Annali di Matematica pura ed applicata
(IV), Vol. CLIX (1991), pp. 1-16

# On the Global Bifurcation for a Class of Degenerate Equations (*).

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**Summary.** – *We consider the nonlinear Dirichlet boundary value problems for the second order equation*

$$-(a(x)|u'(x)|^{p-2}u'(x))' - \lambda c(x)|u(x)|^{p-2}u(x) = g(x, u(x), \lambda),$$

*the fourth order equation*

$$(a(x)|u''(x)|^{p-2}u''(x))'' - \lambda c(x)|u(x)|^{p-2}u(x) = g(x, u(x), u'(x), \lambda),$$

*with $p \geqq 2$, $a \in C^1$, $c \in C$, $a, c > 0$, $g$ Carathéodory's function, and the partial differential equation*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u + g(x, u(x), \lambda) \text{ in } \Omega, u = 0$$

*on $\partial\Omega$, with $p > 1$, $\Omega \subset \mathbb{R}^N$, bounded domain. There is proved a global bifurcation result of Rabinowitz's type using the degree theoretical approach for the mappings acting from the Banach space $X$ into its dual $X^*$.*

## 1. – Introduction.

Let us consider *nonlinear boundary value problems* (BVPs)

$$(1.1)\qquad -(a(x)|u'(x)|^{p-2}u'(x))' - \lambda c(x)|u(x)|^{p-2}u(x) = \\ = g(x, u(x), \lambda), \qquad u(0) = u(\pi) = 0,$$

and

$$(1.2)\qquad (a(x)|u''(x)|^{p-2}u''(x))'' - \lambda c(x)|u(x)|^{p-2}u(x) = \\ = g(x, u(x), u'(x), \lambda), \qquad u(0) = u'(0) = u(\pi) = u'(\pi) = 0.$$

(*) Entrata in Redazione il 2 maggio 1988.

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If the coefficient functions $a$ and $c$ are sufficiently smooth, $p \geqq 2$, then NEČAS [9] proved that the eigenvalues of *the eigenvalue problem*

$$(1.3) \qquad -(a(x)|u'(x)|^{p-2}u'(x))' - \lambda c(x)|u(x)|^{p-2}u(x) = 0, \qquad u(0) = u(\pi) = 0,$$

form a countable discrete set $\{\lambda_i\}_{i=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \dots \lim_{n\to\infty} \lambda_n = \infty$. Moreover, each eigenvalue $\lambda_0$ of (1.3) admits only a finite number of isolated normed eigenfunctions.

The same assertion is proved in KRATOCHVÍL-NEČAS [7] for the eigenvalue problem

$$(1.4) \qquad (a(x)|u''(x)|^{p-2}u''(x))'' - \lambda c(x)|u(x)|^{p-2}u(x) = 0,$$

$$u(0) = u'(0) = u(\pi) = u'(\pi) = 0.$$

It is known that Ljusternik-Schnirelmann theory ensures the existence of an infinite sequence of positive eigenvalues of (1.3) (or (1.4)) but this theory does not give all eigenvalues in general (see FUČÍK et al. [6]). If the eigenvalue $\lambda$ of (1.3) (or (1.4)) is of Ljusternik-Schnirelmann type then under some additional assumptions on the nonlinearity $g$ it is possible to prove that $\lambda$ is *a point of local bifurcation* of (1.1) (or (1.2)) (see [6]). In this paper we intend to prove that if $\lambda$ is any eigenvalue of (1.3) (or (1.4)) to which correspond an odd number of pairs of normed eigenfunctions then $\lambda$ is *a point of global bifurcation* of (1.1) (or (1.2)) in the sense of RABINOWITZ [11]. Our assumptions on nonlinearity $g$ are rather general. In the case of constant coefficients $a$ and $c$ in (1.1) we obtain that from every eigenvalue $\lambda_n$ of the corresponding homogeneous problem two unbounded global branches of nontrivial solutions bifurcate. These branches have the same nodal properties as the eigenfunction corresponding to $\lambda_n$.

We shall also consider nonlinear BVP's for *the partial differential equations*:

$$(1.5) \qquad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u + g(x, u(x), \lambda) \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega,$$

where $\Omega \subset \boldsymbol{R}^N$ is bounded domain with sufficiently smooth boundary $\partial\Omega$ and $p > 1$. We prove that the first eigenvalue $\lambda_1 > 0$ of the nonlinear problem

$$(1.6) \qquad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega,$$

is a point of global bifurcation of (1.5). An analogous result holds also when the Dirichlet boundary data in (1.5) and (1.6) are replaced by the Neumann condition

$$N(u) \equiv |\nabla u|^{p-2}\nabla u \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega,$$

where $\boldsymbol{n}$ is the outer normal of $\partial\Omega$.

Note that the equations with the principal part $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ arise in the theory of quasiregular and quasiconformal mappings or in physics (see *e.g.* [8], [10], [13]).

In order to prove our main results we use functional-theoretical approach. The main tool are the properties of the degree of the mapping $A: X \to X^*$ acting from a Banach space $X$ into its dual $X^*$. The definition of this degree and its basic properties may be found e.g. in SKRYPNIK [12] (cf. also BROWDER-PETRYSHYN [3]).

## 2. – Second order problem.

Let us consider the nonlinear Sturm-Liouville problem (1.1) with $a \in C^1([0, \pi])$, $a(x) > 0$ in $[0, \pi]$, $c \in C([0, \pi])$, $c(x) > 0$ in $[0, \pi]$. Let us suppose that $g = g(x, s, \lambda)$ is a Carathéodory's function, i.e. $g(\cdot, s, \lambda)$ is measurable for all $(s, \lambda) \in R^2$ and $g(x, \cdot, \cdot)$ is continuous for a.e. $x \in (0, \pi)$. We also assume that $g(x, 0, \lambda) = 0$ and

$$\lim_{s \to 0} |s|^{-p+1} g(x, s, \lambda) = 0, \tag{2.1}$$

uniformly for a.e. $x \in (0, \pi)$ and $\lambda$ from bounded intervals.

