Annali di Matematica pura ed applicata
(IV), Vol. CLIX (1991), pp. 117-131

On a Class of Singular Nonlinear Parabolic Variational Inequalities (*).

MARCO LUIGI BERNARDI - GIANNI ARRIGO POZZI (**)

Summary. – *We study a class of singular or degenerate parabolic variational inequalities, containing some nonlinear operators. We prove an existence and uniqueness result for weak solutions, in the framework of suitable Banach weighted spaces.*

Sunto. – *Si studia una classe di disequazioni variazionali paraboliche singolari o degeneri, contenenti operatori non lineari. Si dimostra un risultato di esistenza e unicità per soluzioni deboli, nell'ambito di opportuni spazi di Banach con peso.*

1. – Introduction.

In a previous paper [5], we studied some singular or degenerate parabolic variational inequalities of the form:

$$(1.1)\qquad \begin{cases} u(t) \in K & \text{a.e. on }]0,T[;\\ (tu'(t) + L(t)u(t) - f(t),\ \ v - u(t)) \geqslant 0, & \forall v \in K, \quad \text{a.e. on }]0,T[; \end{cases}$$

where: $0 < T < +\infty$; $V \subseteq H \equiv H^* \subseteq V^*$ is the standard real Hilbert triplet; $(\cdot,\cdot)$ denotes the duality pairing between V^* and V; K is a closed convex subset of V, with $0 \in K$; $f(t)$ is some given V^*-valued function. Moreover, $L(t)$ is, for any $t \in]0,T]$, a *linear and continuous* operator from V into V^*, which is also weakly V-coercive; however, $L(t)$ may be singular or degenerate at $t=0$. In [5], we proved some existence and uniqueness results for weak and strong solutions $u(t)$ of (1.1), in the framework of suitable Hilbert weighted spaces (the weights involve powers of t and the «behaviour» of $L(t)$, as $t \to 0^+$).

(*) Entrata in Redazione il 18 gennaio 1989.
This work was supported in part by the «Istituto di Analisi Numerica del C.N.R.» (Pavia, Italy), the G.N.A.F.A. of the C.N.R. and the Ministero della Pubblica Istruzione (Italy) (through 60% and 40% grants).

(**) Indirizzi degli AA.: M. L. BERNARDI: Dipartimento di Automazione Industriale, Università degli Studi di Brescia, Via Valotti 9, I-25060 Brescia (Italy); G. A. POZZI: Dipartimento di Matematica, Università degli Studi di Pavia, Strada Nuova 65, I-27100 Pavia (Italy).

Our interest in such a problem was motivated by some previous results concerning (in various functional settings) singular or degenerate linear evolution equations of the form

$$(1.2) \qquad tu'(t) + L(t)u(t) = f(t), \qquad \text{for a.e. } t \in]0, T[;$$

see, in particular, BAIOCCHI-BAOUENDI [2] and BERNARDI [3] (also see, e.g., DA PRATO-GRISVARD [6], LEWIS-PARENTI [10], FAVINI [8], DORE-VENNI [7]; other references can be found in [5]).

The aim of the present paper is to extend some of the results given in [5] to the case where, in (1.1), $L(t)$ is *some suitable nonlinear operator* from V to V^* (which now are assumed to be separable, reflexive, and strictly convex Banach spaces).

We give, in the following Section 2, some notation and definitions, and the precise assumptions on $L(t)$; moreover, we give also some preliminary results (in particular, we present an extension of a result in BERNARDI [4], which will be useful in the sequel). In Section 3, we prove an existence and uniqueness theorem for (suitably defined) weak solutions of (1.1); the existence result is obtained by using a suitable procedure of penalization. At the end of Section 3, we present some applications of our «abstract» result (in particular, we give some examples of operators $L(t)$, to which such result applies).

2. – Notation. Assumptions. Some preliminary results.

Let V be a real Banach space and H be a real Hilbert space satisfying:

(2.1) V is separable and reflexive; $V \subseteq H$ continuously and densely.

By identifying H with its dual space, and denoting by V^* the dual space of V, we also get that $H \subseteq V^*$ (continuously and densely). $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ denote respectively the norms in V, H, and V^*, while $(\cdot,\cdot)$ denotes both the scalar product in H and the duality pairing between V^* and V.

In the sequel, it will be useful to assume (without loss of generality; see ASPLUND [1]) that:

(2.2) the norms in V and V^* are strictly convex.

Let now T, p, and a be given, with:

$$(2.3) \qquad 0 < T < +\infty; \qquad 1 < p < +\infty; \qquad a \in \boldsymbol{R},$$

and define $q = p/(p-1)$.

