Divisors of Finite Character (*).

KARL EGIL AUBERT (Oslo, Norway)

« Therefore, when one widens the realm of elements to that of ideals in a given ring, one sometimes gains and sometimes loses. One gets the impression that, generally speaking, the truth lies halfway: if the domain of integers in many cases is too narrow, the domain of ideals is in most cases too wide. »

HERMANN WEYL (in [45], p. 38).

Summary. – *The present paper purports to show that divisors of finite character—also called t-ideals—are the natural building blocks of the general theory of divisibility. Divisors of finite character are here applied to a variety of different arithmetical topics as well as to sectional and functional representation of ordered groups.*

1. – Introduction (1).

In its most general and purest form, the study of the notion of divisibility appears as a strictly multiplicative theory. In spite of this, the majority of the abstract investigations concerning the notion of divisibility have been carried out within the setting of integral domains. The tradition of studying divisibility properties in rings or fields rather than in monoids or groups (2) goes back to the early days of algebraic number theory. Dedekinds ideal concept is a ring-theoretic concept and not a purely multiplicative one (although it turned out later that in the classical case of algebraic integers his ideals *may* be given a purely multiplicative interpretation as « divisorial ideals »). Thus, a somewhat blurring and irrelevant additive ingredient was brought into the general theory of divisibility right from the start.

The true multiplicative liberation came with LORENZEN's thesis [33] in 1939. It is the purpose of the present paper to try to revive and continue some of the work

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(2) The groups, rings and monoids considered in this paper are all commutative with cancellation laws.

of Lorenzen. It seems to us that although his 1939 paper is widely cited it is rather poorly understood. Papers (and also several books such as [12], [18], [19] and [32]) which deal with divisors and multiplicative ideal theory are still being published without taking account of Lorenzen's most basic ideas. Their treatment of several topics is decidedly inferior to what can be extracted (admittedly, sometimes with pain) from Lorenzen's work. Only JAFFARD's monograph [21] seems to us to do full justice to Lorenzen's ideas. This is really a very fine book, but it is written in a style and uses a terminology which may have prevented many from reading it who otherwise could have been attracted by its rich content.

We shall let the present paper revolve around the concept of a divisorial ideal of finite character—called t-ideals for short. Our main objective will be to present some of the evidence which points in favour of t-ideals as the building blocks of a general arithmetic.

In fact, it is not far fetched to say that the t-ideals represent « the truth that lies halfway » which is alluded to in the above quotation of Hermann Weyl. This makes it somewhat hard to understand how the v-ideals (divisors in the terminology of BOURBAKI [12]) or even the ordinary Dedekind ideals (called d-ideals in the sequel) have survived in many multiplicative contexts where the t-ideals turn out to be superior.

In the two last paragraphs of the present paper we shall also show that the notion of a prime t-ideal and that of a t-valuation seem to provide the best foundation for a coherent theory of both sectional and functional representation of ordered groups. This point of view will bring about ameliorations and precisions of earlier work of KEIMEL [25] and FLEISCHER [17] on this topic.

2. – Divisors and t-ideals.

All ordered groups considered in this paper are supposed to be abelian and directed (filtered). The monoid G^+ of positive (integral) elements of the directed group G will be denoted by D. Conversely any monoid D with cancellation law gives rise to a directed factor group G/U—called the *divisibility group* of D—where G is the quotient group of D and U is the group of units in D. Equivalently we may regard the divisibility group as the group of fractional principal ideals ordered by inclusion.

A directed group G is said to be *factorial* if it is isomorphic to an ordered direct sum of copies of $\mathbb{Z}$ (a free abelian group with pointwise order). Such a factorial group is written $\mathbb{Z}^{(I)}$ for some set I and is interpreted as the set of all functions from I to $\mathbb{Z}$, zero outside of a finite set—with pointwise addition and ordering. If G is order-isomorphic to a subgroup of a factorial group we shall say that G is a *préfactorial group*. A unique factorization domain (respectively a Krull domain) is an integral domain whose divisibility group is factorial (respectively préfactorial).

The situation of a préfactorial group exhibits the original arithmetical content of the concept of a «divisor» and a «prime divisor». The divisors which are adjoined in order to achieve unique factorization are conceived of as finite products (or sums) of the canonical generators (the prime divisors) of the free abelian group $\mathbb{Z}^{(I)}$. It is reasonable, however, to restrict the use of the term «divisor» somewhat further. For we are not really interested in «unnecessarily big» extensions with no definite ties between G and $\mathbb{Z}^{(I)}$. It turns out that for a préfactorial group G we can always choose $\boldsymbol{D} = \mathbb{Z}^{(I)}$ in a unique minimal way (i.e. such that $\boldsymbol{D}$ is contained in all factorial groups containing G as an ordered subgroup)—namely as the group of fractional t-ideals of G. Thus the t-ideals—which we are now going to introduce—appear as the true arithmetical divisors.

Let A denote a bounded subset of the directed group G (i.e. there exists an element $g \in G$ such that $gA \subset G^+ = D$). The set

$$A_v = \bigcap_{A \subset (a)} (a)$$

or equivalently $A_v = D:(D:A)$ is then the *divisorial ideal* or the *v-ideal* generated by A. We define the *t-ideal* generated by A as the set-theoretic union of all the v-ideals generated by finite subsets of A:

$$A_t = \bigcup_{\substack{N \subset A \\ N \text{ finite}}} N_v .$$

An important technical difference between v-ideals and t-ideals is given by the fact that the t-generation is of finite character whereas the v-generation is not. The t-system forms the unique coarsest Lorenzen system in G. (As a general source for definitions and results on ideal systems the reader is referred to [2], [3], [21] or [33]. As to the notion of a Lorenzen system (r-system) as opposed to the general notion of an ideal system (x-system) the reader is referred to [3], pp. 523-524.)

If G is a *GCD*-group (3) (= lattice ordered group) with the g.c.d.-operation denoted by $\wedge$, the definition of a t-ideal assumes a more appealing form as the conjunction of the two properties

1) $DA_t \subset A_t$;

2) $a, b \in A_t \Rightarrow a \wedge b \in A_t$.

As opposed to ordinary d-ideals, the presence of a g.c.d. for two (or a finite number of) elements is measured faithfully in terms of t-ideals: Two elements a

(3) In our arithmetical context we prefer the more suggestive term of a *GCD*-goup to that of a lattice ordered group or l-group. This also achieves a uniform terminology which is in harmony with the already established notion of a *GCD*-domain as well as the notion of a *GCD*-functor to be introduced in the next section.

and b have a g.c.d. if and only if the t-ideal generated by a and b is principal. Otherwise expressed: The divisibility group of a monoid D is a GCD-group if and only if D is *t-Bezout* (every finitely generated t-ideal is principal). Already at this elementary level the advantage of t-ideals over d-ideals is hence clear (also apart from the fact that d-ideals only make sense in the case of divisibility groups of integral domains). For a d-ideal (a, b) may fail to be principal also in case a and b have a g.c.d. For a d-ideal (a, b) to be principal it is not only required that a and b have a g.c.d., but that this g.c.d. be a linear combination of a and b. Thus d-ideals bring in an extraneous additive condition which is alien to the purely multiplicative situation at hand.

3. – Lorenzen groups.

We shall now enter a subject which, in spite of being almost entirely neglected, seems to us to form the deepest and most interesting part of the general theory of divisibility.

Exploiting the original ideas of KRONECKER, PRÜFER and especially KRULL defined and used the so-called *Kronecker function rings* in order to study the arithmetic of integral domains. The main virtue of the extension process which leads from an integrally closed domain R to its Kronecker function ring is the fact that the latter is a Bezout domain (finitely generated d-ideals are principal) and hence provide g.c.d.'s. This enables us to get a better grasp of the valuation overrings of R, establishing in particular that these are in one-to-one correspondence with the prime ideals of the corresponding function ring.

The subject of the Kronecker function rings was generalized, clarified and simplified by Lorenzen when he defined the purely multiplicative object of a « Lorenzen group », freeing the initial construction of a Kronecker function ring from any intervention of an additive operation as well as from the Kroneckerian scheme of adjunction of indeterminates. In spite of this face lift, however, the Kronecker function rings have also in their new disguise as Lorenzen groups remained a neglected and poorly understood area. The following presentation of this subject if offered in the hope of contributing to a better understanding of Lorenzen's ideas. We shall do this by stressing functorial properties as well as the universal role which is played by the t-system in this connection.

The main way of motivating the introduction of Lorenzen groups is via the old problem of providing g.c.d.'s by a suitable extension process.

Let G be a directed group equipped with a (fractional) Lorenzen system x. We suppose that G is (*integrally*) *x-closed* in the sense of [2] or [21], i.e. that $A_x : A_x \subset G^+$ for any finite set $A \subset G$. To the given x-system we can associate another fractional ideal system in G which is denoted by x_a and which is determined by

$$A_{x_a} = \{c | cN_x \subset A_x \circ N_x \text{ for some finite } N \subset G\}$$

whenever A is a finite subset of G. The x_a-ideal generated by a (general) bounded subset B of G is then equal to the set-theoretic union of all the x_a-ideals generated by finite subsets of B.

The crucial property of the x_a-system is that the monoid of finitely generated x_a-ideals (under x_a-multiplication) satisfies the cancellation law and hence possesses a group of quotients $\Lambda_x(G) = \Lambda_x$ (see [21], pp. 41-42 for a proof). This group is made into an ordered group by putting $A_{x_a}/B_{x_a} \in \Lambda_x^+$ whenever $A_{x_a} \subset B_{x_a}$ and is as such called the *Lorenzen x-group* associated to G. The main property of the Lorenzen x-group of G is that it is a GCD-group which contains G as an ordered subgroup. It provides the g.c.d.'s which may be missing in G and when the x-system is suitably chosen it does this in the most economical way.