We shall say that $u \in X := \overset{\circ}{W}{}^1_p(0, \pi)$ (the usual Sobolev space for $p \geqq 2$) is *a weak solution* of (1.1) if

$$\int_0^\pi [a(x)|u'|^{p-2}u'v' - \lambda c(x)|u|^{p-2}uv]dx = \int_0^\pi g(x, u(x), \lambda)v dx, \tag{2.2}$$

holds for every $v \in X$.

It is proved in DRÁBEK [5] that whenever $u$ is a weak solution of (1.1) then $u' \in C([0, \pi])$, $a(x)|u'(x)|^{p-2}u'(x)$ is absolutely continuous and (1.1) holds almost everywhere in $(0, \pi)$. From now, we shall speak only about the weak solution bearing on mind its regularity properties mentioned above.

Let us denote by $X^*$ the dual space to $X$, $\|\cdot\|$ and $\|\cdot\|_*$ the norm in $X$ and $X^*$, respectively, $(\cdot, \cdot)$ will be the pairing between $X$ and $X^*$. Define operators $J, S: X \to X^*$ and $G: \mathbb{R} \times X \to X^*$ by

$$(Ju, v) = \int_0^\pi a(x)|u'(x)|^{p-2}u'(x)v'(x)\, dx,$$

$$(Su, v) = \int_0^\pi c(x)|u(x)|^{p-2}u(x)v(x)\, dx,$$

$$(G(\lambda, u), v) = \int_0^\pi g(x, u(x), \lambda)v(x)\, dx,$$

for any $u, v \in X$.

REMARK 1. – With respect to (2.2) the function $u$ is a weak solution of (1.1) if

$$Ju = Su + G(\lambda, u). \tag{2.3}$$

REMARK 2. – The operators $J, S$ and $G$ have the following properties. $J$ is *odd*, $(p-1)$-*homogeneous*, satisfying the *strong monotonicity condition*

$$(Ju - Jv, u - v) \geqq c_0 \|u - v\|^p, \tag{2.4}$$

for any $u, v \in X$ with some constant $c_0 > 0$. Moreover, $J$ is a $(p-1)$-*homeomorphism*, i.e. it is an homeomorphism of $X$ onto $X^*$ and there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|u\|^{p-1} \leqq \|Ju\|_* \leqq c_2 \|u\|^{p-1}, \tag{2.5}$$

for any $u \in X$. The operator $S$ is odd, $(p-1)$-*homogeneous* and *completely continuous*. The operator $G$ is *completely continuous*, $G(\lambda, 0) = 0$ and with respect to (2.1) $G$ satisfies

$$\lim_{\|u\| \to 0} \|u\|^{-p+1} G(\lambda, u) = 0 \tag{2.6}$$

uniformly with respect to $\lambda$ from bounded intervals.

We shall say that $\lambda$ is *an eigenvalue* of (1.3) if

$$Ju - \lambda Su = 0, \tag{2.7}$$

has a solution $u \in X$, $u \neq 0$. Such an element $u$ is called *an eigenfunction*, corresponding to $\lambda$. The eigenvalue $\lambda$ is said to have finite *multiplicity* equal $n$ if there are exactly $n$ pairs $\{(u_i, -u_i)\}_{i=1}^n$ of isolated normed eigenfunctions corresponding to $\lambda$.

REMARK 3. – According to [9] the eigenvalue problem (1.3) has a countable set of eigenvalues of finite multiplicity satisfying

$$0 < \lambda_1 < \lambda_2 < \ldots, \lim_{n \to \infty} \lambda_n = \infty.$$

Let us denote by $u$ any normed eigenfunction of (1.3). It is proved in [9] that for $\rho(x) = |u'(x)|^{p-2}$ the set

$$\overset{\circ}{W}{}^1_{2,\rho} = \left\{ h;\, h(0) = h(\pi) = 0 \text{ and } \int_0^\pi (h')^2 \rho \, dx = \|h\|^2_{1,2,\rho} < \infty \right\}$$

is an Hilbert space imbedded algebraically and topologically into $W^1_q$ when $1 \leqq q < 2(p-1)(2p-3)^{-1}$. Moreover, the system of functions $\{v_n(x)\}_{n=1}^\infty = \{\sin\, nx\}_{n=1}^\infty$ is dense in $\overset{\circ}{W}{}^1_{2,\rho}$. Note that $X \subset \overset{\circ}{W}{}^1_{2,\rho}$ and that $\{v_n\}_{n=1}^\infty$ is dense in $X$. Let $u$ be an eigenfunction of (1.3) and $J'(u)$, $S'(u)$ be the Fréchet derivatives of $J$ and $S$, respectively, at the point $u \in X$, i.e.

$$(J'(u)h, k)_{\overset{\circ}{W}{}^1_{2,\rho}} = (p-1)\int_0^\pi a(x)|u'|^{p-2}h'k'\,dx,$$

$$(S'(u)h, k)_{\overset{\circ}{W}{}^1_{2,\rho}} = (p-1)\int_0^\pi c(x)|u|^{p-2}hk\,dx,$$

for any $h, k \in \overset{\circ}{W}{}^1_{2,\rho}$. Then it is also proved in [9] that the linear eigenvalue problem

$$J'(u)h - \mu S'(u)h = 0 \tag{2.8}$$

has only simple eigenvalues $\mu$ (i.e. with multiplicity 1).

For a fixed eigenfunction $u$ we shall write $H = \overset{\circ}{W}{}^1_{2,\rho}$.

Let $D \subset X$ be a bounded open set with boundary $\partial D$. Assume that $A: X \to X$ is an operator of the type $A = J + K$, where $J$ is strongly monotone and $K$ is completely continuous. Let $A(u) \neq 0$ for any $u \in \partial D$. Set

$$\Phi_n(u) = \sum_{i=1}^n (Au, v_i)v_i,$$

and $V_n = \mathrm{Span}\{v_i\}_{i=1}^n$. Then it is possible to define *the degree of the mapping* $A$ in 0 with respect to $D \subset X$ as the Brouwer degree of $\Phi_n: V_n \to V_n$ at 0 with respect to $D \cap V_n$ for sufficiently large $n$. More precisely,

$$\mathrm{Deg}\,[A; D, 0] = \lim_{n\to\infty} d_B[\Phi_n; D \cap V_n, 0], \tag{2.9}$$

where $d_B$ denotes the Brouwer degree.

