Classical and Weak Solutions for One-Dimensional Pseudo-Parabolic Equations with Typical Boundary Data (*).

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Summary. – *In this paper we consider the pseudo-parabolic equations arising in the filtration of water in media with double porosity and moisture transfer in soil. The existence, uniqueness and stability for both classical and weak solutions are studied.*

1. – Introduction.

We know that the filtration of water in media with double porosity [6], moisture transfer in soil [4], and other similar natural phenomena, can lead to the following typical boundary value problem with a pseudo-parabolic equation:

$$u_{xxt} = Au + F\,, \quad \text{in } Q_T; \tag{1.1}$$

$$u(x, 0) = \varphi(x)\,, \quad 0 \leqslant x \leqslant 1; \tag{1.2}$$

$$u(0, t) = f(t)\,, \quad 0 \leqslant t \leqslant T; \tag{1.3}$$

$$u_x(0, t) = g(t)\,, \quad 0 \leqslant t \leqslant T\,. \tag{1.4}$$

where

$$Au = a_1 u_t + a_2 u_{xt} + a_3 u_{xx} + a_4 u_x + a_5 u\,, \tag{1.5}$$

$Q_T = (0, 1) \times (0, T]$, $T > 0$ and f, g, φ, F and a_i $(i = 1, 2, 3, 4, 5)$ are given functions. This type of boundary value problem (1.1)-(1.4) has been investigated in [3, 6, 7, 15] in the case when $a_2 = 0$ and the other a_i are continuously differentiable in $\bar{Q}_T$. The basic tool in [3, 7, 15] is the Riemann function method which, of course requires appropriate smoothness of the coefficients $a_i = a_i(x, t)$.

The equation of form (1.1) can also arise in studying of nonsteady shearing flow of a second order fluid [2, 14] with the boundary conditions given on both sides of

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$x = 0$ and $x = 1$ or on $\partial\Omega$ in n-dimensional spaces. The initial-boundary value problem is studied in [1, 2, 8, 11, 13, 14]. When $a_1 = a_2 = 0$ in (1.1), SHOWALTER [9, 10] considered the Sobolev equations in Hilbert spaces and obtained abstract weak solutions. Since the solution u of (1.1)-(1.4) is not specified on $x = 1$, problem (1.1)-(1.4) is different from those considered in [1, 2, 8, 9, 10, 11, 13, 14].

In this paper we employ the Banach fixed point theorem to show that a unique solution exists for (1.1)-(1.4) under assumptions on the data which are much weaker than those in [3, 7, 15]. In the case when all a_i are bounded measurable functions we show that the solution exists and satisfies (1.1) in the sense of $L^2(Q_T)$. The method used in this paper is not only different from that employed in [3, 7, 15], but also from that employed in [1, 2, 8, 9, 10, 11, 13, 14].

DEFINITION 1.1. – A function $u(x, t)$ is called a classical solution for (1.1)-(1.4) if

$$u,\, u_x,\, u_{xx},\, u_t,\, u_{xt},\, u_{xxt} \in C(\bar{Q}_T); \tag{1.6}$$

and u satisfies (1.1)-(1.4), where $C(\bar{Q}_T)$ is the class of continuous functions in $\bar{Q}_T$.

REMARK 1.1. – From this definition we see that the solution u satisfies (1.1) not only in Q_T, but also in $\bar{Q}_T$, so it is a very strong solution.

We shall use $C[0, 1]$ and $C[0, T]$ to denote the class of continuous functions in $[0, 1]$ and $[0, T]$ respectively, and

$$C^k[0, 1] = \{\varphi \in C[0, 1]: \varphi^{(l)} \in C[0, 1], 1 \leqslant l \leqslant k\}$$

$$C^k[0, T] = \{f \in C[0, T]: f^{(l)} \in C[0, T], 1 \leqslant l \leqslant k\};$$

let $L^2(0, 1)$ and $L^2(0, T)$ denote classes of the square integrable functions on $(0, 1)$ and $(0, T)$, respectively, and

$$H^k(0, 1) = \{\varphi \in L^2(0, 1): \varphi^{(l)} \in L^2(0, 1), 1 \leqslant l \leqslant k\}$$

$$H^k(0, T) = \{f \in L^2(0, T): f^{(l)} \in L^2(0, T), 1 \leqslant l \leqslant k\}$$

for $k = 1, 2$. We employ the symbol $\|\ \|$ with a subscripted name of the space for the usual norm for that space.

