Existence of Bounded Solutions for Non Linear Elliptic Unilateral Problems (*).

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Summary. – *This paper proves the existence of (at least) one solution of the following variational inequality:*

$$(*)\qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \quad u \geqslant \psi \quad \text{a.e. in } \Omega\,, \\ \langle A(u), v - u\rangle + \int\limits_\Omega H(x, u, Du)(v-u) \geqslant 0\,, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \quad v \geqslant \psi \quad \text{a.e. in } \Omega\,. \end{cases}$$

Here A is an operator of Leray-Lions type acting from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ and H grows like $|Du|^p$. The obstacle ψ is a measurable function with values in $\overline{\mathbf{R}}$, the only hypothesis being $\{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v \geqslant \psi$ a.e. in $\Omega\} \neq \emptyset$. This allows ψ to be $-\infty$, recovering the case where (*) is an equation. Finally there is no smoothness assumptions on the data: Ω is a bounded open set in $\mathbf{R}^N$, A and H are defined from Carathéodory functions.

Résumé. – *Dans cet article nous montrons l'existence d'(au moins) une solution de l'inéquation variationnelle*

$$(*)\qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \quad u \geqslant \psi \quad \text{p.p. dans } \Omega\,, \\ \langle A(u), v - u\rangle + \int\limits_\Omega H(x, u, Du)(v-u) \geqslant 0\,, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \quad v \geqslant \psi \quad \text{pp. dans } \Omega\,, \end{cases}$$

où A est un opérateur de type Leray-Lions défini sur $W_0^{1,p}(\Omega)$, à valeurs dans $W^{-1,p'}(\Omega)$ et où la croissance de H est au plus en $|Du|^p$. L'obstacle ψ est une fonction mesurable à valeurs dans $\overline{\mathbf{R}}$, la seule hypothèse étant que le convexe $\{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v \geqslant \psi$ p.p. dans $\Omega\}$ n'est pas vide: ainsi le cas $\psi = -\infty$ (qui correspond au cas où (*) est une équation) est également traité. Enfin il n'y a aucune hypothèse de régularité sur les données: Ω est un ouvert borné de $\mathbf{R}^N$, et A et H sont définis à partir de fonctions de Carathéodory.

Sunto. – *In questo lavoro si prova un risultato di esistenza di soluzioni della disequazione variazionale*

$$(*)\qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \quad u \geqslant \psi \quad \textit{q.o. in } \Omega\,, \\ \langle A(u), v - u\rangle + \int\limits_\Omega H(x, u, Du)(v-u) \geqslant 0\,, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \quad v \geqslant \psi \quad \textit{q.o. in } \Omega\,, \end{cases}$$

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dove A è un operatore del tipo di Leray-Lions definito su $W_0^{1,p}(\Omega)$ e a valori in $W^{-1,p'}(\Omega)$, e H è una funzione di Carathéodory che cresce al più come $|Du|^p$. La sola ipotesi che si fa su ψ è che $\{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega): v \geq \psi$ q.o. in $\Omega\} \neq \emptyset$; ψ è una funzione misurabile a valori in $\overline{\boldsymbol{R}}$: questo permette $\psi = -\infty$ e in tal caso (*) diventa una equazione. In fine, non viene fatta nessuna ipotesi di regolarità sui dati: Ω è un aperto limitato di $\boldsymbol{R}^N$ ed A e H sono definiti a partire da funzioni di Carathéodory.

1. – Introduction.

This paper is concerned with the existence of bounded solutions of unilateral problems involving non linear operators of the form

$$A(v) + H(x, v, Dv) \tag{1.1}$$

in a bounded domain Ω of $\boldsymbol{R}^N$. The principal part A is a differential operator of second order in divergence form of Leray-Lions type acting from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$

$$A(v) = -\operatorname{div}\big(a(x, v, Dv)\big) + a_0(x, v, Dv) \tag{1.2}$$

and the Hamiltonian H grows at most like $|Dv|^p$.

The main difficulty in proving the existence of a solution stems from the fact that $H(x, v, Dv)$ does not define a mapping from $W_0^{1,p}(\Omega)$ into its dual, but from $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ into $L^1(\Omega)$.

The method used here is to define approximate problems, then to obtain a priori estimates for their solutions using suitable test functions and finally to prove a new compactness property in order to pass to the limit.

2. – Setting of the problem and main result.