It is shown in SKRYPNIK [12] that $d_B[\Phi_n; D \cap V_n, 0]$ is constant for $n \geqq n_0$, for some $n_0 \in \mathbb{N}$.

REMARK 4. – In fact, the degree (2.9) is defined for more general maps $A$ in [12]. For our purposes it will be sufficient to consider $A = J + K$ which guarantee that all assumptions from [12] are satisfied. The properties of the degree defined by (2.9) are similar to properties of the Leray-Schauder degree. The reader may found them in [12].

Using the strong monotonicity of $J$ and the property (2.5) it is easy to see that

$$\mathrm{Deg}\,[A; D, 0] = \deg\,[I + J^{-1}\circ K; D, 0],$$

where «deg» denotes the Leray-Schauder degree. However, $J^{-1}$ is not continuously differentiable and so it is more convenient to use the properties of «Deg» in the proof of our main result.

A point $u_0 \in X$ will be called a critical point of $A$ if $A(u_0) = 0$. We say that $u_0$ is *an* *isolated critical point* of $A$ if there exists sufficiently small $\varepsilon > 0$ such that in $B_\varepsilon(u_0)$ (the open ball centred at $u_0$ with radius $\varepsilon$) $A$ vanishes only in $u_0$. The degree $\mathrm{Deg}[A; B_\varepsilon(u_0), 0]$ will be called the *index of the isolated critical point* $u_0$ (with respect to $A$).

Let us suppose that $\Psi: X \to \mathbb{R}$ is continuously differentaible functional (in the Fréchet sense), and $\Psi'(u) = A(u)$, $u \in X$ where $A$ is of the form $J + K$. The following assertion is an immediate consequence of the above definition of the degree «Deg» and of the finite-dimensional version of the result of AMANN [1].

LEMMA 1. – *Let $u_0$ be a local minimum of $\Psi$ and an isolated critical point of $A$. Then* $\mathrm{Deg}[A; B\varepsilon(u_0), 0] = 1$ *for $\varepsilon > 0$ sufficiently small.*

Let $E = \mathbb{R} \times X$ be equipped with the norm $\|(\lambda, u)\|^2 = |\lambda|^2 + \|u\|^2$. We say that $C = \{(\lambda, u) \in E; (\lambda, u)$ solves (2.3), $u \neq 0\}$ is a *continuum of nontrivial solutions* of (1.1) if it is connected in $E$.

We shall formulate and prove our first bifurcation result.

THEOREM 1. – *Let $\lambda_n$ be an eigenvalue of* (1.3) *of odd multiplicity. Then there exists a continuum $C$ of nontrivial solutions of* (1.1) *which contains the point $(\lambda_n, 0)$, in its closure and it is either unbounded in $E$ or it contains in its closure a point $(\lambda_m, 0)$, where $\lambda_m$ is an eigenvalue of* (1.3), $\lambda_n \neq \lambda_m$.

*The proof* of Theorem 1 will be performed in several steps.

*Step* 1. Let $\delta > 0$ be such that $2\delta < \min\{\lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1}\}$, where $\lambda_{n-1}$ and $\lambda_{n+1}$ are the eigenvalues of (1.3) preceding and following $\lambda_n$, respectively. We prove that

$$\mathrm{Deg}[J - (\lambda_n - \delta)S; B_r(0), 0] \neq \mathrm{Deg}[J - (\lambda_n + \delta)S; B_r(0), 0] \tag{2.10}$$

for any $r > 0$ and $\delta > 0$ small enough. Let us define a twice continuously differentiable function $\psi: \mathbb{R} \to \mathbb{R}$ by the following way:

$$\psi(t) = 0 \qquad \text{for } t \leqq R,$$

$$\psi(t) = (\lambda_n - \lambda_1 + 2\delta)(t - 2R) \qquad \text{for } t \geqq 3R,$$

$\psi$ positive and strictly convex in $(R, 3R)$, where $R > 0$ is fixed number. Define the functional $F_\lambda: X \to \mathbb{R}$ by

$$F_\lambda(u) = p^{-1}(Ju, u) - p^{-1}\lambda(Su, u) + \psi(p^{-1}(Su, u)).$$

Then $F_\lambda$ is twice continuously differentiable and the critical points of $F'_\lambda$ correspond to the solutions of the equation

$$Ju - (\lambda - \psi'(p^{-1}(Su, u)))Su = 0\,. \tag{2.11}$$

Let us suppose that $u_0 \in X$ is a critical point of $F'_\lambda$, $\lambda \in [\lambda_n - \delta, \lambda_n + \delta] \setminus \{\lambda_n\}$. Then either $u_0 = 0$ or $u_0 \neq 0$. In case $u_0 \neq 0$ it follows from (2.11) that

$$\lambda - \psi'(p^{-1}(Su_0, u_0)) = \lambda_k,$$

for some eigenvalue $\lambda_k < \lambda$ and $u_0 = lu_k$ for some $l = l(\lambda) \in \mathbb{R}$ ($u_k$ is a normalized eigenfunction corresponding to $\lambda_k$). Simultaneously, with respect to the definition of $\psi$, for any normed eigenfunction $u_k$ corresponding to $\lambda_k < \lambda$ we find precisely one $l_k = l_k(\lambda) > 0$ such that $u_0 = l_k u_k$ is a critical point of $F'_\lambda$. Hence $F'_\lambda$ has a finite number of isolated critical points of type 0, $l_k u_k$, $-l_k u_k$, where $u_k$ is the eigenfunction corresponding to $\lambda_k < \lambda$. If $\varkappa > 0$ is large enough, we can suppose (with respect to the definition of $\psi$) that all these critical points lie in $B_\varkappa(0)$ and that $(F'_\lambda(u), u) > 0$ for any $u \in \partial B_\varkappa(0)$, $\lambda \in [\lambda_n - \delta, \lambda_n + \delta] \setminus \{\lambda_n\}$. It follows from the properties of the degree (see [12]) that

