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Natural Operations with Projectable Tangent Valued Forms on a Fibred Manifold (*).

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Summary. - *Let $p: E \to B$ be a fibred manifold. Then, we consider the sheaf $\mathfrak{P}(E) = \Omega(B) \otimes \mathcal{P}(E)$ of (local) projectable tangent valued forms on E, where $\Omega(B)$ is the sheaf of (local) differential forms on B and $\mathcal{P}(E)$ is the sheaf of (local) projectable vector fields on E. The Frölicher-Nijenhuis bracket makes $\mathfrak{P}(E)$ to be a sheaf of graded Lie algebras* [18]. *In this paper we study all natural $\mathbb{R}$-bilinear operations on $\mathfrak{P}(E)$ which are of Frölicher-Nijenhuis type. By using the analytical method of* [16], *we prove that there is a three-parameter family of such operators on $\mathfrak{P}(E)$. As a consequence, we obtain a result on the unicity of the covariant differential of tangent valued forms and of the curvature associated with a given connection on E. All manifolds and mappings are assumed to be infinitely differentiable.*

0. - Introduction.

A. FRÖLICHER and A. NIJENHUIS [6, 24] introduced a bracket $[\,,\,]$ in the sheaf

$$\Omega(M, TM) = \bigoplus_{0 \leq r \leq m} \Omega^r(M, TM), \qquad m = \dim M,$$

of (local) tangent valued differential forms on a manifold M and proved that it gives rise to a graded Lie algebra. Namely, the bracket $[\,,\,]$ is an $\mathbb{R}$-bilinear sheaf morphism

$$[\,,\,]: \Omega^r(M, TM) \times \Omega^s(M, TM) \to \Omega^{r+s}(M, TM),$$

satisfying

$$[\varphi, \psi] = (-1)^{rs+1}[\psi, \varphi], \tag{0.1}$$

$$(-1)^{rt}[\varphi, [\psi, \omega]] + (-1)^{rs}[\psi, [\omega, \varphi]] + (-1)^{st}[\omega, [\varphi, \psi]] = 0, \tag{0.2}$$

where $\varphi \in \Omega^r(M, TM)$, $\psi \in \Omega^s(M, TM)$, $\omega \in \Omega^t(M, TM)$. A. FRÖLICHER and A. NIJEN-

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HUIS [6] interpreted this algebra as an algebra of derivations of a certain kind of the graded exterior algebra of ordinary forms. This algebra has been widely applied to the study of complex, almost complex, almost tangent and other structures on a manifold (M. CRAMPIN and L. A. IBORT [2], A. FRÖLICHER and A. NIJENHUIS [7], A. NIJENHUIS [25]).

In some isolated papers the Frölicher-Nijenhuis algebra was linked with the theory of connections. In particular, H. K. NICKERSON [23] has studied principal connections on a principal bundle, T. V. DUC [4] has studied linear connections on a vector bundle and M. CRAMPIN [1] and J. GRIFONE [9] applied this algebra to the study of connections on a manifold. All of them expressed the differential calculus associated with a connection in terms of the Frölicher-Nijenhuis algebra.

L. MANGIAROTTI and M. MODUGNO [17, 18] introduced, by another way, a graded Lie bracket on the sheaf

$$\mathfrak{P}(E) = \bigoplus_{0 \leq r \leq n} \Omega^r(B) \oplus \mathscr{P}(E), \qquad n = \dim B,$$

of (local) projectable tangent valued forms on a fibred manifold $p\colon E \to B$. It can be shown that this algebra turns out to be a distinguished subalgebra of the Frölicher-Nijenhuis algebra $\Omega(E, TE)$ of all tangent valued forms on E. The algebra $\mathfrak{P}(E)$ was interpreted by M. CRAMPIN and L. A. IBORT [2] as an algebra of derivations of the graded exterior algebra of forms on E which preserve the basic forms $p^*\Omega(E)$.

L. MANGIAROTTI and M. MODUGNO [18] showed that the algebra $\mathfrak{P}(E)$ is the natural framework for the study of Ehresmann connections on fibred manifolds and that the Frölicher-Nijenhuis bracket yields a generalization of the standard differential calculus associated with the traditional connections. In particular, if γ is a connection on E, i.e. a projectable tangent valued 1-form which is projected on the identity of TB, then we obtain the covariant differential of a projectable tangent valued form φ

$$d_\gamma \varphi = (1/2)[\gamma, \varphi],$$

the curvature of γ

$$\rho = d_\gamma \gamma$$

and the Bianchi identity

$$d_\gamma \rho = 0.$$

This general approach to the theory of connections on a fibred manifold has been developed by P. MICHOR [20].

M. MODUGNO [21, 22] has developed further this theory including systems of connections and applied it to Lagrangian and gauge theories.

The Frölicher-Nijenhuis bracket on $\Omega(M, TM)$ satisfies the naturality condition

$$f^*[\varphi, \psi] = [f^*\varphi, f^*\psi],$$

(A. FRÖLICHER and A. NIJENHUIS [8]). A natural question arises: there exist other natural operations on $\Omega(M, TM)$? This problem has been studied by P. MICHOR [19] and I. KOLÁŘ and P. MICHOR [14]. I. KOLÁŘ and P. MICHOR gave the full classification of natural $\mathbb{R}$-bilinear operators $\Omega^r(M, TM) \times \Omega^s(M, TM) \to \Omega^{r+s}(M, TM)$. They proved that, for $r \geqslant 2$, $s \geqslant 2$, $r+s < \dim M - 1$, there exists a ten-parameter family of such operators.

The purpose of this paper is to classify the $\mathbb{R}$-bilinear natural operators (sheaf morphisms)

$$\mathfrak{P}^r(E) \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E).$$

The interest for such a problem arises naturally in the contest of the theory of connections on fibred manifolds.

We prove that there is a three-parameter family of such operators. This family is generated by the Frölicher-Nijenhuis bracket and other two operators, which can be easily represented by using the projection, the contraction and the exterior derivative. In particular, if $r = s = 0$ we get the uniquenes of the Lie bracket of two projectable vector fields on E.

Our main result has some consequences for the theory of connections on E introduced by L. MANGIAROTTI and M. MODUGNO [18]. Namely, if γ is a connection on E, then the covariant differential d_γ is the only natural derivation of order 1 of $\mathfrak{P}(E)$ related with γ. Moreover, the curvature is the only natural operator on connections.

The uniqueness of the curvature of a connection on E was proved by KOLÁŘ [13], by using another approach.

For the classification of natural $\mathbb{R}$-bilinear operators on $\mathfrak{P}(E)$ we use the general theory of natural bundles and natural differential operators, J. JANYŠKA [10], I. KOLÁŘ [12], A. NIJENHUIS [26], J. SLOVÁK [27], C. L. TERNG [28]. Our coordinate calculations are based on the method of D. KRUPKA [15], D. KRUPKA and J. JANYŠKA [16].

1. – Tangent valued projectable forms on a fibred manifold.

Let $p\colon E \to B$ be a fibred manifold. We shall use the following notations. TE will be the tangent space of E and $\mathcal{T}(E)$ the sheaf of (local) vector fields on E. $\mathcal{P}(E)$ and $\mathcal{V}(E)$ will be the subsheafs of (local) projectable and vertical vector fields on E, respectively. Moreover, $\Omega(B) = \bigoplus_{0 \leqslant r \leqslant n} \Omega^r(B)$, $n = \dim B$, will be the sheaf of (local) forms on B.

The sheaf of (local) *projectable tangent valued forms* on E is

$$\mathfrak{P}(E) = \bigoplus_{0 \leqslant r \leqslant n} \mathfrak{P}^r(E) = \bigoplus_{0 \leqslant r \leqslant n} \Omega^r(B) \otimes \mathcal{P}(E).$$

Thus, if $\varphi \in \mathfrak{P}^r(E)$, then φ is a (local) section $\varphi\colon E \to \Lambda^r T^* B \otimes TE$ which is projectable on the (local) section $\underline{\varphi}: B \to \wedge^r T^* B \otimes TB$ via the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & \wedge^r T^* B \otimes TE \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle id \otimes Tp} \\ B & \xrightarrow[\underline{\varphi}]{} & \wedge^r T^* B \otimes TB. \end{array}$$

Moreover,

$$\mathfrak{V}(E) = \bigoplus_{0 \leqslant r \leqslant n} \mathfrak{V}^r(E) = \bigoplus_{0 \leqslant r \leqslant n} \Omega^r(B) \otimes \mathcal{V}(E) \subset \mathfrak{P}(E),$$

is the subsheaf of (local) *vertical valued forms* on E, constituted by the (local) projectable tangent valued forms which are projected on zero TB-valued forms on B.