4. – The *GCD*-functor.

Let I denote the category of integrally closed directed groups. An object in this category is a directed (abelian) group G equipped with a Lorenzen system x such that G is (integrally) x-closed. A morphism in I is a morphism of ideal systems $\varphi\colon (G, x) \to (H, y)$ where (G, x) and $(H, y) \in I$ (see [3], p. 523 for the definition of a morphism of ideal systems).

The category I contains in particular two distinguished full subcategories, corresponding to the cases $x = s$ defined by $A_s = G^+A$ and $x = t$ respectively: The category S of all s-closed (semi-closed) directed groups with orderpreserving group homomorphisms as morphisms and the category GCD of all GCD-groups with homomorphisms of GCD-groups as morphisms. The proof of these two facts is simple and we shall content ourselves by treating the case which interests us most. (For an explanation of the term *shadow functor* we refer the reader to [5], p. 39):

LEMMA. – *The t-shadow functor I_t provides a full embedding of the category of GCD-groups into the category of integrally closed directed groups.*

PROOF. – Obviously, any GCD-group is t-closed. It hence suffices to show that the natural map

$$\mathrm{Hom}_{GCD}(G_1, G_2) \to \mathrm{Hom}_I\big((G_1, t), (G_2, t)\big)$$

is a surjection, i.e. any (t, t)-morphism of GCD-groups is really a homomorphism of GCD-groups. First of all, any morphism $\varphi\colon (G, x) \to (H, y)$ between two Lorenzen systems (and hence in particular any (t, t)-morphism) is order preserving. For $a \geqslant b$ is equivalent to $a \in (b)_x$ which implies $\varphi(a) \in \varphi\big((b)_x\big) \subset \big(\varphi(b)\big)_y$ which in turn is equivalent to $\varphi(a) \geqslant \varphi(b)$. On the other hand $\varphi\big((a, b)_t\big) \subset \big(\varphi(a), \varphi(b)\big)_t$ reduces to $\varphi(a\wedge b) \geqslant \varphi(a)\wedge\varphi(b)$. Since $\varphi(a\wedge b) \leqslant \varphi(a)\wedge\varphi(b)$ is a consequence of φ being order preserving, it follows that $\varphi(a\wedge b) = \varphi(a)\wedge\varphi(b)$ and φ is a homomorphism of GCD-groups.

The following theorem could appropriately be termed « *Main theorem of divisibility theory* ». It shows how the Lorenzen groups act as universal objects with respect to the basic arithmetical completion process of providing g.c.d.'s.

THEOREM 1. – *The passage from an x-closed group (G, x) to its Lorenzen group $\Lambda_x(G)$ defines a faithful functor from the category I onto the category GCD such that GCD appears as a full reflective subcategory of I—i.e. the indicated functor is the left adjoint of the t-shadow functor.*

We shall call the functor alluded to here for the *GCD-functor* and denote it by Λ.

PROOF. – So far we have only defined how the functor Λ acts on the objects of I. If $\varphi\colon (G, x) \to (H, y)$ is a morphism in I we define $\Lambda(\varphi) = \Phi$ by putting

$$\Phi\left(\frac{A_{x_a}}{B_{x_a}}\right) = \frac{(\varphi(A))_{y_a}}{(\varphi(B))_{y_a}}. \tag{4.1}$$

When we identify G with its group of principal ideals it is clear that the restriction of Φ to G is just φ, showing that Λ is faithful. To verify ahat Φ is a homomorphism of GCD-groups is routine and we content ourselves by showing that Φ is a lattice homomorphism—the proof that Φ is a group homomorphism being similar. We can assume that the two given quotients have the same denominator and then we get

$$\Phi\left(\frac{A_{x_a}}{C_{x_a}} \wedge \frac{B_{x_a}}{C_{x_a}}\right) = \Phi\left(\frac{A_{x_a} + B_{x_a}}{C_{x_a}}\right) = \frac{(\varphi(A \cup B))_{y_a}}{(\varphi(C))_{y_a}} =$$
$$= \frac{(\varphi(A) \cup \varphi(B))_{y_a}}{(\varphi(C))_{y_a}} = \frac{(\varphi(A))_{y_a}}{(\varphi(C))_{y_a}} \wedge \frac{(\varphi(B))_{y_a}}{(\varphi(C))_{y_a}} = \Phi\left(\frac{A_{x_a}}{C_{x_a}}\right) \wedge \Phi\left(\frac{B_{x_a}}{C_{x_a}}\right)$$

where A and B are finite subsets of G. That Λ is compatible with composition is obvious. We have a commutative diagram

$$\begin{array}{ccc} \Lambda_x(G) & \xrightarrow{\ \Phi\ } & \Lambda_y(H) \\ \uparrow{\scriptstyle \tau_x} & & \uparrow{\scriptstyle \tau_y} \\ (G, x) & \xrightarrow{\ \varphi\ } & (H, y) \end{array} \tag{4.2}$$

where the natural inclusion maps τ_x and τ_y are an (x_a, t)-morphism and a (y_a, t)-morphism, respectively. Since every finitely generated t-ideal in $\Lambda_x(G)$ is principal it suffices to show that $\tau_x^{-1}((c)_t)$ is an x_a-ideal in G whenever $c \in \Lambda_x(G)$. If $b_1, \ldots, b_n \in \tau_x^{-1}((c)_t)$ and $b \in (b_1, \ldots, b_n)_{x_a}$, $\tau_x(b)$ may be identified with the principal ideal it generates in G and hence

$$\tau_x(b) \geqslant (b_1, \ldots, b_n)_{x_a} = \tau_x(b_1) \wedge \ldots \wedge \tau_x(b_n) \geqslant c$$

with respect to the order relation which is defined in $\Lambda_x(G)$. This entails $b \in \tau_x^{-1}((c)_t)$ as required. (Since the x_a-system is coarser than the x-system this shows in particular that τ_x is an (x, t)-morphism.)

By letting H be a GCD-group and putting $y = t$, the diagram 4.2 gives rise to the following one.

(4.3)
$$\begin{array}{ccc} & (\Lambda_x(G), t) & \\ {\scriptstyle\tau_x}\nearrow & & \searrow{\scriptstyle\Phi} \\ (G, x) & \xrightarrow{\ \varphi\ } & (H, t) \end{array}$$

Here φ and τ_x are (x, t)-morphisms, whereas Φ is a homomorphism of GCD-groups, or equivalently a (t, t)-morphism. The diagram (4.3) exhibits the universal role of the Lorenzen group with respect to (x, t)-morphisms into GCD-groups. For Φ is in fact uniquely determined by the formula

(4.4)
$$\Phi\left(\frac{(a_1, \ldots, a_m)_{x_a}}{(b_1, \ldots, b_n)_{x_a}}\right) = \big(\varphi(a_1)\wedge\ldots\wedge\varphi(a_m)\big)\big(\varphi(b_1)\wedge\ldots\wedge\varphi(b_n)\big)^{-1}$$

which is just a particular case of (4.1). We know already that Λ is faithful, such that the above remarks establish an injection

(4.5)
$$\Lambda\colon \mathrm{Hom}_I\big((G, x), I_t(H)\big) \to \mathrm{Hom}_{GCD}\big(\Lambda_x(G), H\big)\,.$$

It remains to be shown that this map is also a surjection, thereby proving that Λ is the left-adjoint of the shadow funtor I_t. Let $\theta \in \mathrm{Hom}_{GCD}\big(\Lambda_x(G), H\big)$ and put $\varphi = \theta\circ\tau_x$. Since τ_x is an (x, t)-morphism, the same is true of φ. Furthermore $\Lambda(\varphi) = \theta$, because there is just *one* extension of φ to a (t, t)-morphism of $\Lambda_x(G)$ (given by the formula (4.4)). ∎

We want to specialize Theorem 1 in such a way as to obtain Lorenzen's main result on the groups $\Lambda_x(G)$ and to establish contact with Krull's researches on the Kronecker function rings. Both of these applications will stress the links with valuation theory.

The natural generalization of the classical notion of a valuation to the setting of ideal systems is the following one: By an *x-valuation* of a directed group G equipped with a Lorenzen system x we understand an (x, t)-morphism of G onto a totally ordered group Γ. (Note that a totally ordered group is characterized by the fact that $s = t$, i.e. it carries only one Lorenzen system (of finite character). We could hence equally well speak of an x-valuation as an (x, s)-morphism onto Γ.)

In the case of the divisibility group of an integral domain, equipped with the d-system, the notion of a d-valuation is nothing but an ordinary Krull valuation. The condition that inverse images of t-ideals are d-ideals is in fact equivalent to the classical inequality $v(a\pm b) \geqslant \mathrm{Min}\,\big(v(a), v(b)\big)$.

COROLLARY 1 (Lorenzen). – *There is a bijection between the x-valuations of an x-closed group G and the t-valuations of the corresponding Lorenzen group $\Lambda_x(G)$. Furthermore these t-valuations are in one-to-one correspondence with the prime t-ideals of $\Lambda_x(G)^+$.* (See [33], Satz 13 and [21], Theorem 4, p. 49.)

The first and main part of this corollary is nothing but a specialization of the bijection (4.5) to the case where H is a totally ordered group. The correspondence between t-valuations and prime t-ideals is not contained in Theorem 1, but is a rather simple matter to which we shall return later in connection with t-localization. It is also a special case of Theorem 8.