Let Ω be a bounded domain in $\boldsymbol{R}^N$ and p, p' be real numbers such that $1 < p, p' < +\infty$; $1/p + 1/p' = 1$. Let us consider the non linear elliptic differential operator of second order in divergence form

$$A(v) = -\operatorname{div}\big(a(x, v, Dv)\big) + a_0(x, v, Dv)$$

where a, a_0 are Carathéodory functions (with values in $\boldsymbol{R}^N$ and $\boldsymbol{R}$) satisfying the following conditions for all $s \in \boldsymbol{R}$, $\xi, \xi^* \in \boldsymbol{R}^N$, and for almost every $x \in \Omega$:

$$|a(x, s, \xi)| \leq \beta[k(x) + |s|^{p-1} + |\xi|^{p-1}] \tag{2.1}$$

(2.2) $$|a_0(x, s, \xi)| \leq \beta[k(x) + |s|^{p-1} + |\xi|^{p-1}] ,$$

(2.3) $$[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*] > 0 , \qquad (\xi \neq \xi^*) ,$$

(2.4) $$a(x, s, \xi)\xi \geq \alpha|\xi|^p ,$$

(2.5) $$a_0(x, s, \xi)s \geq \alpha_0|s|^p ,$$

where α, α_0, β are strictly positive constants and $k(x)$ is a given positive function in $L^{p'}(\Omega)$.

Under these hypotheses, A is a bounded continuous pseudomonotone operator of Leray-Lions type from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ (cf. [LLjl], [Ljl]).

Furthermore let $H(x, s, \xi)$ be a Carathéodory function such that, for all $s \in \mathbf{R}$, $\xi \in \mathbf{R}^N$, and for almost every $x \in \Omega$

(2.6) $$|H(x, s, \xi)| \leq c_0 + b(|s|)|\xi|^p ,$$

where c_0 is a positive constant, $b(s)$ is a positive, increasing, continuous function defined on $\mathbf{R}^+$.

Finally let ψ be a measurable function, with values in $\overline{\mathbf{R}}$, such that the closed convex set

$$K = K(\psi) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$$

has a non empty intersection with $L^\infty(\Omega)$, i.e.

(2.7) $$\exists v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \text{such that } v \geq \psi \text{ a.e. in } \Omega .$$

We shall prove the following existence theorem.

THEOREM 1. – There exists u such that

(2.8) $$\begin{cases} u \in K \cap L^\infty(\Omega) , \\ \langle A(u), v - u\rangle + \int_\Omega H(x, u, Du)(v - u) \geq 0 , \\ \forall v \in K \cap L^\infty(\Omega) . \end{cases}$$

REMARK. – Let us remark explicitely that considering the particular case $\psi = -\infty$ Theorem 1 states the existence of a solution of the equation

(2.8bis) $$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \\ A(u) + H(x, u, Du) = 0 \quad \text{in } D'(\Omega) . \end{cases}$$

The proof of Theorem 1 consists in the following steps. We first define approximate variational inequalities. We then prove a priori estimates in $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$

for the solutions of these approximate problems. Finally we prove that the solutions of the approximate variational inequalities are compact in the strong topology of $W_0^{1,p}(\Omega)$, a result which allows to pass to the limit and to obtain the existence result.

This proof does not require any regularity argument, so no smoothness assumptions have been made neither on Ω nor on the functions a, a_0, H, ψ. We could also consider other types of boundary conditions.

Existence results for problems of type (2.8) or (2.8 bis) in bounded or unbounded domains have already been obtained in several papers with additional assumptions such as the smoothness of the data or the linearity of the operator A with respect to Du (cf. e.g. [AC], [BeFMo], [BeF], [DG1], [DG2], [Lpl1], [Lpl2]; see also the references therein).

In [BMP2] we proved the existence of a solution of equation (2.8 bis) in the present setting, under a slightly reinforced hypothesis: the function k in (2.1) was assumed to be in $L^s(\Omega)$, $s > p'$. This hypothesis allowed to prove a regularity result of Meyers' type on the approximate solutions and on the solutions of (2.8 bis) itself, a result which permitted to pass to the limit in the approximate equations. Although this Meyers' type regularity result also holds true in the case of unilateral problems (see [B]) and then allows to use the same proof as in [BMP2], we prefer to give here a direct proof, close to the proof of [BMP1], with the only hypothesis $k \in L^{p'}(\Omega)$.