$$\text{Deg}\ [F'_\lambda; B_\varkappa(0), 0] = 1. \tag{2.12}$$

Let $u_k^\lambda = l_k u_k$ be any nonzero critical point of $F'_\lambda$. We claim that for $k < n$ and for $\delta > 0$ small enough the index of $u_k^{\lambda_n - \delta}$ (with respect to $F'_{\lambda_n - \delta}$) is equal to the index of $u_k^{\lambda_n + \delta}$ (with respect to $F'_{\lambda_n + \delta}$) and it is either 1 or $-1$. We also claim that the index of $u_n^{\lambda_n + \delta}$ (with respect to $F'_{\lambda_n + \delta}$) is either 1 or $-1$. Suppose for a moment that claims are true. Then for $\lambda = \lambda_n - \delta$ the index of 0 (with respect to $F'_{\lambda_n - \delta}$) is equal to $1 - s$, where $s$ is the sum of indeces of all critical points of $F'_{\lambda_n - \delta}$ different from 0 (see the additivity property of the degree [12] and (2.12)). For $\lambda = \lambda_n + \delta$ the index is $1 - (s + \bar{s})$, where $\bar{s}$ is the sum of indeces of critical points $u_n^{\lambda_n + \delta}$. Since the multiplicity of $\lambda_n$ is odd, it is $\bar{s} \neq 0$. It follows that the index of 0 with respect to $F'_\lambda$ changes when $\lambda$ crosses $\lambda_n$. Because $J$, $S$ are homogeneous and $F'_\lambda(u) = Ju - \lambda Su$, for $u \in B_R(0)$ we have (2.10) with arbitrary $r > 0$ and $\delta > 0$ small enough.

*Step* 2. We prove above claims. Let $u_k$ be the eigenfunction corresponding to $\lambda_k$. We shall take the basis $\{v_i\}_{i=1}^\infty$ in $X$ and in $H$, where $H$ corresponds to $u_k$. According to the definition of the degree the index of $u_k^\lambda$ is equal to

$$d_B[\Phi_{\lambda, q}; B_\varepsilon(u_k^\lambda) \cap V_q, 0],$$

where

$$\Phi_{\lambda, q}(u) = \sum_{i=1}^{q} ((F'_\lambda(u), v_i) v_i,$$

$q \geqq q(\lambda)$ and $\varepsilon > 0$ is small enough. It is

$$(F''_\lambda(u_k^\lambda)w, v) = (J'(u_k^\lambda)w, v) - (\lambda - \psi'(p^{-1}(Su_k^\lambda, u_k^\lambda))) \cdot$$

$$\cdot (S'(u_k^\lambda)w, v) + \psi''(p^{-1}(Su_k^\lambda, u_k^\lambda))(Su_k^\lambda, w)(Su_k^\lambda, v),$$

for any $w, v \in H$. Note that $\lambda - \psi'(p^{-1}(Su_k^\lambda, u_k^\lambda)) = \lambda_k$. Let us suppose that there is $w_0 \in H, w_0 \neq 0$, such that $(F''_\lambda(u_k^\lambda)w_0, v) = 0$ for all $v \in H$. Then for $v = u_k^\lambda$ we get

$$(2.13) \qquad (J'(u_k^\lambda)w_0, u_k^\lambda) - \lambda_k(S'(u_k^\lambda)w_0, u_k^\lambda) + \\ + \psi''(p^{-1}(Su_k^\lambda, u_k^\lambda))(Su_k^\lambda, w_0)(Su_k^\lambda, u_k^\lambda) = 0.$$

Because

$$(J'(u_k^\lambda)w_0, u_k^\lambda) = (p-1)(Ju_k^\lambda, w_0), (S'(u_k^\lambda)w_0, u_k^\lambda) = (p-1)(Su_k^\lambda, w_0)$$

and $(Su_k^\lambda, u_k^\lambda) \neq 0$ it follows from (2.13) that $(Su_k^\lambda, w_0) = 0$. In this case, however,

$$(J'(u_k^\lambda)w_0, v) - \lambda_k(S'(u_k^\lambda)w_0, v) = 0,$$

for any $v \in H$ and since $\lambda_k$ is a simple eigenvalue of the linear problem (2.8) (with $u := u_k^\lambda$), it should be $w_0 = u_k^\lambda$ a contradiction with $(Su_k^\lambda, w_0) = 0$. Hence

$$(2.14) \qquad F''_\lambda(u_k^\lambda)w \neq 0,$$

for any $w \in H$, $w \neq 0$ and for $\lambda \in [\lambda_n - \delta, \lambda_n + \delta]$. if $k < n, \lambda = \lambda_n + \delta$ if $k = n$.

We show that there exists $q_0 = q_0(\delta)$ such that for all $\lambda \in [\lambda_n - \delta, \lambda_n + \delta]$ if $k < n$ and $\lambda = \lambda_n + \delta$ if $k = n$

$$(2.15) \qquad \Phi'_{\lambda, q}(u_k^\lambda)w \neq 0,$$

for any $w \in V_q, w \neq 0, q \geqq q_0$. Let $k < n$. Suppose that (2.15) is not true, i.e. there exist $q \to \infty, \lambda(q) \in [\lambda_n - \delta, \lambda_n + \delta], w_q \in V_q, w_q \neq 0$ such that

$$(2.16) \qquad \Phi'_{\lambda(q), q}(u_k^{\lambda(q)})w_q = 0.$$

We can suppose that $\lambda(q) \to \lambda$, $\|w_q\|_H = 1$ and $w_q \rightharpoonup w_0$ in $H$, for some $w_0 \in H$. Choose $\tilde{w}_q \in V_q$ such that $\tilde{w}_q \to w_0$ in $H$ as $q \to \infty$. Then (2.16) implies

$$(F''_{\lambda(q)}(u_k^{\lambda(q)})w_q, w_q - w_0) = (F''_{\lambda(q)}(u_k^{(q)})w_q, \tilde{w}_q - w_0).$$