We shall use, in the sequel, the following weighted spaces. Given any $m \in \boldsymbol{R}$ (with V, H, T, p, a, satisfying (2.1) and (2.3)), we define:

$$(2.4) \qquad U_m \equiv \{u \mid t^{m+a}u(t) \in L^p(0,T;V);\ t^{mp/2}u(t) \in L^2(0,T;H)\};$$

(2.5) $$Z_m \equiv \{f \,|\, f = f_1 + f_2, \text{ with: } t^{m(p-1)-a} f_1(t) \in L^q(0,T;V^*); \quad t^{mp/2} f_2(t) \in L^2(0,T;H)\};$$

(2.6) $$W_m \equiv \{u \,|\, u(t) \in U_m;\ tu'(t) \in Z_m\}.$$

(Of course, U_m, Z_m, and W_m are, clearly, Banach spaces with respect to their natural norms). The above spaces are particular cases of the ones used by BERNARDI [4]; let us recall at once (see [4]) the following

LEMMA 2.1. – Let (2.1) and (2.3) hold, and let $m \in \boldsymbol{R}$ be given. Then, if $u(t) \in W_m$, it results that:

(2.7) $$u(t) \in C^0(]0,T];\, H), \qquad \text{and moreover } t^{(mp+1)/2}|u(t)| \to 0, \qquad \text{as } t \to 0^+.$$

REMARK 2.1. – Clearly, when $m_1 < m_2$, it results that $U_{m_1} \subset U_{m_2}$ and $Z_{m_1} \subset Z_{m_2}$ (continuously and densely). We observe, moreover, that Z_m can be considered as the dual space of U_m (the duality pairing being expressed in terms of weighted $L^q - L^p$ space, with respect to the measure $d\mu = t^{mp}\, dt$).

We also remark that, when $a \leq 0$ and $p \geq 2$, it results easily that:

$$U_m = \{u \,|\, t^{m+a} u(t) \in L^p(0,T;V)\}; \qquad Z_m = \{f \,|\, t^{m(p-1)-a} f(t) \in L^q(0,T;V^*)\}.$$

Now, we are going to give the assumptions on the operator L, which appears in the variational inequality (1.1). We suppose that $L(t;\cdot) = A(t;\cdot) + B(t;\cdot)$, where $A(t;\cdot)$ and $B(t;\cdot)$ satisfy the following conditions.

$\{A(t)\}$ (for a.e. $t \in]0,T[$) is a family of (possibly nonlinear) operators from V to V^*, such that:

(2.8) $$\text{«}t \to A(t;v(t))\text{» is } V^*\text{-measurable on }]0,T[, \quad \forall v(t) \text{ which is } V\text{-measurable on }]0,T[;$$

(2.9) $$A(t;\cdot) \text{ is hemicontinuous a.e. on }]0,T[;$$

(2.10) $$\exists C_1 > 0 \text{ s.t. } (A(t;v),v) \geq C_1\, t^{ap} \|v\|^p, \qquad \forall v \in V, \text{ a.e. on }]0,T[;$$

(2.11) $$\exists C_2 > 0 \text{ s.t. } \|A(t;v)\|_* \leq C_2\, t^{ap} \|v\|^{p-1}, \qquad \forall v \in V, \text{ a.e. on }]0,T[;$$

(2.12) $$\exists \tilde{l} \in \boldsymbol{R} \text{ s.t. the operator } A(t;\cdot) + \tilde{l} I \text{ is monotone from } V \text{ to } V^*, \quad \text{a.e. on }]0,T[.$$

Clearly, (2.10) is a coercivity hypothesis, while (2.11) is a boundedness assumption (which implies, in particular, that $A(t;0) = 0$, a.e. on $]0,T[$). We refer to the final part of the following Section 3 for some examples, also for the operator $B(t;\cdot)$ we are going to consider hereafter.

$\{B(t)\}$ (for a.e. $t \in]0, T[$) is a family of (possibly nonlinear) operators from H to H, such that:

(2.13) $\quad$ «$t \to B(t; v(t))$» is H-measurable on $]0, T[$,

$\forall v(t)$ which is H-measurable on $]0, T[$;

(2.14) $\quad \exists C_3 > 0$ s.t. $|B(t; u) - B(t; v)| \leq C_3 |u - v|, \quad \forall u, v \in H, \quad$ a.e. on $]0, T[$;

(2.15) $\quad B(t; 0) = 0, \quad$ a.e. on $]0, T[$,

so that, in particular: $|B(t; u)| \leq C_3 |u|$, $\forall u \in H$, a.e. on $]0, T[$. We now define the following function (recall that $L = A + B$):

(2.16) $\quad l_0(t) \equiv \inf\{l | (L(t; u_1) - L(t; u_2), \ u_1 - u_2) + l |u_1 - u_2|^2 \geq 0, \ \forall u_1, u_2 \in V\}$.

It results clearly that $l_0(t) \in L^\infty_{\text{loc}}(]0, T])$. We also define:

(2.17)
$$l_0 \equiv \operatorname{ess} \lim_{t \to 0^+} \inf l_0(t).$$

We have obviously that $l_0 \in [-\infty, +\infty[$ (see, for further details and examples, BERNARDI [4]). Now, let us set:

(2.18) $\quad m_0 \equiv -(2l_0 + 1)/p, \quad$ if $l_0 \in \mathbf{R}$; $\quad m_0 = +\infty, \quad$ if $l_0 = -\infty$.

We can state the following

LEMMA 2.2. - Let (2.1), (2.3), (2.8) ... (2.15) hold. Then, for every $m < m_0$, there exist $r(m) \in \mathbf{R}$ and $c(m) > 0$ such that, for every $r \geq r(m)$:

(2.19) $\quad L(t; \cdot) + \frac{1}{2}(rt - (mp + 1))I$ is strictly monotone from V to V^*,

a.e. on $]0, T[$;

(2.20) $\quad (L(t; v), v) + \frac{1}{2}(rt - (mp + 1))|v|^2 \geq c(m)[t^{\alpha p} \|v\|^p + |v|^2]$,

$\forall v \in V$, a.e. on $]0, T[$.