In order to prove Theorem 1, let us consider the sequence of approximate variational inequalities:

$$(2.9)_\varepsilon \qquad \begin{cases} u_\varepsilon \in K\,, \\ \langle A(u_\varepsilon), v - u_\varepsilon\rangle + \displaystyle\int_\Omega H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)(v - u_\varepsilon)\,dx \geq 0\,, \\ \forall v \in K\,, \end{cases}$$

where

$$(2.10) \qquad H_\varepsilon(x, s, \xi) = \frac{H(x, s, \xi)}{1 + \varepsilon|H(x, s, \xi)|}\,.$$

Note that $|H_\varepsilon| \leq |H|$ and that $|H_\varepsilon| \leq 1/\varepsilon$.

Recall that, since H_ε is bounded, for every fixed $\varepsilon > 0$, there exists at least a solution u_ε of $(2.9)_\varepsilon$ (cf. [LLjl], [KS], [HS], [Bre1], [Ljl]). From the « maximum principle », we know that u_ε belongs to $L^\infty(\Omega)$ (cf. [Bre2]).

3. – A priori estimates.

LEMMA 1. – Let φ_λ be the real functions

$$(3.1) \qquad \varphi_\lambda(t) = t\,e^{\lambda t^2}\,,$$

where λ is a positive real number. If c, d are positive real numbers, and if $\lambda = c^2/4d^2$ the following inequality holds true:

$$d\varphi_\lambda'(t) - c|\varphi_\lambda(t)| \geqslant d/2 , \quad \forall t \in \mathbf{R} . \tag{3.2}$$

PROOF. – For $\lambda \geqslant c^2/4d^2$, we have

$$d\varphi_\lambda'(t) - c|\varphi_\lambda(t)| \geqslant e^{\lambda t^2}(d + 2\lambda\, dt^2 - c|t|) \geqslant e^{\lambda t^2} d/2 \geqslant d/2 .$$

LEMMA 2. – There exists a positive constant c_1 (depending only on the constants c_0, α_0, Sup $\psi(x)$ and p) such that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant c_1 . \tag{3.3}$$

PROOF. – As we already remarked, $u_\varepsilon \in L^\infty(\Omega)$ for each $\varepsilon > 0$. So the function v_ε defined by

$$v_\varepsilon = u_\varepsilon + \varphi_\lambda(z_\varepsilon^-) , \tag{3.4}$$

where

$$\lambda_\varepsilon = \frac{c_\varepsilon^2}{4\alpha^2}, \qquad c_\varepsilon = b(\|u_\varepsilon\|_{L^\infty(\Omega)}) , \qquad z_\varepsilon = u_\varepsilon + \left(\frac{c_0}{\alpha_0}\right)^{1/(p-1)},$$

belongs to K and can be used as a test function in $(2.9)_\varepsilon$, giving

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dz_\varepsilon^- \varphi_{\lambda_\varepsilon}'(z_\varepsilon^-) + \int_\Omega a_0(x, u_\varepsilon, Du_\varepsilon)\varphi_{\lambda_\varepsilon}(z_\varepsilon^-) + \int_\Omega (c_0 + c_\varepsilon|Du_\varepsilon|^p)\, \varphi_{\lambda_\varepsilon}(z_\varepsilon^-) \geqslant 0 ,$$

$$\int_\Omega a(x, u_\varepsilon, Dz_\varepsilon)\, Dz_\varepsilon^- \varphi_{\lambda_\varepsilon}'(z_\varepsilon^-) + \int_\Omega a_0(x, u_\varepsilon, Du_\varepsilon)\varphi_{\lambda_\varepsilon}(z_\varepsilon^-) + \int_\Omega (c_0 + c_\varepsilon|Dz_\varepsilon^-|^p)\, \varphi_{\lambda_\varepsilon}(z_\varepsilon^-) \geqslant 0 ,$$

$$\int_\Omega \{\alpha\varphi_{\lambda_\varepsilon}'(z_\varepsilon^-) - c_\varepsilon \varphi_{\lambda_\varepsilon}(z_\varepsilon^-)\}|Dz_\varepsilon^-|^p \leqslant \int_\Omega [a_0(x, u_\varepsilon, Du_\varepsilon) + c_0]\, \varphi_{\lambda_\varepsilon}(z_\varepsilon^-) . \tag{3.5}$$

Recall that the previous integrals are non zero only in the subset $\{x \in \Omega : u_\varepsilon \leqslant - (c_0/\alpha_0)^{1/(p-1)}\}$. In this subset we have by (2.5),

$$a_0(x, u_\varepsilon, Du_\varepsilon) \leqslant \alpha_0 |u_\varepsilon|^{p-2} u_\varepsilon \leqslant - c_0 .$$

