Stabilization of a Hybrid System of Elasticity by Feedback Boundary Damping (*).

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Summary. – *A hybrid control system is presented as consisting of an elastic beam linked to a rigid body, and the system is asymptotically stabilized through feedback boundary damping. Solutions of the hybrid system are constructed that decay towards zero at nonexponential, even arbitrarily slow, decay rates. This feedback control analysis complements the authors' earlier report on the open-loop controllability of this same hybrid system, which is a simplified model of a space-structure.*

1. – The hybrid system, an elastic space-structure: motivation and a summary of results.

In a previous publication [5] the authors have analysed the controllability of a hybrid dynamical system consisting of a long elastic beam, clamped at rest at one end and clamped (or linked) at the other and to a moveable rigid body upon which the boundary controllers are applied. Such a system provides a simple model for a large-scale space-structure, where the elastic beam is a long flexible mast (M), clamped at one end to a massive space-ship (S) (now at rest after completing some space maneuver), and fastened at the other end to a rigid antenna (A) whereon the control is effected by means of gas-jets. The earlier paper treated the controllability of the system $(M) + (A)$ using open-loop controllers, but in the current work we analyse the stabilization of this system by means of closed-loop controllers specified by feedback laws depending only on the state of a rigid body (A). The goal is the regulation of the system $(M) + (A)$ towards some specified rest state—say, where (M) is represented by a line segment that is orthogonal to the two segments (S) and (A) at its ends.

The dynamical system $(M) + (A)$ is regarded as a hybrid control system in that the elastic vibrations of the mast (M) are governed by a partial differential equation (the PDE of Euler-Bernoulli in linear elasticity theory), whereas the oscillations of the antenna (A) are described by ordinary differential equations (the ODE of Newton-

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Euler in rigid body dynamics). The linkage of the control of the ODE to the boundary conditions of the PDE creates the hybrid system $(M) + (A)$.

Assuming that all motions and forces are restricted to a fixed plane, with x-axis normal to the space-ship (S), and with all vibrations along the y-axis orthogonal to the x-axis, the authors have described the control dynamics of the hybrid system $(M) + (A)$. Namely, the Euler-Bernoulli PDE is

$$w_{tt} + w_{xxxx} = 0 \qquad \text{for } w(x, t) \text{ on } 0 \leqslant x \leqslant 1 , \quad t \geqslant 0$$

and the rigid-body ODE are

$$\begin{aligned} \mu_1 \ddot{y} &= w_{xxx}(1, t) + f_1(t) \\ \mu_2 \ddot{\theta} &= - w_{xx}(1, t) + f_2(t) \qquad \text{on } t \geqslant 0 . \end{aligned}$$

Here $w(x, t)$ is the transverse displacement of (M) satisfying the Euler-Bernoulli PDE and the boundary conditions are:

$$w(0, t) \equiv 0 , \qquad w_x(0, t) \equiv 0 \qquad (\text{clamped at } x = 0)$$

and

$$w(1, t) \equiv y(t) , \qquad w_x(1, t) \equiv \theta(t) \qquad (\text{linked at } x = 1) ,$$

where $y(t)$ and $\theta(t)$ are the linear and angular displacements, respectively, of the rigid antenna (A). The Newton-Euler dynamics of the rigid body (A) are the two ODE, where $w_{xxx}(1, t)$ and $- w_{xx}(1, t)$ are the force and torque, respectively, that (M) exerts on (A)—according to the classical linear theory of elasticity. Further, $f_1(t)$ and $f_2(t)$ are the open-loop controllers that are applied to (A), and hence to the boundary conditions of (M). We have selected the physical units so that the length of (M) is 1, and then the positive constants $\mu_1 > 0$, $\mu_2 > 0$ pertain to the elastic and inertial properties of the hybrid system $(M) + (A)$. We refer to [5] for further descriptions of the physical structure $(M) + (A)$, and for the engineering significance of the various mathematical assumption and conclusions.

The state of the system, at each time $t \geqslant 0$, is:

$$w(\cdot, t), w_t(\cdot, t), y(t), \dot{y}(t), \theta(t), \dot{\theta}(t) ,$$

that is, two real functions (of appropriate smoothness) on $0 \leqslant x \leqslant 1$, and four real numbers. The control problem requires that any prescribed initial state at $t = 0$ be steered to the zero-state (or some pre-assigned target state) by selecting suitable open-loop controllers $f_1(t)$, $f_2(t)$ on some finite duration $0 \leqslant t \leqslant T$. In the previous paper the authors constructed such open-loop controllers $f_1(t)$, $f_2(t)$ on any arbitrarily short duration $T > 0$. Hence the entire hybrid system $(M) + (A)$ is exactly controllable in arbitrarily short times. Moreover this exact control can be accomplished, in an engineering sense, by gas-jets acting on (A) alone.

In this current paper we now demand that the controllers f_1, f_2 be specified through linear feedback laws utilizing only the data of the state $(y, \dot{y}, \theta, \dot{\theta})$ of the rigid antenna (A). In contrast to our earlier results the decay towards rest is only asymptotic as $t \to \infty$, yet the results may be more practical from an engineering viewpoint since closed-loop controllers are employed.

As previously, we can incorporate the hybrid dynamics of $(M) + (A)$ into a single PDE with rather novel boundary control. Namely use the relations $y(t) = w(1, t)$, $\theta(t) = w_x(1, t)$ to obtain

$$w_{tt} + w_{xxxx} = 0 \quad \text{for } w(x, t) \text{ on } 0 \leqslant x \leqslant 1\,, \quad t \geqslant 0\,,$$

$$w(0, t) \equiv 0\,, \quad w_x(0, t) \equiv 0 \qquad (\text{clamped at } x = 0)$$

$$\mu_1 w_{tt} - w_{xxx} = f_1 \qquad (\text{linked at } x = 1)$$

$$\mu_2 w_{xtt} + w_{xx} = f_2$$

We next specify the form of the linear feedback laws

$$f_1 = L_1 w\,, \qquad f_2 = L_2 w \qquad (\text{at } x = 1)\,,$$

where L_1 and L_2 are real linear functions of the four variables $(y, \dot{y}, \theta, \dot{\theta})$. Only certain linear functions L_1 and L_2 will be acceptable in that they inforce a dissipation of the energy of the system, for example $f_1 = -\dot{y} = -w_t(1, t)$ and $f_2 = -\dot{\theta} = -w_{xt}(1, t)$. In this way we utilize the concept of control by *feedback boundary damping*, which produces an asymptotic stabilization of the hybrid system.

In the next section of this paper we characterize the dissipative feedback laws in terms of the system energy E. In the subsequent sections 3 and 4 we reinterpret the feedback control dynamics as a contraction semi-group in an infinite dimensional state-space, the Hilbert space $\mathcal{H}_F$ of all finite energy states. In this analysis the authors benefited from informal communications with M. Slemrod. Within this framework we are able to assert and prove our main Theorem 2 that each initial state in $\mathcal{H}_F$ decays asymptotically to the zero state. We then return to our original description of the physical control system of $(M) + (A)$, with feedback boundary damping, and interpret our Theorem 2 to demonstrate the stabilization of this hybrid system of elastic structures. In the last Section 5 we provide examples of solutions $w(x, t)$ that decay to zero at a rate that is not exponential, but in fact at an arbitrarily slow rate. This result is in contrast to the situation analysed by Chen et. al [9], where the elastic mast is not linked to a boundary mass.

While a briefer proof of Theorem 2 is possible by referring to certain general results of semigroup theory [7], our development, employing Liapounov functionals, illuminates the geometry of the dynamics. Further benefits involve delicate properties concerning the smoothness of the solutions and how this relates to the quantitative decay rate.

2. – Energy dissipation and feedback boundary damping.

Consider the Euler-Bernoulli PDE with boundary control

$$w_{tt} + w_{xxxx} = 0 \quad \text{for } w(x, t) \text{ on } 0 \leqslant x \leqslant 1\,, \quad t \geqslant 0$$
$$w(0, t) \equiv w_x(0, t) \equiv 0 \quad \text{(clamped at } x = 0)$$
$$\mu_1 w_{tt} - w_{xx} = f_1 \quad \text{(linked at } x = 1)$$
$$\mu_2 w_{xtt} + w_{xx} = f_2$$

with feedback boundary control given by

$$f_1 = L_1 w = \alpha_1 y + \beta_1 \dot{y} + \gamma_1 \theta + \delta_1 \dot{\theta}$$
$$f_2 = L_2 w = \alpha_2 y + \beta_2 \dot{y} + \gamma_2 \theta + \delta_2 \dot{\theta}$$

for real constants $\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2}, \delta_{1,2}$-where L_1 and L_2 are the specified linear differential operators on w, but they may also be interpreted as real-valued functions on the state $(y, \dot{y}, \theta, \dot{\theta})$ of the antenna (A).

Each solution $w(x, t)$ of this boundary control problem (suitably smooth as specified in the Proposition asserted later) is likewise a solution of the corresponding hybrid control problem described in Section 1, and vice versa, provided we introduce $y(t) = w(1, t)$, $\theta(t) = w_x(1, t)$. For each such solution $w(x, t)$, at each time $t \geqslant 0$, there is the corresponding state of the hybrid system:

$$w(\cdot, t),\, w_t(\cdot, t),\, y(t) = w(1, t),\, \dot{y}(t),\, \theta(t) = w_x(1, t),\, \dot{\theta}(t)\,,$$

and we then define the energy of such a state

$$E(t) = \tfrac{1}{2}\int_0^1 [w_t(x, t)^2 + w_{xx}(x, t)^2]\, dx + \tfrac{1}{2}[\mu_1 \dot{y}(t)^2 + \mu_2 \dot{\theta}(t)^2]\,.$$

Clearly, the energy $E(t)$, which is a functional of the state of $w(x, t)$, depends continuously on $t \geqslant 0$.

LEMMA. – *The energy $E(t)$, of a suitably smooth solution $w(x, t)$ on $0 \leqslant x \leqslant 1$, $t \geqslant 0$, has a derivative*

$$\dot{E}(t) = \dot{y}(t) L_1 w + \dot{\theta}(t) L_2 w \qquad \text{on } t \geqslant 0$$

so

$$E(t) - E(0) = \int_0^t [\dot{y}(\tau) L_1 w + \dot{\theta}(\tau) L_2 w]\, d\tau\,.$$

In particular if $L_1 w \equiv 0$, $L_2 w \equiv 0$, *then*

$$E(t) \equiv E(0) .$$

PROOF. – Compute the derivative dE/dt on $t \geq 0$ (with smoothness as specified by conditions i) and ii) of Section 3)

$$\dot{E}(t) = \int_0^1 [w_t w_{tt} + w_{xx} w_{xxt}]\, dx + \mu_1 w_t w_{tt} + \mu_2 w_{xt} w_{xtt} ,$$

the latter terms evaluated at $x = 1$, as usual. Integration by parts yields

$$\int_0^1 w_{xx}(x, t) w_{xxt}(x, t)\, dx = w_{xx} w_{xt}\Big]_{x=0}^1 - \int_0^1 w_{xxx} w_{xt}\, dx = w_{xx} w_{xt} - w_{xxx} w_t\Big]_{x=0}^1 + \int_0^1 w_{xxxx} w_t\, dx .$$

Then

$$\dot{E}(t) = w_{xx}(1, t) w_{xt}(1, t) - w_{xxx}(1, t) w_t(1, t) + \mu_1 w_t w_{tt} + \mu_2 w_{xt} w_{xtt} .$$

Hence, writing $L_{1,2}(t)$ for $L_{1,2} w(1, t)$, we find

$$\dot{E}(t) = w_t(1, t) L_1(t) + w_{tx}(1, t) L_2(t)$$

or

$$\dot{E}(t) = \dot{y}(t) L_1(t) + \dot{\theta}(t) L_2(t) ,$$

as required. □

The conclusion of the lemma is understandable from the viewpoint of classical mechanics, since the rate that work is done on the system $(M) + (A)$ (in appropriate units) is the product of linear velocity and force (or angular velocity and torque).

DEFINITION. The linear feedback laws

$$f_1 = L_1 w = \alpha_1 y + \beta_1 \dot{y} + \gamma_1 \theta + \delta_1 \dot{\theta} , \quad f_2 = L_2 w = \alpha_2 y + \beta_2 \dot{y} + \gamma_2 \theta + \delta_2 \dot{\theta} ,$$

with real constant coefficients, are called dissipative in case

$$\dot{y} L_1 w + \dot{\theta} L_2 w \leq 0$$

for all values of $(y, \dot{y}, \theta, \dot{\theta})$ in $\boldsymbol{R}^4$. The corresponding control is called feedback boundary damping.

COROLLARY. *The linear functions* L_1 *and* L_2 *define dissipative feedback laws if and only if*:

$$L_1 w = -a\dot{y} + b\dot{\theta}\,, \qquad L_2 w = c\dot{y} - d\dot{\theta}$$

for constants $a \geqslant 0$, $d \geqslant 0$, $4ad \geqslant (b+c)^2$.

PROOF. – The dissipative condition asserts that, for all real values of the four variables $(y, \dot{y}, \theta, \dot{\theta})$ we find

$$\dot{y}(\alpha_1 y + \beta_1 \dot{y} + \gamma_1 \theta + \delta_1 \dot{\theta}) + \dot{\theta}(\alpha_2 y + \beta_2 \dot{y} + \gamma_2 \theta + \delta_2 \dot{\theta}) \leqslant 0\,,$$

or

$$\beta_1 \dot{y}^2 + \delta_2 \dot{\theta}^2 + (\delta_1 + \beta_2)\dot{y}\dot{\theta} + (\alpha_1 y\dot{y} + \gamma_1 \theta\dot{y} + \alpha_2 y\dot{\theta} + \gamma_2 \theta\dot{\theta}) \leqslant 0\,.$$

First assume that L_1, L_2 are dissipative. If we set $\theta = \dot{\theta} = 0$, then we demand

$$\beta_1 \dot{y}^2 + \alpha_1 y\dot{y} \leqslant 0\,,$$

which is linear in y. Thus we conclude that $\alpha_1 = 0$. Similarly, set $y = \dot{y} = 0$, to demand

$$\delta_2 \dot{\theta}^2 + \gamma_2 \theta\dot{\theta} \leqslant 0\,, \qquad \text{so } \gamma_2 = 0\,.$$

Again, set $y = \dot{\theta} = 0$, to obtain $\gamma_1 = 0$. From $\theta = \dot{y} = 0$, we obtain $\alpha_2 = 0$.

Therefore we conclude that the quadratic form

$$\beta_1 \dot{y}^2 + (\delta_1 + \beta_2)\dot{y}\dot{\theta} + \delta_2 \dot{\theta}^2 \leqslant 0\,.$$

In this case we conclude that

$$\beta_1 \leqslant 0\,, \qquad \delta_2 \leqslant 0\,, \qquad \text{and } (\delta_1 + \beta_2)^2 - 4\beta_1\delta_2 \leqslant 0\,.$$

Now change notation to let

$$a = -\beta_1\,, \qquad d = -\delta_2\,, \qquad b = \delta_1\,, \qquad c = \beta_2$$

to obtain the required result:

$$L_1 w = -a\dot{y} + b\dot{\theta}\,, \qquad L_2 w = c\dot{y} - d\dot{\theta}$$

with constant coefficients satisfying

$$a \geqslant 0\,, \qquad d \geqslant 0\,, \qquad 4ad \geqslant (b+c)^2\,.$$

On the other hand, any such choice of L_1, L_2 is clearly dissipative. □

For any such dissipative laws we can define the energy decay-rate (for state $\dot{y}$, $\dot{\theta}$):

$$|\dot{E}(t)| = -[\dot{y}L_1w + \dot{\theta}L_2w] = a\dot{y}^2 - (b + c)\dot{y}\dot{\theta} + d\dot{\theta}^2 .$$

COROLLARY. – *If* $-[\dot{y}L_1w + \dot{\theta}L_2w]$ *is a positive definite quadratic form in* $(\dot{y}, \dot{\theta})$, *then* L_1, L_2 *are called strictly dissipative feedback laws. This occurs if and only if*

$$a > 0 , \quad d > 0 , \quad 4ad > (b + c)^2 .$$

REMARK. – An expecially interesting case of feedback boundary damping, by dissipative laws, is given by

$$L_1w = -\mu_1\dot{y} + b\dot{\theta} , \quad L_2w = -b\dot{y} - \mu_2\dot{\theta} .$$

In this case

$$\dot{y}L_1w + \dot{\theta}L_2w = -[\mu_1\dot{y}^2 + \mu_2\dot{\theta}^2] ,$$

irrespective of the constant b. [If, for simplicity of exposition, we should take $\mu_1 = \mu_2 = 1$, $b = 0$, then the dissipative feedback boundary damping reduces to $L_1w = -\dot{y}$, $L_2w = -\theta$, which is an illuminating special circumstance.]