PROOF. - We fix any $m < m_0$. Then, thanks to (2.16), (2.17), and (2.18), we obtain that there exists $t_m \in]0, T]$, such that $L(t; \cdot) - ((mp + 1)/2) I$ is strictly monotone (from V to V^*) for a.e. $t \in]0, t_m[$. Now, by using (2.12) and (2.14), we have that there exists l^* such that $L(t; \cdot) + l^* I$ is strictly monotone a.e. on $]0, T[$. Hence, (2.19) follows, by choosing $r(m)$ such that: $r(m) t_m - (mp + 1)/2 \geq l^*$ (e.g. $r(m) > 0$ large enough).

In order to prove (2.20), it suffices to extend slightly the argument used in the proof of Lemma 2.2 in BERNARDI [4], to obtain that (thanks to (2.16), (2.17), (2.18), (2.10), (2.11), (2.14), and (2.15)) there exist $t_m \in]0, T]$ and $\tilde{c}(m) > 0$ such that:

$$(L(t; v), v) - ((mp + 1)/2)|v|^2 \geq \tilde{c}(m)[t^{\alpha p} \|v\|^p + |v|^2], \qquad \forall v \in V, \text{ a.e. on }]0, t_m[.$$

Hence, (2.20) follows, by choosing $r(m)$ such that: $r(m)\,t_m - (mp+1)/2 - C_3 > 0$ (e.g. $r(m) > 0$ large enough).

Now, we can observe that Theorem 3.1 in BERNARDI [4] is easily extended to the present situation (remark that here the operator B is not necessarily linear as in [4]); so, we have the following

THEOREM 2.1. – Let (2.1), (2.3), and (2.8) ... (2.15) hold. Let m_0 be given by (2.18), and take any $m < m_0$. Then, for every $f \in Z_m$, there exists a unique $u \in W_m$, which solves (recall that $L = A + B$):

$$tu'(t) + L(t; u(t)) = f(t), \qquad \text{a.e. on }]0, T[\,. \tag{2.21}$$

(Remark that the solution $u(t) \in W_m$ of (2.21) satisfies the «initial trend» given in (2.7)).

REMARK 2.2. – Theorem 2.1 is easily extended to the more general situation, where $B(t;\cdot)$ satisfies (2.13), (2.14), and

$$B(t; 0) \in Z_m, \tag{2.22}$$

instead of (2.15). In fact, the equation (2.21) is equivalent to

$$tu'(t) + A(t; u(t)) + \widetilde{B}(t; u(t)) = \widetilde{f}(t), \qquad \text{a.e. on }]0, T[\,, \tag{2.23}$$

where $\widetilde{B}(t;\cdot) = B(t;\cdot) - B(t; 0)$ satisfies (2.13) ... (2.15), and $\widetilde{f}(t) = (f(t) - B(t; 0)) \in Z_m$. Remark also that $l_0(t)$ and hence m_0 do not change, if we replace $B(t;\cdot)$ by $\widetilde{B}(t;\cdot)$ in (2.16).

REMARK 2.3. – We observe that Theorem 2.1 is no longer true, in general, if one takes $m \geq m_0$. In fact, even in the case where $L = A + B$ is a linear operator, it may happen that, for some $m \geq m_0$, the existence and/or the uniqueness of a solution $u(t) \in W_m$ of (2.21) (corresponding to some given $f(t) \in Z_m$) fails (see, e.g., BERNARDI [3], [4]).

3. – The main result and some examples.

In this section, we introduce a suitable definition of a weak solution for the variational inequality (1.1), and we prove an existence and uniqueness result for such a solution (in the framework of the weighted spaces defined in the previous Section 2).

In the sequel, given V and H as in (2.1), we shall also take

(3.1) a closed convex subset K of V, such that $0 \in K$.

3.1. – In order to motivate our definition of a weak solution for (1.1), we make the following remark (see, for a similar procedure, BERNARDI-POZZI [5]).

Let us assume that, for some $m \in \mathbf{R}$ and $f(t) \in Z_m$, (1.1) has a solution $u(t) \in W_m$. Fix now any $v(t)$ such that:

$$v(t) \in W_m; \qquad v(t) \in K, \qquad \text{a.e. on }]0,T[; \tag{3.2}$$

fix moreover any $r \in \mathbf{R}$. Taking $v = v(t)$ in (1.1), multiplying both sides of the resulting inequality by $t^{mp} \exp(-2rt)$ and integrating from 0 to s (with $0 < s \leqslant T$), we obtain, with some calculations and using Lemma 2.1, that:

$$\int_0^s t^{mp} \exp(-2rt)(tv'(t) + L(t; u(t)) + \frac{1}{2}(mp + 1 - 2rt)(v(t) - u(t)) - f(t),$$
$$v(t) - u(t))\, dt \geqslant \frac{1}{2} s^{mp+1} \exp(-2rs)|v(s) - u(s)|^2, \qquad \forall s \in]0, T]. \tag{3.3}$$

We take now (3.3) as the weak formulation of (1.1), in the framework of the weighted spaces (2.4), (2.5), (2.6). Our main result is the following