Among the consequences of Corollary 1 is the fact that a group G is x-closed if and only if G^+ is an intersection of x-valuation monoids. We shall have occasion to return to this fact in the next section (Corollary 2 of Theorem 2). Here we specialize Corollary 1 one step further:

COROLLARY 2. – *There is a bijection between the Krull valuations of an integrally closed domain R and the Krull valuations of its corresponding Kronecker function ring.*

The Kronecker function ring $\boldsymbol{K}(R)$ alluded to here is the canonical one corresponding to the d_a-system. In order to derive this corollary from the preceding one we first notice that the monoid $\Lambda_d(G)^+$, where G is the divisibility group of R, is isomorphic to the monoid of the principal and integral d-ideals of $\boldsymbol{K}(R)$. This allows us, in a multiplicative context, to consider a Kronecker function ring as a special case of a Lorenzen group. Having established this identification it remains only to see that any d-valuation of $\boldsymbol{K}(R)$ is in fact a t-valuation. This follows from the fact that $\boldsymbol{K}(R)$ is a Bezout domain, since this implies that finitely generated d-ideals are t-ideals.

5. – Greatest common divisors and integral closure.

The construction of the GCD-functor Λ relies heavily on the condition of integral closure (x-closure). We shall now give a result which clarifies the exact relationship between integral closure and the embeddability in a GCD-group. For this purpose we shall give a few preparatory remarks.

To any morphism of Lorenzen systems $\varphi\colon (G_1, x_1) \to (G_2, x_2)$ we can associate a map Φ between their respective monoids of ideals:

$$\Phi(A_{x_1}) = \big(\varphi(A)\big)_{x_2} \big(= \big(\varphi(A_{x_1})\big)_{x_2}\big) . \tag{5.1}$$

Just as for the functor Λ it is a routine matter to verify that Φ is a morphism of monoids: $\Phi(A_{x_1} \circ B_{x_1}) = \Phi(A_{x_1}) \circ \Phi(B_{x_1})$. A directed group equipped with a Lorenzen system x is said to be *regularly x-closed* if the implication

$$A_x \circ C_x = B_x \circ C_x \Rightarrow A_x = B_x$$

holds true for any finitely generated x-ideal C_x.

With the above notation and terminology we have the following obvious

LEMMA. – *If G_2 is regularly x_2-closed and Φ is injective, then G_1 is regularly x_1-closed.*

With this in mind we can now prove the following

THEOREM 2. – *A directed group G is x-closed if and only if it can be considered as an ordered subgroup of a GCD-group in such a way that the resulting injection is an (x, t)-morphism.*

PROOF. – That an x-closed group can be isomorphically (x, t)-injected into a GCD-group is part of the proof of Theorem 1 where it was established that the canonical injection $G \to \Lambda_x(G)$ is an (x, t)-morphism. That this map identifies G with an ordered subgroup of $\Lambda_x(G)$ is clear.

Assume next that G sits as an ordered subgroup of the GCD-group H in such a way that $A_t \cap G$ is an x-ideal in G for all $A \subset H$. The trace in G, of the t-system in H (i.e. the family of all the sets $A_t \cap G$) is then a Lorenzen system y in G which is coarser than the given x-system. Furthermore, it is clear that the map Φ which is induced from the (y, t)-injection $\varphi\colon G \to H$ is itself injective. This follows from the fact that $\Phi(A_y) = A_t$ and $A_y = A_t \cap G$ for alle $A \subset G$. Since H is a GCD-group and every finitely generated t-ideal is hence principal, it follows that H is regularly t-closed. By the above lemma we infer that G is regularly y-closed, and hence x-closed, since the x-system is finer than the y-system (see [21], Theorem 1, p. 25).

COROLLARY 1. – *A directed group G is x-closed if and only if it can be considered as an ordered subgroup of a direct product of totally ordered groups in such a way that the resulting injection is an (x, t)-morphism.*

This follows immediately from Theorem 2 together with the fact that a GCD-group can be isomorphically (t, t)-injected into a direct product of totally ordered groups (see paragraph 9) and that the composition of an (x, t)-morphism and a (t, t)-morphism is an (x, t)-morphism.

From this follows in turn

COROLLARY 2 (Lorenzen). – *G is x-closed if and only if G^+ is an intersection of x-valuation monoids.*

This is clear since a representation of G^+ as an intersection of x-valuation monoids $v_i^{-1}(\Gamma_i^+)$, where Γ_i is a totally ordered group and v_i is an x-valuation of G into Γ_i

leads to an (x, t)-injection

$$G \to \prod \Gamma_i$$

and vice versa.

As another consequence of Theorem 2 we note the following well-known result

COROLLARY 3. – *G is semi-closed (s-closed) if and only if it is an ordered subgroup of some GCD-group.*

This is a consequence of Theorem 2, simply because the notion of an order-preserving group homomorphism is the same thing as an (s, t)-morphism.

The two following corollaries give specializations to the cases $x = t$ and $x = d$ respectively.

COROLLARY 4. – *G is regularly integrally closed (t-closed) if and only if it can be considered as an ordered subgroup of a GCD-group in such a way that the resulting injection is a (t, t)-morphism.*

Note that the notion of a (t, t)-morphism is the same as what is called a V-homomorphism in [34], p. 5. When Corollary 4 is applied to the divisibility group of an ntegral domain it gives the Corollary 3.3 of [34], p. 8.

COROLLARY 5. – *An integral domain is integrally closed if and only if its divisibility group can be isomorphically (d, t)-injected into a GCD-group.*

This latter corollary is not surprising since the reader will have no difficulty in showing that the morphism condition $\varphi(A_d) \subset (\varphi(A))_t$ for an arbitrary bounded set A is equivalent to the familiar inequality $\varphi(a + b) \geqslant \text{Min}\,(\varphi(a), \varphi(b))$ of a Krull valuation (taking the purely multiplicative condition for granted). Combining this observation with Corollary 1 or 2, we get the usual characterization of an integrally closed domain as an intersection of valuation rings.

6. – Regularly x-closed groups and Prüfer groups.

In his fundamental paper [39], PRÜFER considered two conditions on the divisibility group of a domain, each of which is stronger than integral closure. One of these is Prüfers condition Γ, which by Krull was given the name « arithmetisch brauchbar » or rather « endlich arithmetisch brauchbar ». BOURBAKI ([12], p. 554) introduces this notion only in the case of v-ideals (divisors in his terminology) and then speaks of an integral domain as being « regularly integrally closed ». The general notion is the one introduced above as a *regularly x-closed group.*

A slightly stronger condition is offered by the following definition: G is said to be an *x-Prüfer group* if the finitely generated x-ideals in G form a group under

x-multiplication. For many Lorenzen systems (G, x) there is no difference between the concepts of a regularly x-closed group and an x-Prüfer group. It is for instance well known that in the case $x = d$, a Prüfer domain may be characterized by either of these two properties. A more comprehensive result of this kind will be given in paragraph 10. Here we shall characterize the concepts of a regularly x-closed group and an x-Prüfer group in terms of the map Φ introduced in the preceding paragraphs.

In the following theorem, G is an x-closed group, φ denotes the canonical (x, t)-injection $(G, x) \to (\Lambda_x(G), t)$ and Φ is defined by $\Phi(A_x) = (\varphi(A_x))_t = A_t$ where A is any bounded set in G. If there exists a family V of *valuations* (= s-valuations) of the group G such that for any bounded $A \subset G$,

$$A_x = \bigcap_{v \in V} v^{-1}\big((v(A))_t\big)$$

we say that the given x-system is *defined by a family of valuations.* (See [21], p. 47 and [19], p. 398.)

THEOREM 3. – *The following conditions are equivalent for an x-closed group G:*

1) *G is regularly x-closed.*

2) *The map Φ is injective.*

3) *The x-system in G is the trace of the t-system in some GCD-group which contains G as an ordered subgroup.*

4) *The x-system coincides with the x_a-system in G.*

5) *The x-system is defined by a family of valuations.*

Furthermore the following two conditions are also equivalent

6) *G is an x-Prüfer group.*

7) *The map Φ is bijective.*

8) *G is regularly x-closed and every element of $\Lambda_x(G)$ is of the form* inf N *for a suitable finite subset N of G.*

In case the given x-system is additive, all the above eight conditions are equivalent.

We shall not go into any details with respect to the proof of this theorem since such a proof can be more or less extracted from [21] (especially from Proposition 7, p. 49, Theorem 5, p. 50 and Theorem 3, p. 55). The only statement in the theorem which really needs a proof, is the last one concerning additivity. This will follow, however, from Theorem 6 below. For further elaboration on the properties 5) and 6) in the case $x = d$, the reader should consult [19], p. 303 and Theorem 32.12, p. 402.

7. – Divisors revisited. The axiomatic approach of Borevic-Shafarevic.

We shall now indicate how t-ideals may advantagously be used in order to put the axiomatic introduction of divisors of Borevic-Shafarevic into a slightly different perspective. This will lead to both a generalization and a sharpening of their treatment.

Few introductory books on algebraic number theory take the trouble to explain the notion of a divisor properly. Hasse in his classical « Zahlentheorie » puts considerable emphasis on the concept of a divisor, but without clarifying the most fundamental issues. A step towards such a clarification is taken by Borevic-Shafarevic in Chapter 3 (« The theory of divisibility ») of their book « Number Theory ». Here the notion of « a theory of divisors » is introduced axiomatically as a map φ form the group of divisibility G of an integral domain into a factorial group $\boldsymbol{D}$ verifying the following three conditions:

(1) φ is an isomorphism which identifies G with an ordered subgroup of $\boldsymbol{D}$.

(2) If $\varphi(a)\geqslant\mathfrak{d}$ and $\varphi(b)\geqslant\mathfrak{d}$ then also $\varphi(a\pm b)\geqslant\mathfrak{d}$ (4).