ASSUMPTION (H I):

$$\varphi \in C^2[0, 1], \quad f, g \in C^1[0, T], \quad \varphi(0) = f(0), \quad \varphi'(0) = g(0), \tag{1.7}$$

$$F, a_i \in C(\bar{Q}_T); \quad i = 1, 2, 3, 4, 5. \tag{1.8}$$

REMARK 1.2. – It is clear that assumption (1.8) is much weaker than the requirement of smoothness on the a_i. The compatibility assumption at $(0, 0)$ is necessary from the definition 1.1.

DEFINITION 1.2. – A function $u(x, t)$ is called a weak solution for (1.1)-(1.4) if

$$u,\ u_x,\ u_{xx},\ u_t,\ u_{xt},\ u_{xxt} \in L^2(Q_T); \tag{1.9}$$

and u satisfies (1.1) in the sense of $L^2(Q_T)$ and (1.2)-(1.4) in the trace sense.

For this definition we have

ASSUMPTION (H II):

$$\varphi \in H^2(0,1), \qquad f, g \in H^1(0, T); \tag{1.10}$$

$$F \in L^2(Q_T), \quad \text{and the } a_i, \qquad i = 1, 2, 3, 4, 5, \text{ are bounded measurable}. \tag{1.11}$$

REMARK 1.3. – The classical and weak solutions for (1.1)-(1.4) are defined above, but the proofs of the existence of classical and weak solutions are rather different. When $F, \varphi, \varphi', \varphi'', f, f', g, g'$ are all bounded measurable functions, we can show that the weak solution u exists in the sense of u with all its derivatives u_x, u_{xx}, u_t, u_{xt} and u_{xxt} are bounded measurable in Q_T. In fact it will be seen later that the proofs for this case are exactly the same as those for classical solutions.

In section 2 of this paper we will transform (1.1)-(1.4) to an equivalent integro-differential equation, we also prove some lemmas there which will be needed subsequently. Section 3 contains the proof for the existence of the classical solution under assumption (H I). We shall show, in section 4, that the unique weak solution of (1.1)-(1.4) also exists under the assumption (H II).

2. – An equivalent problem.

Suppose $u(x, t)$ is a classical solution for (1.1)-(1.2), then, we have from integration that

$$u_{xx} = \int_0^t [Au(x,\tau) + F(x,\tau)]\,d\tau + \varphi''(x), \tag{2.1}$$

$$u_x = \int_0^x \int_0^t [Au(\xi,\tau) + F(\xi,\tau)]\,d\tau\,d\xi + \varphi'(x) - \varphi'(0) + g(t), \tag{2.2}$$

$$u = \int_0^x \int_0^\eta \int_0^t [Au(\xi,\tau) + F(\xi,\tau)]\,d\tau\,d\xi\,d\eta + \varphi(x) - \varphi(0) - \varphi'(0)x + g(t)x + f(t), \tag{2.3}$$

$$u_{xt} = \int_0^x [Au(\xi,t) + F(\xi,t)]\,d\xi + g'(t), \tag{2.4}$$

$$u_t = \int_0^x \int_0^\eta [Au(\xi,t) + F(\xi,t)]\,d\xi\,d\eta + g'(t)x + f(t). \tag{2.5}$$

It is appropriate, therefore, to consider the integro-differential equation

$$(2.6)\qquad u(x,t)=\int_0^x\int_0^\eta\int_0^t[Au(\xi,\tau)+F(\xi,\tau)]\,d\tau\,d\xi\,d\eta+G(x,t)\,,\qquad \text{in } Q_T$$

where

$$(2.7)\qquad G(x,t)=\varphi(x)-\varphi(0)-\varphi'(0)x+g(t)x+f(t)\,.$$

DEFINITION 2.1. – A function $u(x,t)$ is called a classical solution of (2.6) if

$$(2.8)\qquad u,\,u_x,\,u_{xx},\,u_t,\,u_{xt}\in C(\bar{Q}_T)\,;$$

and (2.6) is satisfied.