THEOREM 3.1. – Let (2.1), (2.2), (2.3), (2.8) ... (2.12), (2.13) ... (2.15), and (3.1) hold. Let $m_1 < m_0$ be given (where m_0 is defined in (2.18)). Then, for any $f(t) \in Z_{m_1}$, there exists a unique $u(t)$, with

$$\begin{cases} \text{(I)} & u(t) \in U_{m_1}; \\ \text{(II)} & u(t) \in K, \qquad \text{a.e. on }]0, T[; \\ \text{(III)} & t^{(m_1 p + 1)/2} |u(t)| \to 0, \qquad \text{as } t \to 0^+; \\ \text{(IV)} & u(t) \in C^0(]0, T]; H), \end{cases} \tag{3.4}$$

which satisfies (3.3) (where $L = A + B$), with $m = m_1$ and $r = r((m_1)$ (given by Lemma 2.2), for any $v(t)$ satisfying (3.2) (still with $m = m_1$). Moreover, such $u(t)$ satisfies (3.3) also for every $m \in [m_1, m_0[$, and every $r \geqslant r(m)$, for any $v(t)$ as in (3.2). (Recall (see Remark 2.1) that $m > m_1$ implies that $U_{m_1} \subset U_m$ and $Z_{m_1} \subset Z_m$).

REMARK 3.1. – By the same argument we used in Remark 2.2, it is easily seen that Theorem 3.1 still holds, if the assumption (2.15) is replaced by

$$B(t; 0) \in Z_{m_1}. \tag{3.5}$$

REMARK 3.2. – In our weak formulation (3.3), the presence of terms depending on r (e.g. $\exp(-2rt)$) is due to the fact that the operator $L(t; \cdot)$ is only «weakly coercive»; while the presence of terms depending on m (e.g. t^{mp}) obviously comes from our choice of working in the framework of weighted spaces (also see BERNARDI-POZZI [5]).

We could also consider the weak formulation obtained by replacing, in (3.3), s by

T in the left-hand side, and taking 0 as right-hand side. The existence result follows then obviously from Theorem 3.1. On the other hand, in the following subsection 3.2, we shall prove directly the uniqueness result for such a weaker formulation.

In order to prove Theorem 3.1, we need a preliminary result, i.e. the following

LEMMA 3.1. – Let T, p, and a be given as in (2.3). Let any $m \in \boldsymbol{R}$ be fixed, and let K satisfy (3.1). Let moreover $w(t) \in U_m$ be given, with $w(t) \in K$ a.e. on $]0, T[$.

Take now any $h > h_0 \equiv \max(0; (mp+1)/2; (mp+ap+1)/p)$, and define:

$$w_h(t) \equiv h t^{-h} \int_0^t s^{h-1} w(s)\, ds, \qquad 0 < t \leq T. \tag{3.6}$$

Then, we have that:

$$\begin{cases} \text{(I)} & w_h(t) \in K, \qquad 0 < t < T; \\ \text{(II)} & w_h(t) \in U_m; \end{cases} \tag{3.7}$$

$$h^{-1} t w_h'(t) + w_h(t) = w(t), \qquad \text{a.e. on }]0, T[; \tag{3.8}$$

$$t w_h'(t) \in U_m, \qquad \text{and hence } w_h(t) \in W_m \text{ and satisfies (2.7)}; \tag{3.9}$$

$$\begin{cases} \text{(I)} & w_h(t) \to w(t) \text{ (weakly) in } U_m, \text{ and moreover} \\ \text{(II)} & w_h(t) \to w(t) \text{ (strongly) in } \{v | t^{mp/2} v(t) \in L^2(0, T; H)\} \text{ as } h \to +\infty. \end{cases} \tag{3.10}$$

PROOF. – We use here a procedure, which is similar to the one employed in the proof of Lemma 3.1 in BERNARDI-POZZI [5] (which, on the other hand, deals with the case where $p = 2$ and V is a Hilbert space).

Firstly, (3.7) (I) follows from (3.6) (being $w(t) \in K$ a.e. on $]0, T[$). In order to prove (3.7) (II), we use the Hardy inequality (see HARDY-LITTLEWOOD-POLYA [9], Theorem 330), and we obtain:

$$\int_0^T t^{mp} |w_h(t)|^2 dt \leq \frac{4}{(2 - mph^{-1} - h^{-1})^2} \int_0^T t^{mp} |w(t)|^2 dt; \tag{3.11}$$

$$\int_0^T t^{mp+ap} \|w_h(t)\|^p dt \leq \frac{p^p}{(p - mph^{-1} - aph^{-1} - h^{-1})^p} \int_0^T t^{mp+ap} \|w(t)\|^p dt. \tag{3.12}$$

Now, (3.8) follows from (3.6), and (3.9) follows from (3.8) and (3.7) (II).

We remark that, thanks to (3.11) and (3.12), $\{w_h(t); h > h_0\}$ is bounded in U_m.

Then, with the same argument as in the proof of Lemma 3.1 in [5], we can get

(3.10) (I). Finally, we observe that, from (3.11), it follows that;

$$\limsup_{h\to+\infty}\left(\int_0^T t^{mp}\,|w_h(t)|^2\,dt\right)\leqslant\int_0^T t^{mp}\,|w(t)|^2\,dt. \tag{3.13}$$

Hence, (3.10) (II) follows from (3.10) (I) and (3.13). (Remark that, if the Banach space $\{v|t^{m+a}v(t)\in L^p(0,T;V)\}$ is uniformly convex, the above argument, applied to (3.12), gives in fact that: $w_h(h)\to w(t)$ (strongly) in U_m. On the other hand, we observe that, for the proof of Theorem 3.1, we need only (3.10)).