(3) If $\mathfrak{a}$ and $\mathfrak{b}$ are elements in $\boldsymbol{D}$ such that

$$\{g\in G|\varphi(g)\geqslant\mathfrak{a}\}=\{g\in G|\varphi(g)\geqslant\mathfrak{b}\}\qquad\text{then } \mathfrak{a}=\mathfrak{b}\,.$$

The elements of $\boldsymbol{D}$ are called *divisors* and the divisors of the form $\varphi(a)$ are said to be *principal divisors.*

An equivalent formulation of (3) is to say that any $\mathfrak{d}\in\boldsymbol{D}$ is the infimum of a finite number of principal divisors. Both these formulations of (3) express our wish to leave out « unnecessary » divisors—i.e. to consider only minimal factorial extensions of G. By *unicity* we mean that if $(\boldsymbol{D}_1,\varphi_1)$ and $(\boldsymbol{D}_2,\varphi_2)$ are two theories of divisors for G, then there exists an isomorphism between $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ which extends the canonical isomorphism between $\varphi_1(G)$ and $\varphi_2(G)$.

The exposition of Borevic-Shafarevic is in spite of its virtues still blurred by the presence of the additive operation. The additive operation is irrelevant for the general treatment of divisors and should be discarded. But also in case one insists on a ring-theoretic treatment, the axiom (2) of Borevic-Shafarevic is redundant, as was also noticed by L. SKULA in [43]. (An earlier axiomatic treatment of divisors due to KRULL ([29], p. 123), which is essentially equivalent to the one by Borevic-Shafarevic, suffers from the same redundancy.) A significant forerunner of SKULA's purely multiplicative treatment is CLIFFORD's paper [13].

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(4) Since we are here dealing with a divisibility group rather than with the multiplicative group of the given field of quotients it is more accurate to write (a), (b) and $(a\pm b)$ instead of a, b and $a\pm b$.

Our aim here is to look at the axiomatic introduction of divisors in the light of Lorenzen groups and t-ideals. We then define a *theory of divisors* for G in the general situation where G is a directed group, $\boldsymbol{D}$ is a factorial group and the map φ satisfies the above conditions (1) and (3) (thus discarding (2)). In case $\boldsymbol{D}$ is replaced by a GCD-group we shall speak of a *theory of quasi-divisors* for G. In the case where G is the divisibility group of an integral domain it turns out that (2) is in both cases automatically fulfilled, due to the fact that every t-ideal in an integral domain is a d-ideal.

The basic arithmetical extension problem which we have treated so far concerns the embeddability of a directed group G into a GCD-group—the extension process which provides the existence of greatest common divisors. Corollary 3 of Theorem 2 exhibits s-closure as a necessary and sufficient condition for such an embeddability. (It is well known that s-closure may also be characterized by the implication: $a^n \in G^+$ for some integer $n \geqslant 1 \Rightarrow a \in G^+$.)

However, the condition of s-closure does not assure the existence (and unicity) of a *minimal* GCD-extension given by a theory of quasi-divisors for G. The relevant condition for this involves t-ideals:

THEOREM 4 (K. GUDLAUGSSON [47]). – *The directed group G has a theory of quasi-divisors if and only if G is a t-Prüfer group. The group $\boldsymbol{D}$ of quasi-divisors of G is then uniquely determined as the Lorenzen t-group of G which in this case is isomorphic to the group of finitely generated (fractional) t-ideals of G.*

PROOF. – From the remarks given at the bottom of p. 44 in [21] it follows that any minimal GCD-extension $\boldsymbol{D}$ of G must be the Lorenzen x-group $\Lambda_x(G)$ where x denotes the ideal system in G which is the trace (« empreinte », p. 52 in [21]) of the t-system in $\boldsymbol{D}$. By combining 1) and 3) of Theorem 3 it follows that G is not only x-closed (by the fact that integral closure is preserved by trace formation) but also regularly x-closed. Using this together with 6) and 8) of Theorem 3 we conclude that G is an x-Prüfer group. This means that any finitely generated x-ideal is invertible and hence a v-ideal. Since x is of finite character, being the trace of the t-system which is of finite character, it follows that every x-ideal is a t-ideal i.e. $x = t$ showing that the trace of the t-system in $\boldsymbol{D}$ is the t-system in G: $\boldsymbol{D} \simeq \Lambda_t(G)$. The elements of $\Lambda_t(G)$ are of the form

$$\frac{(a_1, \ldots, a_m)_{t_a}}{(b_1, \ldots, b_n)_{t_a}}. \tag{7.1}$$

Since $t = t_a$ this quotient may be identified with the fractional t-ideal $(a_1, \ldots, a_m)_t \circ (b_1, \ldots, b_n)_t^{-1}$. This identification is an isomorphism since the formal quotients (7.1) are multiplied in the same way as the corresponding fractional t-ideals.

Conversely, let G be a t-Prüfer group. Then the finitely generated t-ideals of G form a GCD-group $\boldsymbol{D}$ in which G is injected by the isomorphism $\varphi: a \to (a)_t$. The t-ideal $(a_1, \ldots, a_n)_t$ represents the infimum of the principal ideals $(a_1), \ldots, (a_n)$. This expresses the content of axiom (3) in the present case and shows that φ defines a theory of quasi-divisors for G.

A more suggestive reformulation of the above theorem is perhaps the following

COROLLARY. – *The directed group G admits of a unique minimal GCD-extension if and only if G is a t-Prüfer group and the extension is then given as the group of all finitely generated t-ideals of G.*

The redundancy of the condition (2) in the definition of Borevic-Shafarevic is clear from the proof of Theorem 4 where it was in particular established that the given trace of the t-system in $\boldsymbol{D}$ coincides with the t-system in G. In the case of the divisibility group of an integral domain this gives a strengthening of (2) since (2) simply expresses that the trace of a principal t-ideal of $\boldsymbol{D}$ is a d-ideal in G.

8. – Characterization of préfactorial groups.

The preceding section shows how the existence and unicity of a minimal *GCD*-extension depends fundamentally on the t-system in the given directed group. We shall indicate how the possibility of a factorial extension of a directed group G is also governed by the behaviour of the t-ideals in G. However, in contrast to the *GCD* case the case of factorial extension introduces no discrimination between minimal and non-minimal extensions: If a directed group may at all be embedded in a factorial group it will also admit of a *minimal* factorial extension.

THEOREM 5. – *The following properties are equivalent for a directed group G*

1) *G has a theory of divisors.*
2) *G has a unique theory of divisors.*
3) *G is préfactorial.*
4) *The t-ideals of G form a group under t-multiplication.*
5) *G is t-Prüfer and satisfies the ascending chain condition for integral t-ideals.*
6) *G is t-closed and satisfies the ascending chain condition for integral t-ideals.*
7) *There exists a Lorenzen system x such that G is x-closed and satisfies the ascending chain condition for integral x_a-ideals.*

PROOF. – $1 \Rightarrow 2$: If G has a theory of divisors this is also a theory of quasi-divisors, the unicity of which was established in Theorem 4. $2 \Rightarrow 3$: Obvious. $3 \Rightarrow 4$: This is just one half of Theorem 5 in [21], p. 82 (although Jaffard uses a different terminology). $4 \Rightarrow 5$: Clear since any invertible t-ideal is finitely generated, the t-system being of finite character. $5 \Rightarrow 6$: Obvious. $6 \Rightarrow 7$: Obvious since $t = t_a$. $7 \Rightarrow 3$: If G is x-closed it is also x_a-closed and if every x-$_a$ideal is finitely generated this means that G is completely integrally closed (Theorem 7, p. 29 in [21]) which in turn implies that the v-ideals of G form a group under v-multiplication (Theorem 6, p. 29 in [21]). Since the a.c.c. for x_a-ideals entails the a.c.c. for v-ideals it follows that the GCD-group $\boldsymbol{D}$ is indeed factorial (Theorem 3, p. 8 in [21]). To complete the circle of implications we need only to establish one more implication, say $5 \Rightarrow 1$: This is an immediate consequence of the above Theorem 4 in conjunction with Theorem 3, p. 8 in [21].

If the above theorem is applied to the divisibility group of an integral domain it entails characterizations of Krull domains and suggests that the relationship between t-ideals and Krull domains is to a considerable extent analogous to the relationship between ordinary ideals (d-ideals) and Dedekind domains. This is in particular visible from the characterization 4 which puts Dedekind domains and Krull domains on an equal footing in this respect. It is well known that Dedekind domains are not only characterized by the fact the d-ideals form a group under usual d-multiplication but also by the fact that its integral ideals factorize (uniquely) into prime ideals. This prompted us to pose the following question at the end of the paper [3]: Is a Krull domain characterized by the fact that any of its proper t-ideals can be written as a t-product of prime t-ideals? Recently, K. GUDLAUGSSON [47] has proved that the answer is affirmative. In fact, he proves quite generally that a direted group is préfactorial if and only if its integral t-ideals decompose into products of prime t-ideals. As in the case of d-ideals the unicity of such decompositions follows from the existence.

9. – t-Localization versus the Krull-Kaplansky-Jaffard-Ohm theorem.

In the preceding sections we have dealt with the relevance of t-ideals in connection with the problem of restoring basic arithmetical properties (existence of greatest common divisors and unique factorization) by a suitable extension process.

Another fundamental problem of the theory of divisibility concerns the decomposition of a divisibility relation into a conjunction of linear (total) ones. This issue has already been touched upon above in connection with the topic of Lorenzen groups (Corollaries 1 and 2 of Theorem 1 and Corollaries 1 and 2 of Theorem 2). In ring theory this problem takes the form of writing an integrally closed (d-closed) domain as an intersection of valuation rings. The purely multiplicative problem consists in embedding a GCD-group into a direct product of totally ordered groups-taking for granted that the embedding of a directed group into a GCD-group has already been clarified by Theorem 2 and its corollaries.