The *Frölicher-Nijenhuis bracket* endowes $\mathfrak{P}(E)$ with a canonical structure of a graded Lie R-algebras, which extends the Lie algebras $\mathscr{P}(E)$ of projectable vector fields on E, [18]. Namely, this bracket in $\mathfrak{P}(E)$ can be introduced directly as follows. If $\varphi \in \mathfrak{P}^r(E)$ and $\psi \in \mathfrak{P}^s(E)$, then $[\varphi, \psi]$ is the unique element of $\mathfrak{P}^{r+s}(E)$ such that, for each (local) vector fields $u_1, \ldots, u_{r+s}$ on B, we have

$$(1.1) \qquad [\varphi, \psi](u_1, \ldots, u_{r+s}) = \frac{1}{r!s!} \sum_\sigma \varepsilon(\sigma) \Big\{ [\varphi(u_{\sigma(1)}, \ldots, u_{\sigma(r)}), \psi(u_{\sigma(r+1)}, \ldots, u_{\sigma(r+s)})] -$$
$$-r\varphi(u_{\sigma(1)}, \ldots, u_{\sigma(r-1)}, [u_{\sigma(r)}, \underline{\psi}(u_{\sigma(r+1)}, \ldots, u_{\sigma(r+s)})]) -$$
$$-s\psi([\underline{\varphi}(u_{\sigma(1)}, \ldots, u_{\sigma(r)}), u_{\sigma(r+1)}], u_{\sigma(r+2)}, \ldots, u_{\sigma(r+s)}) +$$
$$+\frac{rs}{2} \varphi(u_{\sigma(1)}, \ldots, u_{\sigma(r-1)}, \underline{\psi}([u_{\sigma(r)}, u_{\sigma(r+1)}], u_{\sigma(r+2)}, \ldots, u_{\sigma(r+s)})) +$$
$$+\frac{rs}{2} \psi(\underline{\varphi}(u_{\sigma(1)}, \ldots, u_{\sigma(r-1)}, [u_{\sigma(r)}, u_{\sigma(r+1)}]), u_{\sigma(r+2)}, \ldots, u_{\sigma(r+s)}) \Big\},$$

where σ is a permutation of $(1, \ldots, r+s)$ and $\varepsilon(\sigma)$ is its sign. It is easy to see that the Frölicher-Nijenhuis bracket defined by (1.1) satisfies the conditions (0.1) and (0.2).

With respect to the Frölicher-Nijenhuis bracket, $\mathfrak{V}(E)$ is a subalgebra in $\mathfrak{P}(E)$.

We shall denote by

$$(x^\lambda, y^i) \qquad \lambda, \mu, \ldots = 1, \ldots, n, \quad i, j, \ldots = 1, \ldots, m,$$

a fibred chart on E, $n = \dim B$, $n + m = \dim E$.

The induced fibred chart on $\wedge^r T^* B \otimes TE$ is

$$(x^\lambda y^i, \varphi^\lambda_{\boldsymbol{\lambda}}, \varphi^i_{\boldsymbol{\lambda}}), \qquad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r), \qquad 1 \leqslant \lambda_1 < \ldots < \lambda_r \leqslant n.$$

Then, any $\varphi \in \mathfrak{P}^r(E)$ can be expressed as

$$(1.2) \qquad \varphi = (\varphi^\mu_{\boldsymbol{\lambda}}(x) \partial_\mu + \varphi^i_{\boldsymbol{\lambda}}(x, y) \partial_i) \otimes d^{\boldsymbol{\lambda}},$$

where $\partial_\mu = \partial/\partial x^\mu$, $\partial_i = \partial/\partial y^i$, $d^\lambda = dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_r}$ and its projection $\underline{\varphi} \in \Omega^r(B, TB)$ can be expressed as

$$\underline{\varphi} = \varphi^\mu_\lambda(x)\partial_\mu \otimes d^\lambda .$$

Moreover, φ is vertical iff $\underline{\varphi} = 0$.

If $\varphi \in \mathfrak{P}^r(E), \psi \in \mathfrak{P}^s(E)$ and

$$\varphi = (\varphi^\mu_{\boldsymbol{\alpha}}\partial_\mu + \varphi^i_{\boldsymbol{\alpha}}\partial_i) \otimes d^{\boldsymbol{\alpha}}, \qquad |\boldsymbol{\alpha}| = r,$$

$$\psi = (\varphi^\mu_{\boldsymbol{\beta}}\partial_\mu + \psi^i_{\boldsymbol{\beta}}\partial_i) \otimes d^{\boldsymbol{\beta}}, \qquad |\boldsymbol{\beta}| = s,$$

then the local coordinate expression of the Frölicher-Nijenhuis bracket is

$$(1.3) \qquad [\varphi, \psi] = \{(\varphi^\rho_{\boldsymbol{\alpha}}\partial_\rho\psi^\mu_{\boldsymbol{\beta}} - \psi^\rho_{\boldsymbol{\beta}}\partial_\rho\varphi^\mu_{\boldsymbol{\alpha}} - r\varphi^\mu_{\underline{\boldsymbol{\alpha}}\rho}\partial_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}} + s\psi^\mu_{\underline{\boldsymbol{\beta}}\rho}\partial_{\beta_s}\varphi^\rho_{\boldsymbol{\alpha}})\partial_\mu +$$
$$+(\varphi^\rho_{\boldsymbol{\alpha}}\partial_\rho\psi^i_{\boldsymbol{\beta}} + \varphi^j_{\boldsymbol{\alpha}}\partial_j\psi^i_{\boldsymbol{\beta}} - \psi^\rho_{\boldsymbol{\beta}}\partial_\rho\varphi^i_{\boldsymbol{\alpha}} - \psi^j_{\boldsymbol{\beta}}\partial_j\varphi^i_{\boldsymbol{\alpha}} - r\varphi^i_{\underline{\boldsymbol{\alpha}}\rho}\partial_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}} + s\psi^i_{\underline{\boldsymbol{\beta}}\rho}\partial_{\beta_s}\varphi^\rho_{\boldsymbol{\alpha}})\partial_i\} \otimes d^\gamma$$

where $|\underline{\boldsymbol{\alpha}}| = r - 1, |\underline{\boldsymbol{\beta}}| = s - 1$ and γ denotes the antisymmetrization of all indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

From the coordinate expression (1.3) it is easy to see that the Frölicher-Nijenhuis bracket is a natural $\mathbb{R}$-bilinear sheaf morphism (differential operator) $\mathfrak{P}^r(E) \times \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E)$, which is of order one. Here, order one means that, for $\forall y \in E$, $[\varphi, \psi](y)$ depends on the first order derivatives (with respect to x^λ, y^i) of φ and ψ at y. Naturality means that, for any (local) fibred diffeomorphism $f: E \to \overline{E}$ projectable on the diffeomorphism $\underline{f}: B \to \overline{B}$ the following condition holds

$$(1.4) \qquad f^*[\varphi, \psi] = [f^*\varphi, f^*\psi],$$

for any $\varphi \in \mathfrak{P}^r(E)$, where $f^*: \mathfrak{P}^r(E) \to \mathfrak{P}^r(\overline{E})$ is defined as

$$(1.5) \qquad f^*\varphi : (\wedge^r T^*\underline{f} \otimes Tf) \circ \varphi \circ f^{-1} : \overline{E} \to \wedge^r T^*\overline{B} \otimes T\overline{E}.$$

In the present paper we shall classify all natural $\mathbb{R}$-bilinear operators

$$B_E : \mathfrak{P}^r(E) \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E).$$

Such classification for the case of TB-valued forms on B was done by I. KOLÁŘ and P. MICHOR [14]. They have deduced that if $\dim B > r + s + 1$, $r \geqslant 2$, $s \geqslant 2$, then there is a ten-parameter family of $\mathbb{R}$-bilinear natural operators of demanded type. In our main theorem it is sufficient to suppose $r \geqslant 1$, $s \geqslant 1$.

2. – The order of natural $\mathbb{R}$-bilinear operators $\mathfrak{P}^r(E) \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E)$.

Local operators are finite order differential operators, by the Peetre theorem, [27]. Then, we can restrict our study to finite order operators.

Let G^k_{n+m} be the group of k-jets of diffeomorphisms $f: \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ with source and target 0. Let $G^k_{n,m} \subset G^k_{n+m}$ be the subgroup of k-jets of diffeomorphisms which preserve the fibration $\mathbb{R}^{n+m} \to R^n$, i.e. whose coordinate expression is $(f^\mu(x^\lambda), f^i(x^\lambda, y^j))$.