We shall characterize this case, among all possible dissipative laws, through an extremal principle. For all appropriate (a, b, c, d) the decay rate is defined by

$$-[\dot{y}L_1w + \dot{\theta}L_2w] = a\dot{y}^2 - (b + c)\dot{y}\dot{\theta} + d\dot{\theta}^2 .$$

Then define the «least decay rate» for (a, b, c, d) compatible with unit energy of the antenna (A):

$$E_A = \tfrac{1}{2}(\mu_1\dot{y}^2 + \mu_2\dot{\theta}^2) = 1 ,$$

that is $R = R(a, b, c, d)$ is given by

$$R = \operatorname*{Min}_{E_A=1}[a\dot{y}^2 - (b + c)\dot{y}\dot{\theta} + d\dot{\theta}^2] .$$

Under the further normalization (fixing the determinant of the quadratic-form for the decay-rate):

$$\Delta = \begin{vmatrix} a & \dfrac{-(b+c)}{2} \\ \dfrac{-(b+c)}{2} & d \end{vmatrix} = \mu_1\mu_2 ,$$

we claim that $R = R(a, b, c, d)$ is maximized only when

$$L_1 w = -\mu_1 \dot{y} + b\dot{\theta}\,, \qquad L_2 w = -b\dot{y} - \mu_2\dot{\theta}\,.$$

The motivation is the wish to select the parameters (a, b, c, d) so as to maximize the decay rate—in terms of the worst case $R(a, b, c, d)$.

Note that we can assume $a > 0$, $d > 0$, for otherwise, say $d = 0$, we have $(b + c) = 0$ and so the least decay rate $R = 0$—which is then eliminated by the demand $\Delta = \mu_1\mu_2$. Further note that in the Euclidean plane with coordinates $\bar{y} = \sqrt{(\mu_1/2)}\,\dot{y}$, $\bar{\theta} = \sqrt{(\mu_2/2)}\,\dot{\theta}$, the locus

$$a\dot{y}^2 - (b + c)\dot{y}\dot{\theta} + d\dot{\theta}^2 = 2\left[\frac{a}{\mu_1}\bar{y}^2 - \frac{(b + c)}{\sqrt{\mu_1\mu_2}}\bar{y}\bar{\theta} + \frac{d}{\mu_2}\bar{\theta}^2\right] = \text{const}$$

defines a family of similar ellipses. Hence the point on the unit circle $E_\Delta = 1$ (or $\bar{y}^2 + \bar{\theta}^2 = 1$) where this quadratic form is minimized, lies on the ellipse

$$\frac{a}{\mu_1}\bar{y}^2 - \frac{(b + c)}{\sqrt{\mu_1\mu_2}}\bar{y}\bar{\theta} + \frac{d}{\mu_2}\bar{\theta}^2 = R/2$$

which is inscribed in the unit circle.

In order to consider the maximization of $R = R(a, b, c, d)$, among the dissipative laws under the normalization of $\Delta = \mu_1\mu_2$, we perform a rotation of coordinates (still called $\bar{y}, \bar{\theta}$) so that the major axis of the ellipse now is horizontal. That is, the ellipse becomes

$$\lambda_1\bar{y}^2 + \lambda_2\bar{\theta}^2 = R/2\,,$$

where λ_1 and λ_2 are the eigenvalues of the matrix

$$\begin{pmatrix} \dfrac{a}{\mu_1} & \dfrac{-(b + c)}{2\sqrt{\mu_1\mu_2}} \\[2ex] \dfrac{-(b + c)}{2\sqrt{\mu_1\mu_2}} & \dfrac{d}{\mu_2} \end{pmatrix}, \qquad \text{so } \lambda_1\lambda_2 = \frac{1}{\mu_1\mu_2}\Delta = 1\,.$$

Further compute the area of this ellipse to be $(\pi/2)(R/\sqrt{\lambda_1\lambda_2}) = \pi R/2$.

Now we choose (a, b, c, d) to maximize the area of the inscribed ellipse, and this maximum value is achieved only for the case where the ellipse is, in fact, the unit circle. Thus, the solution of the extremal problem occurs only for

$$\frac{\pi R}{2} = \pi \quad \text{or } R = 2\,.$$

In the extremal case, the ellipse was the unit circle (even before rotation) so

$$\frac{2a}{\mu_1 R} = 1 , \qquad \frac{2d}{\mu_2 R} = 1 , \qquad (b + c) = 0 .$$

The corresponding extremal feedback laws are

$$L_1 w = - \mu_1 \dot{y} + b\dot{\theta} , \qquad L_2 w = - b\dot{y} - \mu_2 \dot{\theta} ,$$

as asserted.

The motivation of the concept of dissipative feedback laws

$$L_1 w = - a\dot{y} + b\dot{\theta} , \qquad L_2 w = c\dot{y} - d\dot{\theta}$$

$\big(a \geqslant 0,\, d \geqslant 0,\, 4ad \geqslant (b + c)^2\big)$, is the energy decay of any corresponding solution $w(x, t)$, namely

$$\dot{E}(t) = \dot{y} L_1 w + \dot{\theta} L_2 w \leqslant 0 .$$

In such a case $E(t)$ is nonincreasing, and we might anticipate that $\lim_{t\to\infty} E(t) = 0$, and further $\lim_{t\to\infty} w(x, t) = 0$. The demonstration of these results of asymptotic stability constitute the main theorem of this paper, and will be proved in subsequent sections.

3. – Evolutionary dynamics for feedback damping.

We shall interpret the Euler-Bernoulli PDE, with clamped and dissipative boundary conditions, as an evolutionary ODE in an infinite dimensional Hilbert Space. For this purpose consider the PDE system

$$\frac{\partial w}{\partial t} = v , \qquad \frac{\partial v}{\partial t} = - w_{xxxx}$$

for the pair of real functions $w(x, t)$, $v(x, t)$ on $0 \leqslant x \leqslant 1$, $t \geqslant 0$. In order to make precise assertions concerning the differentiability of the solutions we shall assume:

i) $w(x, t)$ and $v(x, t)$ in class C^1 for $0 \leqslant x \leqslant 1$, $t \geqslant 0$.

In addition we shall assume that, for each fixed $t \geqslant 0$,

$$\big(w(\cdot, t), v(\cdot, t)\big) \quad \text{lies in } H^6[0, 1] \times H^4[0, 1] .$$

Here $H^k[0, 1]$ denotes the Sobolev-Hilbert Space of all real functions on $0 \leqslant x \leqslant 1$

with k-th derivative in $L^2[0,1]$, as usual. In this connection we shall make the assumption that:

ii) $t \to (w(\cdot,t), v(\cdot,t)) : [0,\infty) \to H^6[0,1] \times H^4[0,1]$ is a continuous map.

From these two assumptions i) and ii) there are several immediate conclusions about the regularity of the solution $w(x,t)$, $v(x,t)$. Namely we conclude:

1) $w, w_x, w_{xx}, w_{xxx}, w_{xxxx}, w_{xxxxx}$ and v, v_x, v_{xx}, v_{xxx}

are each continuous in x, for each fixed $t \geq 0$. Moreover, since the H^1-norm dominates the uniform norm on $0 \leq x \leq 1$, these functions are each uniformly continuous in t for $0 \leq x \leq 1$. Hence they are each continuous in (x,t) on $0 \leq x \leq 1$, $t \geq 0$.

2) $w_t = v$, $w_{tt} = v_t = -w_{xxxx}$ are continuous in (x,t), and $w_{tt} + w_{xxxx} = 0$ on $0 \leq x \leq 1$, $t \geq 0$.

3) w_x, w_t and $w_{tx} = v_x$ are continuous in (x,t), so $w_{tx} = w_{xt}$.

Also v_x, v_t and $v_{tx} = -w_{xxxxx}$ are continuous in (x,t), so $v_{tx} = v_{xt}$.

4) $w_{txx} = v_{xx}$ so $w_{txx} = w_{xtx} = w_{xxt}$ are continuous in (x,t).

5) $w_{ttx} = v_{tx} = -w_{xxxxx}$, so $w_{ttx} = w_{txt} = w_{xtt}$ are continuous in (x,t).

Also $w_{txxx} = v_{xxx}$, so $w_{txxx} = w_{xtxx} = w_{xxxt}$ are all continuous in (x,t) on $0 \leq x \leq 1$, $t \geq 0$.

It is true that any solution $w(x,t)$, $v(x,t)$ of the PDE boundary value problem, which satisfies condition ii) must necessarily also satisfy i). However, for clarity we shall usually demand both i) and ii).

Under these differentiability assumptions i), ii), the energy calculations of the preceeding section are valid. We next use these concepts to demonstrate a fundamental uniqueness result.

LEMMA. – *Consider a solution $w(x,t)$, $v(x,t)$ of the* PDE *system on* $0 \leq x \leq 1$, $t \geq 0$

$$\frac{\partial w}{\partial t} = v, \qquad \frac{\partial v}{\partial t} = -w_{xxxx}$$

with the boundary conditions

$$w(0,t) \equiv w_x(0,t) \equiv 0 \quad \text{at } x = 0$$

$$\mu_1 w_{tt} - w_{xxx} = L_1 w$$

$$\mu_2 w_{xtt} + w_{xx} = L_2 w \quad \text{at } x = 1,$$

where L_1, L_2 are dissipative feedback laws.

Assume $w(x, t)$, $v(x, t)$ satisfy the regularity conditions i), ii). *Then $w(x, t)$, $v(x, t)$ provide the unique solution (satisfying* i), ii)) *of the* PDE *system with the given boundary conditions, and agreeing with the initial data $w(x, 0)$, $v(x, 0)$.*

PROOF. - Because the boundary value problem is linear and homogeneous, it is sufficient to prove the result for the case of zero initial data.

Then let $w(x, 0) \equiv 0$, $v(x, 0) \equiv 0$ and let $E(t)$ be the energy of this solution, that is,

$$E(t) = \tfrac{1}{2}\int_0^1 [v^2 + w_{xx}^2]\, dx + \tfrac{1}{2}[\mu_1 w_t(1, t)^2 + \mu_2 w_{xt}(1, t)^2]\,.$$

In this case we can calculate (recall $\dot{y} = w_t(1, t)$, $\dot{\theta} = w_{xt}(1, t)$)

$$E(t) - E(0) = \int_0^t [\dot{y} L_1 w + \dot{\theta} L_2 w]\, d\tau\,.$$

But $E(0) = 0$ and $\dot{y}L_1 w + \dot{\theta} L_2 w \leqslant 0$, and hence $E(t) \leqslant 0$ for $t \geqslant 0$. Thus $E(t) \equiv 0$ and so $v(x, t) = w_t(x, t) \equiv 0$ on $t \geqslant 0$. Since $w(x, 0) \equiv 0$ and $w_t(x, t) \equiv 0$, we conclude that $w(x, t) \equiv 0$ and $v(x, t) = w_t(x, t) \equiv 0$. □

Next we define an evolutionary ODE that incorporates the dynamics of the Euler-Bernoulli PDE with the specified boundary conditions. The clamped-end condition $w(0, t) \equiv w_x(0, t) \equiv 0$ at $x = 0$ will be incorporated into the specification of the state space. To account for the feedback boundary damping at $x = 1$, namely,

$$\mu_1 w_{tt} - w_{xxx} = L_1 w\,, \qquad \mu_2 w_{xtt} + w_{xx} = L_2 w\,,$$

we introduce extra state components $p(t)$, $q(t)$ by defining $p(t) = v(1, t)$, $q(t) = v_x(1, t)$ (that is, $p(t) = \dot{y}(t)$ and $q(t) = \dot{\theta}(t)$ in the earlier notation describing the mechanical system). Then the feedback damping at $x = 1$ becomes

$$\mu_1 \frac{dp}{dt} = w_{xxx}(1, t) + L_1 w\,, \qquad \mu_2 \frac{dq}{dt} = -\, w_{xx}(1, t) + L_2 w\,,$$

where

$$L_1 w = -\, ap + bq\,, \qquad L_2 w = cp - dq$$

are the dissipative feedback laws, as before.

In this way we are led formally to the infinite dimensional dynamical system

$$\frac{dw}{dt} = v\,, \quad \frac{dv}{dt} = -\, w_{xxxx}\,, \quad \frac{dp}{dt} = \frac{1}{\mu_1} w_{xxx}(1, t) + \frac{1}{\mu_1} L_1 w\,, \quad \frac{dq}{dt} = \frac{-1}{\mu_2} w_{xx}(1, t) + \frac{1}{\mu_2} L_2 w$$

or, the abstract evolutionary ODE

$$\frac{du}{dt} = Au\,.$$

Here the state $u = \begin{pmatrix} w \\ v \\ q \\ p \end{pmatrix}$, in appropriate Hilbert space $\mathcal{H}_F$ defined subsequently, and the linear operator A, defined on the domain $D(A) \subset \mathcal{H}_F$, is given by:

$$u \to Au = \begin{pmatrix} v \\ -w_{xxxx} \\ \dfrac{1}{\mu_1} w_{xxx}(1, t) + \dfrac{1}{\mu_1} L_1 w \\ \dfrac{-1}{\mu_2} w_{xx}(1, t) + \dfrac{1}{\mu_2} L_2 w \end{pmatrix}.$$

In more technical detail we define the real Hilbert space of *finite energy states for feedback dynamics* by

$$\mathcal{H}_F = \{u = (w, v, p, q) \colon w \in H^2[0, 1],\, v \in L^2[0, 1],\, p \in \boldsymbol{R},\, q \in \boldsymbol{R}\}$$

with the supplementary requirement $w(0) = w_x(0) = 0$. The corresponding inner product is specified by

$$\langle u, \tilde{u}\rangle = \int_0^1 [v\tilde{v} + w_{xx}\tilde{w}_{xx}]\, dx + [\mu_1 p\tilde{p} + \mu_2 q\tilde{q}]\,.$$

Then we can specify the norm $\|u\|_F$ in terms of the energy $E(u)$ of a state $u \in \mathcal{H}_F$

$$\|u\|_F^2 = \langle u, u\rangle = 2E[u]\,.$$

It should also be noted that the norm in $\mathcal{H}_F$ is equivalent to the usual norm inherited from the Sobolev-Hilbert space $H^2[0, 1] \times L^2[0, 1] \times \boldsymbol{R}^2$, since $\|u\|_F \to 0$ implies that $w \to 0$ in the C^1-uniform norm. Also we note that, strictly speaking, the feedback operators should now be written

$$L_1 u = -\,ap + bq\,, \qquad L_2 u = cp - dq$$

as bounded (finite-dimensional) operators on the state $u \in \mathcal{H}_F$. But we shall continue to denote these terms as $L_1 w$ and $L_2 w$, in conformity with earlier notations.