Because the right hand side tends to zero and

$$(F''_{\lambda(q)}(u_k^{\lambda(q)})w_0, w_q - w_0) \to 0,$$

(realize that $u_k^{\lambda(q)} = l_k(q)u_k$ with some $l_k(q)$ from compact interval not containing 0) we get

$$(2.17) \qquad (F''_{\lambda(q)}(u_k^{\lambda(q)})(w_q - w_0), w_q - w_0) \to 0,$$

as $q \to \infty$. It follows from here, from the definition of the norm in $H$, from the compactness of $S'(u_k^\lambda): H \to H$ and from (2.17) that

$$w_q \to w_0 \text{ in } H.$$

Since $\lambda(q) \to \lambda$ implies $u_k^{\lambda(q)} \to u_k^\lambda$ in $X$, we obtain

$$(F_\lambda''(u_k^\lambda)w_0, v) = 0,$$

for any $v \in H$ with $\|w_0\|=1$. But this is a contradiction with (2.14). Analogously we prove (2.15) also for $k = n$.

Because

$$\Phi_{\lambda,q}'(u_k^\lambda)w = \sum_{i=1}^{q} [(J'(u_k^\lambda)w, v_i) - \lambda(S'(u_k^\lambda)w, v_i) + \psi''(p^{-1}(Su_k^\lambda, u_k^\lambda)(Su_k^\lambda, w)(Su_k^\lambda, v_i)]v_i,$$

$w \in V_q$, and $u_k^\lambda$ depends continuously on $\lambda$, det $\Phi_{\lambda,q}'(u_k^\lambda)$ is a continuous function of $\lambda$ and it is

$$\det \Phi_{\lambda,q}'(u_k^\lambda) \neq 0,$$

with respect to (2.15), for all $\lambda \in [\lambda_n - \delta, \lambda_n + \delta]$ if $k < n, q \geqq q_0$. Hence for $k < n, \lambda \in [\lambda_n - \delta, \lambda_n + \delta]$, the Brouwer degree $d_B[\Phi_{\lambda,q}; B_\varepsilon(u_k^\lambda) \cap V_q, 0]$ is a constant function of $\lambda$ either equal to $+1$ for all $q \geqq q_0$ or equal to $-1$ for all $q \geqq q_0$. We obtain directly from (2.15) that

$$d_B[\Phi_{\lambda_n+\delta,q}; B_\varepsilon(u_n^\lambda) \cap V_q, 0] = \pm 1,$$

independently of $q \geqq q_0$. If we take $q(\lambda) = q_0$ in the definition of $\Phi_{\lambda,q}$ the claims are proved.

*Step* 3. Put

$$\mathcal{P}(\lambda, u) = Ju - \lambda Su - G(\lambda, u).$$

Then it follows immediately from (2.6) and from the homotopy invariance property of the degree (see [12]) that

$$\text{Deg}[\mathcal{P}(\lambda_n - \delta, \cdot); B_r(0), 0] \neq \text{Deg}[\mathcal{P}(\lambda_n + \delta, \cdot); B_r(0), 0], \tag{2.18}$$

for $r > 0$ and $\delta > 0$ small enough.

*Step* 4. Following step by step the Rabinowitz's proof [11, Theorem 1.3], using (2.6) and (2.18), the existence of the continuum $C$ of nontrivial solutions of (1.1) with the desired properties can be proved. This completes the proof of Theorem 1.

## 3. – Second order problem with constant coefficients.

Let us consider the special case of (1.1) with $a(x) \equiv c(x) \equiv 1$ on $[0, \pi]$. It is proved in [5] that in this case the eigenvalues of (1.3) are simple and that any eigenfunction corresponding to the $n$-th eigenvalue $\lambda_n$ has precisely $n-1$ simple nodes (i.e. $u'(x_0) \neq 0$ for $u(u_0) = 0$) in $(0, \pi)$. If $g = g(x, u, \lambda)$ is a continuous function then one can prove that $|u'|^{p-2}u' \in C^1([0, \pi])$ for any solution $u$ of (1.1) (see [5]).

Using the assumption (2.1) it follows directly from equation (1.1) that any nontrivial solution of (1.1) has only simple nodes in $[0, \pi]$ (in 0 and in $\pi$ it means that $u'_+(0) \neq 0$ and $u'_-(\pi) \neq 0$, respectively). We have also the following information about the structure of $C$.

LEMMA 2. – *The convergence* $(\lambda_n, u_n) \to (\lambda, u)$ *in* $E$, $(\lambda_n, u_n), (\lambda, u) \in C$, *yields* $u_n \to u$ *in* $C^1([0, \pi])$.

PROOF. – We have

$$-(|u'_n|^{p-2}u'_n)' = \lambda_n |u_n|^{p-2}u_n + g(x, u_n, \lambda_n), \tag{3.1}$$

$$-(|u'|^{p-2}u')' = \lambda |u|^{p-2}u + g(x, u, \lambda), \tag{3.2}$$

and $u_n \to u$ in $C([0, \pi])$ with respect to continuous imbedding $X \subset C([0, \pi])$. We get from (3.1), (3.2)

$$|u'(x)|^{p-2}u'(x) - |u'_n(x)|^{p-2}u'_n(x) =$$

$$= k + \int_0^x [\lambda_n |u_n(t)|^{p-2}u_n(t) - \lambda |u(t)|^{p-2}u(t) + g(t, u_n(t), \lambda_n) - g(t, u(t), \lambda)]dt,$$

i.e.

$$|u'_n(x)|^{p-2}u'_n(x) \to k + |u'(x)|^{p-2}u'(x) \qquad \text{in } C([0, \pi]).$$

But $u_n \to u$ in $X$ implies $u_n \to u$ uniformly on $[0, \pi]$. Hence it follows that $k = 0$. Since the function $s \mapsto |s|^{1/(p-1)} \operatorname{sign} s$ is $(p-1)^{-1}$-Hölder continuous we have $u'_n \to u'$ in $C([0, \pi])$ which concludes the proof.