Hence, the coordinates on $G^k_{n,m}$ are given by

$$a^\mu_\lambda(j^k f(0)) = \partial_\lambda f^\mu(0), \qquad a^i_{\alpha p}(j^k f(0)) = \partial_\alpha \partial_p f^i(0,0), \tag{2.1}$$

where λ, α, p are (symmetric) multiindices such that $|\lambda| = 1, \ldots, k$, $|\alpha| + |p| = 1, \ldots, k$, and $\partial_\lambda = \partial_{\lambda_1} \ldots \partial_{\lambda_{|\lambda|}}$. We shall denote by tilde the coordinates of the element $A^{-1} \in G^k_{n+m}$ inverse of $A \in G^k_{n+m}$ and we shall write shortly

$$A = (a^\mu_\lambda, a^i_{\alpha p}) \text{ and } A^{-1}(\tilde a^\mu_\lambda, \tilde a^i_{\alpha p}).$$

The type fibre of $\wedge^r T^*B \otimes TE$ is

$$S^0_r = (\mathbb{R}^n \otimes \wedge^r \mathbb{R}^{n*}) \times (\mathbb{R}^m \otimes \wedge^r \mathbb{R}^{n*}).$$

Its global coordinates are

$$(\varphi^\mu_\lambda, \varphi^i_\lambda), \qquad |\lambda| = r, 1 \leq \lambda_1 < \ldots < \lambda_r \leq n. \tag{2.2}$$

We obtain an action χ of the group $G^1_{n,m}$ on S^0_r, which is given in coordinates by

$$\bar\varphi^\mu_\lambda \circ \chi = \partial^\mu_\nu \varphi^\nu_\rho \tilde a^\rho_\lambda, \tag{2.3}$$

$$\bar\varphi^i_\lambda \circ \chi = (a^i_\nu \varphi^\nu_\rho + a^i_j \varphi^j_\rho) \tilde a^\rho_\lambda, \tag{2.4}$$

where $\tilde a^\rho_\lambda = \tilde a^{\rho_1}_{\lambda_1} \ldots \tilde a^{\rho_r}_{\lambda_r}$.

Now, let S^k_r be the type fibre of $J^k_E(\wedge^r T^*B \otimes TE)$ (where J^k_E denotes the jet functor over E). It means that S^k_r is the space of k-jets, with source 0, of the maps $\alpha: \mathbb{R}^{n+m} \to S^0_r$ which are projectable on $\underline{\alpha}: \mathbb{R}^n \to \mathbb{R}^n \otimes \wedge^r \mathbb{R}^{n*}$ via the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^{n+m} & \xrightarrow{\alpha} & S^0_r = (\mathbb{R}^n \otimes \wedge^n \mathbb{R}^{n*}) \times (\mathbb{R}^m \otimes \wedge^r \mathbb{R}^{n*}) \\ \downarrow & & \downarrow pr_1 \\ \mathbb{R}^n & \xrightarrow[\underline{\alpha}]{} & \mathbb{R}^n \otimes \wedge^r \mathbb{R}^{n*}. \end{array}$$

The induced coordinates on S^k_r are

$$(\varphi^\mu_{\lambda,\alpha}, \varphi^i_{\lambda,\beta p}), \qquad |\lambda| = r, |\alpha| = 0, \ldots, k, |\beta| + |p| = 0, \ldots, k. \tag{2.5}$$

By using standard jet techniques, the action χ: $G^1_{n,m} \times S^0_r \to S^0_r$ can be prolonged to the action

$$\chi^k : G^{k+1}_{n,m} \times S^k_r \to S^k_r, \tag{2.6}$$

According to the general theory of natural differential operators [10, 12, 28], all natural $\mathbb{R}$-bilinear operators of order k $\mathfrak{P}^r(E) \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E)$ are in bijective correspondence with the $G^{k+1}_{n,m}$-equivariant $\mathbb{R}$-bilinear maps

$$f: S^k_r \times S^k_s \to S^0_{r+s}. \tag{2.7}$$

Hence, the classification of natural $\mathbb{R}$-biliner operators

$$B_E : \mathfrak{P}^r(E) \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E),$$

is reduced to be classification of $\mathbb{R}$-bilinear $G^{k+1}_{n,m}$-equivalent maps (2.7) for certain k. To classify the maps (2.7) we shall use the method of [16] modified for the group $G^{k+1}_{n,m}$. This method is based on the following

LEMMA. – Let U and W be two $G^k_{n,m}$-manifolds and f: $U \to W$ a map. Then, the following conditions are equivalent:

(i) f is a $G^{k(+)}_{n,m}$-equivariant map.

(ii) For each element $\xi \in \mathfrak{g}^k_{n,m}$ (where $\mathfrak{g}^k_{n,m}$ is the Lie algebra of $G^k_{n,m}$) we have

$$\partial_\xi f = 0, \tag{2.8}$$

where ∂_ξ denotes the Lie derivative with respect to ξ and $G^{k(+)}_{n,m}$ is the maximal connected subgroup of $G^k_{n,m}$. ■

This lemma is a simple modification of the lemma for a Lie group G which is proved, for instance, in [16].

THEOREM 1. – All natural $\mathbb{R}$-bilinear operators $\mathfrak{P}^r(E) \times \mathfrak{P}^s(E) \to \mathfrak{P}^{r+s}(E)$ are of order one.

PROOF. – According to the general theory, we have to prove that all $\mathbb{R}$-bilinear $G^{k+1}_{n,m}$-equivariant maps f: $S^k_r \times S^k_s \to S^0_{r+s}$, $k \geq 1$, depend on the coordinates of $S^1_r \times \times S^{1s}$ only.

Let

$$\varphi^\mu_\gamma \circ f = f^\mu_\gamma(\varphi^\lambda_{\alpha,\nu}, \varphi^j_{\alpha,\kappa l}, \psi^\lambda_{\beta,\nu}, \psi^j_{\beta,\kappa l}), \tag{2.9}$$

$$\varphi^i_\gamma \circ f = f^i_\gamma(\varphi^\lambda_{\alpha,\nu}, \varphi^j_{\alpha,\kappa l}, \psi^\lambda_{\beta,\nu}, \psi^j_{\beta,\kappa l}), \tag{2.10}$$

$|\gamma| = r+s$, $|\alpha| = r$, $|\beta| = s$, $|\nu| = 0, \ldots, k$, $|\kappa| + |l| = 0, \ldots, k$, be the coordinate expression of f.

Let $\iota: G^1_n \times G^1_m \to G^{k+1}_{n,m}$ be the canonical group homomorphism. If f is a $G^{k+1}_{n,m}$-equivariant map, then f has to be also a $\iota(G^1_n \times G^1_m)$-equivariant map. The restriction of the action χ^k to the subgroup $\iota(G_n^1 \times G_m^1)$ has the following simple coordinate expression

$$\bar{\varphi}^{\lambda}_{\lambda,\nu} \circ \chi^k = a^{\lambda}_{\beta} \varphi^{\beta}_{\rho,\sigma} \tilde{a}^{\rho}_{\lambda} \tilde{a}^{\sigma}_{\nu}, \qquad |\nu| = 0, \ldots, k, \tag{2.11}$$

$$\bar{\varphi}_{\lambda,\nu l} \circ \chi^k = a^i_j \varphi^j_{\rho,\sigma\mu} \tilde{a}^{\rho}_{\lambda} \tilde{a}^{\sigma}_{\nu} \tilde{a}^{m}_{l} \qquad |\nu| + |l| = 0, \ldots, k, \tag{2.12}$$

Let e^{μ}_{λ}, e^q_p be a base of the Lie subalgebra in $\mathfrak{g}^{k+1}_{n,m}$ corresponding to the subgroup $\iota(G_n^1 \times G_m^1)$ and ξ be an element of this subalgebra. Then

$$\xi = \xi^{\lambda}_{\mu} e^{\mu}_{\lambda} + \xi^p_q e^q_p. \tag{2.13}$$

The fundamental vector field on S^k_r related to the action (2.11) and (2.12) of $\iota(G_n^1 \times G_m^1)$ on S^k_r can be expressed by

$$\Xi(S^k_r) = \Xi^{\mu}_{\lambda}(S^k_r)\xi^{\lambda}_{\mu} + \Xi^q_p(S^k_r)\xi^p_q, \tag{2.14}$$

where $\Xi^{\mu}_{\lambda}(S^k_r), \Xi^q_p(S^k_r)$ are vector fields on S^k_r defined by

$$\Xi^{\mu}_{\lambda}(S^k_r) = \sum_{|\nu|=0}^{k} (\partial \bar{\varphi}^{\alpha}_{\lambda,\nu} / \partial a^{\lambda}_{\mu})_e \partial a^{\lambda,\nu}_{\alpha} + \sum_{|\nu|+|m|=0}^{k} (\partial \bar{\varphi}^j_{\lambda,\nu m} / \partial a^{\lambda}_{\mu})_e \partial^{\lambda,\nu m}_j, \tag{2.15}$$

$$\Xi^q_p(S^k_r) = \sum_{|\nu|+|m|=0}^{k} (\partial \bar{\varphi}^j_{\nu,\nu m} / \partial a^p_q)_e \partial^{\lambda,\nu m}_j, \tag{2.16}$$

where e is the unity in the group $G^{k+1}_{n,m}$, i.e.

$$e = j^{k+1}_0 id_{\mathbb{R}^{n+m}} \text{ and } \partial^{\lambda,\nu}_{\alpha} = \partial/\partial \varphi^{\alpha}_{\lambda,\nu}, \partial^{\lambda,\nu m}_j = \partial/\partial \varphi^j_{\lambda,\nu m}.$$