The domain $D(A)$ of the operator A is a linear subspace of $\mathcal{H}_F$, as defined by:

$$u = (w, v, p, q) \in D(A)$$

in case $u \in \mathcal{H}_F$ so

$$w \in H^2[0, 1], \qquad v \in L^2[0, 1] \quad \text{with } w(0) = w_x(0) = 0;$$

but further conditions are imposed by $Au \in \mathcal{H}_F$,

$$v \in H^2[0, 1], \qquad w_{xxxx} \in L^2[0, 1] \quad \text{with } v(0) = v_x(0) = 0$$

and an additional demand to match the feedback dynamics

$$v(1) = p, \qquad v_x(1) = q.$$

These conditions complete the specification of the linear subspace $D(A) \subset \mathcal{H}_F$.

Similar analyses for the domain of A^2:

$$D(A^2) = \{u \in \mathcal{H}_F \colon u \in D(A) \text{ and } Au \in D(A)\},$$

yield the prior conditions for $u \in D(A)$ and further

$$v \in H^4[0, 1], \qquad w \in H^6[0, 1]$$
$$v(0) = v_x(0) = 0, \qquad w_{xxxx}(0) = w_{xxxxx}(0) = 0$$

and also at $x = 1$,

$$-w_{xxxx}(1) = \frac{1}{\mu_1} w_{xxx}(1) + \frac{1}{\mu_1} L_1 w$$
$$-w_{xxxxx}(1) = -\frac{1}{\mu_2} w_{xx}(1) + \frac{1}{\mu_2} L_2 w.$$

Further studies determine the domain of A^{n+1} for $n = 1, 2, 3, \ldots$

$$D(A^{n+1}) = \{u \in \mathcal{H}_F \colon u \in D(A^{n-1}) \text{ and } Au \in D(A^n)\}.$$

Also define, as usual

$$D(A^\infty) = \bigcap_{n=1} D(A^n),$$

as a linear subspace of $\mathcal{H}_F$.

It is clear that $D(A)$ is dense in $\mathcal{H}_F$, because $H^4[0,1]$ is dense in $H^2[0,1]$ (even accounting for the clamped conditions $w(0) = w_x(0) = 0$) and also $H^2[0,1]$ is dense in $L^2[0,1]$ (even accounting for the arbitrary boundary conditions for v and v_x at $x = 0$ and $x = 1$). We shall later prove that A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on the Hilbert space $\mathcal{H}_F$. From general considerations of semigroup theory it then will follows that the solution $u(t) = S(t)u_0$, from each initial $u_0 \in D(A)$, lies in $D(A)$ for each $t \geqslant 0$; and $u(t)$ has the derivative, in $\mathcal{H}_F$-norm,

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}[u(t + \Delta t) - u(t)] = Au(t) = S(t)(Au_0) ,$$

(of course, $\Delta t > 0$ if $t = 0$). Furthermore $D(A^\infty)$ is dense in $\mathcal{H}_F$, and $S(t)$ carries $D(A^n)$ into itself for each $t \geqslant 0$ and $n = 1, 2, 3, \ldots$.

The relation between the solutions of the Euler-Bernoulli PDE (with feedback boundary damping) and the infinite dimensional evolutionary ODE (as solved by the semigroup $S(t)$ generated by A) is illuminated in the following Proposition.

PROPOSITION. – *Let $w(x, t)$, $v(x, t)$ be real functions on $0 \leqslant x \leqslant 1$, $t \geqslant 0$ where they satisfy*

i) *$w(x, t)$ and $v(x, t)$ in class C^1 in (x, t), and*

ii) *$t \to (w(\cdot, t), v(\cdot, t)) : [0, \infty) \to H^6[0,1] \times H^4[0,1]$ is a continuous map.*

Under these hypothese i), ii) *we further assume*

$$\frac{\partial w}{\partial t} = v , \qquad \frac{\partial v}{\partial t} = -w_{xxxx}$$

and the boundary conditions hold

$$w(0, t) \equiv w_x(0, t) \equiv 0 \qquad (\text{at } x = 0)$$

and

$$\mu_1 w_{tt}(1, t) = w_{xxx}(1, t) + L_1 w$$
$$\mu_2 w_{xtt}(1, t) = -w_{xx}(1, t) + L_2 w \qquad (\text{at } x = 1)$$

for dissipative feedback laws L_1, L_2.

Then $u(t) = (w(\cdot, t), v(\cdot, t), p(t), q(t))$, where $p(t) = v(1, t)$, $q(t) = v_x(1, t)$, lies in $D(A) \subset \mathcal{H}_F$ for each $t \geqslant 0$; and furthermore $u(t) = S(t)u(0)$ is the solution of the evolutionary ODE

$$\frac{du}{dt} = Au \qquad \text{from } u(0) ,$$

in the sense that $S(t)$ is the semigroup with infinitesimal generator A, as defined previously.

On the other hand, under the hypotheses i), ii), *assume that*

$$u(t) = \big(w(\cdot, t), v(\cdot, t), p(t), q(t)\big) = S(t)u(0)$$

lies in $D(A)$ *for each* $t \geqslant 0$, *where* $S(t)u_0$ *is the solution of the evolutionary equation produced by the semigroup* $S(t)$. *Then*

$$\frac{\partial w}{\partial t} = v\,, \qquad \frac{\partial v}{\partial t} = -\,w_{xxxx}$$

and hence $w(s, t)$, $v(x, t)$ *satisfy the Euler-Bernoulli* PDE, *with clamped end at* $x = 0$ *and feedback boundary damping at* $x = 1$.

REMARK. – The Proposition asserts that a solution $\big(w(x, t), v(x, t)\big)$ of the Euler-Bernoulli PDE, with feedback boundary damping, produces a solution $u(t)$ of the evolutionary ODE—and vice versa. Of course, rather special regularity conditions i), ii) are demanded in each case. The Proposition does not guarantee the existence of such solutions, but they are demonstrated later.

PROOF. – The first assertion holds that a suitably smooth solution $\big(w(x, t), v(x, t) = w_t(x, t)\big)$ of the Euler-Bernoulli PDE, with clamped end at $x = 0$ and given feedback boundary damping at $x = 1$, provides a solution $u(t) = \big(w(\cdot, t), v(\cdot, t), p(t), q(t)\big)$ of the evolutionary ODE. Clearly, $w(\cdot, t) \in H^4[0, 1]$, $v(\cdot, t) \in H^2(0, 1]$ for all $t \geqslant 0$, and $w(0, t) \equiv w_x(0, t) \equiv 0$, $v(0, t) \equiv v_x(0, t) \equiv 0$; and then set $p(t) = v(1, t)$, $q(t) = v_x(1, t)$. Hence we find that $u(t) \in D(A)$ for all $t \geqslant 0$.

We must next verify that

$$\begin{aligned}
&\frac{dw}{dt} = v && \text{(in } H^2\text{-norm)}\\
&\frac{dv}{dt} = -\,w_{xxxx} && \text{(in } L^2\text{-norm)}\\
&\frac{dp}{dt} = \frac{1}{\mu_1} w_{xxx}(1, t) + \frac{1}{\mu_1} L_1 w\\
&\frac{dq}{dt} = -\frac{1}{\mu_2} w_{xx}(1, t) + \frac{1}{\mu_2} L_2 w\,.
\end{aligned}$$

Since $w_t = v$ we find that, for each $t \geqslant 0$,

$$\lim_{\Delta t \to 0} \left| \frac{w(x, t + \Delta t) - w(x, t)}{\Delta t} - v(x, t) \right| = 0\,,$$

with uniform convergence on $0 \leqslant x \leqslant 1$. Similarly, uniform convergence holds for the limits

$$\lim_{\Delta t \to 0} \left| \frac{\Delta w_x}{\Delta t} - v_x \right| = 0\,, \qquad \lim_{\Delta t \to 0} \left| \frac{\Delta w_{xx}}{\Delta t} - v_{xx} \right| = 0\,.$$

Therefore,

$$\lim_{\Delta t\to 0}\int_0^1 \left|\frac{\Delta w}{\Delta t}-v\right|^2+\left|\frac{\Delta w_x}{\Delta t}-v_x\right|^2+\left|\frac{\Delta w_{xx}}{\Delta t}-v_{xx}\right|^2 dx = 0 ,$$

so

$$\frac{dw}{dt}=v , \quad \text{in the sense of the } H^2\text{-norm} .$$

In the same way use the uniform convergence

$$\lim_{\Delta t\to 0}\left|\frac{v(x, t+\Delta t)-v(x,t)}{\Delta t}+w_{xxxx}\right| = 0 ,$$

to prove that

$$\frac{dv}{dt}=-w_{xxxx} , \quad \text{in the sense of the } L^2\text{-norm} .$$

The last two equations of the evolutionary system for $u=(w, v, p, q)$ are merely restatements of the feedback boundary damping conditions for the PDE. Hence we obtain $du/dt = Au$.

On the other hand, assume that $w(x, t)$, $v(x, t)$ (satisfying i), ii)) and $p(t)=v(1, t)$, $q(t)=v_x(1, t)$ define the $\mathcal{H}_F$-valued function $u(t)=\big(w(\cdot, t), v(\cdot, t), p(t), q(t)\big)$ which lies in $D(A)$ for each $t\geqslant 0$. Also assume that $u(t)$ is a solution of the evolutionary equation in the sense

$$u(t)=S(t)u(0) ,$$

so

$$\frac{du}{dt}=Au .$$

That is assume that for each $t\geqslant 0$ we have

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}[S(t+\Delta t)u(0)-S(t)u(0)] = Au(t) ,$$

with the limit in the $\mathcal{H}_F$-norm.

In this case we obtain

$$\frac{dw}{dt}=v , \quad \text{in } H^2\text{-norm}$$

so

$$\frac{\partial w}{\partial t}=v , \quad \text{in uniform and hence pointwise sense} .$$

From the second equation of $du/dt = Au$ we compute

$$\frac{dv}{dt} = -w_{xxxx}\,, \qquad \text{in } L^2\text{-norm}\,,$$

or

$$\lim_{\Delta t\to 0}\int_0^1 \left|\frac{v(x, t+\Delta t) - v(x,t)}{\Delta t} + w_{xxxx}(x,t)\right|^2 dx = 0\,.$$

But then, for a subsequence $\Delta t_k \to 0$ we have pointwise convergence for almost all $0 \leqslant x \leqslant 1$, and so (recalling that $v(x, t)$ lies in the class C^1)

$$\frac{\partial v}{\partial t} = -w_{xxxx}\,.$$

Hence $w(x, t)$, $v(x, t)$ is a solution of the Euler-Bernoulli PDE. The boundary conditions are also satisfied at $x = 0$

$$w(0, t) \equiv w_x(0, t) \equiv 0 \qquad (\text{definition of } \mathcal{H}_F)\,,$$

and the feedback boundary damping conditions at $x = 1$ are explicitly specified by the last two equations of the evolutionary ODE. □

We now turn to the question of the existence of the strongly continuous semigroup $S(t)$ generated by A in the Hilbert space $\mathcal{H}_F$. For this purpose we first show that the densely defined linear operator A is dissipative (that is, $-A$ is monotone).

LEMMA. – *The linear operator A is dissipative in the sense*

$$\langle Au, u\rangle \leqslant 0\,, \qquad \text{for each } u \in D(A)\,.$$

PROOF. – From the definition of the inner product in $\mathcal{H}_F$

$$\langle Au, u\rangle = \int_0^1 [v_{xx}w_{xx} - w_{xxxx}v]\,dx + \\ + [w_{xxx}(1,t) + L_1 w]v(1,t) + [-w_{xx}(1,t) + L_2 w]v_x(1,t)\,.$$

The integration by parts yields

$$\langle Au, u\rangle = v_x w_{xx}]_0^1 - \int_0^1 v_x w_{xxx}\,dx - v w_{xxx}]_0^1 + \int_0^1 v_x w_{xxx}\,dx + \\ + [w_{xxx} + L_1 w]v(1,t) + [-w_{xx} + L_2 w]v_x(1,t)\,.$$

Thus

$$\langle Au, u\rangle = v(1,t)L_1 w + v_x(1,t)L_2 w \leqslant 0\,,$$

since L_1, L_2 are dissipative feedback laws. □

REMARK. – It is interesting to relate the dissipative behavior of A with the decrease in the energy of a solution $u(t) = S(t)u_0$, for some $u_0 \in D(A)$. Then, the energy becomes

$$E(u(t)) = \tfrac{1}{2}\langle u(t), u(t)\rangle ,$$

so

$$\dot{E}(t) = \tfrac{1}{2}\langle Au, u\rangle + \tfrac{1}{2}\langle u, Au\rangle$$

and

$$\dot{E}(t) = \langle Au, u\rangle \leqslant 0 .$$

We are now in a position to prove that A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on $\mathcal{H}_F$. In fact, we shall prove that $S(t)$ is a contraction semigroup in that the operator norm

$$\|S(t)\|_F \leqslant 1 \qquad \text{for all } t \geqslant 0 .$$

We follow the Lumer-Phillips Theorem [6] which guarantees the existence of this contraction semigroup $S(t)$ provided three conditions hold:

1) $D(A)$ is dense in $\mathcal{H}_F$.

2) The range $R(\lambda_0 I - A) = \mathcal{H}_F$, for some real $\lambda_0 > 0$.

3) A is dissipative.

We have already verified that $\overline{D(A)} = \mathcal{H}_F$, and that the linear operator A is dissipative. There remains only the condition

2) $R(\lambda_0 I - A) = \mathcal{H}_F$, for some real $\lambda_0 > 0$.

THEOREM 1. *Consider the real Hilbert space $\mathcal{H}_F$ with the dissipative linear operator A, with domain $D(A)$ dense in $\mathcal{H}_F$, as before. Then A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on $\mathcal{H}_F$, and $S(t)$ is contractive*

$$\|S(t)\|_F \leqslant 1 \qquad \textit{for all } t \geqslant 0 .$$

PROOF. – With reference to the Lumer-Phillips Theorem [6], and the previous lemmas, we need only verify that the range

$$R(\lambda_0 I - A) = \mathcal{H}_F , \qquad \text{for some real } \lambda_0 > 0 .$$

For this purpose assign an arbitrary element

$$\nu = (\varphi, \psi, \varrho, \sigma) \in \mathcal{H}_F \text{ and seek } u = (w, v, p, q) \in D(A) \text{ so } (\lambda_0 I - A)u = \nu$$

or

$$\lambda_0 w - v = \varphi\,, \qquad \lambda_0 v + w_{xxxx} = \psi\,,$$

$$\lambda_0 p - \frac{1}{\mu_1} w_{xxx} - \frac{1}{\mu_1} L_1 w = \varrho\,, \qquad \lambda_0 q + \frac{1}{\mu_2} w_{xx} - \frac{1}{\mu_2} L_2 w = \sigma\,.$$

We shall solve this differential system for $u = (w, v, p, q)$, where $\lambda_0 > 0$ is specified later. From the first two equations eliminate v to write

$$\frac{d^4 w}{dx^4} + \lambda_0^2 w = \psi + \lambda_0 \varphi$$

and note that $h(x) = \psi + \lambda_0 \varphi \in L^2[0, 1]$. We seek a solution $w(x) \in H^4[0, 1]$ with with $w(0) = w_x(0) = 0$. Then we shall define $v(x) = \lambda_0 w(x) - \varphi(x) \in H^2[0, 1]$ with $v(0) = v_x(0) = 0$ and $v(1) = p$, $v_x(1) = q$. Thus the first two equations in the system are satisfied, and the last two also hold (while determing $p = v(1)$, $q = v_x(1)$) provided at $x = 1$,

$$\lambda_0 p - \frac{1}{\mu_1} w_{xxx} - \frac{1}{\mu_1} L_1 w = \varrho$$

and

$$\lambda_0 q + \frac{1}{\mu_2} w_{xx} - \frac{1}{\mu_2} L_2 w = \sigma\,.$$

That is, replace

$$p = v(1) = \lambda_0 w(1) - \varphi(1)\,, \qquad q = v_x(1) = \lambda_0 w'(1) - \varphi'(1)$$

and

$$L_1 w = -\,a v(1) + b v_x(1)\,, \qquad L_2 w = c v(1) - d v_x(1)$$

with $a \geqslant 0$, $d \geqslant 0$, $4ad \geqslant (b + c)^2$, to obtain boundary conditions on $w(x)$ at $x = 1$:

$$-\frac{1}{\mu_1} w_{xxx} + \left(\lambda_0 + \frac{a}{\mu_1}\right)\lambda_0 w - \frac{b}{\mu_1}\lambda_0 w' = \varrho + \left(\lambda_0 + \frac{a}{\mu_1}\right)\varphi - \frac{b}{\mu_1}\varphi'$$

$$\frac{1}{\mu_2} w_{xx} - \frac{c}{\mu_2}\lambda_0 w + \left(\lambda_0 + \frac{d}{\mu_2}\right)\lambda_0 w' = \sigma - \frac{c}{\mu_2}\varphi + \left(\lambda_0 + \frac{d}{\mu_2}\right)\varphi'\,.$$

Therefore the process of solution for $u = (w, v, p, q)$ can be summarized:
Solve the ODE

$$w'''' + \lambda_0^2 w = h(x)\,, \qquad \text{for } h(x) \in L^2[0, 1]$$

to obtain $w(x) \in H^4[0, 1]$ satisfying the boundary conditions

$$w(0) = w'(0) = 0 \qquad \text{at } x = 0$$

and the two conditions at $x=1$

$$-w_{xxx}+(\mu_1\lambda_0+a)\lambda_0 w-b\lambda_0 w'=\alpha\,,\qquad w_{xx}-c\lambda_0 w+(\mu_2\lambda_0+d)\lambda_0 w'=\beta$$

for arbitrarily prescribed real numbers (α,β).