In connection with this question some authors have advocated a point of view which may be said to be strictly opposite to the one which underlies the present paper. These authors have tried to solve problems concerning GCD-groups by reducing them to ring theory via the so-called Krull-Kaplansky-Jaffard-Ohm theorem (see in particular [34]). This theorem tells us that any *GCD*-group is order isomorphic to the divisibility group of a suitably chosen Bezout domain. In this way the general theory of *GCD*-groups can profit from what is known about Bezout domains. This method can in particular be used in order to realize the embedding of a *GCD*-group into a direct product of totally ordered groups (a result which was first obtained by Lorenzen). For if G is a *GCD*-group which is the divisibility group of a Bezout domain R we can argue as follows: Being a Bezout domain, R is in particular integrally closed (d-closed) and as such equal to an intersection of valuation rings V_i sitting in the quotient field of R. If Γ_i denotes the totally ordered divisibility group of V_i then

$$G \rightarrow \prod_e \Gamma_i \tag{9.1}$$

gives an embedding of the desired type.

This is simple enough, once the K-K-J-O-theorem has been proved. Still, it is fair to say that this proof procedure succeeds—not because of its relevance for the problem at hand, but rather in spite of its irrelevance. It seems far fetched to use d-ideals, d-closure and d-valuations in connection with this purely multiplicative problem, just because the sufficient amount of commutative algebra happens to be readily available in the d-case. The recipe should rather be to use the concept of a t-ideal which matches the multiplicative situation perfectly—and develop the relevant piece of commutative algebra in the t-case. In fact, only the bare rudiments of a theory of t-localization is all that is needed. This was already recognized by Lorenzen although he did not develop any systematic theory of localization for ideal systems. The general globalization formula for localization in ideal systems (see [4]) gives in the case $x = t$:

$$G^+ = \bigcap_{S_i} S_i^{-1} G^+ \tag{9.2}$$

with $S_i = G^+ - P_t^{(i)}$ running over all the complements of maximal (prime) t-ideals $P_t^{(i)}$ in G^+.

Let us now elucidate the relationship between localization in *GCD*-groups and the t-shadow functor.

Let G be a *GCD*-group with $D = G^+$ as its monoid of integral elements and let S be a submonoid of D. According to the general procedure described in [4] we can, on the basis of the Lorenzen system (D, t), form the localized ideal system $(S^{-1}D, t_S)$. This integral ideal system is a Lorenzen system and will hence define a fractional ideal system in G where the new order relation in G is having $S^{-1}D$ as its monoid of integral elements. It is easy to see that the corresponding ordered group is isomorphic to the factor group

$$G/S^{-1}D \cap SD^{-1}$$

and is hence again a *GCD*-group since $S^{-1}D \cap SD^{-1}$ is an 1-ideal (absolutely convex subgroup) of G. This fact can also be seen by explicitely computing the g.c.d.'s relative to the new «localized» ordering, according to the formula

$$\frac{d_1}{s_1} \wedge \frac{d_2}{s_2} = \frac{d_1 s_2 \wedge d_2 s_1}{s_1 s_2}. \tag{9.3}$$

Using (9.3) we also see that the t_s-system defined in $S^{-1}D$ is the same as the t-system in $S^{-1}D$ defined intrinsically in terms of the order relation given by (9.3). By (9.3) the t_s-ideal $S^{-1}A_t$ $(A \subset D)$ is a t-ideal in $S^{-1}D$ and for any t-ideal B_t in $S^{-1}D$ we have $B_t = S^{-1}(B_t \cap D)$ where $B_t \cap D$ is a t-ideal in D.

The contents of these remarks may be summarized as follows: We have a localization procedure going on at two levels—one for *GCD*-groups and one for ideal systems (the t-system). These localization procedures are linked by the t-shadow functor in such a way that we obtain an obvious commutative diagram.

Let Γ_i denote the ordered group which is associated to the préordering of G, given by specifying $S_i^{-1}D$ as the monoid of integral elements. The injectivity of (9.1) then follows from (9.2) and the fact that (9.1) is a morphism of *GCD*-groups follows from the map $D \to S^{-1}D$ being a (t, t_s)-morphism by construction (see [4]), together with the fullness of the t-shadow functor (see the Lemma of paragraph 4). Finally each Γ_i is totally ordered, due to the fact that $S_i^{-1}D$ is a t-local (préordered) monoid in the sense that it contains a unique maximal t-ideal $M_t = S_i^{-1}P_t^{(i)}$ which in the associated ordered group simply consists of all elements $> e$. Since M_t is closed under intersection this means that we have the implication $a > e$ and $b > e \Rightarrow$ $\Rightarrow a \wedge b > e$ and this is characteristic of a *GCD*-group which is totally ordered.

One of the features of the duality between prime t-ideals and prime 1-ideals in *GCD*-groups is that the localization with respect to a prime t-ideal is order isomorphic to the factor group with respect to the dual prime 1-ideal. Alternatively one may therefore obtain the embedding (9.1) by replacing (9.2) by the fact that the intersection of all prime 1-ideals in a *GCD*-group reduces to the identity element and that any factor group modulo a prime 1-ideal is totally ordered. It seems to us, however, that the method of localization may have an advantage because of its broader perspective. This will come up again in connection with sheaf representation.

10. – Additive ideal systems and a counterexample of Dieudonné.

The relative strength between the notions of an x-closed group, a regularly x-closed group and an x-Prüfer group has been touched upon in paragraph 6. For the t-system we have already noticed that a t-closed group and a regularly t-closed group is one and the same thing, simply due to the fact that the t-system is the coarsest Lorenzen system (of finite character) which exists in a directed group—and hence $t = t_a$. Theorem 5 shows that « t-closed » is even equivalent to « t-Prüfer » in the presence of the ascending chain condition for integral t-ideals. However, it was shown by LORENZEN (in [33], p. 551) that there exist directed groups which are t-closed, but which are not t-Prüfer groups. DIEUDONNÉ, (in [16]), sharpened this result by showing that there is a distinction between these two notions also within the more restricted realm of divisibility groups of integral domains.

Our interest in this question comes from the general theory of additive ideal systems (see [3]). As we see it, it is in the light of the below Theorem 6 that the counterexamples of Lorenzen and Dieudonné acquire some additional interest by exhibiting the *reason* for the existence of these examples—namely the lack of additivity.

Theorem 6 will generalize a result of Prüfer to the effect that a regularly d-closed domain is a Prüfer domain. Our proof will closely follow the proof of this result as given in [21], pp. 26-28. In this generality the theorem was first proved by H. BIE LORENTZEN in [9].

LEMMA 1. – *G is an x-Prüfer group if and only if every x-ideal with two generators is invertible.*

PROOF. – Assume that we have shown that any x-ideal with *less* than $n+1$ generators is invertible and let $A_x = (a_1, \ldots, a_{n+1})_x$ with $n \geqslant 2$. We then have finitely generated x-ideals B_x, C_x and D_x such that

(10.1) $$(a_1, \ldots, a_n)_x \circ B_x = (e)$$

(10.2) $$(a_2, \ldots, a_{n+1})_x \circ C_x = (e)$$

(10.3) $$(a_1, a_{n+1})_x \circ D_x = (e) .$$

By putting $E_x = a_1 B_x \circ D_x + a_{n+1} C_x \circ D_x$, a computation, using an easy consequence of the continuity axiom for ideal systems (see [2]) as well as the equations (10.1-3), shows that $A_x \circ E_x = (e)$ as desired. (See [21], p. 27 for details in the case $x = d$.)

LEMMA 2. – *Let (G, x) be an additive Lorenzen system and assume that G is x-closed. G will then be an x-Prüfer group if and only if $a \in (e, a^2)_x$ for all $a \in G$.*

PROOF. – We are here mainly interested in proving the « if »-part. (The proof of the « only if »-part is contained in the proof of Theorem 6.) By Lemma 1 it suffices to show that any x-ideal of the form $(b, c)_x$ is invertible. Since $(b, c)_x = (b)_x \circ \circ (e, c/b)_x$ it is in turn sufficient to show that $(e, a)_x$ is invertible for any $a \in G$. From the assumption $a \in (e, a^2)_x$ it follows by additivity that

(10.4) $$(g, a^2)_x = (a, a^2)_x$$

for suitable $g \in G^+$. In particular $a \in (g, a^2)_x$ which by additivity gives

$$(g, a)_x = (g, ha^2)_x \tag{10.5}$$

for some $h \in G^+$. Putting $A_x = (ga^{-1}, h)_x \circ (e, a)_x = (ga^{-1}, g, h, ah)_x$ it will be sufficient to show that $A_x = G^+$. From (10.5) we infer that

$$(ga^{-1}, e)_x = (ga^{-1}, ha)_x \tag{10.6}$$

which entails $e \in (ga^{-1}, ha)_x \subset A_x$ showing that $G^+ \subset A_x$. It remains to be shown that ga^{-1} and ha belong to G^+ since this will give $A_x \subset G^+$. We get $g(e, a)_x = (g, ga)_x \subset (g, a)_x = (g, ha^2)_x \subset (g, a^2)_x = (a, a^2)_x = a(e, a)_x$ using (10.4) and (10.5) as well as the fact that g and h are integral elements of G. From $g(e, a)_x \subset a(e, a)_x$ we get $ga^{-1} \in G^+$ since G is x-closed. Together with (10.6) this also yields $ha \in G^+$.

THEOREM 6. – *Any regularly x-closed group is an x-Prüfer group provided that the given fractional x-system is additive.*

PROOF. – By Lemma 2 it is sufficient to show that the property of regular x-closure implies that $a \in (e, a^2)_x$ for all $a \in G$. We have

$$(a)_x \circ (e, a)_x = (a, a^2)_x \subset (e, a^2)_x \circ (e, a)_x$$

and $(a)_x \subset (e, a^2)_x$ results by cancellation (noting that cancellation with respect to equalities is equivalent to cancellation with respect to inclusions).