The condition (2.8) is then equivalent to the f-relation of vector fields $\Xi(S^k_r) + \Xi(S^k_s)$ and $\Xi(S^0_{r+s})$. From the first part of the coordinate expression of f, given by (2.9), we obtain, for $\lambda = \mu$, $p = q$, the following systems of partial differential equations

$$\sum_{|\nu|=0}^{k} (1 - r - |\nu|) \varphi^{\beta}_{\lambda,\nu} \partial^{\lambda,\nu}_{\beta} f^{\alpha}_{\gamma} + \sum_{|\nu|+|m|=0}^{k} (-r - |\nu|) \varphi^j_{\lambda,\nu m} \partial^{\lambda,\nu m}_j f^{\alpha}_{\gamma} + \sum_{|\nu|=0}^{k} (1 - s - |\nu|) \psi^{\beta}_{\rho,\nu} \bar{\partial}^{\rho,\nu}_{\beta} f^{\alpha}_{\gamma} + \sum_{|\nu|+|m|=0}^{k} (-s - |\nu|) \psi^j_{\rho,\nu m} \bar{\partial}^{\rho,\nu m}_j f^{\alpha}_{\gamma} = (1 - r - s) f^{\alpha}_{\gamma}, \tag{2.17}$$

$$\sum_{|\nu|+|m|=0}^{k} (1 - |m|) \varphi^j_{\lambda,\nu m} \partial^{\lambda,\nu m}_j f^{\alpha}_{\gamma} + \sum_{|\nu|+|m|=0}^{k} (1 - |m|) \psi^j_{\rho,\nu m} \bar{\partial}^{\rho,\nu m}_j f^{\alpha}_{\gamma} = 0, \tag{2.18}$$

where $|\lambda| = r$, $|\rho| = s$ and $\bar{\partial}^{\rho,\nu}_{\beta} = \partial/\partial \psi^{\beta}_{\rho,\nu}$, $\bar{\partial}^{\rho,\nu m}_j = \partial/\partial \psi^j_{\rho,\nu m}$. We are interested in bilinear and hence polynomial solutions of (2.17) and (2.18). Let us denote as $a_{|\nu|}$ the degree of f with respect to $\varphi^{\beta}_{\lambda,\nu}$, as $a_{|\nu||m|}$ the degree of f with respect to $\varphi^j_{\lambda,\nu m}$ and similarly as $b_{|\nu|}$ the degree with respect to $\psi^{\beta}_{\rho,\nu}$ and as $b_{|\nu||m|}$ the degree with respect to $\psi^j_{\rho,\nu m}$. Then, according to [16], the degrees have to satisfy the following system of linear equations

$$(2.19)\qquad \sum_{|\nu|=0}^{k} a_{|\nu|}(1-r-|\nu|) + \sum_{|\nu|+|m|=0}^{k} a_{|\nu||m|}(-r-|\nu|) + \\ + \sum_{|\nu|=0}^{k} b_{|\nu|}(1-s-|\nu|) + \sum_{|\nu|+|m|=0}^{k} b_{|\nu||m|}(-s-|\nu|) = 1-r-s,$$

$$(2.20)\qquad \sum_{|\nu|+|m|=0}^{k} a_{|\nu||m|}(1-|m|) + \sum_{|\nu|+|m|=0}^{k} b_{|\nu||m|}(1-|m|) = 0.$$

It is easy to see that there are only four solutions (in $\{0\} \cup N$) which correspond to bilinear maps. They are:

$$(2.21)\qquad \begin{array}{lll} a_0 = 1, & b_{01} = 1 & \text{and the other variables vanish,} \\ a_0 = 1, & b_1 = 1 & \text{and the other variables vanish,} \\ a_{01} = 1, & b_0 = 1 & \text{and the other variables vanish,} \\ a_1 = 1, & b_0 = 1 & \text{and the other variables vanish.} \end{array}$$

It implies that f^{α}_{γ} is defined on $S^1_r \times S^1_s$ only.

By using the same method for the second part of f, given by (2.10), we obtain the following system of linear equations for the degrees

$$(2.22)\qquad \sum_{|\nu|=0}^{k} a_{|\nu|}(1-r-|\nu|) + \sum_{|\nu|+|m|=0}^{k} a_{|\nu||m|}(-r-|\nu|) + \\ + \sum_{|\nu|=0}^{k} b_{|\nu|}(1-s-|\nu|) + \sum_{|\nu|+|m|=0}^{k} b_{|\nu||m|}(-s-|\nu|) = -r-s,$$

$$(2.23)\qquad \sum_{|\nu|+|m|=0}^{k} a_{|\nu||m|}(1-|m|) + \sum_{|\nu|+|m|=0}^{k} b_{|\nu||m|}(1-|m|) = 1.$$

There are only six solutions which correspond to bilinear maps. They are:

$$(2.24)\qquad \begin{array}{lll} a_0 = 1, & b_{10} = 1 & \text{and the other variables vanish,} \\ a_{10} = 1, & b_0 = 1 & \text{and the other variables vanish,} \\ a_{00} = 1, & b_{01} = 1 & \text{and the other variables vanish,} \\ a_{01} = 1, & b_{00} = 1 & \text{and the other variables vanish,} \\ a_{00} = 1, & b_1 = 1 & \text{and the other variables vanish,} \\ a_1 = 1, & b_{00} = 1 & \text{and the other variables vanish.} \end{array}$$

Hence also f^{α}_{γ} is defined on $S^1_r \times S^1_s$ only which proves our Theorem 1. ∎

3. – Classification of $\mathbb{R}$-bilinear natural operators from $\mathfrak{P}^r(E)\times\mathfrak{P}^s(E)$ to $\mathfrak{P}^{r+s}(E)$.

The Theorem 1 implies that we can restrict our study to the first order $\mathbb{R}$-bilinear operators only. Such operators are in bijective correspondence with $\mathbb{R}$-bilinear $G^2_{n,m}$-equivariant maps $f\colon S^1_r\times S^1_s\to S^0_{r+s}$. The action χ^1 of the group $G^2_{n,m}$ on S^1_r has, together with (2.3) and (2.4), the following coordinate expression

$$\bar\varphi^\alpha_{\boldsymbol\lambda,\rho}\circ\chi^1 = a^\alpha_{\beta\gamma}\tilde a^\gamma_\rho\varphi^\beta_{\boldsymbol\sigma}\tilde a^{\boldsymbol\sigma}_{\boldsymbol\lambda} + a^\alpha_\beta\varphi^\beta_{\boldsymbol\sigma,\gamma}\tilde a^\beta_{\rho,\gamma}\tilde a^\gamma_\rho\tilde a^{\boldsymbol\sigma}_{\boldsymbol\lambda} + a^\alpha_\beta\varphi^\beta_{\boldsymbol\sigma}(\tilde a^{\sigma_1}_{\lambda_1\rho}\tilde a^{\underline{\boldsymbol\sigma}}_{\underline{\boldsymbol\lambda}} + \ldots + \tilde a^{\underline{\boldsymbol\sigma}}_{\underline{\boldsymbol\lambda}}\tilde a^{\sigma_r}_{\lambda_r\rho}), \tag{3.1}$$

$$\bar\varphi^i_{\boldsymbol\lambda,\rho}\circ\chi^1 = \{(a^i_{j\gamma}\tilde a^\gamma_\rho + a^i_{jm}\tilde a^m_\rho)\varphi^j_{\boldsymbol\sigma} + a^i_j(\varphi^j_{\boldsymbol\sigma,\gamma}\tilde a^\gamma_\rho + \varphi^j_{\boldsymbol\sigma,m}\tilde a^m_\rho) + \tag{3.2}$$
$$+(a^i_{\beta\gamma}\tilde a^\gamma_\rho + a^i_{\beta m}\tilde a^m_\rho)\varphi^\beta_{\boldsymbol\sigma} + a^i_\beta(\varphi^\beta_{\boldsymbol\sigma,\gamma}\tilde a^\gamma_\rho + \varphi^\beta_{\boldsymbol\sigma,m}\tilde a^m_\rho)\}\tilde a^{\boldsymbol\sigma}_{\boldsymbol\lambda} + (a^i_j\varphi^j_{\boldsymbol\sigma} + a^i_\beta\varphi^\beta_{\boldsymbol\sigma})(\tilde a^{\sigma_1}_{\lambda_1\rho}\tilde a^{\underline{\boldsymbol\sigma}}_{\underline{\boldsymbol\lambda}} + \ldots + \tilde a^{\underline{\boldsymbol\sigma}}_{\underline{\boldsymbol\lambda}}\tilde a^{\sigma_r}_{\lambda_r\rho}),$$

$$\bar\varphi^i_{\boldsymbol\lambda,j}\circ\chi^1 = (a^i_{km}\tilde a^m_j\varphi^k_{\boldsymbol\sigma} + a^i_k\varphi^k_{\boldsymbol\sigma,m}\tilde a^m_j + a^i_{\beta m}\tilde a^m_j\varphi^\beta_{\boldsymbol\sigma})\tilde a^{\boldsymbol\sigma}_{\boldsymbol\lambda}, \tag{3.3}$$

where $\underline{\boldsymbol\sigma}$ and $\underline{\boldsymbol\lambda}$ arise from $\boldsymbol\sigma$ and $\boldsymbol\lambda$ by leaving out one index and the summation runs over $\boldsymbol\sigma$ and $(\sigma_i,\underline{\boldsymbol\sigma})$.