So, by Lemma 1, for $\lambda = \lambda_\varepsilon = c_\varepsilon^2/4\alpha^2$, we obtain $z_\varepsilon^- = 0$, i.e.:

$$u_\varepsilon \geqslant -\left(\frac{c_0}{\alpha_0}\right)^{1/(p-1)} . \tag{3.6}$$

Consider now the test function v_ε defined by

$$v_\varepsilon = u_\varepsilon - \delta_\varepsilon \varphi_{\lambda_\varepsilon}(z_\varepsilon^+) , \tag{3.7}$$

where

$$\lambda_\varepsilon = \frac{c_\varepsilon^2}{4\alpha^2}, \qquad c_\varepsilon = b(\|u_\varepsilon\|_{L^\infty(\Omega)}), \qquad z_\varepsilon = u_\varepsilon - c_2, \qquad c_2 = \sup\left\{\left(\frac{c_0}{\alpha_0}\right)^{1/(p-1)};\ \sup \psi(x)\right\},$$

$$\delta_\varepsilon = \exp\{-\lambda_\varepsilon \|u_\varepsilon\|^2_{L^\infty(\Omega)}\}.$$

Note that ψ is bounded from above, since $K \cap L^\infty(\Omega) \neq \emptyset$. It is easy to see that $0 \leqslant \delta_\varepsilon \varphi_{\lambda_\varepsilon}(z_\varepsilon^+) \leqslant z_\varepsilon^+$, and that $v_\varepsilon \in K$. So $(2.9)_\varepsilon$ yields

$$\delta_\varepsilon \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dz_\varepsilon^+ \varphi'_{\lambda_\varepsilon}(z_\varepsilon^+) + \delta_\varepsilon \int_\Omega a_0(x, u_\varepsilon, Du_\varepsilon) \varphi_{\lambda_\varepsilon}(z_\varepsilon^+) \leqslant \delta_\varepsilon \int_\Omega (c_0 + c_\varepsilon |Du_\varepsilon|^p)\, \varphi_{\lambda_\varepsilon}(z_\varepsilon^+),$$

which implies

$$\int_\Omega \{\alpha \varphi'_{\lambda_\varepsilon}(z_\varepsilon^+) - c_\varepsilon \varphi_{\lambda_\varepsilon}(z_\varepsilon^+)\} |Dz_\varepsilon^+|^p + \int_\Omega [a_0(x, u_\varepsilon, Du_\varepsilon) - c_0] \varphi_{\lambda_\varepsilon}(z_\varepsilon^+) \leqslant 0 .$$

Recall that the previous integrals are non zero only in the subset $\{x \in \Omega : u_\varepsilon \geqslant c_2\}$. In this subset we have, by (2.5),

$$a_0(x, u_\varepsilon, Du_\varepsilon) \geqslant \alpha_0 |u_\varepsilon|^{p-2} u_\varepsilon \geqslant \alpha_0 (c_2)^{p-1} \geqslant c_0 .$$

So, by Lemma 1, $z_\varepsilon^+ = 0$, i.e.:

$$u_\varepsilon \leqslant c_2 . \tag{3.8}$$

The inequalities (3.6) and (3.8) give the desired estimate (3.3).

LEMMA 3. – There exists a positive constant c_3 (depending only on the data in the hypotheses) such that

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \leqslant c_3 . \tag{3.9}$$

PROOF. – Let us denote $c_4 = b(c_1)$, where c_1 is defined in (3.3). For w given in $K \cap L^\infty(\Omega)$ (recall that $K \cap L^\infty(\Omega) \neq \emptyset$), let us define v_ε by

$$v_\varepsilon = [1 - \delta \exp\{\lambda |u_\varepsilon - w|^2\}] u_\varepsilon + \delta \exp\{\lambda |u_\varepsilon - w|^2\} w , \tag{3.10}$$

where

$$\lambda = \frac{c_4^2}{4\alpha^2}, \qquad \delta = \exp - \{\lambda [c_1 + \|w\|_{L^\infty(\Omega)}]^2\} .$$

Using v_ε, which belongs to K, as test function in $(2.9)_\varepsilon$ we obtain, since $v_\varepsilon - u_\varepsilon = - \delta\varphi_\lambda(u_\varepsilon - w)$:

$$- \delta\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)D(u_\varepsilon - w)\varphi'_\lambda(u_\varepsilon - w) - \delta\int_\Omega a_0(x, u_\varepsilon, Du_\varepsilon)\varphi_\lambda(u_\varepsilon - w) - \delta\int_\Omega H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\varphi_\lambda(u_\varepsilon - w) \geqslant 0 ,$$

$$\alpha\int_\Omega |Du_\varepsilon|^p\varphi'_\lambda(u_\varepsilon - w) \leqslant \int_\Omega |a_0(x, u_\varepsilon, Du_\varepsilon)||\varphi_\lambda(u_\varepsilon - w)| + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dw\,\varphi'_\lambda(u_\varepsilon - w) + \int_\Omega (c_0 + c_4|Du_\varepsilon|^p)|\varphi_\lambda(u_\varepsilon - w)| .$$

