section over the point w along the cell E_1, til the preferred point v.

We look for a chain

$$\gamma(C, F^2) = \sum_{i=1}^{4} \varepsilon_i \tilde{E}_i, \qquad \varepsilon_i \in \mathbb{C},$$

such that $\partial(\gamma(C, F^2))$ does not contain edges with indices 2 and 3 (so it only contains some combination of edges whose real projection intersects C). By computing the boundary of $\gamma(C, F^2)$ one obtains that the linear system $\varepsilon_i + \varepsilon_{i+1} = 0$, $i = 1, \ldots 4$, is to be satisfied (the indices are mod 4). So the general solution is $\varepsilon(F^2) \cdot \gamma(C, F^2)$,

$\varepsilon(F^2) \in \mathbb{C}$, where $\gamma(C, F^2) = \tilde{E}_1 - \tilde{E}_2 + \tilde{E}_3 - \tilde{E}_4$ and it follows

$$\partial(\varepsilon(F^2)\cdot\gamma(C, F^2)) = \varepsilon(F^2)\{(\tilde{e}_1 + \tilde{e}_1')(1-\tau_2) - (\tilde{e}_4 + \tilde{e}_4')(1-\tau_1)\}.$$

Now looking for a cycle of the kind

$$\gamma(C) = \sum_{F^2 \subset \text{closure } (C)} \varepsilon(F^2)\,\gamma(C, F^2),$$

one obtains (computing the boundary) another linear system in the $\varepsilon(F^2)$, which gives formula (*).

(Note that the fact that C is bounded is essential to the vanishing of the boundary of $\gamma(C)$.) Q.E.D.

It is not hard to prove directly that (under genericity conditions for the τ_i) the above constructed cycles are independent.

THEOREM 4. – Let us assume now that $\mathcal{M}$ has also triple points as singularities, but not higher order singularities. Let $\mathcal{L}_\tau$ be a local system on the complement Y of $\mathcal{M}$. Then to each bounded chamber C of $\mathcal{M}$ a 2-cycle $\gamma(C)$ with coefficients in $\mathcal{L}_\tau$ is associated, where

$$(**)\qquad \gamma(C) = \sum_{F^2 \in \text{closure}(C)} \left(\prod_{C > M_i,\, F^2 \notin M_i} (1-\tau_i)\right)\left(\prod_{\substack{f^2 \in \text{closure}(C)\\ f^2 \neq F^2,\, O(f^2)=3}} k(C, f^2)\right)\cdot\gamma(C, F^2).$$

Here $O(f^2)$ is the order of the singularity f^2 of $\mathcal{M}$ and $\gamma(C, F^2)$ is as in theorem 3 if $O(F^2) = 2$; if $O(F^2) = 3$ it is given by the 2-chain

$$(***)\qquad \gamma(C, F^2) = \frac{1}{\tau_\alpha\tau_\gamma - 1}((\tau_\alpha\tau_\gamma - 1)\tilde{E}_1 + (1-\tau_\gamma)\tilde{E}_2 +$$

$$+(1-\tau_\alpha)\tau_\gamma\tilde{E}_3 + (\tau_\alpha\tau_\gamma - 1)\tilde{E}_4 + (1-\tau_\gamma)\tau_\alpha\tilde{E}_5 + (1-\tau_\alpha)E_6),$$

where $E_1, \dots, E_6$ are the six 2-cells of X whose real part is dual to F^2 (see picture 7). For each $F^2 \in$ closure (C), α, β, γ denote the indices of the lines passing through F^2, in the anticlockwise order (and so that $M_\beta \cap$ closure $(C) = F^2$), and

$$(****)\qquad k(C, F^2) = (\tau_\alpha\tau_\beta\tau_\gamma - 1)/(\tau_\alpha\tau_\gamma - 1).$$

PROOF. – If F^2 is a triple point of a chamber C which contains w as 0-cell of X, there are six 2-cells $E_i = e^2_{F^2}(w_i)$, $i = 1, \dots, 6$ $(w_1 = w)$, and twelve 1-cells $e_1, e_1', \dots, e_6, e_6'$ around F^2, where each 2-cell is attached over six 1-cells ([8]; see fig. 7).