Let $\xi\in\mathfrak{g}^2_{n,m}$. Then

$$\xi = \xi^\lambda_\mu e^\mu_\lambda + \xi^p_q e^q_p + \xi^p_\mu e^\mu_p + \xi^\lambda_{\mu\nu}e^{\mu\nu}_\lambda + \xi^p_{\mu\nu}e^{\mu\nu}_p + \xi^p_{q\mu}e^{q\mu}_p + \xi^p_{qr}e^{qr}_p, \tag{3.4}$$

where $e^\mu_\lambda, e^q_p, e^\mu_p, e^{\mu\nu}_\lambda e^{\mu\nu}_p, e^{q\mu}_p, e^{qr}_p$ is a system of generators of $\mathfrak{g}^2_{n,m}$. The fundamental vector field on S^1_r generated by ξ related to the action χ^1 of $G^2_{n,m}$ on S^1_r is

$$\Xi(S_r) = \Xi^\mu_\lambda(S^1_r)\xi^\lambda_\mu + \Xi^q_p(S^1_r)\xi^p_q + \Xi^\mu_p(S^1_r)\xi^p_\mu + \Xi^{\mu\nu}_\lambda(S^1_r)\xi^\lambda_{\mu\nu} + \tag{3.5}$$
$$+\Xi^{\mu\nu}_p(S^1_r)\xi^p_{\mu\nu} + \Xi^{q\mu}_p(S^1_r)\xi^\rho_{q\mu} + \Xi^{qr}_p(S^1_r)\xi^p_{qr},$$

where $\Xi^\mu_\lambda(S^1_r)$ and $\Xi^q_p(S^1_r)$ are given by (2.15) and (2.16), respectively, and

$$\Xi^\mu_p(S^1_r) = (\partial\bar\varphi^m_{\boldsymbol\lambda}/\partial a^p_\mu)_e\partial^{\boldsymbol\lambda}_m + (\partial\bar\varphi^m_{\boldsymbol\lambda,\rho}/\partial a^p_\mu)_e\partial^{\boldsymbol\lambda,\rho}_m, \tag{3.6}$$

$$\Xi^{\mu\nu}_\lambda(S^1_r) = (\partial\bar\varphi^\alpha_{\boldsymbol\lambda,\rho}/\partial a^\lambda_{\mu\nu})_e\partial^{\boldsymbol\lambda,\rho}_\alpha + (\partial\bar\varphi^m_{\boldsymbol\lambda,\rho}/\partial a^\lambda_{\mu\nu})_e\partial^{\boldsymbol\lambda,\rho}_m, \tag{3.7}$$

$$\Xi^{\mu\nu}_p(S^1_r) = (\partial\bar\varphi^m_{\boldsymbol\lambda,\rho}/\partial a^p_{\mu\nu})_e\partial^{\boldsymbol\lambda,\rho}_m, \tag{3.8}$$

$$\Xi^{q\mu}_p(S^1_r) = (\partial\bar\varphi^m_{\boldsymbol\lambda,\rho}/\partial a^p_{q\mu})_e\partial^{\boldsymbol\lambda,\rho}_m + (\partial\bar\varphi^m_{\boldsymbol\lambda,k}/\partial a^p_{q\mu})_e\partial^{\boldsymbol\lambda,k}_m, \tag{3.9}$$

$$\Xi^{qs}_p(S^1_r) = (\partial\bar\varphi^m_{\boldsymbol\lambda,k}/\partial a^p_{qs})_e\partial^{\boldsymbol\lambda,k}_m, \tag{3.10}$$

where e is the unity in $G^2_{n,m}$. The vector fields (2.15), (2.16) and (3.6)-(3.10) span the Lie algebra of fundamental vector fields on S^1_r. Now, we are in position to prove the main theorem

THEOREM 2. – All natural $\mathbb{R}$-bilinear operators

$$\mathfrak{P}^r(E)\times\mathfrak{P}^s(E)\to\mathfrak{P}^{r+s}(E)\qquad \dim B > r+s,\ r\geqslant 1,\ s\geqslant 1,$$

form a vector space over $\mathbb{R}$ generated by the following three operators

$$[\varphi, \psi], \qquad p^* dC\underline{\varphi} \wedge \psi, \qquad \varphi \wedge p^* dC\underline{\psi},$$

where C is the contraction operator, d is the exterior derivative and $\wedge$ is the exterior product of basic differential forms and tangent valued forms on E.

PROOF. – To prove our theorem we have to find all $\mathbb{R}$-bilinear $G^2_{n,m}$-equivariant maps $S^1_r \times S^1_s \to S^0_{r+s}$. During the proof of Theorem 1 we have proved ((2.21) and (2.24)) that the coordinate expressions of such maps are

$$f^\lambda_\gamma = A^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}\rho}_{\gamma\beta\gamma}\, \varphi^\beta_{\boldsymbol{\alpha}} \psi^\gamma_{\boldsymbol{\beta},\rho} + \overline{A}^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}\rho}_{\gamma\beta\gamma}\, \varphi^\beta_{\boldsymbol{\alpha},\rho} \psi^\gamma_{\boldsymbol{\beta}} + B^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}r}_{\gamma\beta m}\, \psi^\beta_{\boldsymbol{\alpha}} \psi^m_{\boldsymbol{\beta},r} + \overline{B}^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}r}_{\gamma\beta m}\, \varphi^m_{\boldsymbol{\alpha},r} \psi^\beta_{\boldsymbol{\beta}}, \tag{3.11}$$

$$\begin{aligned} f^i_\gamma = {} & C^{i\boldsymbol{\alpha}\boldsymbol{\beta}\rho}_{\gamma\gamma m}\, \varphi^\gamma_{\boldsymbol{\alpha}} \psi^m_{\boldsymbol{\beta},\rho} + \overline{C}^{i\boldsymbol{\beta}\boldsymbol{\alpha}\rho}_{\gamma\gamma m}\, \psi^\gamma_{\boldsymbol{\beta}} \varphi^m_{\boldsymbol{\alpha},\rho} + D^{i\boldsymbol{\alpha}\boldsymbol{\beta}s}_{\gamma m r}\, \varphi^m_{\boldsymbol{\alpha}} \psi^r_{\boldsymbol{\beta},s} + \\ & + \overline{D}^{i\boldsymbol{\beta}\boldsymbol{\alpha}s}_{\gamma m r}\, \psi^m_{\boldsymbol{\beta}} \varphi^r_{\boldsymbol{\alpha},s} + E^{i\boldsymbol{\alpha}\boldsymbol{\beta}\rho}_{\gamma m\gamma}\, \varphi^m_{\boldsymbol{\alpha}} \psi^\gamma_{\boldsymbol{\beta},\rho} + \overline{E}^{i\boldsymbol{\beta}\boldsymbol{\alpha}\rho}_{\gamma m\gamma}\, \psi^m_{\boldsymbol{\beta}} \varphi^\gamma_{\boldsymbol{\alpha},\rho}, \end{aligned} \tag{3.12}$$

where, from the invariancy condition, all real coefficients $A^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}\rho}_{\gamma\beta\gamma}, \ldots, \overline{E}^{i\boldsymbol{\beta}\boldsymbol{\alpha}\rho}_{\gamma m\gamma}$ are absolute invariant tensors, i.e. linear combinations of Kronecker symbols, [3], [12], [16].

The condition (2.8) is equivalent to the f-relation of vector fields

$$\Xi(S^1_r) + \Xi(S^1_r) \text{ and } \Xi(S^0_{r+s}).$$

By using the vector fields (3.7)-(3.10), we obtain that both (3.11) and (3.12) have to satisfy the following systems of partial differential equations

$$\begin{aligned} & \varphi^\mu_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},\nu}_{\lambda} f + \varphi^\nu_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},\mu}_{\lambda} f - r\varphi^\gamma_{\lambda\underline{\boldsymbol{\lambda}}} \partial^{\nu\underline{\boldsymbol{\lambda}},\mu}_{\gamma} f - r\varphi^\gamma_{\lambda\underline{\boldsymbol{\lambda}}} \partial^{\mu\underline{\boldsymbol{\lambda}},\nu}_{\lambda} f - r\varphi^m_{\lambda\underline{\boldsymbol{\lambda}}} \partial^{\nu\underline{\boldsymbol{\lambda}},\mu}_{m} f - r\varphi^m_{\lambda\underline{\boldsymbol{\lambda}}} \partial^{\mu\underline{\boldsymbol{\lambda}},\nu}_{m} f + \\ & + \psi^\mu_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},\nu}_{\lambda} f + \psi^\nu_{\sigma} \overline{\partial}^{\boldsymbol{\sigma},\mu}_{\lambda} f - s\psi^\gamma_{\lambda\underline{\boldsymbol{\sigma}}} \overline{\partial}^{\nu\underline{\boldsymbol{\sigma}},\mu}_{\gamma} f - s\psi^\gamma_{\lambda\underline{\boldsymbol{\sigma}}} \overline{\partial}^{\mu\underline{\boldsymbol{\sigma}},\nu}_{\gamma} f - s\psi^m_{\lambda\underline{\boldsymbol{\sigma}}} \overline{\partial}^{\nu\underline{\boldsymbol{\sigma}},\mu}_{m} f - s\psi^m_{\lambda\underline{\boldsymbol{\sigma}}} \overline{\partial}^{\mu\underline{\boldsymbol{\sigma}},\nu}_{m} f = 0, \end{aligned} \tag{3.13}$$

$$\varphi^\nu_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},\mu}_{p} f + \varphi^\mu_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},\nu}_{p} f + \psi^\nu_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},\mu}_{p} f + \psi^\mu_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},\nu}_{p} f = 0, \tag{3.14}$$

$$\varphi^q_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},\mu}_{p} f + \varphi^\mu_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},q}_{p} f + \psi^q_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},\mu}_{p} f + \psi^\mu_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},q}_{p} f = 0, \tag{3.15}$$

$$\varphi^q_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},r}_{p} f + \varphi^r_{\boldsymbol{\lambda}} \partial^{\boldsymbol{\lambda},q}_{p} f + \psi^q_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},r}_{p} f + \psi^r_{\boldsymbol{\sigma}} \overline{\partial}^{\boldsymbol{\sigma},q}_{p} f = 0, \tag{3.16}$$