Although (2.6) does not give any information on initial-boundary values explicitly, they have been built in (2.6) via the function $G(x,t)$. Moreover, (2.6) is a problem which is equivalent to (1.1)-(1.4), owing to the following lemmas, and is easier to study.

LEMMA 2.1. – If $u(x,t)$ is a solution of (2.6), then u_{xxt} exists and is in $C(\bar{Q}_T)$.

PROOF. – Differentiate (2.6).

LEMMA 2.2. – (1.1)-(1.4) is equivalent to (2,6) under assumption (H I) (classical sense).

PROOF. – If u is a classical solution of (2.6), then it satisfies (1.2)-(1.4). From lemma 2.1 we see that it is a classical solution for (1.1)-(1.4). The other part of lemma follows from the formulation of (2.6).

With these two lemma in hand we shall study (2.6) instead of (1.1)-(1.4) since if we have a classical solution for (2.6), then it is automatically a classical solution for (1.1)-(1.4). It is also easy to see that the uniqueness and continuous dependence of the solution of (2.6) will carry over to (1.1)-(1.4).

DEFINITION 2.2. – Define the Banach space $B(M)$, where M is a closed set, by

$$(2.9)\qquad B(M)=\{u\in C(M):\,u_x,\,u_{xx},\,u_t,\,u_{xt}\in C(M)\}$$

with norm defined by

$$\|u\|_B=\|u\|_\infty+\|u_x\|_\infty+\|u_{xx}\|_\infty+\|u_t\|_\infty+\|u_{xt}\|_\infty\,.$$

where $\|\cdot\|_\infty$ is the usual sup-norm on $C(M)$.

3. – Classical solutions.

Let $K = [0, x'] \times [0, t']$ with $x' \in (0, 1]$ and $t \in (0, T]$, and

$$B_L(K) = \{u \in B(K): \|u\|_{B(K)} \leqslant L, u(x, 0) = \varphi(x), \quad u(0, t) = (f, t) \text{ and } u_x(0, t) = g(t)\}, \tag{3.1}$$

where L is a positive constant which will be defined later. The mapping S on $B_L(K)$ is defined by

$$Su = \int_0^x \int_0^\eta \int_0^t [Au(\xi, \tau) + F(\xi, \tau)]\, d\tau\, d\xi\, d\eta + G(x, t), \quad \text{in } Q_T, \tag{3.2}$$

where $G(x, t)$ is defined in (2.7). We know from (3.2) that for any $u \in B_L(K)$ we have

$$(Su)(x, 0) = \varphi(x), \quad (Su)(0, t) = f(t) \text{ and } (Su)_x(0, t) = g(t). \tag{3.3}$$

Now we shall show that

$$S: B_L(K) \to B_L(K)$$

for some appropriate choices of x', t' and L, and that S is a contraction mapping on $B_L(K)$. It is easy to deduce from (3.2) and (2.1)-(2.5) that

$$\|Su\|_{B(K)} \leqslant C + C^* Max\{x', t'\} \|u\|_{B(K)} \tag{3.5}$$

where $C = C(f, g, \varphi, F, T)$ and $C^* = C^*(a_i, T)$ are two positive constants. If we let

$$L = 2C, \quad \sigma = (2C^* + 1)^{-1} \quad \text{and } K_\sigma = [0, \sigma] \times [0, \sigma] \tag{3.6}$$

then we can obtain from (3.5) that

$$\|Su\|_{B_L(K_\sigma)} \leqslant L, \quad \text{for all } u \in B_L(K_\sigma)$$

and

$$\|Su - Sv\|_{B_L(K_\sigma)} \leqslant r \|u - v\|_{B_L(K_\sigma)}, \quad \text{for all } u, v \in B_L(K_\sigma). \tag{3.8}$$

where $r = 2C^*(2C^* + 1)^{-1} < 1$. Combining (3.7)-(3.8) and assumption (H I) we see that we have established

THEOREM 3.1. – Under assumption (H I), (2.6) has a unique «local» classical solution in K_σ.