REMARK 5. – In other words Lemma 2 states that the topology on $C$ induced by the norm on $E$ is equivalent to the topology on $C$ induced by the norm

$$\|(\lambda, u)\|_1^2 = |\lambda|^2 + \|u\|^2_{C^1([0,\pi])}.$$

Let us suppose that $\lambda_n$ is the $n$-th eigenvalue of (1.1) and $u_n$ is the corresponding normalized eigenfunction satisfying $u'_n(0) > 0$. Using the same argument as in [11, Lemma 1.24, Theorem 1.25] it is possible to show that there are two maximal connected subsets of $C$ ($C$ being from Theorem 1) bifurcating from $(\lambda_n, 0)$ in the directions $u_n$ and $-u_n$. More precisely, let $\tau \in (0, 1)$ and $K_\tau^+ = \{(\lambda, u) \in E; (\varphi^*, u) > \tau\|u\|\}$, $K_\tau^- = \{(\lambda, u) \in E; (\varphi^*, u) < -\tau\|u\|\}$, where $\varphi^* \in X^*$ is a fixed element such that $(\varphi^*, u_n) = 1$. Then there are maximal connected sets $C_n^+, C_n^-$ such that $C_n^+ \cup C_n^- \subset C$, $C_n^\pm$ contain $(\lambda_n, 0)$ in its closure and $C_n^+ \cap \mathcal{B}_s(\lambda_n, 0) \subset K_\tau^+$, $C_n^- \cap \mathcal{B}_s(\lambda_n, 0) \subset K_\tau^-$ with some $s = s(\tau) > 0$ small enough (here $\mathcal{B}_s(\lambda_n, 0)$ is the ball in $E$ with radius $s > 0$ centred at $(\lambda_n, 0)$).

It follows immediately from Lemma 2 that $C_n^+ \cap C_n^- = \emptyset$. Moreover, if $m \neq n$ then

$C_m^\nu \cap C_n^\nu = \emptyset$ for any $\nu \in \{+,-\}$ with respect to the nodal properties of the eigenfunctions and Lemma 2.

THEOREM 2. – *Let $\lambda_n$ be an eigenvalue of* (1.1) *with constant coefficients* $a(x) \equiv c(x) \equiv 1$. *Let the function $g = g(x, u, \lambda)$ be continuous and satisfy* (2.1). *Then there are two unbounded continua of nontrivial solutions $C_n^+$ and $C_n^-$ bifurcating from $(\lambda_n, 0)$ with the property that if $(\lambda, u) \in C_n$ then $u$ has $n-1$ simple nodes in $(0, \pi)$ and sign $u'(0) = \nu$, $\nu \in \{+,-\}$.*

PROOF. – It follows from the above considerations that the continuum $C_n$ bifurcating from $(\lambda_n, 0)$ which can be decomposed as $C_n^+ \cup C_n^-$ does not satisfy the second alternative of Theorem 1. According to Lemma 2 both $C_n^+$ and $C_n^-$ save the nodal properties of $u_n$ and $-u_n$, respectively. Hence neither $C_n^+$ nor $C_n^-$ contains the points of the form $(\lambda, u)$, $(\lambda, -u)$. Then following the proof of Theorem 1.27 in [11] we get that $C_n$ is unbounded for both $\nu =$ «+» and $\nu =$ «−».

## 4. – Fourth order problem.

Let us consider the nonlinear problem (1.2) with the same assumptions on $a$ and $c$ as in Section 2. We suppose that $g = g(x, s, r, \lambda)$ is a Carathéodory's function, i.e. $g(\cdot, s, r, \lambda)$ is measurable for all $(s, r, \lambda) \in \mathbb{R}^3$ and $g(x, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in (0, \pi)$. We also assume that $g(x, 0, 0, \lambda) = 0$ and

$$\lim_{(s,r)\to(0,0)} |s^2 + r^2|^{(-p+1)/2} g(x, s, r, \lambda) = 0,$$

uniformly for a.e. $x \in (0, \pi)$ and $\lambda$ from bounded intervals. Set $X := \overset{\circ}{W}{}_p^2(0, \pi)$ and define $J, S: X \to X^*$ and $G: \mathbb{R} \times X \to X^*$ as follows

$$(Ju, v) = \int_0^\pi a(x)|u''(x)|^{p-2} u''(x) v''(x)\, dx,$$

$$(Su, v) = \int_0^\pi c(x)|u(x)|^{p-2} u(x) v(x)\, dx,$$

$$(G(\lambda, u), v) = \int_0^\pi g(x, u(x), u'(x), \lambda) v(x)\, dx,$$

for any $u, v \in X$.

The operators $J$, $S$ and $G$ have the same properties as these in Section 2 (cf. Remarks 1, 2).

It is proved in KRATOCHVÍL-NEČAS [7] that the eigenvalue problem (2.7) with $J$

and $S$ defined above has a countable set of eigenvalues of finite multiplicity

$$0<\lambda_1<\lambda_2<\ldots,\ \lim_{n\to\infty}\lambda_n=\infty.$$

Moreover, if $u$ is any normalized eigenfunction then the set

$$\overset{\circ}{W}{}^2_{2,\rho}=\left\{h;\, h(0)=h'(0)=h(\pi)=h'(\pi)=0 \ \text{ and } \int_0^\pi (h'')^2\rho\, dx\equiv\|h\|^2_{2,2,\rho}<\infty\right\},$$

where $\rho(x)=|u''(x)|^{p-2}$, is the Hilbert space imbedded algebraically and topologically into $W^2_q$ when $1\leqq q<2(p-1)(2p-3)^{-1}$. It is also proved in [7] that the linear eigenvalue problem

$$\int_0^\pi a(x)|u''(x)|^{p-2}h''(x)k''(x)\,dx-\mu\int_0^\pi c(x)|u(x)|^{p-2}h(x)k(x)\,dx=0,$$

for any $k\in\overset{\circ}{W}{}^2_{2,\rho}$, has only simple eigenvalues.