$|\boldsymbol{\lambda}| = r$, $|\boldsymbol{\sigma}| = s$, $|\underline{\boldsymbol{\lambda}}| = r - 1$, $|\underline{\boldsymbol{\sigma}}| = s - 1$. Let us discuss first the map f^λ_γ given by (3.11). By putting (3.11) into (3.16), we get $B^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}r}_{\gamma\beta m} = 0$, $\overline{B}^{\lambda\boldsymbol{\alpha}\boldsymbol{\beta}r}_{\gamma\beta m} = 0$. Hence, we can rewrite (3.11) in the form

$$\begin{aligned} f^\lambda_\gamma = {} & A_1\, \varphi^\lambda_{\boldsymbol{\alpha}} \psi^\gamma_{\boldsymbol{\beta},\gamma} + A_2\, \varphi^\lambda_{\gamma\underline{\boldsymbol{\alpha}}} \psi^\gamma_{\boldsymbol{\beta},\alpha_r} + A_3\, \varphi^\lambda_{\boldsymbol{\alpha}} \psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\beta_s} + A_4\, \varphi^\gamma_{\boldsymbol{\alpha}} \psi^\lambda_{\boldsymbol{\beta},\gamma} + A_5\, \varphi^\gamma_{\boldsymbol{\alpha}} \psi^\lambda_{\gamma\underline{\boldsymbol{\beta}},\beta_s} + \\ & + A_6\, \varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}}} \psi^\lambda_{\boldsymbol{\beta},\alpha_r} + A_7\, \delta^\lambda_{\alpha_r} \varphi^\rho_{r\underline{\boldsymbol{\alpha}}} \psi^\gamma_{\boldsymbol{\beta},\gamma} + A_8\, \delta^\lambda_{\alpha_r} \varphi^\rho_{r\underline{\boldsymbol{\alpha}}} \psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\beta_s} + A_9\, \delta^\lambda_{\alpha_r} \varphi^\rho_{\rho\gamma\hat{\underline{\boldsymbol{\alpha}}}} \psi^\gamma_{\underline{\boldsymbol{\beta}},\alpha_{r-1}} + A_{10}\, \delta^\lambda_{\beta_s} \varphi^\rho_{\alpha} \psi^\gamma_{\rho\gamma\hat{\underline{\boldsymbol{\beta}}},\beta_{s-1}} + \\ & + A_{11}\, \delta^\lambda_{\beta_s} \varphi^\rho_{\boldsymbol{\alpha}} \psi^\gamma_{\rho\underline{\boldsymbol{\beta}},\gamma} + A_{12}\, \delta^\lambda_{\alpha_r} \varphi^\rho_{\gamma\underline{\boldsymbol{\alpha}}} \psi^\gamma_{\rho\underline{\boldsymbol{\beta}},\beta_s} + A_{13}\, \delta^\lambda_{\beta_s} \varphi^\rho_{\boldsymbol{\alpha}} \psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\rho} + A_{14}\, \delta^\lambda_{\alpha_r} \varphi^\rho_{\gamma\underline{\boldsymbol{\alpha}}} \psi^\gamma_{\underline{\boldsymbol{\beta}},\rho} + \overline{A}_1\, \psi^\lambda_{\boldsymbol{\beta}} \varphi^\gamma_{\boldsymbol{\alpha},\gamma} + \\ & + \overline{A}_2\, \psi^\lambda_{\gamma\underline{\boldsymbol{\beta}}} \varphi^\gamma_{\boldsymbol{\alpha},\beta_s} + \overline{A}_3\, \psi^\lambda_{\boldsymbol{\beta}} \varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r} + \overline{A}_4\, \psi^\gamma_{\boldsymbol{\beta}} \varphi^\lambda_{\boldsymbol{\alpha},\gamma} + \overline{A}_5\, \psi^\gamma_{\boldsymbol{\beta}} \varphi^\lambda_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r} + \overline{A}_6\, \psi^\gamma_{\gamma\underline{\boldsymbol{\beta}}} \varphi^\lambda_{\boldsymbol{\alpha},\beta_s} + \end{aligned} \tag{3.17}$$

$$+\overline{A}_7\delta^\lambda_{\beta_s}\psi^\rho_{\rho\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\boldsymbol{\alpha},\gamma}+\overline{A}_8\delta^\lambda_{\beta_s}\psi^\rho_{\rho\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\underline{\boldsymbol{\alpha}},\alpha_r}+\overline{A}_9\delta^\lambda_{\beta_s}\psi^\rho_{\rho\gamma\underline{\hat{\boldsymbol{\beta}}}}\varphi^\gamma_{\boldsymbol{\alpha},\beta_{s-1}}+\overline{A}_{10}\delta^\lambda_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}}\varphi^\gamma_{\rho\gamma\underline{\hat{\boldsymbol{\alpha}}},\alpha_{r-1}}+$$

$$+\overline{A}_{11}\delta^\lambda_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}}\varphi^\gamma_{\rho\underline{\boldsymbol{\alpha}},\gamma}+\overline{A}_{12}\delta^\lambda_{\beta_s}\psi^\rho_{\gamma\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\rho\underline{\boldsymbol{\alpha}},\alpha_r}+\overline{A}_{13}\delta^\lambda_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}}\varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\rho}+\overline{A}_{14}\delta^\lambda_{\beta_s}\psi^\rho_{\gamma\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\boldsymbol{\alpha},\rho},$$

where $|\hat{\boldsymbol{\alpha}}|=r-2$, $|\hat{\boldsymbol{\beta}}|=s-2$ and $\boldsymbol{\gamma}$ is the antisymmetrization of all indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. At the moment we have to suppose $r\geqslant 2$, $s\geqslant 2$, but this assumption can be omitted later. By putting (3.17) into (3.13), we obtain after long and tedious calculations

$$A_1=\overline{A}_1=A_5=\overline{A}_5=A_6=\overline{A}_6=A_7=\overline{A}_7=A_{11}=\overline{A}_{11}=A_{14}=\overline{A}_{14}=0,$$

$$\overline{A}_4=-A_4,\ A_2=(-1)^r rA_4,\ \overline{A}_2=-(-1)^s sA_4,\ A_{12}=(-1)^{s-1}\overline{A}_{12},$$

$$A_9=(-1)^s\overline{A}_{10}+(-1)^{r+s}(r-1)\overline{A}_{13},\ \overline{A}_9=(s-1)A_{13}+(-1)^r A_{10},$$

and $A_3,\overline{A}_3,A_4,A_8,\overline{A}_8,A_{10},\overline{A}_{10},\overline{A}_{12},A_{13},\overline{A}_{13}$ are arbitrary. Hence, we get

$$\text{(3.18)}\qquad f^\lambda_{\boldsymbol{\gamma}}=A_4(\varphi^\gamma_{\boldsymbol{\alpha}}\psi^\lambda_{\boldsymbol{\beta},\gamma}-\psi^\gamma_{\boldsymbol{\beta}}\varphi^\lambda_{\boldsymbol{\alpha},\gamma}-s(-1)^s\psi^\lambda_{\gamma\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\boldsymbol{\alpha},\beta_s}+r(-1)^r\varphi^\lambda_{\gamma\underline{\boldsymbol{\alpha}}}\psi^\gamma_{\boldsymbol{\beta},\alpha_r})+A_3\varphi^\lambda_{\boldsymbol{\alpha}}\psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\beta_s}+$$

$$+\overline{A}_3\psi^\lambda_{\boldsymbol{\beta}}\varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r}+A_8\delta^\lambda_{\alpha_r}\varphi^\rho_{\rho\underline{\boldsymbol{\alpha}}}\psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\beta_s}+\overline{A}_8\delta^\lambda_{\beta_s}\varphi^\rho_{\rho\underline{\boldsymbol{\beta}}}\psi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r}+A_{10}(\delta^\lambda_{\beta_s}\varphi^\rho_{\boldsymbol{\alpha}}\psi^\gamma_{\rho\gamma\underline{\hat{\boldsymbol{\beta}}},\beta_{s-1}}+(-1)^r\delta^\lambda_{\beta_s}\psi^\rho_{\rho\gamma\underline{\hat{\boldsymbol{\beta}}}}\varphi^\gamma_{\boldsymbol{\alpha},\beta_{s-1}})+$$