In [16] DIEUDONNÉ gives an example of an integral domain which is regularly t-closed but not t-Prüfer (regularly integrally closed but not pseudo-Prüfer in BOURBAKI's terminology [12], p. 554 and 561). When this is combined with the above Theorem 6 we get the following

COROLLARY 1. – *There exists an integral domain where the divisorial ideals of finite character do not form an additive ideal system.*

A sharpening of this result is the following

COROLLARY 2. – *There exists a t-closed divisibility group where no x_a-system is additive.*

PROOF. – If the directed group G is t-closed it is x-closed for any Lorenzen system x in G. If an x_a-system in G were additive for some x it would follow from Theorem 6 that G is x_a-Prüfer, hence also t-Prüfer (according to [21], Theorem 1, p. 25) contradicting Theorem 6.

A more explicit result in the same direction is the following corollary which exhibits an abundance of non-additive ideal systems.

COROLLARY 3. – *The s_a-system in a GCD-group G is additive if and only if G is totally ordered.*

PROOF. – If G is totally ordered, all ideal systems in G coincide with the s-system which is additive. Assume conversely that G is a GCD-group which is not totally ordered. There then exist strictly positive elements $a, b \in G^+$ such that $a \wedge b = e$. This entails $(a, b)_t = (e)$ and $(a, b)_{s_a} \neq (e)$. The latter fact follows from a result of LORENZEN ([33], p. 538) and shows that $(a, b)_{s_a}$ cannot be invertible since it as such would be a t-ideal, contradicting $(a, b)_{s_a} \neq (a, b)_t$.

11. – Sheaf representation over the t-spectrum.

Among the most important types of ordered groups are on the one hand the multiplicative groups arising from the theory of divisibility (divisibility groups, groups of ideals, groups of divisors, Lorenzen groups, etc.), and on the other hand additive groups of real-valued functions. Although these two types of ordered abelian groups arise in different contexts, the preceding paragraphs have shown that there is a common meeting ground for them within the theory of divisibility. In fact, the most satisfactory arithmetical situations arise exactly when either the divisibility group itself or a suitable group of ideals form a nice function-group like an additive group of integer-valued functions vanishing outside of finite sets.

Viewing factorial and prefactorial groups from the point of view of a functional representation of these groups over the family of prime t-ideals, this suggests a more general representation theory for ordered groups which closely parallels the well-known sectional representation of commutative rings.

We shall here content ourselves by giving the full sectional representation of the integral part of a GCD-group. This also accomplishes a sectional representation of a semi-closed group via the embedding into its Lorenzen s-group.

Let $D = G^+$ denote the monoid of integral (positive) elements of a GCD-group G. By the *t-spectrum* of D, denoted by $X = \mathrm{Spec}_t D$ (or $\mathrm{Spec}_t G$), we understand the family of all prime t-ideals of D, equipped with the usual spectral topology where the basic open sets are given by the sets of the form $D(a) = \{P_t | a \notin P_t\}$. Whenever S is a submonoid of D we can form the usual monoid of quotients $S^{-1}D$ with $D \subset S^{-1}D \subset G$. As explained earlier the monoid $S^{-1}D$ induces a preorder in G, and it is the restriction of this preorder to $S^{-1}D$ which will be considered in the sequel. This makes $S^{-1}D$ into a preordered GCD-monoid according to (9.3). The particular case where S is of the form $S_a = \{e, a, a^2, ...\}$ gives rise to a presheaf of preordered GCD-monoids over $\mathrm{Spec}_t D$. For $D(b) \subset D(a)$ is by the Krull-Stone theorem for x-ideals ([2], Theorem 12) equivalent to $b \in \sqrt{(a)}$. By putting $b^n = ga$ this gives rise to a well-defined homomorphism of GCD-monoids

$$\varphi_b^a : S_a^{-1}D \to S_b^{-1}D$$

where $\varphi_b^a(d/a^m) = dg^m/b^{m\cdot n}$. Obviously $\varphi_c^b \circ \varphi_b^a = \varphi_c^a$ whenever $D(c) \subset D(b) \subset D(a)$. In this way the assignment $D(a) \to S_a^{-1}D$ defines a presheaf of *GCD*-monoids on the basis $\{D(a),\, a \in D\}$ and hence determines a presheaf $\boldsymbol{T}_X$ on $X = \mathrm{Spec}_t D$. In much the same way as for commutative rings we can prove the following

THEOREM 7. – *The presheaf $\boldsymbol{T}_X$ is a sheaf. In particular there is an isomorphism of GCD-monoids $D \simeq \Gamma(X, \boldsymbol{T}_X)$. Furthermore the stalk of $\boldsymbol{T}_X$ at P_t is isomorphic to the totally preordered monoid $S^{-1}D$ where $S = D \setminus P_t$.*

PROOF. – As usual one must verify that the presheaf $\boldsymbol{T}_X$ satisfies the two defining properties of a sheaf. These two properties correspond, respectively, to the injectivity and the surjectivity of the natural map $D \to \Gamma(X, \boldsymbol{T}_X)$. The injectivity is obvious in this case, since we operate within a group where cancellation is available. Let us show the surjectivity, i.e. that any global section of the given presheaf comes from an element in D. By the (quasi) compactness of X ([2], p. 35) the problem reduces to the following one: Given a finite covering of X by basic open sets $X = D(a_1) \cup D(a_2) \cup \ldots \cup D(a_k)$ and given a corresponding family of elements $s_i \in S_a^{-1}D$ such that s_i and s_j have the same «restriction» to $D(a_i) \cap D(a_j) = D(a_i a_j)$—we want to exhibit an element $d \in D$ whose «restriction» to $D(a_i)$ is s_i.

Since we are dealing with a finite covering we can adjust the representation of s_i as a quotient in such a way that the exponent in the denominator is independent of i, i.e. $s_i = d_i/a_i^n$ for all i. The fact that s_i and s_j by the presheaf restriction maps are mapped onto the same element in $S_{a_i a_j}^{-1}D$ gives rise to the equations

$$a_j^n d_i = a_i^n d_j\,. \tag{11.1}$$

Using the equality $D(a_i^n) = D(a_i)$ and the fact that the sets $D(a_i)$ form a covering of X we deduce the identity

$$(a_1^n, \ldots, a_k^n)_t = (a_1^n \wedge \ldots \wedge a_k^n) = D$$

which simply means that

$$a_1^n \wedge \ldots \wedge a_k^n = e\,. \tag{11.2}$$

Putting $d = d_1 \wedge \ldots \wedge d_k$ and using (11.1) and (11.2) we get

$$a_i^n d = a_i^n(d_1 \wedge \ldots \wedge d_k) = a_i^n d_1 \wedge \ldots \wedge a_i^n d_k = a_1^n d_i \wedge \ldots \wedge a_k^n d_i = (a_1^n \wedge \ldots \wedge a_k^n)\, d_i = d_i\,.$$

This shows that $d = d/e = d_i/a_i^n = s_i$ when compared in $S_{a_i}^{-1}D$ and thus proves that $d \in D$ gives rise to the given section $s \in \Gamma(X, \boldsymbol{T}_X)$.

The verification of the isomorphism $\varinjlim S_a^{-1}D \simeq S^{-1}D$ is routine and may be left to the reader. (Here $S = D \setminus P_t$ and the inductive limit is taken with respect to all $a \notin P_t$.)

By replacing each stalk $S^{-1}D$ in the sheaf $\boldsymbol{T}_X$ by the group G equipped with the preordering which is induced by choosing $S^{-1}D$ as the monoid of integral elements—we can easily extend the above sheaf representation from D to G. In fact, any element $g \in G$ may be written uniquely in the form $g = g^+(g^-)^{-1}$ where $g^+ = g \vee e$ and $g^- = g^{-1} \vee e$ both belong to $D = G^+$. The section s_g corresponding to g is then defined by

$$s_g(P_t) = s_{g^+}(P_t)\big(s_{g^-}(P_t)\big)^{-1}.$$

This will indeed be a section if we extend the definition of the topology on the disjoint union of the stalks by declaring all sets which may be written as a union of sets of the form

$$\{s_g(P_t) | P_t \in D(a)\}$$

as open.

We have thus obtained a sheaf representation of a *GCD*-group in terms of a sheaf which is built up of totally preordered groups as stalks. From there on we can easily go one step further by passing from the preorder to the associated order in each stalk, i.e. to pass from G to the (totally) ordered factorgroup $G_S = G/SS^{-1}$ and redefine the sections accordingly. We may formulate this as

COROLLARY 1. – *Every (ordered) GCD-group G may be represented as the GCD-group of all sections in a sheaf of totally ordered groups over the quasi-compact space* $\mathrm{Spec}_t\, G$.

Let us also give a more special corollary concerning representations by «real-valued» sections. By a *real group* we shall understand an ordered subgroup of the ordered additive group of real numbers. We shall further say that a *GCD*-group G is *regular* if every prime t-ideal in G^+ is maximal.

COROLLARY 2. – *Any regular GCD-group G is isomorphic to the GCD-group of all sections in a sheaf of real groups over the quasi-compact space* $\mathrm{Spec}_t\, G$.

According to Theorem 7 and earlier remarks the stalk at P_t is isomorphic to the factor group $G/H_P = G/SS^{-1}$ where H_P is the prime 1-ideal corresponding to P_t. If every prime t-ideal of G is maximal, it will also be minimal. Hence, each H_P will be maximal and the corresponding factor group will be totally ordered and archimedian, thus a real group.

Corollary 1 gives a sharpening of the purely algebraic embedding (9.1) of a *GCD*-group into a direct product of totally ordered groups. Using a language which corresponds to the one which we used in connection with divisors we may say that the « principal sections » corresponding to the image of G in the general and « discontinuous » representation

$$G \to \prod \Gamma_i$$

of paragraph 9 are here characterized (selected) as the *continuous* ones with respect to the topological restrictions imposed by the given sheaf.