To obtain the required function $w(x)$ take the solution $w_p(x)$ of $w''''+\lambda_0^2 w=h(x)$ from the initial data $w_p(0)=w_p'(0)=0$, $w_p''(0)=w_p'''(0)=1$. Then we seek a solution $\hat{w}(x)$ of the homogeneous differential equation $w''''+\lambda_0^2 w=0$ with

$$\hat{w}(0)=\hat{w}'(0)=0\qquad\text{at } x=0\,,$$

and further

$$-\hat{w}'''+(\mu_1\lambda_0+a)\lambda_0\hat{w}-b\lambda_0\hat{w}'=\hat{\alpha}\,,\qquad \hat{w}''-c\lambda_0\hat{w}+(\mu_2\lambda_0+d)\lambda_0\hat{w}'=\hat{\beta}$$

(where $\hat{\alpha},\hat{\beta}$ are arbitrarily prescribed real numbers). Then $w(x)=w_p(x)+\hat{w}(x)$ will provide the required function in $H^4[0,1]$, from which $v(x)=\lambda_0 w(x)-\varphi(x)\in H^2[0,1]$, and $p=v(1)$, $q=v_x(1)$, can then be found to obtain $u=(w,v,p,q)\in D(A)$.

The general solution of the homogeneous linear differential equation

$$w''''+\lambda_0^2 w=0\,,\qquad w(0)=w'(0)=0$$

is given by

$$\hat{w}(x)=P\left[\cosh\sqrt{\frac{\lambda_0}{2}}\,x\sin\sqrt{\frac{\lambda_0}{2}}\,x-\sinh\sqrt{\frac{\lambda_0}{2}}\,x\cos\sqrt{\frac{\lambda_0}{2}}\,x\right]+Q\sinh\sqrt{\frac{\lambda_0}{2}}\,x\sin\sqrt{\frac{\lambda_0}{2}}\,x$$

for real constants P and Q. Now we must find constants P,Q (not both zero) so that $\hat{w}(1),\hat{w}'(1),\hat{w}''(1),\hat{w}'''(1)$ satisfy the two boundary conditions at $x=1$, for arbitrary constants $\hat{\alpha},\hat{\beta}$.

The two boundary conditions at $x=1$ lead to two linear equations for the unknowns P and Q. It is sufficient to show that the determinant Δ of this linear system is nonvanishing. Since $\Delta=\Delta(\lambda_0)$ is an analytic function of $\sqrt{\lambda_0}$, we need only prove that $\Delta(\lambda_0)$ is not identically zero (for all $\lambda_0>0$), and then there exist positive numbers $\hat{\lambda}_0$ where $\Delta(\hat{\lambda}_0)\neq 0$.

It is also clear that the determinant Δ consists of a polynomial in powers of $\lambda_0^{1/2}$, with coefficients which are themselves polynomials in

$$\left(\cosh\sqrt{\frac{\lambda_0}{2}}\,,\ \sinh\sqrt{\frac{\lambda_0}{2}}\,,\ \cos\sqrt{\frac{\lambda_0}{2}}\,,\ \sin\sqrt{\frac{\lambda_0}{2}}\right).$$

If $\Delta(\lambda_0)\equiv 0$, then the coefficient of $\lambda_0^{1/2}$ (and of each of the powers of $\lambda_0^{1/2}$) must vanish identically.

In the first equation ($-\hat{w}''' + ... = \hat{\alpha}$) the highest power is $\lambda_0^{4/2}$; and in the second equation ($\hat{w}'' + ... = \hat{\beta}$) the highest power is $\lambda_0^{5/2}$. The coeficient of $\lambda_0^{9/2}$ in $\Delta(\lambda_0)$ can easily be computed to be (arguments $\sqrt{\lambda_0/2}$ omitted)

$$\begin{vmatrix} \mu_1[\cosh\sin - \sinh\cos] & \mu_1 \sinh\sin \\ \mu_2\sqrt{2}\sinh\sin & \dfrac{\mu_2}{\sqrt{2}}[\cosh\sin + \sinh\cos] \end{vmatrix}.$$

which equals $(\mu_1\mu_2)/\sqrt{2}\,\{\sin^2\sqrt{\lambda_0/2} - \sinh^2\sqrt{\lambda_0/2}\} \not\equiv 0$.

Hence $\Delta(\lambda_0) \not\equiv 0$ and there exists a value $\hat{\lambda}_0 > 0$ where $\Delta(\hat{\lambda}_0) \neq 0$. Fix this value of $\hat{\lambda}_0$. Now solve the boundary value equations at $x = 1$ to compute the required P, Q (not both zero), and hence define the required solution $w(x) = w_p(x) + \hat{w}(x)$. As indicated earlier we now obtain $u = (w, v, p, q) \in D(A)$. Therefore $R(\hat{\lambda}_0 I - A) = \mathcal{H}_F$.

From the Lumer-Phillips Theorem we conclude that the linear operator A is the infinitesimal generator of a contraction semigroup $S(t)$ on $\mathcal{H}_F$. □

NOTATION. – In the real Hilbert space $\mathcal{H}_F$ of finite energy states $u = (w, v, p, q)$ the dissipative operator

$$A = \begin{pmatrix} v \\ -w_{xxxx} \\ \dfrac{1}{\mu_1} w_{xxx}(1, t) + \dfrac{1}{\mu_1} L_1 w \\ \dfrac{-1}{\mu_2} w_{xx}(1, t) + \dfrac{1}{\mu_2} L_2 w \end{pmatrix}$$

and the contraction semigroup $S(t)$ are called the Euler-Bernoulli feedback operators for the dissipative boundary damping.

4. – Stabilization by feedback boundary damping.

In this section we prove our main result that feedback boundary damping enforces asymptotic stability of the Euler-Bernoulli dynamical system. That is,

$$\lim_{t\to\infty} u(t) = 0$$

for each solution $u(t) = S(t)u_0$ initiating at any state u_0 in the Hilbert space $\mathcal{H}_F$ of all finite energy states.

Recall that the finite energy states $u \in \mathcal{H}_F$ are

$$u = \begin{pmatrix} w \\ v \\ p \\ q \end{pmatrix}, \qquad \text{for } w \in H^2[0,1],\ v \in L^2[0,1],\ p \in \mathbf{R},\ q \in \mathbf{R} \text{ with } w(0) = w_x(0) = 0\,.$$

The energy $E[u]$ specifies the inner product and norm of $u \in \mathcal{H}_F$

$$E[u] = \tfrac{1}{2}\langle u, u\rangle = \tfrac{1}{2}\|u\|_F^2 = \tfrac{1}{2}\int_0^1 [v^2 + w_{xx}]\,dx + \tfrac{1}{2}[\mu_1 p^2 + \mu_2 q^2]\,.$$

The Euler-Bernoulli evolutionary ODE in $\mathcal{H}_F$ is

$$\frac{du}{dt} = Au = \begin{pmatrix} v \\ -w_{xxxx} \\ \dfrac{1}{\mu_1} w_{xxx}(1,t) + \dfrac{1}{\mu_1} L_1 w \\ \dfrac{-1}{\mu_2} w_{xx}(1,t) + \dfrac{1}{\mu_2} L_2 w \end{pmatrix},$$

for prescribed dissipative feedback laws at $x = 1$

$$L_1 w = -av + bv_x\,, \qquad L_2 w = cv - dv_x$$

for constants $a \geqslant 0$, $d \geqslant 0$, $4ad \geqslant (b+c)^2$.

The domain of A is the dense linear subspace $D(A) \subset \mathcal{H}_F$ defined by the usual conditions on $u \in \mathcal{H}_F$ and the additional demands: $w \in H^4[0,1]$, $v \in H^2[0,1]$ and $v(0) = v_x(0) = 0$, $v(1) = p$, $v_x(1) = q$.

Under these circumstances we have demonstrated in Theorem 1 that the dissipative operator A generates a contraction semigroup $S(t)$ on $\mathcal{H}_F$. In the terminology of our main Theorem 2, which will be demonstrated later in this section,

$$\lim_{t\to\infty} S(t) u_0 = 0\,, \qquad \text{for each } u_0 \in \mathcal{H}_F\,.$$

Our method involves the energy $E[u] = \frac{1}{2}\|u\|_F^2$ as a Liapounov functional for the evolutionary dynamics in $\mathcal{H}_F$. But because $\mathcal{H}_F$ is infinite dimensional we shall need to modify the classical elementary techniques of stability theory to incorporate the methods of the LaSalle Invariance Principle [3]. Furthermore, at this final stage of the theory, it will be necessary to assume that the feedback laws are strictly dissipative—but these additional concepts and hypotheses will be discussed in detail later.

LEMMA 1. – *Let $u_0 \in D(A^2)$ and consider the solution of the evolutionary* ODE *in $\mathcal{H}_F$*

$$u(t) = S(t)u_0 = \begin{pmatrix} w(x,t) \\ v(x,t) \\ p(t) \\ q(t) \end{pmatrix} \quad on\ t \geqslant 0 .$$

Then $u(t) \in D(A^2)$ for all $t \geqslant 0$, and further

i) *$w(x,t)$, $v(x,t)$ lie in class C^1 on $0 \leqslant x \leqslant 1$, $t \geqslant 0$*

and

ii) $t \to (w(\cdot,t), v(\cdot,t)) : [0, \infty) \to H^6[0,1] \times H^4[0,1]$

is a continuous map.

Moreover, in accord with the earlier Proposition,

$$w_t = v , \qquad v_t = -w_{xxxx}$$

so $w(x,t)$ is a solution of the Euler-Bernoulli PDE

$$w_{tt} + w_{xxxx} = 0 \quad on\ 0 \leqslant x \leqslant 1 , \quad t \geqslant 0 ,$$

with the boundary conditions

$$w(0,t) \equiv w_x(0,t) \equiv 0 \qquad \text{(clamped at } x = 0)$$

and

$$\begin{aligned} \mu_1 w_{tt}(1,t) &= w_{xxx}(1,t) + L_1 w \qquad \text{(dissipative feedback at boundary } x = 1) . \\ \mu_2 w_{xtt}(1,t) &= -w_{xx}(1,t) + L_2 w \end{aligned}$$

PROOF. – The semigroup $S(t)$ carries $D(A)$ into $D(A)$, and $D(A^2)$ into $D(A^2)$. Hence, since $u_0 \in D(A^2)$,

$$u(t) = S(t)u_0 \quad \text{lies in } D(A^2) \text{ for all } t \geqslant 0 .$$

Recall the formulas

$$Au(t) = \begin{pmatrix} v(x,t) \\ -w_{xxxx}(x,t) \\ \dfrac{1}{\mu_1} w_{xxx}(1,t) + \dfrac{1}{\mu_1} L_1 w \\ -\dfrac{1}{\mu_2} w_{xx}(1,t) + \dfrac{1}{\varphi_2} L_2 w \end{pmatrix} \text{with } v \in H^2[0,1],\ w \in H^4[0,1]$$

and

$$A^2u(t)=\begin{pmatrix} -w_{xxxx}(x,t) \\ -v_{xxxx}(x,t) \\ \dfrac{1}{\mu_1}v_{xxx}+\dfrac{1}{\mu_1}L_1v \\ -\dfrac{1}{\mu_2}v_{xx}+\dfrac{1}{\mu_2}L_2v \end{pmatrix} \text{ with } w\in H^6[0,1],\ v\in H^4[0,1],$$

for given dissipative feedback laws L_1, L_2: Thus, for each $t\geqslant 0$, $w\in H^6[0,1]$ and $v\in H^4[0,1]$. In addition

$$w(0,t)\equiv w_x(0,t)\equiv 0\,,\qquad w_{xxxx}(0,t)\equiv w_{xxxxx}(0,t)\equiv 0\,,$$

and $v(0,t)\equiv v_x(0,t)\equiv 0$.

Since $u(t)$ and also $Au(t)=AS(t)u_0=S(t)(Au_0)$ are solutions of the evolutionary ODE in $\mathcal{H}_F$,

$$\frac{dw}{dt}=v \qquad \text{in } H^2$$
$$\frac{dv}{dt}=-w_{xxxx} \qquad \text{in } L^2$$

and

$$\frac{dv}{dt}=-w_{xxxx} \qquad \text{in } H^2$$
$$-\frac{dw_{xxxx}}{dt}=-v_{xxxx} \qquad \text{in } L^2\,.$$

Therefore we conclude that

$$\frac{\partial w}{\partial t}=v\,,\qquad \frac{\partial v}{\partial t}=-w_{xxxx}$$

pointwise in $0\leqslant x\leqslant 1$, $t\geqslant 0$. From these relations, and assuming the continuity of the map ii), it easily follows that i) $w(x,t)$, $v(x,t)$ are in C^1 in $0\leqslant x\leqslant 1$, $t\geqslant 0$.

There remains the demonstration of the continuity of the map

ii) $t\to\big(w(\cdot,t),v(\cdot,t)\big)\colon [0,\infty)\to H^6[0,1]\times H^4[0,1]$.

From the continuity of $u(t)$, $Au(t)=S(t)(Au_0)$, and $A^2u(t)=A^2S(t)u_0=S(t)(A^2u_0)$ we obtain continuous maps:

$$t\to\big(w(\cdot,t),v(\cdot,t)\big)\colon[0,\infty)\to H^2[0,1]\times L^2[0,1]$$
$$t\to\big(v(\cdot,t),-w_{xxxx}(\cdot,t)\big)\colon[0,\infty)\to H^2[0,1]\times L^2[0,1]$$
$$t\to\big(-w_{xxxx}(\cdot,t),-v_{xxxx}(\cdot,t)\big)\colon[0,\infty)\to H^2[0,1]\times L^2[0,1]\,.$$

Clearly the integrals

$$\int_0^1 [w^2 + w_x^2 + w_{xx}^2]\, dx \quad \text{and} \quad \int_0^1 [w_{xxxx}^2 + w_{xxxxx}^2 + w_{xxxxxx}^2]\, dx$$

both vary continuously with $t \geqslant 0$. The only question involves $\int_0^1 w_{xxx}^2\, dx$, but this result follows from elementary calculus inequalities. In brief, write $\Delta w = w(x, t + \Delta t) - w(x, t)$ (for fixed $t \geqslant 0$). Since

$$\lim_{\Delta t \to 0} \Delta w_{xxxx} = 0 \quad \text{in } L^2[0, 1]\,,$$

it is not difficult to show that

$$\lim_{\Delta t \to 0} \Delta w_{xxx} = 0 \quad \text{in } L^2[0, 1]\,,$$

in fact, even under uniform convergence on $0 \leqslant x \leqslant 1$, as $\Delta t \to 0$.