In fact, if we observe that the constant C^* defined in (3.5) is independent of the initial-boundary data and free term F, we can obtain

THEOREM 3.2. – Under assumption (H I), (2.6) has a unique global classical solution in Q_T for any given $T > 0$.

PROOF. – Let $K_{2\sigma} = [\sigma, 2\sigma] \times [0, \sigma]$ and

$$(3.9) \qquad B_{L'}(K_{2\sigma}) = \\ = \{v \in B(K_{2\sigma})\colon \|v\|_{B(K_{2\sigma})} \leqslant L',\ v(x, 0) = \varphi(x),\ v(\sigma, t) = u(\sigma, t) \text{ and } v_x(\sigma, t) = u_x(\sigma, t)\}$$

where σ is defined in (3.6) and u is the unique «local» solution obtained in theorem 3.1. We can define a mapping similar to that above and the arguments above tell us that there exists a unique classical solution in $K_{2\sigma}$. Now repeating the previous arguments finitely many times we see that there exists a unique solution in $[0, 1] \times [0, \sigma]$. Then, repeating all of the procedure above finitely many times we have that the unique classical solution of (2.6) exists in Q_T.

Now let prove a theorem which concerns the continuous dependence upon the data.

THEOREM 3.3. – If u^k, $(k = 1, 2)$ are two solutions with the data f^k, g^k, φ^k and F^k, then we have

$$(3.10) \qquad \|u^1 - u^2\|_{B(Q_T)} \leqslant C(T)\{\|\varphi^1 - \varphi^2\|_2 + \|g^1 - g^2\|_1 + \|f^1 - f^2\|_1 + \|F^1 - F^2\|_\infty\}$$

where $C = C(T)$ is a positive constant depending upon a_i and T; and

$$(3.11) \qquad \|\psi\|_1 = \|\psi\|_\infty + \|\psi'\|_\infty\,, \quad \text{for } \psi \in C^1[0, T]\,,$$

and

$$(3.12) \qquad \|\psi\|_2 = \|\psi\|_\infty + \|\psi'\|_\infty + \|\psi''\|_\infty\,, \quad \text{for } \psi \in C^2[0, 1]\,.$$

PROOF. – In K_σ, where σ is defined in (3.6), we have from (2.1)-(2.5) that

$$(3.13) \qquad \|u^1 - u^2\|_{B(K_\sigma)} \leqslant C\{\|f^1 - f^2\|_1 + \|g^1 - g^2\|_1 + \|\varphi^1 - \varphi^2\|_2 + \|F^1 - F^2\|_\infty\}\,.$$

Here and in what follows C denote the various constants which depend upon T and a_i or finite combination of such constants.

Now in $K_{2\sigma}$, we see that

$$(3.14) \qquad \|u^1 - u^2\|_{B(K_{2\sigma})} \leqslant \\ \leqslant C\{\|\varphi^1 - \varphi^2\|_2 + \|F^1 - F^2\|_\infty + \|u^2(\sigma, \cdot) - u^2(\sigma, \cdot)\|_1 + \|u_x^1(\sigma, \cdot) - u_x^2(\sigma, \cdot)\|_1\} \leqslant \\ \leqslant C\{\|\varphi^1 - \varphi^2\|_2 + \|L^1 - F^2\|_\infty + \|u^1 - u^2\|_{B(K_\sigma)}\}\,.$$

In general we let

$$K_{m\sigma} = [(m-1)\sigma, m\sigma]\times[0,\sigma], \qquad m = 1, 2, \ldots, \leqslant [1/\sigma]+1$$

and

$$d = \|\varphi^1 - \varphi^2\| + \|F^1 - F^2\|_\infty + \|f^1 - f^2\|_1 + \|g^1 - g^2\|_1 .$$

Then it follows that

$$y_m \leqslant C\{y_{m-1} + d\}, \qquad m = 1, 2, \ldots, \leqslant [1/\sigma]+1, \tag{3.15}$$

where

$$y_m = \|u^1 - u^2\|_{B(K_{m\sigma})} \quad \text{for } m = 1, 2, \ldots, \leqslant [1/\sigma]+1 \text{ and } y_0 = 0 .$$