By Lemma 1, and the choice of λ, we have

$$(\alpha/2)\int_\Omega |Du_\varepsilon|^p \leqslant \int_\Omega \{|a(x, u_\varepsilon, Du_\varepsilon)| + c_0\}|\varphi_\lambda(u_\varepsilon - w)| + \int_\Omega |a(x, u_\varepsilon, Du_\varepsilon)||Dw||\varphi'_\lambda(u_\varepsilon - w)| .$$

The estimate (3.9) follows from the hypotheses (2.1), (2.2) and from the $L^\infty(\Omega)$ bound (3.3).

REMARK. – In the previous proof, we used the $L^\infty(\Omega)$ bound (3.3) of u_ε and the hypotheses (2.1), (2.2), (2.4), (2.6). We did not use explicitely hypothesis (2.5), but only its consequence, the $L^\infty(\Omega)$ bound (3.3). This means that Lemma 3 holds true whenever one can obtain an estimate in $L^\infty(\Omega)$ independently of the way this estimate obtained. The same remark applies to Lemma 4 below.

4. – Compactness of approximate solutions and proof of existence.

From Lemma 2 and 3 there exists $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and some subsequence (still denoted by u_ε) such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^\infty(\Omega) \text{ weak } * , \tag{4.1}$$

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega , \tag{4.2}$$

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \text{ weak} . \tag{4.3}$$

LEMMA 4. – The sequence u_ε is relatively compact for the strong topology of $W_0^{1,p}(\Omega)$.

PROOF. – We choose as test function in $(2.9)_\varepsilon$ the v_ε defined in (3.10), with $w = u$.

Note that $u \in K \cap L^\infty(\Omega)$. We have

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) D(u_\varepsilon - u)\varphi'_\lambda(u_\varepsilon - u) + \int_\Omega a_0(x, u_\varepsilon, Du_\varepsilon)\varphi_\lambda(u_\varepsilon - u) \leqslant \\ \leqslant \int_\Omega (c_0 + c_4|Du_\varepsilon|^p)|\varphi_\lambda(u_\varepsilon - u)|\,,$$

$$\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]D(u_\varepsilon - u)\varphi'_\lambda(u_\varepsilon - u) \leqslant \\ \leqslant -\int_\Omega a(x, u_\varepsilon, Du)D(u_\varepsilon - u)\varphi'_\lambda(u_\varepsilon - u) + \\ + \int_\Omega [|a_0(x, u_\varepsilon, Du_\varepsilon)| + c_0]|\varphi_\lambda(u_\varepsilon - u)| + (c_4/\alpha)\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon|\varphi_\lambda(u_\varepsilon - u)|\,,$$

and then

$$(4.4)\quad \begin{cases} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]D(u_\varepsilon - u)\{\varphi'_\lambda(u_\varepsilon - u) - (c_4/\alpha)|\varphi_\lambda(u_\varepsilon - u)|\} \leqslant \\ \leqslant -\int_\Omega a(x, u_\varepsilon, Du)D(u_\varepsilon - u)\varphi'_\lambda(u_\varepsilon - u) + (c_4/\alpha)\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Du|\varphi_\lambda(u_\varepsilon - u)| + \\ + (c_4/\alpha)\int_\Omega a(x, u_\varepsilon, Du)D(u_\varepsilon - u)|\varphi_\lambda(u_\varepsilon - u)| + \int_\Omega [|a_0(x, u_\varepsilon, Du_\varepsilon) + c_0]|\varphi_\lambda(u_\varepsilon - u)|\,. \end{cases}$$

Because of the choice of $\lambda = c_4^2/(4\alpha^2)$ and of Lemma 1, the left hand side is greater than

$$\tfrac{1}{2}\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]D(u_\varepsilon - u)\,;$$

using (4.1), (4.2), (4.3), (2.1), (2.2) and Lebesgue's dominated convergence Theorem, we see that the right hand side converges to zero. So inequality (4.4) gives

$$(4.5)\qquad \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]D(u_\varepsilon - u) \to 0\,.$$

The compactness result of Lemma 4 is then a consequence of (4.5) and of the following Lemma 5 which is proved in [Bro], p. 27. We give here a simple proof of it for sake of completeness. We use the method of [LLjl], p. 104, and we complete it to obtain the strong convergence of u_ε, since hypothesis (2.4) is slightly stonger than Leray-Lions' original coercivity assumption.