We have proved that the map

$$t \to w(\cdot, t)\colon [0, \infty) \to H^6[0, 1]$$

is continuous. In the same way we conclude that the map

$$t \to v(\cdot, t)\colon [0, \infty) \to H^4[0, 1]$$

is continuous. Therefore the map

$$\text{ii)}\ t \to \big(w(\cdot, t), v(\cdot, t)\big)\colon [0, \infty) \to H^6[0, 1] \times H^4[0, 1]$$

is continuous, as required.

The rest of the conclusions of the lemma then follow immediately from the regularity assertions 1-5) and the Proposition occurring at the beginning of Section 3. □

REMARK. – According to the Lemma before the Proposition of Section 3, $w(x, t)$ and $v(x, t) = w_t(x, t)$ constitute the unique solution (with regularity conditions i), ii)) of the Euler-Bernoulli PDE with the given boundary and initial data.

In addition, an extension of our arguments of Lemma 1 shows that if $u_0 \in D(A^\infty)$, then $w(x, t)$ is in class C^∞ in $0 \leqslant x \leqslant 1$, $t \geqslant 0$.

LEMMA 2. – *Let $u_0 \in D(A^3)$ and consider the solution of the evolutionary* ODE *in $\mathcal{H}_F$*

$$u(t) = S(t)\, u_0 = \begin{pmatrix} w(x, t) \\ v(x, t) \\ p(t) \\ q(t) \end{pmatrix} \quad \textit{on } t \geqslant 0.\,.$$

Then the energy for $u(t)$, and for $Au(t)$, are each nonincreasing on $t \geq 0$. Furthermore, the trajectory $\{u(t): t \geq 0\}$ lies within a compact subset of $\mathcal{H}_F$.

PROOF. – Note that $u(t) = S(t)u_0 \in D(A^3)$, and $Au(t) = S(t)(Au_0) \in D(A^2)$, are both solutions of the evolutionary ODE, and they satisfy all the regularity specifications of the prior Lemma 1, and the Proposition of Section 3. Recall that the energy of the solution $u(t)$ is

$$E(u(t)) = \tfrac{1}{2}\|u(t)\|_F^2 = \tfrac{1}{2}\int_0^1 [v^2 + w_{xx}^2]\,dx + \tfrac{1}{2}[\mu_1 p^2 + \mu_2 q^2]\,,$$

and this has the derivative

$$\dot{E}(u(t)) = v(1, t)L_1 w + v_x(1, t)L_2 w \leq 0$$

(because of the dissipative feedback laws L_1, L_2). Thus $E(u(t))$ is nonincreasing on $t \geq 0$. But $Au(t)$ is also a solution, and so its energy $E(Au(t))$ is also nonincreasing on $t \geq 0$.

Since $E(u(t))$ is nonincreasing $u = (w, v, p, q)$ is bounded in $\mathcal{H}_F$ for $t \geq 0$. In particular, $w(\cdot, t)$ is bounded in $H^2[0, 1]$, $v(\cdot, t)$ is bounded in $L^2[0, 1]$, and $(p(t), q(t))$ is bounded in $\boldsymbol{R}^2$. We shall prove that $w(\cdot, t)$ and $v(\cdot, t)$ are also bounded in higher Sobolev norms.

Recall that the solution $Au(t)$ is given by

$$S(t)(Au_0) = Au(t) = \begin{pmatrix} v(x, t) \\ -w_{xxxx}(x, t) \\ \dfrac{1}{\mu_1} w_{xxx}(1, t) + \dfrac{1}{\mu_1} L_1 w \\ -\dfrac{1}{\mu_2} w_{xx}(1, t) + \dfrac{1}{\mu_2} L_2 w \end{pmatrix}$$

so that the energy of $Au(t)$ can be computed

$$E(Au(t)) = \frac{1}{2}\int_0^1 [w_{xxxx}^2 + v_{xx}^2]\,dx + \frac{\mu_1}{2}\left[\frac{1}{\mu_1} w_{xxx} + \frac{1}{\mu_1} L_1 w\right]^2 + \frac{\mu_2}{2}\left[-\frac{1}{\mu_2} w_{xx} + \frac{1}{\mu_2} L_2 w\right]^2 .$$

Since $E\big(A(u(t))\big)$ is bounded on $t \geq 0$, we conclude that $\int_0^1 [w_{xxxx}^2 + v_{xx}^2]\,dx$ is also bounded.

Since $\int_0^1 v_{xx}^2\,dx$ is bounded for all $t \geq 0$, and since $v(0, t) \equiv v_x(0, t) \equiv 0$, we find that $v(\cdot, t)$ is bounded in the H^2-norm. Since $\int_0^1 w_{xxxx}^2\,dx$ is bounded for all $t \geq 0$, we find that $\max_{0 \leq x \leq 1} |w_{xxx}(x, t)|$ is bounded (just as in the calculus argument in the proof of prior Lemma 1). Therefore $w(\cdot, t)$ is bounded in the norm of $H^4[0, 1]$, for all $t \geq 0$.

From the well-known Rellich's Lemma [1] we conclude that the trajectory $\{u(t)|t\geqslant 0\}\subset\mathcal{H}_F$ lies within a compact subset of $H^2[0,1]\times L^2[0,1]\times\boldsymbol{R}^2$. But $\mathcal{H}_F$ is a closed subset of $H^2[0,1]\times L^2[0,1]\times\boldsymbol{R}^2$, and also has the norm $\|u\|_F$ equivalent to the induced norm. Hence we conclude that $\{u(t)\colon t\geqslant 0\}$ lies within a compact subset of $\mathcal{H}_F$. That is, the $\mathcal{H}_F$-closure of $\{u(t)\colon t\geqslant 0\}$ is a compact subset of $\mathcal{H}_F$: □

REMARK. – Since $D(A^3)$ is dense in $\mathcal{H}_F$, every $u_0\in\mathcal{H}_F$ determines a solution $u(t)=S(t)u_0$ along which the energy is nonincreasing.

LEMMA 3. – *Let $u_0\in D(A^3)$ and consider the solution $u(t)=S(t)u_0$ of the evolutionary* ODE *in $\mathcal{H}_F$.*

Assume now:

1) *$E(u(t))$ is constant on $t\geqslant 0$ and*

2) *the feedback laws L_1, L_2 are* strictly dissipative,

$$L_1 w = -av + bv_x\,,\qquad L_2 w = cv - dv_x$$

with $a>0$, $d>0$, $4ad>(b+c)^2$.

Then $u(t)\equiv 0$ on $t\geqslant 0$.

PROOF. – Since $u(t)=S(t)u_0\in D(A^3)$, we note that $Au(t)=S(t)(Au_0)\in D(A^2)$ is a solution of the evolutionary ODE, and it satisfies the regularity conditions of the prior Lemma 1.

Recall that solution $u(t)$ and $Au(t)$ are given by

$$u(t)=\begin{pmatrix} w(x,t)\\ v(x,t)\\ p(t)\\ q(t)\end{pmatrix}$$

with $w(x,t)\in H^2[0,1]$ $w(0,t)\equiv w_x(0,t)\equiv 0$ (and $v\in L^2[0,1]$) and also

$$Au(t)=\begin{pmatrix} v(x,t)\\ -w_{xxxx}(x,t)\\ \dfrac{1}{\mu_1}w_{xxx}(1,t)+\dfrac{1}{\mu_1}L_1 w\\ -\dfrac{1}{\mu_2}w_{xx}(1,t)+\dfrac{1}{\mu_2}L_2 w\end{pmatrix}$$

with $v(x,t)\in H^2[0,1]$, $v(0,t)\equiv v_x(0,t)\equiv 0$ and in addition, $p(t)=v(1,t)$, $q(t)=v_x(1,t)$. (and $w_{xxxx}\in L^2[0,1]$).

Since $E(u(t))$ is constant,

$$\dot{E}(u(t)) = v(1, t) L_1 w + v_x(1, t) L_2 w \equiv 0 \ .$$

But L_1 and L_2 are *strictly dissipative* feedback laws, so

$$v(1, t) \equiv 0 \ , \qquad v_x(1, t) \equiv 0 \ ,$$

and hence $L_1 w \equiv 0$, $L_2 w \equiv 0$ on $t \geqslant 0$. Therefore $p(t) \equiv 0$, $q(t) \equiv 0$ for all $t \geqslant 0$. Furthermore the last two components of the evolutionary ODE $du/dt = Au$ assert that

$$\mu_1 \frac{dp}{dt} = w_{xxx}(1, t) + L_1 w$$

$$\mu_2 \frac{dq}{dt} = - w_{xx}(1, t) + L_2 w$$

so we conclude that

$$w_{xx}(1, t) \equiv 0 \ , \qquad w_{xxx}(1, t) \equiv 0 \ .$$

In summary, an analysis of the solution $Au(t)$ shows that (see Proposition in Section 3)

i) $v(x, t)$ and $- w_{xxxx}(x, t)$ are in C^1 for $0 \leqslant x \leqslant 1$, $t \geqslant 0$

ii) $t \to v(\cdot, t)$: $[0, \infty) \to H^6[0, 1]$ is continuous.

Moreover,

$$v_t = - w_{xxxx} \ , \qquad - w_{xxxxt} = - v_{xxxx}$$

so $v(x, t)$ is a solution of the Euler-Bernoulli PDE

$$v_{tt} + v_{xxxx} = 0 \qquad \text{on } 0 \leqslant x \leqslant 1 \ , \quad t \geqslant 0 \ .$$

The boundary conditions for $v(x, t)$ are

$$v(0, t) \equiv 0 \ , \quad v_x(0, t) \equiv 0 \ , \quad v_{xx}(1, t) \equiv 0 \ , \quad v_{xxx}(1, t) \equiv 0$$

and the extra conditions

$$v(1, t) \equiv 0 \ , \qquad v_x(1, t) \equiv 0 \ .$$

ASIDE. – The regularity of $w(x, t)$ and $v(x, t)$ follow from the prior Lemma 1, and the earlier Proposition and listed properties at the beginning of Section 3, as applied to $u(t) \in D(A^3)$ and $Au(t) \in D(A^2)$. In particular, besides the properties 1-5) of Section 3 for $w(x, t)$ we further note that

$$v(x, t) \in C^2 \quad \text{in } 0 \leqslant x \leqslant 1\,, \quad t \geqslant 0$$
$$v_{txx} = v_{xtx} = v_{xxt} \quad \text{and } v_{ttx} = v_{txt} = v_{xtt}$$

are continuous in (x, t) for $0 \leqslant x \leqslant 1$, $t \geqslant 0$.

We return to the theme of the proof which analyzes the solution $v(x, t)$ of the Euler-Bernoulli PDE, with the six boundary conditions at $x = 0$ and $x = 1$. It is known by the Holmgren Uniqueness Theorem [4] that any such solution must vanish identically, that is,

$$v(x, t) \equiv 0 \quad \text{on } 0 \leqslant x \leqslant 1\,, \quad t \geqslant 0\,.$$

But this means that

$$w(x, t) = w(x, 0) + \int_0^t v(x, \tau)\, d\tau = w(x, 0)$$

is independent of t. Further

$$w_{tt} + w_{xxxx} = 0$$

implies that $w_{xxxx}(x, 0) \equiv 0$ so that $w(x, 0)$ is a cubic polynomial in x. However, $w(0, 0) = w_x(0, 0) = 0$ and $w_{xx}(1, 0) = w_{xxx}(1, 0) = 0$. This implies that $w(x, 0) \equiv 0$, and therefore

$$w(x, t) \equiv 0\,, \quad v(x, t) \equiv 0 \quad \text{on } 0 \leqslant x \leqslant 1\,, \quad t \geqslant 0\,.$$

Since we have already demonstrated that $p(t) \equiv q(t) \equiv 0$, we conclude that $u(t) \equiv 0$ on $t \geqslant 0$, as required. □

REMARK. – It is only necessary to assume $\dot{E}(u(t)) \equiv 0$ on an open interval, from which the conclusion $u(t) \equiv 0$ on $t \geqslant 0$ then follows. We are now in a position to state and prove our main Theorem 2.

THEOREM 2. – *Consider the real Hilbert space* $\mathcal{H}_F$ *of all states* $u = \begin{pmatrix} w \\ v \\ p \\ q \end{pmatrix}$ *with finite energy, for the Euler-Bernoulli evolutionary* ODE,

$$\frac{du}{dt} = Au = \begin{pmatrix} v(x, t) \\ -w_{xxxx}(x, t) \\ \dfrac{1}{\mu_1} w_{xxx}(1, t) + \dfrac{1}{\mu_1} L_1 w \\ -\dfrac{1}{\mu_2} w_{xx}(1, t) + \dfrac{1}{\mu_2} L_2 w \end{pmatrix}$$

with strictly dissipative feedback laws L_1, L_2. *Let* $S(t)$ *be the contraction semigroup generated by* A *in* $\mathcal{H}_F$, *as before.*

Then, for each $u_0 \in \mathcal{H}_F$ *the solution* $u(t) = S(t)u_0$ *tends asymptotically towards the origin as* $t \to \infty$, *that is,*

$$\lim_{t\to\infty} u(t) = 0 \quad \text{in } \mathcal{H}_F .$$

PROOF. – We first prove the required result for an initial state $u_0 \in D(A^\infty)$, that is, we shall show that the solution

$$u(t) = S(t)u_0 = \begin{pmatrix} w(x,t) \\ v(x,t) \\ p(t) \\ q(t) \end{pmatrix}$$

approaches the origin in $\mathcal{H}_F$, as $t \to \infty$. Later we shall note that this special case is sufficient to prove the general case asserted in the Theorem.

The energy of this solution is

$$E(u(t)) = \tfrac{1}{2}\|u(t)\|_F^2 \leqslant E(u_0) \quad \text{for all } t \geqslant 0$$

and

$$\frac{d}{dt}E(u(t)) = v(1,t)L_1 w + v_x(1,t)L_2 w \leqslant 0 .$$

Therefore $E(u(t))$ is nonincreasing and

$$\lim_{t\to\infty} E(u(t)) = E_\infty$$

exists for some limit $E_\infty \geqslant 0$. In geometric terms, $u(t)$ approaches the sphere of radius $(2E_\infty)^{1/2}$, centered at the origin of the Hilbert Space $\mathcal{H}_F$. We seek to prove that $E_\infty = 0$, which would imply that $\lim_{t\to\infty} u(t) = 0$.

Suppose $E_\infty > 0$ and seek a contradiction. Since $u_0 \in D(A^\infty) \subset D(A^3)$, the trajectory $\{u(t): t \geqslant 0\}$ lies in a compact subset of $\mathcal{H}_F$, according to Lemma 2. We now consider the positive limit (or ω-limit) set of u_0,

$$\omega(u_0) = \bigcap_{\tau>0} \overline{\bigcup_{t\geqslant\tau} u(t)} ,$$

where the closure of the future trajectory $\{\bigcup_{t\geqslant\tau} u(t)\}$ is within $\mathcal{H}_F$. Then $\omega(u_0)$ is a connected compact nonempty subset of $\mathcal{H}_F$. Furthermore the set $\omega(u_0)$ lies on the energy sphere $E = E_\infty$, that is, $E(u_\infty) = E_\infty$ for each point $u_\infty \in \omega(u_0)$.