From (3.15) we have

$$\begin{aligned} y_m &\leqslant C y_{m-1} + Cd \leqslant C^2 y_{m-2} + C^2 d + Cd \leqslant \ldots \leqslant \\ &\leqslant C^m y_1 + Cd\Big(\sum_{l-1}^{l=m} C^l\Big) \leqslant \Big\{C^{[1/\sigma]+1} + C\Big(\sum_{l=1}^{[1/\sigma]+1} C^l\Big)\Big\} d \end{aligned} \tag{3.16}$$

since $y_1 \leqslant Cd$. Hence, we have

$$\|u^1 - u^2\|_{B([0,1]\times[0,\sigma])} \leqslant \sum_{l=1}^{[1/\sigma]+1} y_l \leqslant ([1/\sigma]+1)\Big\{C^{[1/\sigma]+1} + C\sum_{l=1}^{[1/\sigma]+1} C^l\Big\} d \leqslant C(\sigma)\, d . \tag{3.17}$$

Thus we see from (2.17) that

$$\|u^1 - u^2\|_{B([0,1]\times[\sigma,2\sigma])} \leqslant C\{d + \|u^1 - u^2\|_{B([0,1]\times 0,\sigma])}\} \leqslant C(\sigma)\, d , \tag{3.18}$$

Hence (3.10) follows from a similar argument to that above applied at most $[T/\sigma]+1$ times.

4. – Weak solutions.

In this section we shall study (1.1)-(1.4) where some of the data is not necessarily smooth and where the smoothness of the rest the data is not enough to provide a classical solution. We begin by defining

$$W(Q_T) = \{u \in L^2(Q_T) : u_x, u_{xx}, u_{xt}, u_t \in L^2(Q_T)\} \tag{4.1}$$

with norm

$$\|u\|^2_{W(Q_T)} = \|u\|^2_{L^2(Q_T)} + \|u_x\|^2_{L^2(Q_T)} + \|u_{xx}\|^2_{L^2(Q_T)} + \|u_{xt}\|^2_{L^2(Q_T)} + \|u_t\|^2_{L^2(Q_T)} ; \tag{4.2}$$

DEFINITION 4.1. – A function $u(x, t)$ is called a weak solution for (2.6) if $u \in W(Q_T)$ and satisfies (2.6) in $L^2(Q_T)$.

LEMMA 4.1. – Under assumption (H II), if u is a weak solution of (2.6), then $u_{xxt} \in L^2(Q_T)$ and u satisfies (1.2)-(1.4) in the trace sense.

LEMMA 4.2. – Under assumption (H II), (1.1)-(1.4) is equivalent to (2.6) in the weak sense.

The proofs of these two lemmas are elementary and, therefore, omitted. We now approximate our weak solution u in the following way: For $\varepsilon > 0$ small, let

$$u^\varepsilon(x, t) = \varphi(x)\,, \qquad \text{for } t \leqslant 0 \tag{4.3}$$

and for $t \geqslant 0$, $u^\varepsilon(x, t)$ is defined by

$$u^\varepsilon(x, t) = \int_0^x \int_0^\eta \int_0^t [Au^\varepsilon(\xi, \tau - \varepsilon) + F(\xi, \tau)]\, d\tau\, d\xi,\, d\eta + G(x, t) \tag{4.4}$$

where G is defined in (2.7). In fact, u^ε is defined in (4.4) by retarding arguments on t. For u^ε we have

LEMMA 4.3. – Under assumption (H II), for each sufficiently small $\varepsilon > 0$, $u^\varepsilon(x, t) \in W(Q_T)$ and $u^\varepsilon_{xxt} \in L^2(Q_T)$ with

$$u^\varepsilon(x, 0) = \varphi(x)\,, \qquad u^\varepsilon(0, t) = f(t) \text{ and } u^\varepsilon_x(0, t) = g(t)\,. \tag{5.4}$$

PROOF. – It follows from the definition of u^ε and lemma 4.1.