Set $H:=\overset{\circ}{W}{}^2_{2,\rho}$ for fixed eigenfunction $u$ and as $\{v_n\}_{n=1}^\infty$ take the eigenfunctions of the problem

$$u^{IV}-\lambda u=0,\qquad u(0)=u'(0)=u(\pi)=u'(\pi)=0.$$

Then $\{v_n\}_{n=1}^\infty$ is dense both in $X$ and in $H$. By a weak solution of (1.2) we shall mean a function $u\in X$ satisfying (2.3). Similarly as in Section 2 we define a continuum $C$ of nontrivial weak solutions of (1.2). Using essentially the same approach as in Section 2 we get.

THEOREM 3. – *Let $\lambda_n$ be an eigenvalue of homogeneous fourthorder problem* (1.4) *of odd multiplicity. Then there exists a continuum $C$ of nontrivial weak solutions of* (1.2) *which contains the point $(\lambda_n, 0)$ in its closure and it is either unbounded in $E$ or it contains in its closure the point $(\lambda_m, 0)$ where $\lambda_m$ is an eigenvalue of* (1.4), $\lambda_m\neq\lambda_n$.

## 5. – Partial differential equations.

In this section we consider the bifurcation problem (1.5). We shall suppose that the nonlinear function $g=g(x,s,\lambda)$ satisfies the assumptions from Section 2 with $[0,\pi]$ replaced by $\Omega\subset\mathbb{R}^N$. Set $X:=\overset{\circ}{W}{}^1_p(\Omega)$ and define $J$, $S$: $X\to X^*$ and $G$: $\mathbb{R}\times X\to X^*$ as follows

$$(Ju,v)=\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx,$$

$$(Su, v) = \int_{\Omega} |u|^{p-2} uv \, dx,$$

$$(G(\lambda, u), v) = \int_{\Omega} g(x, u(x), \lambda) \, v(x) \, dx,$$

for any $u, v \in X$, with $p > 1$.

The operators $J$, $S$ and $G$ have the same properties as those defined in Section 2 (cf. Remarks 1, 2) for $p \geqq 2$. In the case $1 < p < 2$ the operator $J$ is not strongly monotone in the sense of (2.4) but it satisfies condition ($\alpha$) from SKRYPNIK [12] (cf. Remark 4).

A function $u \in X$ is said to be *a weak solution* of (1.5) and (1.6) if it satisfies

$$Ju = \lambda Su + G(\lambda, u) \tag{5.1}$$

and

$$Ju = \lambda Su, \tag{5.2}$$

respectively.

REMARK 6. – If a weak solution $u$ of (1.5) (or (1.6)) satisfies apriori $u \in L_\infty$ then it follows from the result of TOLKSDORF [13] that the derivatives of $u$ are already Hölder continuous. Particularly, if $g$ is bounded, we obtain the regularity in this sense in the case $p > N$ with respect to the Sobolev's imbedding theorem.

As in Section 2 we define a continuum $C$ of nontrivial weak solutions of (1.5).

The following assertion concerning the first eigenvalue to the problem (1.6) is proved in ANANE [2].

LEMMA 3. – *Let us suppose that the boundary $\partial\Omega$ is of class $C^{2,\beta}$. Then there exists the first eigenvalue $\lambda_1 > 0$ of* (1.6) *which is simple and isolated. A corresponding eigenfunction $u_1 \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, can be chosen such that $u_1 > 0$ in $\Omega$ and $\partial u_1/\partial \boldsymbol{n} < 0$ on $\partial\Omega$.*

We use this and the approach from the proof of Theorem 1 in order to prove the following global bifurcation result.

THEOREM 4. – *Let us suppose that all assumptions stated above are fulfilled. Then there exists a continuum $C$ of nontrivial weak solutions of* (1.5) *which contains the point $(\lambda_1, 0) \in E$ in its closure and it is either unbounded in $E$ or it contains in its closure a point $(\lambda_0, 0)$, where $\lambda_0 > \lambda_1$ is an eigenvalue of* (1.6).

PROOF. – We prove that

$$\operatorname{Deg}[J-(\lambda_1-\delta)S; B_r(0), 0] \neq \operatorname{Deg}[J-(\lambda_1+\delta)S; B_r(0), 0], \tag{5.3}$$

for $r>0$ and $\delta>0$ small enough.

Similarly as in the proof of Theorem 1 we define the function $\psi\colon \mathbb{R}\to\mathbb{R}$ by

$$\psi(t)=0, \quad \text{for } t\leqq R,$$

$$\psi(t)=2\delta(t-2R), \quad \text{for } t\geqq 3R,$$

$\psi$ positive and strictly convex in $(R, 3R)$, where $R>0$ is fixed number and $\delta>0$ is chosen in such a way that $(\lambda_1, \lambda_1+\delta]$ does not contain any eigenvalue of (1.6) (cf. Lemma 3). Following the notations from the proof of Theorem 1, we obtain that $F_\lambda$ is at least once continuously differentiable (we have, now, $p>1$) and that the critical points of $F'_\lambda$ correspond to the solutions of the equation (2.11). We obtain also that for $\lambda=\lambda_1-\delta$ the point $0\in X$ is the only critical point of $F'_\lambda$ while for $\lambda=\lambda_1+\delta$ there are precisely three isolated critical points of $F'_\lambda: 0, u_1, -u_1$, where $u_1>0$ is an eigenfunction corresponding to $\lambda_1$. Since 0 is the minimum of $F_{\lambda_1-\delta}$, the index of 0 with respect to $F'_{\lambda_1-\delta}$ is 1, by Lemma 1, i.e.

$$\operatorname{Deg}\,[J-(\lambda_1-\delta)S; B_r(0), 0]=1, \tag{5.4}$$

with arbitrary $r>0$.

The points $u_1$ and $-u_1$ are minima of $F_{\lambda_1+\delta}$ and hence, by Lemma 1, their index with respect to $F'_{\lambda_1+\delta}$ is 1. Simultaneously, $\operatorname{Deg}[F'_{\lambda_1+\delta}; B_\varkappa(0), 0]=1$, for $\varkappa>0$ large enough, with respect to the definition of $\psi$. Combining these two facts we get that the index of 0 with respect to $F'_{\lambda_1+\delta}$ is $-1$, i.e.

$$\operatorname{Deg}[J-(\lambda_1+\delta)S; B_r(0), 0]=-1, \tag{5.5}$$

with arbitrary $r>0$.