3.2. – We proceed now to the *proof of Theorem* 3.1. We begin with the *proof of the uniqueness*. In fact, we prove a stronger uniqueness result, which we now state.

Firstly, let us write the following inequality (obtained by weakening (3.3)):

$$\int_0^T t^{mp}\exp(-2rt)\cdot \tag{3.14}$$

$$\cdot\left(tv'(t)+L(t;u(t))+\frac{1}{2}(mp+1-2rt)(v(t)-u(t))-f(t),\;\; v(t)-u(t)\right)dt\geqslant 0.$$

Under the assumptions in Theorem 3.1, let now $m_1<m_0$ and $f(t)\in Z_{m_1}$ be given.

Assume that $u_1(t)$ and $u_2(t)$ both satisfy (3.4), and for a given $m\in[m_1,m_0[$ also satisfy (3.14), for every $v(t)$ as in (3.2), and for some $r\geqslant r(m)$ (given by Lemma 2.2). We can prove that $u_1=u_2$. To do this, we suitably modify the procedure used in subsection 3.2 of BERNARDI-POZZI [5].

Firstly, we define

$$w(t)\equiv\frac{1}{2}(u_1(t)+u_2(t)). \tag{3.15}$$

Then, from $w(t)$ (which clearly satisfies (3.4)), we can consider the family $\{w_h(t)\}(h>h_0)$ defined as in Lemma 3.1. Hence, $\{w_h(t)\}$ satisfies (3.7) ... (3.10) (and, in particular, (3.2)). Now, we take (3.14), with $u(t)=u_1(t)$ and $v(t)=w_h(t)$, and then we also take (3.14), but with $u(t)=u_2(t)$ and $v(t)=w_h(t)$; we obtain, by adding the resulting inequalities and using (3.8):

$$\int_0^T t^{mp}\exp(-2rt)\Big\{(L(t;u_1(t)),\;\; w_h(t)-u_1(t))+(L(t;u_2(t)),\;\; w_h(t)-u_2(t))+ \tag{3.16}$$

$$+\frac{1}{2}(mp+1-2rt)[|w_h(t)-u_1(t)|^2+|w_h(t)-u_2(t)|^2]-2(f(t),w_h(t)-w(t))\Big\}\,dt\geqslant 0\,.$$

We now let $h \to +\infty$ in (3.16), and we use (3.10); then, in the resulting inequality, we substitute $w(t)$ with its expression (3.15); so, we get:

$$0 \geq \int_0^T t^{mp} \exp(-2rt)\Big\{(L(t; u_1(t)) - L(t; u_2(t)),\ u_1(t) - u_2(t)) + \frac{1}{2}(2rt - mp + 1)|u_1(t) - u_2(t)|^2\Big\}\,dt. \tag{3.17}$$

Hence, being $m < m_0$ and $r \geq r(m)$, we deduce from Lemma 2.2 (in particular, from (2.19)), that $u_1(t) = u_2(t)$ on $]0, T]$.

3.3. – We now *prove the existence result in Theorem* 3.1. We proceed by several steps: some of them are suitable generalizations of the ones in subsection 3.3 of [5].

a) Firstly, let us define

$$\beta(v) \equiv J(v - P_K v), \qquad \forall v \in V, \tag{3.18}$$

where P_K is the projection operator from V to K, and J is the duality mapping from V to V^*, connected with the function $\Phi(r) = r^{p-1}$, $r \geq 0$. Thanks to (2.2), it is well known (see, e.g., LIONS [11], chap. 3, n. 5) that β is a penalty operator connected with K, i.e.

(3.19) $\beta\colon V \to V^*$ is a bounded hemicontinuous monotone operator, with $\ker(\beta) = K$;

moreover, since $0 \in K$, it results that:

$$(\beta(v), v) \geq 0 \quad \forall v \in V; \qquad \|\beta(v)\|_* \leq \|v\|^{p-1}, \qquad \forall v \in V. \tag{3.20}$$

We also remark that «$v(t) \to t^{ap}\beta(v(t))$» is, for every $m \in \mathbf{R}$, a penalty operator connected with the closed convex set

$$\{v(t) \mid t^{m+a} v(t) \in L^p(0, T; V);\ v(t) \in K \text{ a.e. on }]0, T[\}.$$

b) We now consider, for every integer $k > 0$, the penalized equation

$$t u_k'(t) + L(t; u_k(t)) + k t^{ap} \beta(u_k(t)) = f(t), \qquad \text{a.e. on }]0, T[. \tag{3.21}$$

Hence, Theorem 2.1 applies here to (3.21) (by taking of course, $L(t; \cdot)$ in (2.21) as $L(t; \cdot) + k t^{ap}\beta(\cdot)$ in (3.21)). We thus have that (3.21) has a unique solution $u_k(t) \in W_{m_1}$ (and hence $u_k(t) \in W_m$ for every $m \geq m_1$). We fix any $m \in [m_1, m_0[$ and any $r \geq r(m)$; then we «multiply» (in the duality pairing between V^* and V) both sides of (3.21) by $t^{mp}\exp(-2rt)\, u_k(t)$ and we integrate from 0 to T. Thanks to Lemma 2.1 (see (2.7)) and to Lemma 2.2 (see (2.20)), we obtain that:

$$c(m)\int_0^T t^{mp}\exp(-2rt)[t^{ap}\|u_k(t)\|^p+|u_k(t)|^2]\,dt+ \tag{3.22}$$

$$+k\int_0^T t^{(m+a)p}\exp(-2rt)(\beta(u_k(t)),\,u_k(t))\,dt\leqslant\int_0^T t^{mp}\exp(-2rt)|(f(t),\,u_k(t))|\,dt.$$