LEMMA 4.4. – There exists a positive constant $C > 0$ such that

$$\|u^\varepsilon\|_{W(Q_T)} \leqslant C \tag{4.6}$$

where C is independent of ε.

PROOF. – Recall that $u^\varepsilon \in W(Q_T)$, using (2.1)-(2.5) and integrating on $[0, x] \times [0, t]$ for $0 < x \leqslant 1$ and $0 < t \leqslant T$, we obtain from assumption (H II) that

$$\begin{aligned}\int_0^x \int_0^t (|u^\varepsilon|^2 + |u^\varepsilon_x|^2 + |u^\varepsilon_{xx}|^2) &\leqslant C + C \int_0^x \int_0^t \int_0^\tau (|u^\varepsilon|^2 + |u^\varepsilon_x|^2 + |u^\varepsilon_{xx}|^2 + |u^\varepsilon_t|^2 + |u^\varepsilon_{xt}|^2) + \\ &\quad + C \int_0^x \int_0^t \int_{-\varepsilon}^0 (|u^\varepsilon|^2 + |u^\varepsilon_x|^2 + |u^\varepsilon_{xx}|^2 + |u^\varepsilon_t|^2 + |u^\varepsilon_{xt}|^2) \leqslant \\ &\leqslant C + C \int_0^t \int_0^x \int_0^\tau (|u^\varepsilon|^2 + |u^\varepsilon_x|^2 + |u^\varepsilon_{xx}|^2 + |u^\varepsilon_t|^2 + |u^\varepsilon_{xt}|^2)\,.\end{aligned} \tag{4.7}$$

Here and in what follows, we denote the various constants by C which are independent of ε, but depend on the data and T. From Gronwall's inequality, we see that

$$\int_0^x\int_0^t(|u^\varepsilon|^2+|u^\varepsilon_x|^2+|u^\varepsilon_{xx}|^2)\leqslant C+C\int_0^t\int_0^x\int_0^\tau(|u^\varepsilon_t|^2+|u^\varepsilon_{xt}|^2)\,. \tag{4.8}$$

A similar argument will lead to

$$\int_0^t\int_0^x(|u^\varepsilon_t|^2+|u^\varepsilon_{xt}|^2)\leqslant C+C\int_0^x\int_0^\eta\int_0^t(|u^\varepsilon|^2+|u^\varepsilon_x|^2+|u^\varepsilon_{xx}|^2)\,. \tag{4.9}$$

Thus (4.6) follows from combining (4.8) and (4.9) together and using Gronwall's inequality twice for x and t.

We know from lemma 4.4 that there exists $u\in W(Q_T)$ such that

$$u^\varepsilon \to u \quad \text{weakly in } L^2(Q_T)\,; \tag{4.10}$$

$$u^\varepsilon_x \to u_x \quad \text{weakly in } L^2(Q_T)\,; \tag{4.11}$$

$$u^\varepsilon_{xx} \to u_{xx} \quad \text{weakly in } L^2(Q_T)\,; \tag{4.12}$$

$$u^\varepsilon_t \to u_t \quad \text{weakly in } L^2(Q_T)\,; \tag{4.13}$$

$$u^\varepsilon_{xt} \to u_{xt} \quad \text{weakly in } L^2(Q_T)\,. \tag{4.14}$$

Obviously, we have

$$\|u^\varepsilon_{xxt}\|_{L^2(Q_T)}\leqslant C+C\|u^\varepsilon\|_{W(Q_T)} \tag{4.15}$$

for some $C>0$. So that

$$u^\varepsilon_{xxt} \to u_{xxt} \quad \text{weakly in } L^2(Q_T)\,. \tag{4.16}$$

If we write the operator $A=A(x,t)$, then we see that

$$u^\varepsilon_{xxt}(x,t)=A(x,t)\,u^\varepsilon(x,t-\varepsilon)+F(x,t)\qquad\text{for all } \varepsilon>0 \text{ small}\,. \tag{4.17}$$