$$+\overline{A}_{10}(\delta^\lambda_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}}\varphi^\gamma_{\rho\gamma\underline{\hat{\boldsymbol{\alpha}}},\alpha_{r-1}}+(-1)^s\delta^\lambda_{\alpha_r}\psi^\rho_{\boldsymbol{\beta},\alpha_{r-1}}\varphi^\gamma_{\rho\gamma\underline{\hat{\boldsymbol{\alpha}}}})+\overline{A}_{12}(\delta^\lambda_{\beta_s}\psi^\rho_{\gamma\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\rho\underline{\boldsymbol{\alpha}},\alpha_r}+(-1)^{s-1}\delta^\lambda_{\alpha_r}\varphi^\rho_{\gamma\underline{\boldsymbol{\alpha}}}\psi^\gamma_{\rho\underline{\boldsymbol{\beta}},\beta_s})+$$

$$+A_{13}(\delta^\lambda_{\beta_s}\varphi^\rho_{\boldsymbol{\alpha}}\psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\rho}+(s-1)\delta^\lambda_{\beta_s}\varphi^\rho_{\boldsymbol{\alpha},\beta_{s-1}}\psi^\gamma_{\gamma\rho\underline{\hat{\boldsymbol{\beta}}}})+\overline{A}_{13}(\delta^\lambda_{\alpha_r}\psi^\rho_{\boldsymbol{\beta}}\varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\rho}+(r-1)\delta^\lambda_{\alpha_r}\psi^\rho_{\rho\gamma\underline{\hat{\boldsymbol{\alpha}}}}\psi^\gamma_{\boldsymbol{\beta},\alpha_{r-1}}).$$

It is easy to see that (3.18) satisfies (3.14) and (3.15) identically.

The map (3.12) can be rewritten in the form

$$\text{(3.19)}\qquad f^i_{\boldsymbol{\gamma}}=C_1\varphi^\gamma_{\boldsymbol{\alpha}}\psi^i_{\boldsymbol{\beta},\gamma}+C_2\varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}}}\psi^i_{\boldsymbol{\beta},\alpha_r}+C_3\varphi^\gamma_{\boldsymbol{\alpha}}\psi^i_{\gamma\underline{\boldsymbol{\beta}},\beta_s}+\overline{C}_1\psi^\gamma_{\boldsymbol{\beta}}\varphi^i_{\boldsymbol{\alpha},\gamma}+$$

$$+\overline{C}_2\psi^\gamma_{\gamma\underline{\boldsymbol{\beta}}}\varphi^i_{\boldsymbol{\alpha},\beta_s}+\overline{C}_3\psi^\gamma_{\boldsymbol{\beta}}\varphi^i_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r}+D_1\varphi^i_{\boldsymbol{\alpha}}\psi^m_{\boldsymbol{\beta},m}+D_2\varphi^m_{\boldsymbol{\alpha}}\psi^i_{\boldsymbol{\beta},m}+\overline{D}_1\psi^i_{\boldsymbol{\beta}}\varphi^m_{\boldsymbol{\alpha},m}+$$

$$+\overline{D}_2\psi^m_{\boldsymbol{\beta}}\varphi^i_{\boldsymbol{\alpha},m}+E_1\varphi^i_{\boldsymbol{\alpha}}\psi^\gamma_{\boldsymbol{\beta},\gamma}+E_2\varphi^i_{\gamma\underline{\boldsymbol{\alpha}}}\psi^\gamma_{\boldsymbol{\beta},\alpha_r}+E_3\varphi^i_{\boldsymbol{\alpha}}\psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\beta_s}+\overline{E}_1\psi^i_{\boldsymbol{\beta}}\varphi^\gamma_{\boldsymbol{\alpha},\gamma}+\overline{E}_2\psi^i_{\gamma\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\boldsymbol{\alpha},\beta_s}+\overline{E}_3\psi^i_{\boldsymbol{\beta}}\varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r}.$$

By putting (3.19) into (3.13)-(3.16), we get

$$C_2=\overline{C}_2=C_3=\overline{C}_3=E_1=\overline{E}_1=0,$$

$$\overline{D}_2=D_2,\ C_1=D_2,\ \overline{C}_1=-D_2,\ E_2=-(-1)^{r-1}rD_2,\ \overline{E}_2=(-1)^{s-1}sD_2$$

and $D_2,E_3,\overline{E}_3$ are arbitrary. Hence $f^i_{\boldsymbol{\gamma}}$ has the form

$$\text{(3.20)}\qquad f^i_{\boldsymbol{\gamma}}=D_2(\varphi^\gamma_{\boldsymbol{\alpha}}\psi^i_{\boldsymbol{\beta},\gamma}+\varphi^m_{\boldsymbol{\alpha}}\psi^i_{\boldsymbol{\beta},m}-\psi^\gamma_{\boldsymbol{\beta}}\varphi^i_{\boldsymbol{\alpha},\gamma}-\psi^m_{\boldsymbol{\beta}}\varphi^i_{\boldsymbol{\alpha},m}-$$

$$-(-1)^{r-1}r\varphi^i_{\gamma\underline{\boldsymbol{\alpha}}}\psi^\gamma_{\boldsymbol{\beta},\alpha_r}+(-1)^{s-1}s\psi^i_{\gamma\underline{\boldsymbol{\beta}}}\varphi^\gamma_{\boldsymbol{\alpha},\beta_s})+E_3\varphi^i_{\boldsymbol{\alpha}}\psi^\gamma_{\gamma\underline{\boldsymbol{\beta}},\beta_s}+\overline{E}_3\psi^i_{\boldsymbol{\beta}}\varphi^\gamma_{\gamma\underline{\boldsymbol{\alpha}},\alpha_r}.$$

We have not yet used the vector field (3.6). By using it, we get

$$\varphi^{\mu}_{\lambda}\,\partial^{\lambda}_{p}f^{i}_{\gamma}+\varphi^{\mu}_{\lambda,\rho}\,\partial^{\lambda,\rho}_{p}f^{i}_{\gamma}-\varphi^{m}_{\lambda,p}\,\partial^{\lambda,\mu}_{m}f^{i}_{\gamma}+\psi^{\mu}_{\sigma}\,\overline{\partial}^{\sigma}_{p}f^{i}_{\gamma}+\psi^{\mu}_{\sigma,\rho}\,\overline{\partial}^{\sigma,\rho}_{p}f^{i}_{\gamma}-\psi^{m}_{\sigma,p}\,\overline{\partial}^{\sigma,\mu}_{m}f^{i}_{\gamma}=\delta^{i}_{p}f^{\mu}_{\gamma}. \tag{3.21}$$

By putting (3.18) and (3.20) into (3.21), we get

$$A_4=D_2\,,\ A_3=E_3\,,\ \overline{A}_3=\overline{E}_3\,,\ A_8=\overline{A}_8=A_{10}=\overline{A}_{10}=\overline{A}_{12}=A_{13}=\overline{A}_{13}=0, \tag{3.22}$$

which proves our Theorem 2, since (3.18) and (3.20), where (3.22) is satisfied, give $G^{2(+)}_{n,m}$-equivariant maps. It is easy to see that they are also $G^{2}_{n,m}$-equivariant and that the corresponding natural operators are

$$A_4[\varphi,\psi]+A_3\,\varphi\wedge p^*\,dC\underline{\psi}+\overline{A}_3\,p^*\,dC\underline{\varphi}\wedge\psi. \qquad \blacksquare$$

REMARK. – During the proof of Theorem 2 we have proved that, for any natural $\mathbb{R}$-bilinear operator $B_E:\mathfrak{P}^r(E)\times\mathfrak{P}^s(E)\to\mathfrak{P}^{r+s}(E)$, its projection $\underline{B_E}(\underline{\varphi},\underline{\psi})$ depends on $\underline{\varphi}$, $\underline{\psi}$ (and their 1st order derivatives) only. Hence, our result (3.18) for $\underline{B_E}(\underline{\varphi},\underline{\psi})$ is exactly the same as the result due to I. KOLÁŘ and P. MICHOR [14]. Among ten operators deduced by them there are two operators which are defined only if $r\geq 2$, $s\geq 2$. These operators correspond to our parameters A_{10}, $\overline{A}_{10}$. But these operators have no role if we compare the underlying operators on the base with the vertical part of $B_E(\varphi,\psi)$. That is why it is sufficient to suppose $r\geq 1$, $s\geq 1$ in our Theorem 2.

There are several corollaries which follow immediately from Theorem 2.

COROLLARY 1. – The Frölicher-Nijenhuis bracket is the only (up to a multiplicative constant) natural graded $\mathbb{R}$-bilinear operator $\mathfrak{P}(E)\times\mathfrak{P}(E)\to\mathfrak{P}(E)$.