The above approach seems to give the simplest and most general access to a full sectional representation of GCD-groups by means of totally ordered groups. It is based on a Grothendieck approach in terms of localization rather than on a Gelfand-like approach in terms of factor formation. The sheaf-representation of various classes of lattice ordered groups and rings has been extensively studied by KLAUS KEIMEL ([10], [24] and [25]) who has preferred to use a Gelfand-type of approach. As far as we can see this seems to have some slight disadvantages in the case of GCD-groups: (1) It is less simple than the approach in terms of localization. (2) It is less general in the sense that it requires extra conditions on the given GCD-group in order to obtain a full representation over a quasi-compact space. (3) The stalks are not in general totally ordered and hence less simple and appealing. This latter disadvantage may be compensated for in Keimels approach by passing to the subspace of minimal prime 1-ideals which is in addition Hausdorff and zero-dimensional (but generally not compact). We shall return to a somewhat closer comparison with Keimels approach in the next paragraph.

In a sense, localization and factor formation are dual procedures. In ring theory the « self-dual » case (where $R_P \simeq R/P$ for all prime ideals P) is represented by the class of von Neumann regular rings. In this case the two representation procedures coincide as far as the stalks are concerned. The classical representation theory of Boolean rings may thus be considered from either point of view, although it is the Grothendieck approach which allows us to extend Stone's theory to general commutative rings. A similar advantage of the approach in terms of localization also prevails in the case of GCD-groups. These groups bear in fact a considerable resemblance to regular rings in that they exhibit a similar duality, although this duality for GCD-groups involves *two* different ideal systems rather than *one*. We have already alluded to the bijection between the prime t-ideals and the prime 1-ideals of a GCD-group and the correspondence which it induces between localization with respect to a prime t-ideal and the factor formation with respect to the corresponding prime 1-ideal. One aspect of this duality which is of particular relevance to functional and sectional representation of GCD-groups is the fact that the « semisimplicity » for the 1-system (the Krull-Stone theorem [2], p. 17 applied to the zero-ideal) corresponds to the globalization formula (9.2) for the t-system. (In terms of our notation the bijection between prime t-ideals and prime 1-ideals is given by $P_t \to H_P = SS^{-1}$ where $S = G^+ - P_t$. See remarks at the end of paragraph 9.)

We shall now further clarify the relative virtues of the different candidates for a notion of a «*spectrum*» for a partially ordered group. As we have indicated, the prime t-ideals are superior to the prime 1-ideals even in the case of GCD-groups although this is more visible in connection with sectional representation than in the functional case. We shall next show that the applicability of the prime t-spectrum for a sectional representation of partially ordered groups, which are not necessarily GCD-groups, is in a certain precise sense limited to the Prüfer groups. For integrally closed groups which are not Prüfer groups one preferably passes to a spectrum consisting of x-valuations. Again it is the GCD-functor and Lorenzens theorem (Corollary 1 of Theorem 1) which gives the clue to this insight. Thus it is the concept of an x-valuation which turns out to have the widest scope when it comes to the problem of picking the points of the representation space.

DEFINITION. – The topological space $\mathrm{Specval}_x G$ (called the *x-valuation spectrum* of G) consists of all (equivalence classes of) x-valuations of an x-closed group G with the sets $D(a) = \{v|v(a) = e,\ a \in G^+\}$ as basic open sets. (The notion of equivalence of x-valuations extends in an obvious way the usual notion of equivalence between Krull-valuations.)

For every x-closed group G we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec\,val}_t\,(\Lambda_x(G)) & \xrightarrow{\beta} & \mathrm{Spec}_t\,(\Lambda_x(G)) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\delta}\downarrow \\ \mathrm{Spec\,val}_x\, G & \xrightarrow{\gamma} & \mathrm{Spec}_x\, G \end{array} \tag{11.3}$$

where α is the restriction map related to Lorenzens theorem (Corollary 1 of Theorem 1), δ is the map $P_t \to P_t \cap G$ and γ is the map $v \to v^{-1}((\mathrm{Im}\, v)^+ \setminus \{e\})$. Finally β is just the specialization of γ to the case $x = t$.

By Lorenzens theorem, α is a bijection. This bijection is obviously continuous, but seemingly not in general a homeomorphism. It follows from the following theorem, however, that α is surely a homeomorphism when G is an x-Prüfer group. This theorem also shows that β is a homeomorphism for any x-closed group G. The maps γ and δ are both continuous but in general not bijective. If they are bijective they are also homeomorphisms. More precisely:

THEOREM 8. – *The following conditions are equivalent for an x-closed group G.*

1) *G is an x-Prüfer group.*

2) *Every localization at a prime x-ideal of G^+ yields an x-valuation monoid in G.*

3) *The map* γ: $\mathrm{Spec\,val}_x G \to \mathrm{Spec}_x G$ *is a (surjective) homeomorphism.*

4) *The map* δ: $\mathrm{Spec}_t\,(\Lambda_x(G)) \to \mathrm{Spec}_x G$ *is a (surjective) homeomorphism.*

PROOF. – We first show that 1) and 2) are equivalent. If G is x-Prüfer it is clear that G is also x_S-Prüfer where S is the complement of a prime x-ideal P_x in $D = G^+$.

It is sufficient to observe that the equality $A_x \circ B_x = D$ entails the equality $A_{x_S} \circ \circ B_{x_S} = S^{-1}D$. (We have quite generally that $S^{-1}(A_x \circ B_x) = S^{-1}A_x \circ S^{-1}B_x$ where the latter $\circ$ denotes the x_S-multiplication.) In order to establish the implication $1 \Rightarrow 2$ it is hence sufficient to show that an x-local and x-Prüfer monoid is an x-valuation monoid (observing that $S^{-1}D$ is an x_S-local monoid in the sense that the set $S^{-1}P_x$ of all non-units of $S^{-1}D$ forms a maximal x_S-ideal of $S^{-1}D$). The fact that $S^{-1}D$ produces a total order in G is proved in the case $x = d$ in Proposition 4, p. 67 in [21] and this proof carries over to the general case without change. By an *x-valuation monoid* in G we understand a set of the form $v^{-1}(\Gamma^+)$ where $v: G \to \Gamma$ is an x-valuation of G, (see Corollary 2 of Theorem 2). In the present situation the canonical map $v: G \to G/SS^{-1} = \Gamma$ will in fact be an x-valuation with $S^{-1}D$ as corresponding valuation monoid. For if $\{a_1, \ldots, a_n\} \subset S^{-1}D$, there exists an element $s \in S$ such that $s\{a_1, \ldots, a_n\} \subset D$ and hence also $s\{a_1, \ldots, a_n\}_x \subset D$ since D is (by definition) an x-ideal in G. Thus $\{a_1, \ldots, a_n\}_x \subset S^{-1}D$ and $v^{-1}(\Gamma^+)$ is an x-ideal in G. By «translation» it follows that inverse images of principal ideals in Γ are x-ideals in G. Since the given x-system is supposed to be of finite character we conclude that $v^{-1}(A_t)$ is an x-ideal in G for any bounded set $A \subset \Gamma$.

In order to show that $2 \Rightarrow 1$ it is (according to Lemma 1 in paragraph 10) enough to prove that every x-ideal of the form $(a, b)_x$ is invertible. By the fact that every localization at a prime x-ideal gives rise to a total order, we must have

$$(a)_{x_S} \subset (b)_{x_S} \quad \text{or} \quad (b)_{x_S} \subset (a)_{x_S} .$$

This entails easily that

$$S^{-1}\big((a)_x \circ (b)_x\big) = S^{-1}\Big((a, b)_x \circ \big((a)_x \cap (b)_x\big)\Big)$$

which by the globalization formula of [4] gives

$$(ab)_x = (a)_x \circ (b)_x = (a, b)_x \circ \big((a)_x \cap (b)_x\big) .$$

Since a principal x-ideal is invertible, it follows that $(a, b)_x$ is invertible.

By assuming 2) we see that the map γ has an inverse, as constructed in the first part of the proof. In fact, γ is then a homeomorphism because the basic open sets in the two topologies correspond to each other as follows:

$$\{v | v(a) = e\} \leftrightarrow \{\gamma(v) | a \notin \gamma(v)\} .$$

That 3) implies 2) is obvious. From the implication $1 \Rightarrow 3$ and the fact that a *GCD*-group is always a t-Prüfer group it follows that there is a bijection between the t-valuations and the prime t-ideals in such a group. This establishes of course that β is a homeomorphism for any x-closed group G. It follows that γ is bijective if and only if δ is bijective. This shows in particular that $4 \Rightarrow 1$ (since the bijectivity of γ implies 1)). On the other hand if G is an x-Prüfer group (i.e. γ is bijective) then δ will be bijective. More precisely, it follows in conjunction with the equivalence of 6) and 7) in Theorem 3 that δ and the map Φ of that theorem are inverses of each other when Φ is restricted to $\operatorname{Spec}_x G$. From this we can infer that a basic open set $D(a) = \{P_t | a \notin P_t\} \subset \operatorname{Spec}_t \Lambda(G)$ by δ corresponds to an open set in $\operatorname{Spec}_x G$. For $a \in G^+$ this is obvious since then $\delta(D(a)) = \{P_x | a \notin P_x\}$. In case $a \in \Lambda(G)^+ \setminus G^+$ we can prove that

$$\delta(D(a)) = \{P_x | (a) \cap G^+ \not\subset P_x\} \tag{11.4}$$

or equivalently

$$a \notin P_t \Leftrightarrow (a) \cap G^+ \not\subset \delta P_t) . \tag{11.5}$$

Since $a \in P_t \Rightarrow (a) \cap G^+ \subset P_t \cap G^+$ the implication $\Leftarrow$ in (11.5) is clear. Conversely, since Φ is the inverse of δ it follows that the t-ideal in $\Lambda_t(G)$ which is generated from $(a) \cap G^+$ is (a). If $(a) \cap G^+ \subset P_t \cap G^+$ we therefore obtain $a \in P_t$ as desired.