Take a point $u_\infty \in \omega(u_0)$ so

$$\lim_{k\to\infty} u(t_k) = u_\infty ,$$

for some increasing sequence of times $t_k \to \infty$. We shall prove that the solution

$$u_\infty(t) = S(t)u_\infty \quad \text{lies always in } D(A^\infty) .$$

Note that $Au(t) \in D(A^\infty)$ describes a trajectory $\{Au(t): t \geq 0\}$ which itself lies within some compact subset of $\mathcal{H}_F$, and so $Au(t_{k_i})$ converges in $\mathcal{H}_F$ for some subsequence of times $t_{k_i} \to \infty$. Since A is a closed operator, we know [6] that $u_\infty \in D(A)$ and

$$Au(t_{k_i}) \to Au_\infty \quad \text{in } \mathcal{H}_F .$$

Repeat this argument, with further subsequences (now denoted simply as t_k) to find that

$$Au_\infty \in D(A) \quad \text{and } A^2u(t_k) \to A^2u_\infty .$$

Similarly $u_\infty \in D(A^3)$ with $A^3u(t_k) \to A^3u_\infty$, etc. Hence the results of Lemma 2 and Lemma 3 apply to the solution $u_\infty(t) = S(t)u_\infty \in D(A^\infty)$ on $t \geq 0$.

Under the supposition that $E_\infty > 0$, we can examine the two alternative cases:

a) either $E(u_\infty(t))$ eventually decreases to $E_\infty - \varepsilon$ (for some positive $\varepsilon > 0$) or

b) $E(u_\infty(t))$ is constant on all $t \geq 0$.

In the first case *a*) $E(u_\infty(t)) \leq E_\infty - \varepsilon$ for all suitably large t, say for $t \geq T_\varepsilon$. But since $u(t_k) \to u_\infty$, and since $\|S(t)\| \leq 1$, we find that eventually $E(u(t)) \leq E_\infty - \varepsilon/2$. This is impossible from the definition of $E_\infty = \lim_{t\to\infty} E(u(t))$. In the second case *b*) $E(u_\infty(t))$ is constant. But then Lemma 3 asserts that $u_\infty(t) \equiv 0$ so $E(u_\infty) = E_\infty = 0$. Again this conclusion contradicts the supposition that $E_\infty > 0$. Hence in each of the alternative cases we obtain a contradiction, and therefore we have proved that $E_\infty = 0$.

Because $\lim_{t\to\infty} E(u(t)) = \lim_{t\to\infty} \frac{1}{2}\|u(t)\|_F^2 = 0$, we conclude that

$$\lim_{t\to\infty} u(t) = 0 \quad \text{on } \mathcal{H}_F ,$$

as required for $u_0 \in D(A^\infty)$.

Finally turn to the general case for an arbitrary initial state $\hat{u}_0 \in \mathcal{H}_F$. We must prove that $\hat{u}(t) = S(t)\hat{u}_0$ tends to the origin in $\mathcal{H}_F$. Pick any $\varepsilon > 0$ and consider the ε-sphere about the origin in $\mathcal{H}_F$. We shall show that $\hat{u}(t)$ eventually enters, and thereafter remains, within this chosen ε-sphere.

Since $D(A^\infty)$ is dense in $\mathcal{H}_F$, take some point $u_0 \in D(A^\infty)$ with

$$\|\hat{u}_0 - u_0\|_F < \varepsilon/3 .$$

Since $\|S(t)\| \leqslant 1$ for all $t \geqslant 0$, we calculate that

$$\|S(t)(\hat{u}_0 - u_0)\|_F = \|\hat{u}(t) - u(t)\|_F < \varepsilon/3 \quad \text{for } t \geqslant 0 .$$

But for some time $T_\varepsilon < \infty$ we know that

$$\|u(t)\|_F < \varepsilon/3 \quad \text{on } t \geqslant T_\varepsilon .$$

Then certainly

$$\|\hat{u}(t)\|_F < 2\varepsilon/3 < \varepsilon \quad \text{for } t \geqslant T_\varepsilon .$$

In this way we conclude that

$$\lim_{t\to\infty} \hat{u}(t) = 0 \quad \text{in } \mathcal{H}_F ,$$

as required for each $\hat{u}_0 \in \mathcal{H}_F$. □

As a last topic in this section 4 let us re-interpret Theorem 2 for the Euler-Bernoulli PDE with strictly dissipative feedback boundary damping.

COROLLARY. – *Consider the Euler-Bernoulli* PDE

$$w_{tt} + w_{xxxx} = 0 \qquad \textit{for } w(x, t) \textit{ on } 0 \leqslant x \leqslant 1 , \quad t \geqslant 0$$

with the boundary conditions

$$w(0, t) \equiv w_x(0, t) \equiv 0 \qquad (\textit{clamped at } x = 0)$$

and

$$\mu_1 w_{tt}(1, t) = w_{xxx}(1, t) + L_1 w \qquad (\textit{feedback boundary damping at } x = 1) ,$$
$$\mu_2 w_{xtt}(1, t) = -w_{xx}(1, t) + L_2 w$$

for strictly dissipative feedback laws L_1, L_2, as before.

Give initial data for $w(x, 0)$ and $v(x, 0) = w_t(x, 0)$ satisfying the conditions of $D(A^2)$, that is

$$w(x, 0) \in H^6[0, 1] , \qquad v(x, 0) \in H^4[0, 1] ,$$

with

$$w(0, 0) = w_x(0, 0) = 0\,, \qquad v(0, 0) = v_x(0, 0) = 0$$

$$w_{xxxx}(0, 0) = w_{xxxxx}(0, 0) = 0 \qquad (\text{at } x = 0)$$

and also

$$-w_{xxxx}(1, 0) = \frac{1}{\mu_1} w_{xxx}(1, 0) + \frac{1}{\mu_1} L_1 w$$

$$-w_{xxxxx}(1, 0) = -\frac{1}{\mu_2} w_{xx}(1, 0) + \frac{1}{\mu_2} L_2 w \qquad (\text{at } x = 1)\,.$$

Then there exists a solution $w(x, t)$, $v(x, t) = w_t(x, t)$ *of this boundary-initial value problem so that*

i) $w(x, t)$ *and* $v(x, t)$ *are in class* C^1 *on* $0 \leqslant x \leqslant 1$, $t \geqslant 0$

ii) $t \to (w(\cdot, t), v(\cdot, t)) : [0, \infty) \to H^6[0, 1] \times H^4[0, 1]$ *is a continuous map.*

Moreover $w(x, t)$ *is the unique solution of this boundary-initial value problem satisfying the regularity conditions* i), ii).

Furthermore this unique solution decays asymptotically to zero,

$$\lim_{t\to\infty} w(x, t) = 0\,, \qquad \lim_{t\to\infty} w_x(x, t) = 0$$

with each convergence uniform on $0 \leqslant x \leqslant 1$.

The proof of the Corollary follows directly from Theorem 2, and the prior Lemma 1, in view of the Proposition of Section 3 relating the solutions of the Euler-Bernoulli PDE with the corresponding solutions of the corresponding evolutionary ODE in the Hilbert space $\mathcal{H}_F$.

If we further assume that the initial state $w(x, 0)$, $v(x, 0)$ satisfies the conditions of $D(A^3)$, then we further conclude

$$\lim_{t\to\infty} v(x, t) = 0\,, \qquad \lim_{t\to\infty} v_x(x, t) = 0\,,$$

uniformly on $0 \leqslant x \leqslant 1$. The conditions for $u = \begin{pmatrix} w \\ v \\ p \\ q \end{pmatrix} \in \mathcal{H}_F$ to lie in $D(A^3)$ are all those conditions already obtained for $D(A^2)$, and in addition

$$v \in H^6[0, 1]\,, \qquad w \in H^8[0, 1]$$

with

$$v_{xxxx}(0) = v_{xxxxx}(0) = 0 \qquad (\text{at } x = 0)$$

and

$$-v_{xxxxx}(1)=\frac{1}{\mu_1}v_{xxx}(1)+\frac{1}{\mu_1}L_1v \qquad (\text{at } x=1)$$

$$-v_{xxxxx}(1)=-\frac{1}{\mu_2}v_{xx}(1)+\frac{1}{\mu_2}L_2v\,.$$

Of course, if we take the corresponding initial state $u_0\in D(A^\infty)$, then $w(x,t)$, $v(x,t)$ are in class C^∞ on $0\leqslant x\leqslant 1$, $t\geqslant 0$ and higher order derivatives also decay to zero as $t\to\infty$. In particular, it is evident that $w(x,t)$, $v(x,t)$ and their x-derivatives of order 4, 8, 12, 16, etc. as well as their t-derivatives of all orders—are in class C^1 in (x,t) on $0\leqslant x\leqslant 1$, $t\geqslant 0$. The intermediate order derivatives can be shown to be continuous by routine arguments involving generalized derivatives.

Finally we further reinterpret the stability Theorem 2 in terms of our original control problem for the elastic mast (M) and the rigid antenna (A), as described in Section 1 of this paper.

Consider the control system for the elastic mast (M)

$$w_{tt}+w_{xxxx}=0 \qquad \text{for } w(x,t) \text{ on } 0\leqslant x\leqslant 1\,,\quad t\geqslant 0$$

and for the rigid antenna (A)

$$\mu_1\ddot{y}=w_{xxx}(1,t)+L_1w \qquad \mu_2\ddot{\theta}=-\,w_{xx}(1,t)+L_2w\,,$$

with strictly dissipative feedback boundary damping

$$L_1w=-\,a\dot{y}+b\dot{\theta}\,,\qquad L_2w=c\dot{y}-d\dot{\theta} \qquad a>0\,,\quad d>0\,,\quad 4ad>(b+c)^2\,.$$

Assume the boundary conditions, as usual,

$$w(0,t)\equiv w_x(0,t)\equiv 0 \qquad (\text{clamped at } x=0)$$

and

$$w(1,t)\equiv y(t)\,,\qquad w_x(1,t)\equiv\theta(t) \qquad (\text{linked at } x=1)\,.$$

Give any initial state for

$$w(\cdot,t),\ v(\cdot,t)=w_t(\cdot,t),\ y(t),\ \dot{y}(t),\ \theta(t),\ \dot{\theta}(t)$$

satisfying the initial conditions for $D(A^2)$; that is, $w(x,0)$, $v(x,0)$ as in the preceeding Corollary and the additional conditions

$$w(1,0)=y(0)\,,\qquad w_x(1,0)=\theta(0)\,,\qquad v(1,0)=\dot{y}(0)\,,\qquad v_x(1,0)=\dot{\theta}(0)\,.$$

Then there exists a solution of the feedback control system $(M)+(A)$ (unique within the regularity specified in the Corollary), and furthermore

$$\lim_{t\to\infty} w(x,t)=0\,,\quad \lim_{t\to\infty} y(t)=0\,,\quad \lim_{t\to\infty}\theta(t)=0\,.$$

If we further demand that the initial data satisfy the conditions of $D(A^3)$, then

$$\lim_{t\to\infty} w_t(x,t)=0\,,\quad \lim_{t\to\infty} \dot y(t)=0\,,\quad \lim_{t\to\infty}\dot\theta(t)=0\,.$$

It is of interest to examine the rate of decay towards zero for $w(x,t)$ in this asymptotically stable feedback system. Since the differential equations involved are all linear with constant coefficients, it might seem plausible to suppose that this decay rate is uniformly exponential. However, this is definitely not the case. In fact, there exist such solutions $w(x,t)$ which approach towards zero more slowly than any preassigned function. In the next section of this paper we clarify these questions by some unusual examples.

5. – Decay rate under feedback boundary damping. Nonexponential and slow decay.

In this final section we shall examine the rate of decay of a solution $w(x,t)$ of the Euler-Bernoulli PDE, under dissipative feedback boundary damping. Since the dynamical system is described by linear differential equations with constant coefficients it might seem plausible to suppose that the asymptotic decay of $w(x,t)$ is uniformly exponential. This is definitely not the case, and we shall demonstrate this lack of uniform exponential decay by means of a sequence of examples $\{w_n(x,t)\}$ with decay rates slower than any preassigned exponential function. In fact, our method will construct examples $u_n(t)=S(t)u_{0n}$ in $\mathcal{H}_F$ with arbitrarily slow decay towards zero.

In order to make our constructions as explicit and as transparent as possible, we shall set the mass constants $\mu_1=\mu_2=1$, and choose the special, but typical feedback laws

$$L_1 w=-\,w_t(1,t)\,,\qquad L_2 w=-\,w_{xt}(1,t)\,,$$

which are strictly dissipative in the sense of the definitions of Section 2, and of Theorem 2 of Section 4.

Thus consider the Euler-Bernoulli PDE

$$w_{tt}+w_{xxxx}=0\qquad \text{for } w(x,t) \text{ on } 0\leqslant x\leqslant 1\,,\quad t\geqslant 0$$

with the boundary conditions

$$w(0,t) \equiv w_x(0,t) \equiv 0 \qquad \text{(clamped at } x=0)$$

and

$$\begin{aligned} w_{tt}(1,t) &= w_{xxx}(1,t) - w_t(1,t) \qquad \text{(dissipative damping at } x=1)\,. \\ w_{xtt}(1,t) &= -\,w_{xx}(1,t) - w_{xt}(1,t) \end{aligned}$$

We seek smooth solutions $w(x,t)$ (also defining solutions $u(t) = S(t)u_0 \in D(A^\infty)$ of the evolutionary ODE in $\mathcal{H}_p$), with slow decay rate governed by $\exp[-\tau t]$, for small positive $\tau > 0$. That is, we first find complex-valued solutions of the form

$$\mathcal{W}(x,t) = \exp[ist]\Psi(x) \qquad (s = \sigma + i\tau)$$

for $\Psi(x)$ satisfying the corresponding ODE

$$\Psi_{xxxx}(x) - s^2\Psi(x) = 0\,,$$

with boundary conditions

$$\Psi(0) = \Psi_x(0) = 0\,.$$

This demand produces the most general solution

$$\Psi(x) = A(\cosh\lambda x - \cos\lambda x) + B(\sinh\lambda x - \sin\lambda x)$$

for the complex parameter $\lambda = \mu + i\nu$ and complex coefficients A, B. The relation between λ and s is specified $s^2 = \lambda^4$, and we shall choose

$$s = \lambda^2 \qquad \text{or } \sigma = \mu^2 - \nu^2\,, \qquad \tau = 2\mu\nu\,.$$

Later we shall let $w(x,t) = \operatorname{Re}\mathcal{W}(x,t)$ to obtain a real solution of the real linear boundary value problem.

The remaining two boundary conditions at $x = 1$ will impose a determinantal demand on λ, in order that nontrivial constants A, B can be found. In more detail, the two boundary conditions at $x = 1$ yield the linear homogeneous equations for A, B:

$$\begin{aligned} &A\{-\lambda^3(\sinh\lambda - \sin\lambda) - s^2(\cosh\lambda - \cos\lambda) + is(\cosh\lambda - \cos\lambda)\} + \\ &\qquad + B\{-\lambda^3(\cosh\lambda + \cos\lambda) - s^2(\sinh\lambda - \sin\lambda) + is(\sinh\lambda - \sin\lambda)\} = 0 \end{aligned}$$

and

$$A\{\lambda^2(\cosh\lambda+\cos\lambda)-s^2\lambda(\sinh\lambda+\sin\lambda)+is\lambda(\sinh\lambda+\sin\lambda)\}+$$
$$+B\{\lambda^2(\sinh\lambda+\sin\lambda)-s^2\lambda(\cosh\lambda-\cos\lambda)+is\lambda(\cosh\lambda-\cos\lambda)\}=0\,.$$

The determinant $\Delta(\lambda)$ of these two linear equations is (upon setting $s=\lambda^2$):

$$\Delta(\lambda)=(-2\lambda^4s^2+2is\lambda^4+2s^2\lambda^2-2is\lambda^2)\sinh\lambda\cos\lambda+$$
$$+(-2s^4\lambda+4is^3\lambda+4\lambda^5)\cosh\lambda\cos\lambda+$$
$$+(-2s^2\lambda^2+2is\lambda^2-2\lambda^4s^2+2is\lambda^4)\cosh\lambda\sin\lambda+2(\lambda^9-2is\lambda^5)+O(\lambda^5)\,.$$

Here the last term $O(\lambda^5)$ consists of a polynomial of degree 5 in λ, with coefficients that are complex polynomials in $(\cos\lambda,\sin\lambda)$—that is, these are trigonometric polynomials of an explicit format.