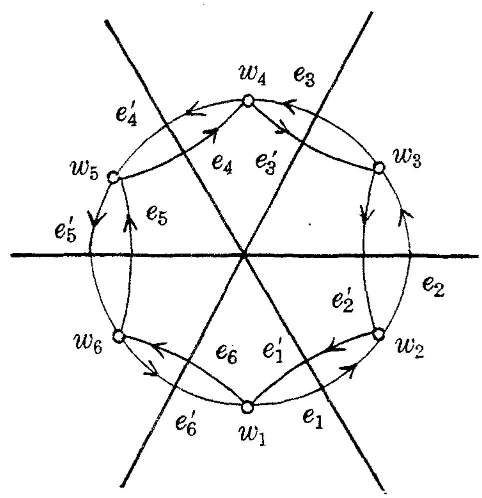

Fig. 7.

As in Theorem 4, we look for a chain

$$\gamma(C,F^2)=\sum_{i=1}^{6}\varepsilon_i\widetilde{E}_i,\qquad \varepsilon_i\in\mathbb{C},$$

such that $\partial(\gamma(C,F^2))$ does not contain edges of index different from 1 and 6. If α, β, γ are the indices of the lines containing F^2 and crossing respectively the real projections of e_1, e_2, e_3 one obtains that the following linear system is to be solved:

$$\begin{aligned}
&\varepsilon_1+\varepsilon_2+\tau_\gamma\varepsilon_6=0,\\
&\varepsilon_1+\varepsilon_2+\varepsilon_3=0,\\
&\varepsilon_3+\tau_\gamma(\varepsilon_4+\varepsilon_5)=0,\\
&\varepsilon_4+\varepsilon_5+\varepsilon_6=0,\\
&\varepsilon_2+\varepsilon_3+\varepsilon_4=0,\\
&\varepsilon_5+\tau_\alpha(\varepsilon_3+\varepsilon_4)=0,\\
&\varepsilon_1+\varepsilon_5+\varepsilon_6=0,\\
&\tau_\alpha\varepsilon_2+\varepsilon_1+\varepsilon_6=0,
\end{aligned}$$

which gives as general solution $\varepsilon(F^2)\cdot\gamma(C,F^2)$, $\varepsilon(F^2)\in\mathbb{C}$, where $\gamma(C,F^2)$ is as in formula (***) (it is obviously assumed $\tau_\alpha\tau_\gamma\neq 1$). One has

$$\partial(\gamma(C,F^2))=k(C,F^2)\cdot[(1-\tau_\gamma)(\tilde{e}_1+\tilde{e}_1')-(1-\tau_\alpha)(\tilde{e}_6+\tilde{e}_6')],$$

where $k(C,F^2)$ is the rational function in the τ's .given by (****).

For every bounded chamber C we look as above for a cycle of the kind $\gamma(C)=\sum_{F^2\in\,\mathrm{closure}(C)}\varepsilon(F^2)\cdot\gamma(C,F^2)$ where now F^2 can be either a double point or a triple point. By computing the boundary we obtain another linear system which yields the expression given in formula (**). Q.E.D.

Again one can prove directly that the cycles $\gamma(C)$ are independent (for generic τ's). When $\mathcal{M}$ has more complicated singularities one has to work harder to obtain an explicit expression for a basis of $H_N(\mathcal{M})$ since chains with analog properties to the above $\gamma(C, F^2)$ cannot be found. However the above strategy can be generalized by «grouping» bounded chambers togheter. For instance note that if $F^2 \in \text{closure}(C')$ is a 4-fold point and C' is bounded there always exists another bounded chamber C'' adjacent to C' and with $F^2 \in \text{closure}(C'')$. We shall probably return to this in the future.

NOTE. – By using different methods in [5] a result analog to Theorem 2 is proved for any arrangement under genericity conditions of L_τ (expressed in terms of connexions) which are more restrictive than ours for fundamental strata.

REMARK. – By using the semplification of $X(\mathcal{M})$ given in [9] (which is in turn based on [8]) we implemented a computer program (LISP language) which given whatever arrangement $\mathcal{M}$ and local system L_τ returns the matrix associated to the boundary operators, and computes from it the dimensions of $H_i(Y; L_\tau)$ and their bases (in this case the difficulties are in dimension two). The «experimental» results obviously agree with the theoretical fact of [5] that for generic L_τ one has $H_1(\mathcal{M}) = H_0(\mathcal{M}) = 0$. Experimental results give also indications about the interesting problem (on which we have some partial results) of determining in a combinatorial fashion the polynomial equations in the τ's whose solutions lower the dimension of H_2.

Observe that one could use directly the complex $X(\mathcal{M})$: for any dimension, it is in theory possible to write a computer program which returns the dimension of the homology groups and their bases. In practice the simplified complex given in [9] is very useful having in general much less cells than X.

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