So, we obviously get that:

$$\{u_k(t)\}\ \text{is bounded in}\ U_m; \tag{3.23}$$

$$\left\{k\int_0^T t^{(m+a)p}(\beta(u_k(t)),u_k(t))\,dt\right\}\ \text{is bounded}. \tag{3.24}$$

Then, from $\{u_k(t)\}$ we can take a subsequence (still denoted by $\{u_k(t)\}$), such that, for every $m\in[m_1,m_0[$, as $k\to+\infty$:

$$u_k(t)\to u(t)\quad\text{(weakly) in}\ U_m; \tag{3.25}$$

$$L(t;u_k(t))\to\chi(t)\quad\text{(weakly) in}\ Z_m; \tag{3.26}$$

$$t^{ap}\beta(u_k(t))\to\psi(t)\quad\text{(weakly) in}\ Z_m; \tag{3.27}$$

$$\int_0^T t^{(m+a)p}(\beta(u_k(t)),\ u_k(t))\,dt\to 0. \tag{3.28}$$

Of course, in (3.25) ... (3.27), $u(t)$, $\chi(t)$ and $\psi(t)$ do not depend on $m\in[m_1,m_0[$, thanks to the continuous imbeddings $U_{m_1}\subset U_m$ and $Z_{m_1}\subset Z_m$.

Now, from (3.23), the properties of $L(t;\cdot)$, and (3.21), we can deduce that, in (3.27), $\psi(t)=0$ a.e. on $]0,T[$. Hence, thanks also to (3.28) and to the pseudomonotonicity of $t^{ap}\beta(\cdot)$: $U_m\to Z_m$, we can get that:

$$\beta(u(t))=0\quad\text{a.e. on}\ ]0,T[,\qquad\text{i.e.}\ u(t)\in K\ \text{a.e. on}\ ]0,T[. \tag{3.29}$$

c) We prove now that, for every $m\in[m_1,m_0[$, $u(t)$ satisfies (3.3), for any $v(t)$ as in (3.2), and for every $r\geqslant r(m)$. We fix any such m, any $r\geqslant r(m)$, and any $v(t)$ satisfying (3.2). Then, we can get that (by using: (3.21); the fact that $\beta(v(t))=0$ a.e. on

$]0, T[$; the monotonicity of $\beta(\cdot)$; the initial trend in (2.7) for $v(t)$ and $u_k(t)$):

$$\int_0^s t^{mp}\exp(-2rt)\Big(tv'(t)+L(t;\,u_k(t))+\frac{1}{2}(mp+1-2rt)(v(t)- \tag{3.30}$$

$$-u_k(t))-f(t),\ \ v(t)-u_k(t)\Big)\,dt \geqslant \frac{1}{2}s^{mp+1}\exp(-2rs)|v(s)-u_k(s)|^2\geqslant 0, \qquad 0<s\leqslant T.$$

Hence, thanks to (3.25) and (3.26), (3.30) gives that:

$$\limsup_{k\to+\infty}\left\{\int_0^s t^{mp}\exp(-2rt)\Big(L(t;\,u_k(t))+\frac{1}{2}(2rt-mp-1)u_k(t),u_k(t)\Big)\,dt\right\}\leqslant \tag{3.31}$$

$$\leqslant\int_0^s t^{mp}\exp(-2rt)\left\{\Big(\chi(t)+\frac{1}{2}(2rt-mp-1)\,u(t),v(t)\Big)+\right.$$

$$\left.+\Big(tv'(t)+\frac{1}{2}(mp+1-2rt)\,v(t)-f(t),\ \ v(t)-u(t)\Big)\right\}dt.$$

To proceed in our proof, we need now to remark the following fact.

(3.32) Let $m, r\in\boldsymbol{R}$, and let $w(t)\in U_m$ with $w(t)\in K$ a.e. on $]0,T[$. Consider, from $w(t)$, the family $\{w_h(t)\}$ $(h>h_0)$, defined as in Lemma 3.1. Then, in addition to (3.7)...(3.10), the following property holds:

$$\limsup_{h\to+\infty}\left\{\int_0^s t^{mp}\exp(-2rt)\cdot\right.$$

$$\left.\cdot\Big(tw_h'(t)+\frac{1}{2}(mp+1-2rt)\,w_h(t),w_h(t)-w(t)\Big)\,dt\right\}\leqslant 0, \qquad \forall s\in[0,T].$$

In fact, to get (3.32), it suffices to remark that, thanks to (3.8),

$$\int_0^s t^{mp}\exp(-2rt)\Big(tw_h'(t)+\frac{1}{2}(mp+1-2rt)\,w_h(t),w_h(t)-w(t)\Big)\,dt= \tag{3.33}$$

$$=-h^{-1}\int_0^s t^{mp}\exp(-2rt)|tw_h'(t)|^2\,dt+\int_0^s t^{mp}\exp(-2rt)\cdot$$

$$\cdot\frac{1}{2}(mp+1-2rt)(w_h(t),w_h(t)-w(t))\,dt,$$

and clearly, at the right-hand side of (3.33), the first term is $\leqslant 0$, while the second term goes to 0, as $h \to +\infty$ (thanks to (3.10) (II)).