We have for any $\psi\in L^2(Q_T)$ that

$$\iint_{Q_T}u^\varepsilon_{xxt}\,\psi=\iint_{Q_T}[A(x,t)\,u^\varepsilon(x,t-\varepsilon)]\,\psi(x,t)+\iint_{Q_T}F\psi= \tag{4.18}$$
$$=\int_0^1\int_0^{T-\varepsilon}\psi(x,t+\varepsilon)[A(x,t+\varepsilon)\,u^\varepsilon(x,t)]+\iint_{Q_T}F\psi+\int_0^1\int_{-\varepsilon}^0\psi(x,t+\varepsilon)[A(x,t+\varepsilon)\,u^\varepsilon(x,t)]\,.$$

Obviously, the last term on the right hand side of (4.18) goes to zero as $\varepsilon \to 0$. For the first term on the right hand side, since

$$(4.19) \quad \int_0^1 \int_0^{T-\varepsilon} \psi(x, t+\varepsilon)[A(x, t+\varepsilon)u^\varepsilon(x, t)] =$$
$$= \int_0^1 \int_0^{T-\varepsilon} \psi(x, t)A(x, t)u^\varepsilon(x, t) + \int_0^1 \int_0^{T-\varepsilon} [\psi(x, t+\varepsilon)A(x, t+\varepsilon) - \psi(x, t)A(x, t)]u^\varepsilon(x, t) = I + J$$

and

$$|J| \leqslant C \sum_{i=1}^{5} \sqrt{\iint_{Q_{T-\varepsilon}} |a_i(x, t+\varepsilon)\psi(x, t+\varepsilon) - a_i(x, t)\psi(x, t)|^2 \, dx\, dt} \, \|u^\varepsilon\|_{W(Q_T)}$$

which goes to zero as $\varepsilon \to 0$ because of the continuity of averages of $L^2(Q_T)$ functions and the boundedness of u^ε, we have

$$(4.20) \qquad \iint_{Q_T} u_{xxt}\psi = \iint_{Q_T} Au\psi + \iint_{Q_T} F\psi$$

for all $\psi \in L^2(Q_T)$. Thus, u satisfies (1.1) in the sense of $L^2(Q_T)$ and (1.2)-(1.4) in the trace sense. Let us summarize above analysis:

THEOREM 4.1. – Under assumption (H II), there exists a unique weak solution u for (1.1)-(1.4), which satisfies

$$(4.21) \qquad \|u\|^2_{W(Q_T)} + \|u_{xxt}\|^2_{L^2(Q_T)} \leqslant C\{\|\varphi\|^2_{H^2(0,1)} + \|f^2_{H^1(0,T)}\| + \|g\|^2_{H(0,T)} + \|F\|^2_{L^2(Q_T)}\}$$

where $C = C(T)$ is a positive constant independent of the data.

PROOF. – The existence of the weak solution follows from the argument given above and (4.21) can be drived from (2.1)-(2.5), from which the uniqueness follows.

REMARK 4.1. – The method developed in this paper works well for the general linear case, moreover, it appears that the method can be modified to be applicable to non-linear cases, at least for some local results.

We have shown that the problem (1.1)-(1.4) is well-posed for any data including a_1 and a_3 of arbitrary sign. This is not surprising since when $(a_1)_t$ and $(a_3)_t$ exist, an integration with respect to the time t yields a non-classical second order ordinary differential equation (2.1) with initial data $u(0, t) = f(t)$ and $u_x(0, t) = g(t)$ for which in general the well-posedness is not a problem. Therefore, our method can be viewed as a general method for studying this kind of problem.

For the regularity of the weak solutions, since $\varphi \in H^2(0, 1)$ and $f, g \in H^1(0, T)$, it follows that $\varphi' \in C^{\frac{1}{2}}[0, 1]$ and $f, g \in C^{\frac{1}{2}}[0, T]$. Thus, it is easy to see from (2.2)-(2.3) that we can obtain

THEOREM 4.2. – The weak solution u of (1.1)-(1.4) is such that

$$u,\ u_x \in C^{\frac{1}{2},\frac{1}{2}}(\bar{Q}_T)\ , \tag{4.2}$$

where $C^{\frac{1}{2}}$ and $C^{\frac{1}{2},\frac{1}{2}}$ are Holder classes of one and two variables with exponent $\frac{1}{2}$.

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