Our program then consists in solving the transcendental equation

$$\Delta(\lambda)=0$$

for roots $\lambda_n=\mu_n+i\nu_n$ with $\sigma_n=\mu_n^2-\nu_n^2\to\infty$ and $\tau_n=2\mu_n\nu_n\searrow 0$. In terms of the geometry of the complex plane, we seek roots λ_n that lie in the open first quadrant with $\mu_n\to+\infty$, and $\nu_n\searrow 0$, so that λ_n lies below any preassigned hyperbola $\mu_n\nu_n=\text{const}>0$. Later we interpret these constraints with respect to the decay rate for $w(x,t)=\operatorname{Re}\mathcal{W}(x,t)$ (in terms of the H^2-norm, and the corresponding $\mathcal{H}_F$-norm, and also for other significant norms).

In order to solve $\Delta(\lambda)=0$ for roots λ_n, we simplify the expression for this determinant by setting

$$\cosh\lambda=\frac{e^\lambda+e^{-\lambda}}{2},\qquad \sinh\lambda=\frac{e^\lambda-e^{-\lambda}}{2},$$

and collect terms in power of λ to obtain:

$$\Delta(\lambda)=-e^\lambda[\lambda^8(\lambda+1)\cos\lambda+\lambda^8\sin\lambda-Q_7(\lambda)]+[Q_9(\lambda)e^{-\lambda}+Q_9(\lambda)]\,,$$

where the $Q(\lambda)$ are polynomials of the indicated degree in the complex variable λ, with coefficients that are explicit trigonometric polynomials in $\sin\lambda$, $\cos\lambda$.

We shall seek complex solutions λ_n for

$$\Delta(\lambda)=0$$

in the strip S along the positive real axis:

$$S=\{\lambda=\mu+i\nu\colon |\mu|\geqslant 10,\ |\nu|\leqslant 1\}\,,$$

wherein $|\cos\lambda|\leqslant 10$, $|\sin\lambda|\leqslant 10$ and $e^{-\lambda}\to 0$ as $\lambda\to\infty$. Define the analytic functions $G(\lambda)$ and $\varphi(\lambda)$ in the strip S by:

$$G(\lambda)=\cos\lambda+\frac{\sin\lambda}{\lambda+1}\,,\qquad \frac{e^{-\lambda}\Delta(\lambda)}{\lambda^8(\lambda+1)}=\varphi(\lambda)-G(\lambda)\,.$$

Then

$$\varphi(\lambda)=\frac{e^{-\lambda}}{\lambda^8(\lambda+1)}\,[Q_9(\lambda)\,e^{-\lambda}+Q_9(\lambda)]+\frac{Q_7(\lambda)}{\lambda^8(\lambda+1)}=O\left(\frac{1}{\lambda^2}\right),$$

that is,

$$|\varphi(\lambda)|\leqslant\frac{\text{const}}{|\lambda|^2}\qquad\text{for } |\lambda|\to\infty \text{ in } S\,.$$

We first solve $G(\lambda)=0$, and then study the perturbations caused by the term $\varphi(\lambda)$, for large $|\lambda|$ in the strip S. Elementary analysis shows that there is a sequence of real positive roots $\tilde\lambda_n$ of

$$G(\lambda)=\cos\lambda+\frac{\sin\lambda}{\lambda+1}=0\,,\qquad\text{or } (\lambda+1)=-\tan\lambda\,,$$

namely $\tilde\lambda_n$ with

$$\left[\tilde\lambda_n-\left(\frac{\pi}{2}+n\pi\right)\right]\searrow 0\qquad\text{for } n=0,1,2,3,\dots,\,.$$

Moreover there are only these real roots since

$$|(\lambda+1)\cos\lambda+\sin\lambda|^2=[(\mu+1)^2+\nu^2+1]\sinh^2\nu+\nu^2\cos^2\mu+$$
$$+[(\mu+1)\cos\mu+\sin\mu]^2+2\nu\sinh\nu\cosh\nu>2\nu\sinh\mu\cosh\nu>0\qquad\text{for } \nu\neq 0\,.$$

Since

$$G'(\lambda)=-\sin\lambda+\frac{(\lambda+1)\cos\lambda-\sin\lambda}{(\lambda+1)^2}=-\sin\lambda+O\left(\frac{1}{\lambda}\right),$$

we find that

$$\lim_{n\to\infty}|G'(\tilde\lambda_n)|=1\,,$$

and the roots $\tilde\lambda_n$ are all simple — at least for all sufficiently large n. Hence we anticipate that perturbation techniques will be effective in locating roots λ_n of $G(\lambda)-\varphi(\lambda)$ nearby to $\tilde\lambda_n$.

Around each complex point $\tilde\lambda_n$ we construct a circle $\tilde C_n$ of radius $\tilde R_n=n^{-3/2}$, so that the disk $|\lambda-\tilde\lambda_n|\leqslant\tilde R_n$ lies within the strip S. We shall prove the existence of a zero $\lambda_n = \mu_n + i\nu_n$ of the analytic function $G(\lambda) - \varphi(\lambda)$ within each such disk, at least for all suitably large n. In such a case

$$\mu_n \to \frac{\pi}{2} + n\pi\,, \qquad 0 < \nu_n \leqslant n^{-3/2}$$

and so $\tau_n = 2\mu_n\nu_n \to 0$ as as $n \to \infty$, [note: $\nu_n \leqslant 0$ is impossible since this would imply the existence of a solution $w(x, t)$ of the PDE whose energy fails to decay towards zero—as will be re-emphasized later].

At the center $\tilde{\lambda}_n$ of the circle $\tilde{C}_n$ we have

$$G(\tilde{\lambda}_n) = 0\,,$$

and we now seek a lower bound for $|G(\lambda)|$ on $\tilde{C}_n$ in order to apply Rouché's Theorem. As noted before

$$\lim_{\lambda \to \infty} [G'(\lambda) + \sin \lambda] = 0\,,$$

and hence

$$\operatorname{Re} G'(\lambda) \geqslant \tfrac{1}{2} \text{ in the disk } |\lambda - \tilde{\lambda}_n| \leqslant n^{-3/2}$$

for n even—otherwise $\operatorname{Re} G'(\lambda) \leqslant -\frac{1}{2}$.

Consider $G(\hat{\lambda})$ at some point $\hat{\lambda}$ on $\tilde{C}_n$ and join $\hat{\lambda}$ to the center $\tilde{\lambda}_n$ by a radial segment

$$\lambda = \tilde{\lambda}_n + r(\hat{\lambda} - \tilde{\lambda}_n) \qquad \text{for } 0 \leqslant r \leqslant 1\,.$$

Then

$$G(\hat{\lambda}) - G(\tilde{\lambda}_n) = \int_0^1 G'(\lambda(r)) \cdot (\hat{\lambda} - \tilde{\lambda}_n)\, dr$$

so

$$G(\hat{\lambda}) = \Big[\int_0^1 G'(\lambda(r))\, dr\Big] \cdot (\hat{\lambda} - \tilde{\lambda}_n)\,.$$

Therefore there is a complex constant

$$\gamma = \int_0^1 G'(\lambda(r))\, dr\,,$$

that lies in the closed convex hull of the value set $\{G'(\lambda(r))\}$, see [2]. Since

$$|\operatorname{Re} \gamma| \geqslant \tfrac{1}{2}\,, \qquad |G(\hat{\lambda})| = |\gamma| \tilde{R}_n \geqslant \tfrac{1}{2} n^{-3/2}$$

for all $\hat{\lambda} \in \tilde{C}_n$.

Our estimates show that on the circle $\tilde{C}_n$, for all suitably large n,

$$|G(\lambda)| \geqslant \frac{1}{2} n^{-3/2} \quad \text{and} \quad |\varphi(\lambda)| \leqslant \frac{\text{const}}{n^2},$$

so

$$|G(\lambda)| > |\varphi(\lambda)| .$$

The Theorem of Rouché now guarantees the existence of a root $\lambda_n = \mu_n + i\nu_n$ within the disk $|\lambda - \tilde{\lambda}_n| \leqslant \tilde{R}_n$ for the equation

$$G(\lambda) = \varphi(\lambda)$$

or equally well for the determinantal equation

$$-\Delta(\lambda) = e^{\lambda}\lambda^8(\lambda + 1)[G(\lambda) - \varphi(\lambda)] = 0 .$$

We shall use these roots $\lambda_n = \mu_n + i\nu_n$ of $\Delta(\lambda) = 0$, with

$$\left|\mu_n - \left(\frac{\pi}{2} + n\pi\right)\right| \to 0 , \qquad 0 < \nu_n \leqslant n^{-3/2}$$

so

$$\sigma_n = \mu_n^2 - \nu_n^2 \to \infty \quad \text{and } \tau_n = 2\mu_n\nu_n \searrow 0 \quad \text{as } n \to \infty .$$

For such complex roots λ_n we have the desired complex solutions of the Euler-Bernoulli PDE, with the prescribed dissipative boundary conditions,

$$\mathfrak{W}_n(x, t) = \exp[-\tau_n t][\cos\sigma_n t + i \sin\sigma_n t]\Psi_n(x)$$

($s_n = \sigma_n + i\tau_n$ so $s_n = \lambda_n^2 = \mu_n^2 - \nu_n^2 + 2i\mu_n\nu_n$), where

$$\Psi_n(x) = A_n(\cosh\lambda_n x - \cos\lambda_n x) + B_n(\sinh\lambda_n x - \sin\lambda_n x)$$

with

$$\lambda_n = \mu_n + i\nu_n , \qquad \tau_n = 2\mu_n\nu_n \searrow 0 , \qquad \sigma_n = \mu_n^2 - \nu_n^2 \to \infty .$$

The complex constants $(A_n, B_n) \neq (0, 0)$ can be normalized, up to ratio, in any convenient manner. Since $\mathfrak{W}_n(x, t)$ is a nontrivial solution of the boundary value problem,

$$\operatorname{Re}\mathfrak{W}_n(x, t) = \exp[-\tau_n t][\cos\sigma_n t(\operatorname{Re}\Psi_n) - \sin\sigma_n t(\operatorname{Im}\Psi_n)] \not\equiv 0 ,$$

for otherwise we obtain the contradiction

$$\operatorname{Re}\Psi_n(x) \equiv \operatorname{Im}\Psi_n(x)x) \equiv 0 \quad \text{and } \Psi_n(x) \equiv 0 \quad \text{on } 0 \leqslant x \leqslant 1 .$$

Next we analyse the rates of decay of the real solutions of the PDE boundary value problem

$$w_n(x, t) = \mathrm{Re}\, \mathcal{W}_n(x, t) ,$$

and of the corresponding solution of the evolutionary ODE

$$u_n(t) = \begin{pmatrix} w_n(x, t) \\ v_n(x, t) \\ p_n(t) \\ q_n(t) \end{pmatrix} = S(t)\, u_n(0)$$

where $v_n(x, t) = \partial w_n/\partial t$, $p_n(t) = v_n(1, t)$, $q_n(t) = (\partial v_n/\partial x)(1, t)$ are defined as usual.

Suppressing the index n, we write the real solution of the boundary value problem

$$w(x, t) = \mathrm{Re}\, \mathcal{W}(x, t) \not\equiv 0$$

so

$$w(x, t) = \exp[-\tau t][(\cos \sigma t)\Psi_R(x) - (\sin \sigma t)\Psi_I(x)]$$

(where $\Psi(x) = \Psi_R(x) + i\Psi_I(x)$).

Direct computations yield

$$w_t = -\tau \exp[-\tau t][(\cos \sigma t)\Psi_R(x) - (\sin \sigma t)\Psi_I(x)] + \\ + \sigma \exp[-\tau t][(-\sin \sigma t)\Psi_R(x) - (\cos \sigma t)\Psi_I(x)]$$

and

$$w_{xx} = \exp[-\tau t][(\cos \sigma t)\Psi_R''(x) - (\sin \sigma t)\Psi_I''(x)]$$

so the corresponding energy is

$$E(t) = \exp[-2\tau t][\alpha_2 \cos 2\sigma t + \alpha_1 \cos \sigma t + \beta_2 \sin 2\sigma t + \beta_1 \sin \sigma t + \alpha_0]$$

for real constants $\alpha_2, \alpha_1, \beta_2, \beta_1, \alpha_0$.

On a period of duration $2\pi/\sigma$ we let

$$M = \max_{0 \leqslant t \leqslant 2\pi/\sigma} [\alpha_2 \cos 2\sigma t + \alpha_1 \cos \sigma t + \beta_2 \sin 2\sigma t + \beta_1 \sin \sigma t + \alpha_0]$$

so that at a discrete periodic set of times $t_k \nearrow \infty$

$$E(t_k) = M \exp[-2\tau t_k] .$$

Moreover, since $E(t) > 0$ is nonincreasing, we conclude that for each $t \geqslant 0$

$$E(t) \geqslant M \exp[-2\tau t_k] \geqslant M \exp[-2\tau(t + 2\pi/\sigma)]$$

where $t_{k-1} \leqslant t < t_k$ on the t-axis. Similarly, $(M \exp[-4\pi\tau/\sigma]) \exp[-2\tau t] \leqslant E(t) \leqslant$ $\leqslant M \exp[4\pi\tau/\sigma] \exp[-2\tau t]$. But as $n \to \infty$, $\tau_n \searrow 0$ and $\sigma_n \to \infty$

$$\exp[-4\pi\tau_n/\sigma_n] \nearrow 1 .$$

We conclude that, for each $\varepsilon > 0$, the energy of $w_n(x, t)$ for all large n satisfies bounds

$$(1 - \varepsilon) M_n \exp[-2\tau_n t] \leqslant E_n(t) \leqslant (1 + \varepsilon) M_n \exp[-2\tau_n t] \quad \text{on } t \geqslant 0 ,$$

and has the exponential decay rate

$$\lim_{t\to\infty} \left| \frac{1}{t} \ln E_n(t) \right| = 2\tau_n .$$

Hence, for each prescribed positive decay rate $\hat{\tau} > 0$, there exists $\hat{n} = n(\hat{\tau})$ so that the exponential decay rate $2\tau_n < \hat{\tau}$ for all suitably large $n \geqslant \hat{n}$. In this sense the sequence of real solutions $\{w_n(x, t)\}$ of the given boundary value problem displays arbitrarily slow exponential decay rates, and so the Euler-Bernoulli semigroup $S(t)$ cannot have a fixed positive exponential decay rate, uniformly on the state space $\mathcal{H}_F$.

REMARK. – As another and different measure of the rate of decay of a solution $w_n(x, t)$ we can consider $\|w_n(x, t)\| = \sup_{0 \leqslant x \leqslant 1} |w_n(x, t)|$ and then define

$$R_n(t) = \sup_{[t-1, t+1]} \|w_n(x, t)\| , \qquad r_n(t) = \inf_{[t-1, t+1]} \|w_n(x, t)\|$$

and consider n large so that the period $2\pi/\sigma_n$ is much less than 1. With the earlier expression for $w(x, t)$, it is elementary to demonstrate that there are constants $0 \leqslant r_n < R_n$ so that $R_n(t) \leqslant \mathrm{R}_n \exp[-\tau_n t]$, $r_n(t) \geqslant r_n \exp[-\tau_n t]$ and

$$r_n \exp[-\tau_n t] \leqslant \|w_n(x, t)\| \leqslant R_n \exp[-\tau_n t] , \quad \text{for all } t \geqslant 0 .$$

Henceforth we consider only the energy norms for the solutions $w(x, t)$.