Lemma 5. – Assume that

$$(4.6)\qquad u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \text{ weak and a.e. in } \Omega\,,$$

and

$$\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]D(u_\varepsilon - u) \to 0 . \tag{4.7}$$

Then

$$u_\varepsilon \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ strongly} . \tag{4.8}$$

PROOF. – Let D_ε be defined by

$$D_\varepsilon = [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du)]D(u_\varepsilon - u) .$$

Then by (2.3) D_ε is a positive function and (4.7) implies

$$D_\varepsilon \to 0 \quad \text{in } L^1(\Omega) \text{ strongly} .$$

Extracting a subsequence (still denoted by u_ε), we have

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega ,$$
$$D_\varepsilon \to 0 \quad \text{a.e. in } \Omega .$$

Then there exists a subset Z of Ω, of zero measure, such that, for $x \in \Omega - Z$,

$$|u(x)| < \infty , \qquad |Du(x)| < \infty , \qquad |k(x)| < \infty ,$$

and

$$u_\varepsilon(x) \to u(x) , \qquad D_\varepsilon(x) \to 0 .$$

Defining

$$\xi_\varepsilon = Du_\varepsilon(x) , \qquad \xi = Du(x) ,$$

we have

$$D_\varepsilon(x) = \big[a\big(x, u_\varepsilon(x), \xi_\varepsilon\big) - a\big(x, u_\varepsilon(x), \xi\big)\big](\xi_\varepsilon - \xi) ,$$

and using (2.4) and (2.1)

$$D_\varepsilon(x) \geqslant \alpha|\xi_\varepsilon|^p - c(x)[1 + |\xi_\varepsilon|^{p-1} + |\xi_\varepsilon|] ,$$

where $c(x)$ is a constant which depends on x, but does not depend on ε. Thus, $|\xi_\varepsilon|$ is bounded uniformly with respect to ε, since $D_\varepsilon(x) \to 0$.

Let ξ^* be a cluster point of ξ_ε. We have $|\xi^*| < \infty$ and

$$\big[a\big(x, u(x), \xi^*\big) - a\big(x, u(x), \xi\big)\big](\xi^* - \xi) = 0 ,$$

because of the continuity of a with respect to s and ξ. Now (2.3) implies that $\xi^* = \xi$. The uniqueness of the cluster point means that for the whole sequence

$$Du_\varepsilon(x) \to Du(x)\,, \qquad \forall x \in \Omega - Z\,,$$

i.e.

$$Du_\varepsilon \to Du \qquad \text{a.e. in } \Omega\,.$$

Remark that

$$a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon \geqslant 0\,, \tag{4.9}$$

$$a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon \to a(x, u, Du)\, Du \qquad \text{a.e. in } \Omega\,. \tag{4.10}$$

On the other hand, using (4.6), (4.7), Vitali's Theorem and $a(x, u_\varepsilon, Du_\varepsilon) \to a(x, u, Du)$ in $(L^{p'}(\Omega))^N$ weak and a.e. in Ω, we obtain:

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon \to \int_\Omega a(x, u, Du)\, Du\,. \tag{4.11}$$

We set

$$y_\varepsilon = a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon\,, \qquad y = a(x, u, Du)\, Du\,.$$

So, from (4.9), (4.10), (4.11),

$$y_\varepsilon \geqslant 0\,, \qquad y_\varepsilon \to y \qquad \text{a.e. in } \Omega\,, \qquad y \in L^1(\Omega), \qquad \int_\Omega y_\varepsilon \to \int_\Omega y\,.$$

Now

$$\int_\Omega |y_\varepsilon - y| = 2 \int_{0 \leqslant y_\varepsilon \leqslant y} (y - y_\varepsilon) + \int_\Omega (y_\varepsilon - y)\,.$$

Using Lebesgue's dominated convergence Theorem, we obtain

$$y_\varepsilon \to y \qquad \text{in } L^1(\Omega)\,,$$

i.e.

$$a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon \to a(x, u, Du)\, Du \qquad \text{in } L^1(\Omega)\,. \tag{4.12}$$

Now from (4.12) and hypothesis (2.4) we deduce, using Vitali's Theorem

$$Du_\varepsilon \to Du \qquad \text{in } (L^p(\Omega))^N \text{ strong}\,. \tag{4.13}$$