Now, we take $w(t) = u(t)$ (and then we set $w_h(t) \equiv u_h(t)$) in (3.32). Hence, by choosing $v(t) = u_h(t)$ in (3.31) and making $h \to +\infty$, we get that (3.31) gives (thanks to (3.32) and (3.26)):

$$(3.34) \qquad \limsup_{k \to +\infty} \left\{ \int_0^s t^{mp} \exp(-2rt) \Big(L(t; u_k(t)) + \frac{1}{2}(2rt - mp - 1) u_k(t), u_k(t) - u(t) \Big) dt \right\} \leqslant 0, \qquad \forall s \in [0, T].$$

We now recall we are working with some fixed $m \in [m_1, m_0[$ and $r \geqslant r(m)$. Hence, thanks to Lemma 2.2 (in particular, to (2.19)) and to the properties of $L = A + B$, we have that $[L(t; \cdot) + (rt - (mp+1)/2)I]$: $U_m \to Z_m$ is a pseudomonotone operator. We thus obtain, from (3.34), that:

$$(3.35) \quad \liminf_{k \to +\infty} \left\{ \int_0^s t^{mp} \exp(-2rt) \Big(L(t; u_k(t)) + \frac{1}{2}(2rt - mp - 1) u_k(t), u_k(t) - v(t) \Big) dt \right\} \geqslant$$

$$\geqslant \int_0^s t^{mp} \exp(-2rt) \Big(L(t; u_k(t)) + \frac{1}{2}(2rt - mp - 1) u(t), \ \ u(t) - v(t) \Big) dt ,$$

$$\forall s \in [0, T], \qquad \forall v(t) \in U_m .$$

Now, from (3.30), taking any $\varphi(t) \in C^0([0, T])$, with $\varphi(t) \geqslant 0$ on $[0, T]$, we get that (recall that any $v(t)$ satisfying (3.2) was previously fixed):

$$(3.36) \quad \int_0^T \varphi(s)\, ds \int_0^s t^{mp} \exp(-2rt) \Big(tv'(t) + \frac{1}{2}(mp + 1 - 2rt) v(t) - f(t), v(t) - u_k(t) \Big) dt \geqslant$$

$$\geqslant \int_0^T \varphi(s)\, ds \int_0^s t^{mp} \exp(-2rt) \cdot \Big(L(t; u_k(t)) + \frac{1}{2}(2rt - mp - 1) u_k(t), u_k(t) - v(t) \Big) dt +$$

$$+ \frac{1}{2} \int_0^T \varphi(s)\, s^{mp+1} \exp(-2rs) |v(s) - u_k(s)|^2 .$$

Hence, by taking in (3.36) the lim inf, for $k \to +\infty$, and using (3.25) and (3.35), we obtain that $u(t)$ satisfies:

$$(3.37)\qquad \int_0^T \varphi(s)\,ds \int_0^s t^{mp}\exp(-2rt)\cdot$$

$$\cdot\Big(tv'(t)+L(t;u(t))+\frac{1}{2}(mp+1-2rt)(v(t)-u(t))-f(t),\ v(t)-u(t)\Big)\,dt \geqslant$$

$$\geqslant \frac{1}{2}\int_0^T \varphi(s)\,s^{mp+1}\exp(-2rs)|v(s)-u(s)|^2\,ds\,,$$

$$\forall \varphi(t)\in C^0([0,T]),\ \text{ with } \varphi(t)\geqslant 0 \text{ on } [0,T].$$

(3.37) thus gives that, for every $m\in[m_0\, m_1[$ and every $r\geqslant r(m)$, $u(t)$ *satisfies* (3.3) *for a.e.* $s\in]0,T[$, and every $v(t)$ as in (3.2).

d) Now, we have only to prove that $u(t)$ satisfies (3.4) (III) and (IV) (and this also gives that $u(t)$ satisfies (3.3) *for every* $s\in]0,T[$).

Since $u(t)$ has the properties (3.4) (I) and (II), we can consider, from $u(t)$, the family $\{u_h(t)\}$ defined as in Lemma 3.1 (where $m=m_1$ and $h>h_0$). Hence, $\{u_h(t)\}$ satisfies (3.7) ... (3.10) (with $m=m_1$ and $r\geqslant r(m_1)$); by using (3.8) (where $w_h(t)$ is replaced by $u_h(t)$), we obtain that (recall the conclusion in the preceding step *c*)):

$$(3.38)\qquad \frac{1}{2}s^{m_1p+1}\exp(-2rs)|u_h(s)-u(s)|^2 \leqslant \int_0^s t^{m_1p}\exp(-2rt)\cdot$$

$$\cdot\Big\{\Big(L(t;u(t))-f(t),\ u_h(t)-u(t)\Big)+\frac{1}{2}(m_1p+1-2rt)|u_h(t)-u(t)|^2\Big\}\,dt,$$

$$\text{for } a.e.\ s\in]0,T].$$

Hence, thanks to the properties of $\{u_h(t)\}$ (in particular, (3.9) and (3.10)), (3.38) gives clearly that $u(t)$ satisfies (3.4) (III) and (IV).