The relation (5.3) then follows from (5.4) and (5.5). To complete the proof we proceed by the same way as in Steps 3 and 4 of the proof of Theorem 1.

REMARK 7. – The proof of Theorem 4 works also without the assumption of simplicity of $\lambda_1$. In order to establish (5.3) (and hence to prove Theorem 4) it is sufficient to know that the set of normalized eigenfunctions corresponding to $\lambda_1$ is finite. The simplicity of $\lambda_1$ (Lemma 3) allows us to strengthen the assertion of Theorem 4 in the following sense.

THEOREM 5. – *Let the assumptions of Theorem 4 be fulfilled. Moreover, suppose that there exists $\delta>0$ such that*

$$Ju \neq \lambda_1 Su + G(\lambda_1, u), \tag{5.6}$$

*for $0<\|u\|\leqq\delta$. Then there are maximal connected subsets $C^+, C^-$, of $C$ containing*

$(\lambda_1, 0) \in E$ *in their closure,* $C^{\pm} \cap \mathcal{B}_s(\lambda_1, 0) \subset K_{\tau}^{\pm}$ *(for* $K_{\tau}^{\pm}, \mathcal{B}_s(\lambda_1, 0)$ *see Section 2,* $n = 1$*) and such that either*

(i) *both* $C^+, C^-$ *are unbounded in* $E$*, or*

(ii) *both* $C^+, C^-$ *contain in their closure a point different from* $(\lambda_1, 0) \in E$.

*The proof* of this Theorem may be performed step by step as the proof of Theorem 2 in DANCER [4]. In fact, with respect to our hypothesis (5.6), we need only Lemma 1 and 2 from [4], where the linearity of the principal part of the equation is not essential.

Let us consider the BVP with Neumann boundary data

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2} u + g(x, u(x), \lambda) & \text{in } \Omega, \\ N(u) \equiv |\nabla u|^{p-2}\nabla u \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Setting $X := W_p^1(\Omega)$ and defining the operators $J$, $S$ and $G$ like at the beginning of this section we define a weak solution of (5.7) as a function $u \in X$ satisfying (5.1). The following analogue of Lemma 3 holds.

LEMMA 4. – *The first eigenvalue* $\lambda_1 = 0$ *of the homogeneous problem*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2} u & \text{in } \Omega, \\ N(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

*is simple and isolated. There exists precisely one normalized positive eigenfunction* $u_1 \equiv (\text{meas } \Omega)^{-1/p}$ *corresponding to* $\lambda_1 = 0$.

PROOF. – The simplicity of $\lambda_1$ follows from the property

$$0 = \lambda_1 = \min_{u \in X, u \neq 0} \left( \int_{\Omega} |\nabla u|^p \, dx \right) \left( \int_{\Omega} |u|^p \, dx \right)^{-1}.$$

Let us suppose that there are $\lambda_n \to 0$ (the eigenvalues of (5.8)) with corresponding normalized eigenfunctions $u_n$. Then with respect to complete continuity of $S$ we can suppose that $u_n \to u$ in $X$ and $(Ju, u) = 0$. Since $\|u\| = 1$, it should be either $u \equiv (\text{meas } \Omega)^{-1/p}$ or $u \equiv -(\text{meas } \Omega)^{-1/p}$. Simultaneously, taking $v \equiv 1$ in

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla v \, dx = \lambda \int_{\Omega} |u_n|^{p-2} u_n v dx,$$

we get

$$\int_{\Omega} |u_n|^{p-2} u_n \, dx = 0,$$

which is a contradiction.

If we use Lemma 4 instead of Lemma 3 and substitute everywhere $\lambda_1 := 0$, we can prove the same assertions as Theorems 4, 5 concerning the global bifurcation for the Neumann BVP (5.7).

## REFERENCES

[1] H. AMANN, *A note on the degree theory for gradient mappings*, Proc. Amer. Math. Society, **85** (1982), pp. 591-595.

[2] A. ANANE, *Simplicité et isolation de la premier valeur propre du p-laplacian avec poids*, C. R. Acad. Sci. Paris, **305**, Sér. 1 (1987), pp. 725-728.

[3] F. E. BROWDER - W. F. PETRYSHYN, *Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces*, J. Functional Analysis, **3** (1969), pp. 217-245.

[4] E. N. DANCER, *On the structure of the solutions of non-linear eigenvalue problems*, Indiana Univ. Math. J., **23** (11) (1974), pp. 1069-1076.

[5] P. DRÁBEK, *Ranges of a-homogeneous operators and their perturbations*, Čas. Pěst. Mat., **105** (1980), pp. 167-183.

[6] S. FUČÍK - J. NEČAS - J. SOUČEK - V. SOUČEK, *Spectral Analysis of Nonlinear Operators*, Lecture Notes in Math., **346**, Springer, Berlin, 1973.

[7] A. KRATOCHVÍL - J. NEČAS, *On the discretness of the spectrum of nonlinear forth-order Sturm-Liouville problem* (in Russian), Comment. Math. Univ. Carolinae, **12** (1971), pp. 639-653.

[8] L. LIBOURTY, *Traité de Glaceologie*, Masson and Lie, Paris (I) 1964 et (II) 1965.

[9] J. NEČAS, *On the discretness of the spectrum of nonlinear second-order Sturm-Liouville problem* (in Russian), Dokl. Akad. Nauk SSSR, **201** (5) (1971), pp. 1045-1048.

[10] M. C. PÉLLISIER - L. REYNAUD, *Étude d'un modéle mathématique d'écoulement de glacier*, C. R. Acad. Sci. Paris, Sér. A, **279** (1979), pp. 531-534.

[11] R. H. RABINOWITZ, *Some global results for nonlinear eigenvalue problem*, J. Functional Analysis, **7** (1971), pp. 487-513.

[12] I. V. SKRYPNIK, *Nonlinear Elliptic Equations of Higher Order* (in Russian), Naukovaja Dumka, Kyjev, 1973.

[13] P. TOLKSDORF, *Regularity of a more general class of quasi-linear elliptic equations*, J. Differential Equations, **51** (1984), pp. 126-150.