Proof of theorem 1. – From Lemmas 2, 3, 4, we know that (for some subsequence)

$$u_\varepsilon \to u \qquad W_0^{1,p}(\Omega) \text{ strong}\,, \tag{4.14}$$

$$u_\varepsilon \rightharpoonup u \qquad L^\infty(\Omega) \text{ weak } *\,. \tag{4.15}$$

Now Vitali's Theorem and the inequality

$$|H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \leqslant |H(x, u_\varepsilon, Du_\varepsilon)| \leqslant c_0 + c_4 |Du_\varepsilon|^p ,$$

show that

$$H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to H(x, u, Du) \quad \text{in } L^1(\Omega) . \tag{4.16}$$

So it is easy to pass to the limit in $(2.9)_\varepsilon$, for any given $v \in K \cap L^\infty(\Omega)$ and to obtain that u is a solution of problem (2.8).

5. – Extension.

In this section we present the problem (2.8) in a different framework, replacing hypothesis (2.5) by the existence of a sub- and a super-solution as in [BMP2]. On the other hand, we will assume that:

$$a_0(x, s, \xi) = 0 , \tag{5.1}$$

and that (2.6) is replaced by

$$|H(x, s, \xi)| \leqslant b(|s|)(1 + |\xi|^p) . \tag{5.2}$$

Definition. – The function φ is a sub-solution of the unilateral problem (2.8) if

$$\begin{cases} \varphi \in W^{1,\infty}(\Omega) , \quad \varphi \leqslant 0 \quad \text{on } \partial\Omega , \\ \langle A(\varphi), (\varphi - v)^+ \rangle + \int_\Omega H(x, \varphi, D\varphi)(\varphi - v)^+ \leqslant 0 \\ \text{for all } v \in K . \end{cases} \tag{5.3}$$

The function θ is a super-solution of the unilateral problem (2.8) if

$$\begin{cases} \theta \in W^{1,\infty}(\Omega) , \quad \theta \geqslant 0 \quad \text{on } \partial\Omega , \quad \theta \geqslant \psi \quad \text{on } \Omega , \\ \langle A(\theta), v \rangle + \int_\Omega H(x, \theta, D\theta) v \geqslant 0 , \\ \text{for all} \quad v \in W_0^{1,p}(\Omega) , \quad v \geqslant 0 \quad \text{on } \Omega . \end{cases} \tag{5.4}$$

Theorem 2. – Assume that (2.1), (2.2), (2.3), (2.4), (2.7), (5.1) and (5.2) hold true and that there exist a sub-solution φ and a super-solution θ of problem (2.8), with $\varphi \leqslant \theta$ a.e. in Ω. Then there exists a solution u of (2.8) such that $\varphi \leqslant u \leqslant \theta$ a.e. in Ω.

REMARK. – If $\psi = -\infty$, the above definition coincides with the classical definition of the sub-and super-solution for an equation. Then, Theorem 2 gives essentially the same result as Theorem 2.1 in [BMP2], the difference being that here k just belongs to $L^{p'}(\Omega)$.

Note that definition (5.4) of a super-solution θ for the unilateral problem (2.8) is equivalent to:

$$\begin{cases} \theta \in W^{1,\infty}(\Omega)\,, \quad \theta \geqslant 0 \quad \text{on } \partial\Omega\,, \\ \theta \geqslant \psi \quad \text{a.e. in } \Omega \quad \textit{and} \quad A(\theta) + H(x, \theta, D\theta) \geqslant 0 \quad \text{in } D'(\Omega)\,. \end{cases}$$

In contrast, when the obstacle ψ and the function φ are smooth, the above definition of a sub-solution heuristically coincides with

$$\begin{cases} \varphi \leqslant 0 \quad \text{on } \partial\Omega\,, \\ \textit{either} \quad \varphi \leqslant \psi \quad \textit{or} \quad A(\varphi) + H(x, \varphi, D\varphi) \leqslant 0 \quad \text{a.e. in } \Omega\,. \end{cases}$$

OUTLINE OF THE PROOF. – We first modify the problem, using the same truncation as in [BMP2], section 3.1. Let us define

$$\tilde{a}(x, s, \xi) = \begin{cases} a(x, \varphi(x), \xi) & \text{if } s \leqslant \varphi(x) \\ a(x, s, \xi) & \text{if } \varphi(x) < s < \theta(x) \\ a(x, \theta(x), \xi) & \text{if } s \geqslant \theta(x)\,, \end{cases}$$

$$\tilde{H}(x, s, \xi) = \begin{cases} H(x, \varphi(x), D\varphi(x)) & \text{if } s \leqslant \varphi(x)\,, \\ H(x, s, \xi) & \text{if } \varphi(x) < s < \theta(x)\,, \\ H(x, \theta(x), D\theta(x)) & \text{if } s \geqslant \theta(x)\,. \end{cases}$$