PROOF. – It is easy to see that, in the 3-parameter family of natural $\mathbb{R}$-bilinear operators of Theorem 2, only multiples of the Frölicher-Nijenhuis bracket satisfy the conditions (0.1) and (0.2). ■

COROLLARY 2. – The Lie bracket is the only (up to a multiplicative constant) natural $\mathbb{R}$-bilinear operator $\mathcal{P}(E)\times\mathcal{P}(E)\to\mathcal{P}(E)$. ■

COROLLARY 3. – The Frölicher-Nijenhuis bracket is the only (up to a multiplicative constant) natural $\mathbb{R}$-bilinear graded operator $\mathfrak{V}(E)\times\mathfrak{V}(E)\to\mathfrak{V}(E)$. ■

4. – Connections on a fibred manifold.

According to [18] we define a (local) connection on E as a (local) tangent valued 1-form

$$\gamma: E\to T^*B\otimes TE, \tag{4.1}$$

which is projectable onto

$$\underline{\gamma} = \mathbf{1}: B \to T^* B \otimes TB.$$

Its coordinate expression is

$$\gamma = \partial_\lambda \otimes dx^\lambda + \gamma^i_\lambda (x, y) \partial_i \otimes dx^\lambda \,. \tag{4.2}$$

The *covariant differential* with respect to a connection γ is then defined as the $\mathbb{R}$-linear sheaf morphism

$$d_\gamma : \mathfrak{P}^r(E) \to \mathfrak{P}^{r+1}(E) : \varphi \mapsto d_\gamma \varphi = 1/2[\gamma, \varphi]. \tag{4.3}$$

The formula (0.2) yields the following property for d_γ

$$d_\gamma [\varphi, \psi] = [d_\gamma \varphi, \psi] + (-1)^r [\varphi, d_\gamma \psi], \tag{4.4}$$

$\varphi \in \mathfrak{P}^r(E), \psi \in \mathfrak{P}^s(E)$. Thus, d_γ is a derivation of degree 1 of $\mathfrak{P}(E)$. From the coordinate expression, it is easy to see that $d_\gamma \varphi \in \mathfrak{V}(E)$ for any $\varphi \in \mathfrak{P}(E)$.

THEOREM 3. – Let γ be a connection. Then, covariant differential d_γ is the only (up to a multiplicative constant) derivation $D_\gamma : \mathfrak{P}(E) \to \mathfrak{P}(E)$ of degree 1, which satisfies the naturality condition

$$f^* (D_\gamma \varphi) = D_{f^* \gamma} (f^* \varphi).$$

PROOF. – Theorem 2 implies that there is a two-parameter family of $\mathbb{R}$-linear sheaf morphisms $\mathfrak{P}^r(E) \to \mathfrak{P}^{r+1}(E)$ related with a given connection γ. This family is generated by d_γ and $\varphi \mapsto p^* dC\underline{\varphi} \wedge \gamma$. It is easy to see that only multiples of d_γ satisfy (4.4). ■

The *curvature* of a given connection γ is defined as

$$\rho = d_\gamma \gamma = 1/2[\gamma, \gamma]. \tag{4.5}$$

The curvature is a natural sheaf morphism $\mathcal{C}(E) \to \mathfrak{V}^2(E)$, where $\mathcal{C}(E) \subset \mathfrak{P}(E)$ is the subsheaf constituted by the projectable tangent valued 1-forms which are projected on the identity of TB. The coordinate expression of ρ is

$$\rho = (\partial_\lambda \gamma^i_\mu + \gamma^j_\lambda \partial_j \gamma^i_\mu) dx^\lambda \wedge dx^\mu \otimes \partial_i \,. \tag{4.6}$$

THEOREM 4. – All natural operators which associate a vertical valued 2-form with a given connection γ are of the form $k\rho$, with $k \in \mathbb{R}$.

PROOF. – The proof of Theorem 4 has the same steps as the proof of Theorem 2. First, from [27] it follows that all natural operator from $\mathcal{C}(E)$ to $\mathfrak{V}^2(E)$ are of finite order. Let $\overline{S}{}^0_1 = \mathbb{R}^m \otimes \mathbb{R}^{n*}$ be the type fibre of $\mathcal{C}(E)$ with global coordinates γ^i_λ and the action $\bar{\chi}$ of the group $G^1_{n,m}$ on $\bar{S}^0_1$ given by

$$\bar{\gamma}^i_\gamma \circ \bar{\chi} = a^i_m \gamma^m_\nu \tilde{a}^\nu_\lambda + a^i_\nu \tilde{a}^\nu_\lambda .$$

Let $B_E : \mathcal{C}(E) \to \mathfrak{V}^2(E)$ be a k-order operator for some $k \geq 1$. Then, we get the corresponding $G^{k+1}_{n,m}$-equivariant map $f: \bar{S}^k_1 \to \bar{S}^0_2 = \mathbb{R}^m \otimes \wedge^2 \mathbb{R}^{n*}$ where $\bar{S}^k_1 = J^k_0(\mathbb{R}^{n+m}, \bar{S}^0_1)$ and $\bar{S}^0_2$ is the vertical part of S^0_2. The action $\bar{\chi}$ can be prolonged to the action $\chi^k : G^{k+1}_{n,m} \times \bar{S}^k_1 \to \bar{S}^k_1$. By using the same method as in the proof of Theorem 1, we prove that all $G^{k+1}_{n,m}$-equivariant globally defined maps from $\bar{S}^k_1$ to $\bar{S}^0_2$ are polynomials of degrees $a_{|\nu||\boldsymbol{m}|}$ with respect to $\gamma^i_{\lambda,\nu\boldsymbol{m}}$ such that

$$\sum_{|\nu|+|\boldsymbol{m}|=0}^{k} a_{|\nu||\boldsymbol{m}|}(-1-|\nu|) = -2, \qquad \sum_{|\nu|+|\boldsymbol{m}|=0} a_{|\nu||\boldsymbol{m}|}(1-|\boldsymbol{m}|) = 1. \tag{4.7}$$

The equations (4.7) have only two solutions in $\{0\} \cup \boldsymbol{N}$:

$$\begin{aligned} &a_{10} = 1 && \text{and the other variables vanish,} \\ &a_{00} = 1, a_{01} = 1 && \text{and the other variables vanish.} \end{aligned} \tag{4.8}$$

This implies that f is defined on $\bar{S}^1_1$ only, i.e. all operators from $\mathcal{C}(E)$ to $\mathfrak{V}^2(E)$ are of order one.

The formula (4.8) yields the following coordinate expression of f

$$f^i_{\lambda\mu} = A^{i\alpha\beta}_{\lambda\mu m} \gamma^m_{\alpha,\beta} + B^{i\alpha\beta r}_{\lambda\mu m p} \gamma^m_\alpha \gamma^p_{\beta,r}, \tag{4.9}$$

and, by using the equivariancy condition, we can rewrite (4.9) in the form

$$f^i_{\lambda\mu} = A_1 \gamma^i_{\lambda,\mu} + A_2 \gamma^i_{\mu,\lambda} + B_1 \gamma^i_\mu \gamma^m_{\mu,m} + B_2 \gamma^i_\mu \gamma^m_{\lambda,m} + B_3 \gamma^m_\lambda \gamma^i_{\mu,m} + B_4 \gamma^m_\mu \gamma^i_{\lambda,m}. \tag{4.10}$$

Now, by using the vector fields (3.6)-(3.10), we get

$$A_1 = -A_2, \;\; B_3 = A_2, \;\; B_4 = -A_2, \;\; B_1 = B_2 = 0,$$

which proves our Theorem 4. ■

This theorem was proved by I. KOLÁŘ [13], by another approach.

Let γ_0, γ_1 be two connections on E. We can define a pencil

$$\gamma_t = (1-t)\gamma_0 + t\gamma_1, \qquad t \in \mathbb{R}, \tag{4.11}$$

of connections on E. The curvature ρ_t of γ_t is then

$$\rho_t = (1-t)^2 \rho_0 + 2(t-t^2) d_{\gamma_0} \gamma_1 + t^2 \rho_1, \tag{4.12}$$

where ρ_0 and ρ_1 are the curvatures of γ_0 and γ_1, respectively.

$d_{\gamma_0}\gamma_1 = 1/2[\gamma_0, \gamma_1]$ is a natural operator $\mathcal{C}(E) \times \mathcal{C}(E) \to \mathfrak{V}^2(E)$. By using the same methods as in the proofs of Theorem 2 and 4, we can easily prove

THEOREM 5. – All natural operators $\mathcal{C}(E) \times \mathcal{C}(E) \to \mathfrak{V}^2(E)$ form a three-parameter family $k_1 \rho_0 + k_2 \rho_1 + k_3 d_{\gamma_0} \gamma_1$ with real coefficients k_i, $i = 1, 2, 3$. ■

I. KOLÁŘ [13] has remarked that this theorem is true if we replace the operator $d_{\gamma_0}\gamma_1$ by the *mixed curvature* $\varkappa(\gamma_0, \gamma_1)$, [11]. In our notation

$$\varkappa(\gamma_0, \gamma_1) = 2d_{\gamma_0}\gamma_1 - 2\rho_0 . \tag{4.13}$$

In the term of the mixed curvature we can rewrite (4.12) as

$$\rho_t = (1-t^2)\rho_0 + (t-t^2)\varkappa(\gamma_0, \gamma_1) + t^2\rho_1 , \tag{4.14}$$

which coincides with [11].

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