Thus, Theorem 3.1 is completely proved.

3.4. – We finish this paper, by presenting some applications of our abstract results (Theorems 2.1 and 3.1): in particular, we give some examples of operators $A(t;\cdot)$ and $B(t;\cdot)$, to which such results apply.

As an introductory remark, we firstly present an example concerning ordinary differential equations and inequalities (also see BERNARDI [4]).

EXAMPLE 3.1. – Let us take:

$$V=H=V^*=\boldsymbol{R};\ B(t;u)=bu,\ A(t;u)=t^{\alpha p}|u|^{p-1}\operatorname{sgn}(u)$$

where $b, a \in \boldsymbol{R}$, and $p > 1$. Hence, (2.8) ... (2.15) hold trivially; in particular, $A(t; \cdot)$ is monotone for every $t \in]0, T]$ (and nonlinear, if $p \neq 2$). So, our abstract results apply here to this situation. We observe (also see [4]) that it results here: $m_0 = (2b - 1)/p$, if $p \neq 2$; $m_0 = \left(b - (1/2) + \lim_{t \to 0^+} t^{2a}\right)$ if $p = 2$. More generally, our results apply as well to the following situation. Take: $B(t; u) = b(t)\gamma(u)$, where $b(t) \in L^\infty(0, T)$ and $\gamma(u)$ is uniformly Lipschitz continuous on $\boldsymbol{R}$, with $\gamma(0) = 0$ (recall however Remarks 2.2 and 3.1). Take moreover $A(t; u) = t^{ap}\lambda(u)$ where: $a \in \boldsymbol{R}$, $p > 1$; $\lambda(u) \in C^0(\boldsymbol{R})$, and satisfies:

$$\text{(3.39)} \qquad \begin{cases} \exists c_1 > 0\colon u\lambda(u) \geqslant c_1 |u|^p, & \forall u \in \boldsymbol{R}; \\ \exists c_2 > 0\colon |\lambda(u)| \leqslant c_2 |u|^{p-1}, & \forall u \in \boldsymbol{R}; \end{cases}$$

$$\text{(3.40)} \qquad \exists l^* \in \boldsymbol{R}\colon (\lambda(u) - \lambda(v))/(u - v) \geqslant l^*, \qquad \forall u, \ v \in \boldsymbol{R}, \quad \text{with } u \neq v.$$

Hence, (2.8) ... (2.15) hold trivially. Remark however that here, unlike the previous case, $A(t; \cdot)$ is not necessarily monotone.

EXAMPLE 3.2. – Let now Ω be a bounded open subset of $\boldsymbol{R}^n$ and let $p > 1$ such that $1/p - 1/n \leqslant 1/2$. Take $V = W_0^{1,p}(\Omega)$, and $H = L^2(\Omega)$ (hence $V^* = W^{-1,q}(\Omega)$). Then, (2.1) and (2.2) clearly hold. Take now (for some $a \in \boldsymbol{R}$):

$$\text{(3.41)} \qquad A(t; u) = -t^{ap} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right),$$

and, for instance, $B(t; u) = b(t)\gamma(u)$, where $b(t)$ and $\gamma(u)$ are as in the previous Example 3.1. Hence, (2.8) ... (2.15) clearly hold. So, our abstract results apply here as well. Remark that the operator $A(t; \cdot)$, given in (3.41), is monotone, for every $t \in]0, T]$. However, we recall that our results need only the «weaker monotonicity assumption» (2.12).

REFERENCES

[1] E. ASPLUND, *Averaged norms*, Israel J. Math., 5 (1967), pp. 227-233.

[2] C. BAIOCCHI - M. S. BAOUENDI, *Singular evolution equations*, J. Funct. Anal., 25 (1977), pp. 103-120.

[3] M. L. BERNARDI, *Su alcune equazioni d'evoluzione singolari*, Boll. Un. Mat. Ital., (5), 13-B (1976), pp. 498-517.

[4] M. L. BERNARDI, *On some singular nonlinear evolution equations*, Proc. Conf. on Diff. Equations in Banach Spaces (ed. by A. Favini and E. Obrecht), *Lecture Notes in Math.*, Springer, Vol. 1223 (1986), pp. 12-24.

[5] M. L. BERNARDI - G. A. POZZI, *On some singular or degenerate parabolic variational inequalities*, Houston J. Math., 15 (1989), pp. 163-192.

[6] G. DA PRATO - P. GRISVARD, *On an abstract singular Cauchy problem*, Comm. in P.D.E., 3 (1978), pp. 1077-1082.

[7] G. DORE - A. VENNI, *On a singular evolution equation in Banach spaces*, J. Funct. Anal., 64 (1985), pp. 227-250.

[8] A. FAVINI, *Degenerate and singular evolution equations in Banach spaces*, Math. Annalen, 273 (1985), pp. 17-44.

[9] G. H. HARDY - J. E. LITTLEWOOD - G. POLYA, *Inequalities*, University Press, Cambridge, 1952.

[10] J. E. LEWIS - C. PARENTI, *Abstract singular parabolic equations*, Comm. in P.D.E., 7 (1982), pp. 279-324.

[11] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier Villars, Paris, 1969.