We set

$$\tilde{A}(v) = -\operatorname{div}\left(\tilde{a}(x, v, Dv)\right)\,.$$

Observe that for any $v \in W^{1,p}(\Omega)$

$$|\tilde{H}(x, v, Dv)| \leqslant c_5(1 + |Dv|^p)\,,$$

where c_5 is a positive constant which depends on φ and θ.

We consider the auxiliary problem

$$\text{(5.5)} \qquad \begin{cases} z \in K \cap L^\infty(\Omega)\,, \\ \langle \tilde{A}(z), v - z\rangle + \int_\Omega \tilde{H}(x, z, Dz)(v - z) \geqslant 0\,, \\ \forall v \in K \cap L^\infty(\Omega)\,. \end{cases}$$

With the same proof as in [BMP2], Proposition 3.3, we can prove that every solution z of (5.5) (if it exists) satisfies

$$\varphi \leqslant z \leqslant \theta \quad \text{a.e. in } \Omega , \tag{5.6}$$

and actually is a solution of (2.8). Next we define

$$\tilde{H}_\varepsilon(x, v, Dv) = \begin{cases} \tilde{H}(x, v, Dv) & \text{if } |\tilde{H}(x, v, Dv)| \leqslant \dfrac{1}{\varepsilon} \\ \dfrac{1}{\varepsilon} \dfrac{\tilde{H}(x, v, Dv)}{|\tilde{H}(x, v, Dv)|} & \text{if } |\tilde{H}(x, v, Dv)| > \dfrac{1}{\varepsilon} . \end{cases}$$

Note that $|\tilde{H}_\varepsilon| \leqslant |\tilde{H}|$ and $|\tilde{H}_\varepsilon| \leqslant 1/\varepsilon$. Since $\tilde{H}_\varepsilon$ is bounded, there exists a solution z_ε of the problem

$$(5.7)_\varepsilon \qquad \begin{cases} z_\varepsilon \in K \cap L^\infty(\Omega) , \\ \langle \tilde{A}(z_\varepsilon), v - z_\varepsilon \rangle + \displaystyle\int_\Omega \tilde{H}_\varepsilon(x, z_\varepsilon, Dz_\varepsilon)(v - z_\varepsilon) \geqslant 0 , \\ \forall v \in K . \end{cases}$$

Note that the existence results in the litterature do not apply directly to $(5.7)_\varepsilon$, since the function $\tilde{H}_\varepsilon$ are not Carathéodory functions, but $v \to \tilde{H}_\varepsilon(x, v, Dv)$ is a continuous mapping from $W^{1,p}(\Omega)$ into $L^q(\Omega)$ for all finite q, and the usual proofs adapt immediately. It is easy to see that for ε small enough $\big(1/\varepsilon \geqslant \sup\{|H(x, \varphi(x), D\varphi(x))|, |H(x, \theta(x), D\theta(x))|\}\big)$, we have

$$\tilde{H}_\varepsilon(x, \varphi, D\varphi) = \tilde{H}(x, \varphi, D\varphi) = H(x, \varphi, D\varphi) ,$$
$$\tilde{H}_\varepsilon(x, \theta, D\theta) = \tilde{H}(x, \theta, D\theta) = H(x, \theta, D\theta) .$$

Thus φ and θ are sub- and super-solution of the unilateral problem $(5.7)_\varepsilon$. It can be proved, as in [BMP2], that every solution of $(5.7)_\varepsilon$ satisfies $\varphi \leqslant z_\varepsilon \leqslant \theta$ a.e. in Ω. This implies

$$\|z_\varepsilon\|_{L^\infty(\Omega)} \leqslant c_6$$

for a suitable constant c_6.

The end of the proof of Theorem 2 exactly follows the proof of Theorem 1, proving first that z_ε is bounded in $W_0^{1,p}(\Omega)$ and then that z_ε is relatively compact in this space (see the Remark at the end of section 3). Passing to the limit in $(5.7)_\varepsilon$ proves the existence of a solution of (5.5), thus of (2.8), in the framework of the present section.

Added in proof. – In the recent paper [RT] the L^∞ bound is derived by using a rearrangement technique if H satisfies a sign condition.

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