We now return to the construction of a solution $w(x, t)$ of the Euler-Bernoulli boundary value problem with an arbitrarily slow decay rate. We shall consider the real solutions, for $n = 1, 2, 3, \ldots$,

$$w_n(x, t) = \exp[-\tau_n t] [(\cos \sigma_n t) \Psi_{nR}(x) - (\sin \sigma_n t) \Psi_{nI}(x)]$$

where $\tau_1 > \tau_2 > \tau_k > \dots \to 0$, and $\sigma_n \to +\infty$, as before. Each such solution defines a continuous function

$$u_n(t) = S(t)\,u_n(0) = \begin{pmatrix} w_n(x,t) \\ v_n(x,t) \\ p_n(t) \\ q_n(t) \end{pmatrix}, \qquad t \to u_n(\cdot)\colon [0,\infty) \to \mathcal{H}_F\,,$$

which is a solution of the corresponding evolutionary ODE in the finite-energy Hilbert space $\mathcal{H}_F$. Clearly the energy norm of $u_n(t)$ is

$$[2E(u_n(t))]^{1/2} = \|u_n(t)\|_F\,.$$

According to the exponential decay rate that we have already established,

$$\tfrac{1}{2}\exp[-\tau_n t] \leqslant \|u_n(t)\|_F \leqslant \exp[-\tau_n t] \qquad \text{on } 0 \leqslant t < \infty$$

(after discarding a finite initial set of these $u_n(t)$, and multiplication of each $u_n(t)$ be a suitable scaling constant, and then making the corresponding notational modifications).

LEMMA. – *Let $\{z_n(t)\}$ be a sequence of continuous functions*

$$t \to z_n(\cdot)\colon [0,\infty) \to \mathcal{B}$$

for some real Banach space $\mathcal{B}$.

Assume for each $n = 1, 2, 3, \dots$ the norms

$$\|z_n(t)\| \qquad \text{on } 0 \leqslant t < \infty$$

satisfy

$$\tfrac{1}{2}\exp[-\tau_n t] \leqslant \|z_n(t)\| \leqslant \exp[-\tau_n t]$$

for $\tau_1 > \tau_2 > \tau_k > \dots \to 0$.

Let $\psi(t)$ be a real continuous function on $0 \leqslant t < \infty$, positive and strictly decreasing so

$$\psi(0) = 1\,, \qquad \lim_{t\to\infty} \psi(t) = 0\,,$$

Then there exists a subsequence $\{z_{n_k}(t)\}$ so

$$z(t) = \sum_{k=1}^{\infty} \frac{a_k}{k!} z_{n_k}(t) \qquad (\text{constants } 0 < a_k \leqslant 1)$$

is continuous

$$t \to z(\cdot): [0, \infty) \to \mathcal{B}.$$

Moreover $\|z(t_k)\| > \psi(t_k)$ *at some sequence of times* $t_k \to \infty$.

PROOF. – The proof consists in a construction of $z(t)$ by choices of $\{\tau_{n_k}\}$, $\{a_k\}$, and intervals I_k, which contain the times $t_k \in I_k$.

STAGE 1. – Take $n_1 \geqslant 1$, compact interval $I_1 \subset [0, \infty)$, and positive constant $a_1 = 1$ so that:

$$\psi(t) < \frac{1}{10} \qquad \text{on } I_1$$

and

$$\frac{3}{10} \leqslant \|z_{n_1}(t)\| \leqslant \frac{7}{10} \qquad \text{on } I_1.$$

[Take the time $\hat{t}_1$ when $\psi(\hat{t}_1) = 1/10$. Then select n_1 so that

$$\|z_{n_1}(\hat{t}_1)\| \geqslant \frac{1}{2} \exp[-\tau_{n_1} \hat{t}_1] > \frac{3}{10}.$$

Since $\lim_{t \to \infty} \|z_{n_1}(t)\| = 0$, we can find a later compact interval I_1 whereon

$$\frac{3}{10} \leqslant \|z_{n_1}(t)\| \leqslant \frac{7}{10}.]$$

STAGE 2. – Take $n_2 > n_1$, compact interval $I_2 \subset [0, \infty)$ after I_1 (following and disjoint from I_1), and positive constant $0 < a_2 \leqslant 1$ so that:

i) $\|z_{n_1}(t)\| < \dfrac{1}{2!}\dfrac{1}{100}$ on I_2

ii) $\psi(t) < \dfrac{1}{2!}\dfrac{1}{100}$ on I_2

and

iii) $\dfrac{3}{100} \leqslant a_2 \|z_{n_2}(t)\| \leqslant \dfrac{7}{100}$ on $I_1 \cup I_2$.

[First choose I_2 so i) and ii) hold, then choose n_2 so $\exp[-\tau_{n_2} t]$ is « suitably flat » out to I_2, then choose a_2 so iii) holds.]

STAGE 3. – Take $n_3 > n_2$, compact interval $I_3 \subset [0, \infty]$ after I_2, and positive constant $0 < a_3 \leqslant 1$ so that:

$$\text{i) } \|z_{n_1}(t)\| + \frac{a_2}{2!}\|z_{n_2}(t)\| < \frac{1}{3!}\frac{1}{1000} \quad \text{on } I_3$$

$$\text{ii) } \psi(t) < \frac{1}{3!}\frac{1}{1000} \quad \text{on } I_3$$

and

$$\text{iii) } \frac{3}{1000} \leqslant a_3\|z_{n_3}(t)\| \leqslant \frac{7}{1000} \quad \text{on } I_1 \cup I_2 \cup I_3 .$$

Continue this selection process to stage l.

Take $n_l > n_{l-1}$, compact interval $I_l \subset [0, \infty)$ after I_{l-1}, and positive constant $0 < a_l \leqslant 1$ so that:

$$\text{i) } \sum_{k=1}^{l-1}\left\|\frac{a_k}{k!} z_{n_k}(t)\right\| < \frac{1}{l!}\frac{1}{10^l} \quad \text{on } I_l$$

$$\text{ii) } \psi(t) < \frac{1}{l!}\frac{1}{10^l} \quad \text{on } I_l$$

and

$$\text{iii) } \frac{3}{10^l} \leqslant a_l\|z_{n_l}(t)\| \leqslant \frac{7}{10^l} \quad \text{on } I_1 \cup I_2 \cup \ldots \cup I_l .$$

Then define

$$z(t) = \sum_{k=1}^{\infty}\frac{a_k}{k!} z_{n_k}(t) \quad \text{on } 0 \leqslant t < \infty .$$

It is clear that this series is uniformly absolutely convergent on $0 \leqslant t < \infty$, so the $\mathcal{B}$-value function $z(t)$ is continuous as required.

We now show that

$$\|z(t)\| \geqslant \psi(t) \quad \text{on } I_1 \cup I_2 \cup \ldots ,$$

and in particular, this inequality holds at the midpoint t_k of I_k. Clearly on the interval I_l

$$\frac{a_l}{l!}\|z_{n_l}(t)\| \geqslant \frac{1}{l!}\frac{3}{10^l} > \frac{1}{l!}\frac{1}{10^l} > \psi(t) .$$

We must show that the consideration of the infinitely many other terms in the series for $z(t)$ does not destroy the force of this inequality.

Now on I_l we note

$$\|z(t)\| \geq \frac{a_l}{l!}\|z_{n_l}(t)\| - \sum_{k\neq l}^{\infty}\left\|\frac{a_k}{k!}z_{n_k}(t)\right\| .$$

But

$$\sum_{k<l}\left\|\frac{a_k}{k!}z_{n_k}(t)\right\| < \frac{1}{l!}\frac{1}{10^l} \qquad (\text{each } l>1)\,,$$

and

$$\sum_{k>l}\left\|\frac{a_k}{k!}z_{n_k}(t)\right\| < \frac{1}{(l+1)!}\frac{7}{10^{l+1}} + \frac{1}{(l+2)!}\frac{7}{10^{l+2}} + \ldots <$$

$$< \frac{1}{(l+1)!}\frac{7}{10^{l+1}}\cdot\left[1 + \frac{1}{10} + \frac{1}{100} + \ldots\right] < \frac{1}{(l+1)!}\frac{8}{10^{l+1}} < \frac{1}{l!}\frac{1}{10^l}\,.$$

Hence on I_l

$$\|z(t)\| \geq \frac{1}{l!}\frac{3}{10^l} - \frac{1}{l!}\frac{1}{10^l} - \frac{1}{l!}\frac{1}{10^l} = \frac{1}{l!}\frac{1}{10^l} > \psi(t)\,.$$

In particular, at the midpoint t_l of I_l we have

$$\|z(t_l)\| > \psi(t_l) \qquad \text{for } l = 1, 2, 3, \ldots\,. \qquad \square$$

COROLLARY. – *For each positive* $\tau > 0$

$$\|z(t)\| > \exp[-\tau t]$$

at a sequence of times $t_k \nearrow \infty$. *Hence the exponential decay rate (Liapunov exponent)*

$$\varliminf_{t\to\infty}\left|\frac{1}{t}\log\|z(t)\|\right| \leq \tau\,.$$

Therefore the Liapunov exponent is zero,

$$\varliminf_{t\to\infty}\left|\frac{1}{t}\log\|z(t)\|\right| = 0\,.$$

THEOREM 3. – *Consider the Hilbert space* $\mathcal{H}_F$ *of all states* $u = \begin{pmatrix} w \\ v \\ p \\ q \end{pmatrix}$ *with finite energy, for the Euler-Bernoulli evolutionary* ODE,

$$\frac{du}{dt} = Au = \begin{pmatrix} v(x,t) \\ -w_{xxxx}(x,t) \\ w_{xxx}(1,t) - w_t(1,t) \\ -w_{xx}(1,t) - w_{xt}(1,t) \end{pmatrix}$$

with the strictly dissipative feedback laws

$$L_1 w = -w_t(1, t), \qquad L_2 w = -w_{xt}(1, t)$$

(and mass constants μ_1, μ_2 *set equal to* 1*). Let* $S(t)$ *be the contraction semigroup generated by* A *in* $\mathcal{H}_F$*, as before.*

Then, there exist solutions $u(t)$ *with arbitrarily slow decay towards zero. In more detail, let* $\psi(t)$ *be a real continuous function on* $0 \leqslant t < \infty$*, positive and strictly decreasing so*

$$\psi(0) = 1, \quad \lim_{t \to \infty} \psi(t) = 0 .$$

Then there exists an initial state $u_0 \in \mathcal{H}_F$ *and corresponding solution*

$$u(t) = S(t) u_0 \quad \text{on } 0 \leqslant t < \infty ,$$

so that

$$\|u(t)\|_F > \psi(t)$$

at some sequence of times $t_k \nearrow \infty$.

PROOF. – Consider the sequence of solutions of the Euler-Bernoulli boundary value problem:

$$w_n(x, t) = \exp[-\tau_n t][(\cos \sigma_n t)\Psi_{nR}(x) - (\sin \sigma_n t)\Psi_{nI}(x)] ,$$

where $\tau_1 > \tau_2 > \tau_k > \ldots \to 0$, and $\sigma_n \to \infty$, as before. Each such solution defines a continuous function

$$u_n(t) = S(t) u_n(0) = \begin{pmatrix} w_n(x, t) \\ v_n(x, t) \\ p_n(t) \\ q_n(t) \end{pmatrix}$$

(for $v_n = \partial w_n / \partial t$, $p_n(t) = v_n(1, t)$, $q_n(t) = (\partial v_n / \partial x)(1, t)$)

$$t \to u_n(\cdot) : [0, \infty) \to \mathcal{H}_F ,$$

which is a solution of the evolutionary ODE

$$\frac{du}{dt} = Au \quad \text{in } \mathcal{H}_F .$$

Furthermore, since $\tau_n \searrow 0$, we can assume, after a convenient renumbering and rescaling of $u_n(t)$, that

$$\tfrac{1}{2}\exp[-\tau_n t] \leqslant \|u_n(t)\|_F \leqslant \exp[-\tau_n t] \quad \text{on } 0 \leqslant t < \infty .$$

Then, as in the preceeding lemma,

$$u(t) = \sum_{k=1}^{\infty} \frac{a_k}{k!} u_{n_k}(t) \quad \text{on } 0 \leqslant t < \infty ,$$

for some suitable subsequence $\{u_{n_k}(t)\}$, and constants $0 < a_k \leqslant 1$, is a continuous function

$$t \to u(\cdot) \colon [0, \infty) \to \mathcal{H}_F .$$

Moreover

$$\|u(t_k)\| > \psi(t_k)$$

for some sequence of times $t_k \to \infty$.

It remains to demostrate that $u(t)$ is a solution of the evolutionary ODE, that is, we must prove that

$$u(t) = S(t)u_0$$

where

$$u_0 = \sum_{k=1}^{\infty} \frac{a_k}{k!} u_{n_k}(0) \quad \text{in } \mathcal{H}_F .$$

Consider the continuous curve $S(t)u_0$ in $\mathcal{H}_F$, and the continuous curve $u(t)$ in $\mathcal{H}_F$, on $0 \leqslant t < \infty$. Now the partial sums converge absolutely

$$\sum_{k=1}^{N} \frac{a_k}{k!} u_{n_k}(t) \to u(t) ,$$

uniformly on $0 \leqslant t < \infty$, since $\|u_n(t)\| \leqslant 1$. Also, for each $t \geqslant 0$,

$$S(t)\left(\sum_{k=1}^{N} \frac{a_k}{k!} u_{n_k}(0)\right) = \sum_{k=1}^{N} \frac{a_k}{k!} S(t) u_{n_k}(0) = \sum_{k=1}^{N} \frac{a_k}{k!} u_{n_k}(t) .$$

Hence, for each $t \geqslant 0$, since $\|S(t)\| \leqslant 1$,

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{a_k}{k!} u_{n_k}(t) = S(t)u_0 .$$

Therefore $\sum_{k=1}^{\infty} (a_k/k!)\, u_{n_k}(t)$ converges in $\mathcal{H}_F$ to $S(t)u_0$ and also to $u(t)$. Thus we conclude that

$$u(t) = S(t)u_0 \quad \text{for all } 0 \leqslant t < \infty\,,$$

and the Theorem is proved. □

REMARK. – Each solution $u_n(t)$ is real-analytic on $0 \leqslant t < \infty$, that is, $u_n(t)$ is given by an absolutely convergent power series in t, with coefficients in $\mathcal{H}_F$. Further,

$$u(t) = \sum_{k=1}^{\infty} \frac{a_k}{k!} u_{n_k}(t)$$

converges absolutely uniformly on $0 \leqslant t < \infty$, but the behavior for complex values of t is not apparent.

Moreover the corresponding components

$$u(t) = \begin{pmatrix} w(x, t) \\ v(x, t) \\ p(t) \\ q(t) \end{pmatrix},$$

are not guaranteed to be smooth in x on $0 \leqslant x \leqslant 1$. Certainly $w(x, t)$ lies in $H^2[0, 1]$ for each $t \geqslant 0$, but we cannot be assured that w_{xx}, w_{xxx}, or $w_{xxxx}(x, t)$ are continuous in (x, t), without further investigations on the nature of the convergence of the series for $u(t)$. In particular, it seems likely that one might require better quantitative estimates for $\tau_n = 2\mu_n \nu_n$ given by,

$$\left|\mu_n - \left(\frac{\pi}{2} + n\pi\right)\right| \to 0$$

and

$$0 < \nu_n \leqslant n^{-3/2} \quad \text{for all large integers } n\,.$$

We do not pursue these questions further at this time.

For a nonconstructive proof of an absraction of Theorem 3, see [17].

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