Since the right-hand side of (11.4) is evidently a union of basic open sets in $\operatorname{Spec}_x G$ it follows that δ is an open map and this completes the proof of the theorem. ∎

It is clear from the above proof that the mere bijectivity of either of the maps γ or δ is sufficient to assure that G is an x-Prüfer group. In case of γ the bicontinuity follows immediately from the bijectivity whereas our proof of the openness of δ relies on Theorem 3.

We spell out two special cases.

Corollary 1. – *An integrally closed domain R is a Prüfer domain if and only if the map δ induces a homeomorphism between the prime spectra of R and its Kronecker function ring* $\mathbf{K}(R)$.

(See Corollary 2 of Theorem 1 and succeeding remarks.)

Corollary 2. – *A t-closed group G is a t-Prüfer group if and only if the map δ gives a homeomorphism between the prime t-spectra of G and its Lorenzen t-group.*

We shall say that a subgroup G of a GCD-group $\mathbf{D}$, as on p. 12-13 is *dense* if the axiom (3) of « a theory of quasi-divisors » is satisfied. As a joint corollary of Theorems 7 and 8 we get

Corollary 3. – *Every x-Prüfer group G may be represented as a dense subgroup of the GCD-group of all sections in a sheaf of totally ordered groups over the quasi-compact space* $\operatorname{Spec}_x G$.

In fact, the axiom (3) of paragraph 8 amounts to the condition that any element in the GCD-group is an infimum of a finite number of elements of the given dense subgroup. In the case of a pair $G \to \Lambda_x(G)$ the latter denseness property is by Theorem 3 equivalent to G being an x-Prüfer group.

In all the cases where the map α (in the commutative diagram (11.3)) is a homeomorphism we obtain a sheaf representation of the group G over Spec $\text{val}_x G$, simply by restricting the full sectional representation of $\Lambda_x(G)$ to G. In case of an arbitrary x-closed group we can obtain the same type of representation by transferring the topology of Spec $\text{val}_t(\Lambda_x(G))$ to Spec $\text{val}_x G$ via the bijection α. It seems reasonable to conjecture that α is a homeomorphism if and only if G is an x-Prüfer group. When trying to prove that α is an open map one encounters a problem which is analogous to the one in connection with the openness of δ. By the very definition of the GCD-functor (see (4.4)) we get

$$(11.6)\qquad \alpha\left(D\left(\frac{A_{x_a}}{B_{x_a}}\right)\right) = \alpha\{v \in \text{Spec val}_t(\Lambda_x(G))\,|\,v(a_1)\wedge\ldots\wedge v(a_m)[v(b_1)\wedge\ldots\wedge v(b_n)]^{-1} = e\}$$

where $A_{x_a} = (a_1, \ldots, a_m)_{x_a} \subset (b_1, \ldots, b_n)_{x_a} = B_x$.

Without any further hypothesis it is not clear how the set (11.6) can be written as a union of basic open sets $D(a) \subset \text{Spec val}_x G$ with $a \in G^+$. If G is an x-Prüfer group, however, we know that an element in $\Lambda_x(G)^+$ may be identified with an integral and finitely generated x-ideal $C_x = (c_1, \ldots, c_b)_x$ (i.e. with all $c_i \in G^+$). In this case

$$\alpha(D(C_x)) = \{v \in \text{Spec val}_x G\,|\,v(c_1)\wedge\ldots\wedge v(c_k) = e\} = D(c_1) \cup \ldots \cup D(c_k)$$

and α is hence an open map.

Although this seems to reconfirm that the openness of α depends on the x-Prüfer condition we have not been able to prove the converse: α is open $\Rightarrow G$ is an x-Prüfer group.

12. – Germinal ideals and real representations.

We shall now relate the material of the preceding paragraph to Keimel's sectional representation theory for GCD-groups. His approach is based on the notion of a germinal 1-ideal which in a purely algebraic form imitates the analytical notion of an ideal of vanishing germs at a given point. Without using Keimels general machinery this notion will quickly lead us to a quite satisfactory sectional representation theorem for regular GCD-groups with a formal unit (bearing in fact a considerable resemblance to Stone's representation theorem for Boolean algebras).

The 1-ideals of a GCD-group form an ideal system with respect to the «multiplication» $a \circ b = |a| \wedge |b|$. Let $\mathrm{Spec}_1 G$ denote the family of prime 1-ideals P equipped with the spectral topology where the basic open sets are given by $E(a) = \{P \in \mathrm{Spec}_1 G | a \notin P\}$. (For simplicity we are dropping the subscript 1 in the prime 1-ideals, thereby also avoiding any confusion with t-ideals.) For any subset $A \subset G$, $E(A)$ denotes the open set $\{P | A \not\subset P\} = \bigcup_{a \in A} E(a)$.

We now fix $P \in \mathrm{Spec}_1 G$ and let U denote an open neighbourhood of P. We put

$$O_U = \bigcap_{Q \in U} Q \quad \text{and} \quad O_P = \bigcup O_U$$

(where the latter union is taken over all open neighbourhoods U of P).

The set O_P is an 1-ideal contained in P which is called the *germinal 1-ideal* associated with P. A sheaf of GCD-groups may now be defined over $\mathrm{Spec}_1 G$ by choosing G/O_P as the stalk corresponding to P. Every element $g \in G$ will give rise to a «section» $\hat{g}$ in the disjoint union F of these stalks by putting

$$\hat{g}(P) = g_P$$

where g_P denotes the residue class in G/O_P to which g belongs. This induces a projection map $\pi\colon F \to \mathrm{Spec}_1 G$ by putting $\pi(\hat{g}(P)) = P$. In order to make $(\mathrm{Spec}_1 G, F, \pi)$ into a sheaf of GCD-groups we equip F with the finest topology making all the maps $\hat{g}$ continuous.

An alternative approach, leading to the same sheaf, is to start out with the presheaf $U \to G/O_U = G(U)$ where every inclusion $V \subset U$ gives rise to a canonical homomorphism of GCD-groups $G/O_U \to G/O_V$.

In case G has a formal unit (i.e. an element u such that $\{u\}_1 = G$) Keimel proves that the map $g \to \hat{g}$ gives an isomorphism of G onto the GCD-group $\Gamma(\mathrm{Spec}_1 G, F)$ consisting of all global sections of F. As already indicated, this sectional representation has the disadvantage that the stalks need not be totally ordered. A natural condition which assures this is the condition that every prime 1-ideal is identical with its associated germinal 1-ideal: $P = O_P$. This condition is in turn equivalent to the condition that every prime 1-ideal is maximal. This equivalence results from the fact that O_P equals the intersection of all (minimal) prime 1-ideals contained in P (see Proposition 6.6 in [25]).

Whereas our approach yields quasi-compactness of the base space and total order of the stalks for general GCD-groups, the corresponding properties are obtained in Keimels approach only when G has a formal unit and the germinal 1-ideal which is associated to a prime 1-ideal is itself prime. (See Theorem 10.6.2 in [10] and its corollaries.) For regular GCD-groups the two approaches give sectional representations which bear a certain resemblance to each other in that they both have real groups as stalks. But apart from this there are marked differences, stemming above all from the different topological properties of $\mathrm{Spec}_t G$ and $\mathrm{Spec}_1 G$.

It should be noted, however, that Keimel is able to dispense with the condition that O_P is a prime 1-ideal and still obtain a sheaf representation with totally ordered stalks. This is done by restricting the given sheaf to $\operatorname{Spec\,min}_1 G$ consisting of the minimal prime 1-ideals with the subspace topology induced from $\operatorname{Spec}_1 G$. For a minimal prime 1-ideal is always identical with its associated germinal 1-ideal and the stalk is hence totally ordered. It seems, however, that the restriction to $\operatorname{Spec\,min}_1 G$ further damages the fullness of the representation. Without a formal unit Keimel can only claim that sections with quasi-compact support on $\operatorname{Spec}_1 G$ come from elements in G. When restricting the sheaf to $\operatorname{Spec\,min}_1 G$ even this is no longer true.

Although this is somewhat of a digression from the main theme of the present paper we shall close these considerations on sheaf representation of GCD-groups by proving the following rather specialized representation theorem (which in spirit comes close to Stones topological representation of Boolean algebras).

THEOREM 9. – *Every regular GCD-group with a formal unit is isomorphic to the GCD-group of all sections in a sheaf of real groups over a totally disconnected, compact Hausdorff space.*

PROOF. – We shall give a direct proof of this theorem which is based on the notion of a germinal 1-ideal but which avoids any use of the material in Chapter 10 of [10]. In particular we shall avoid the use of Keimel's « standard construction » (10.4.7, p. 212 in [10]) and the succeeding main theorem 10.6.2. Instead we shall base the proof on the consideration of the presheaf $\boldsymbol{L}_Y$ defined over the space $Y = \operatorname{Spec}_1 G$ by the assignment $U \to G/O_U = G(U)$ and combine this with the use of NAKANO's chinese remainder theorem for 1-ideals [36].

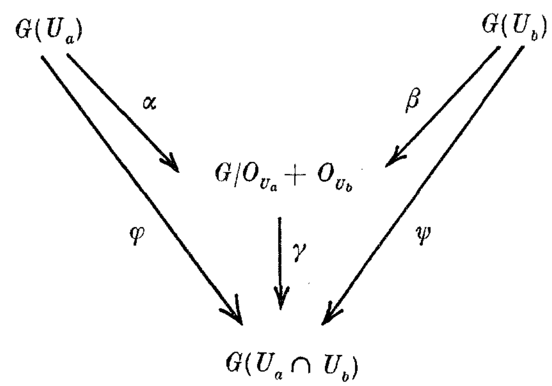
Let us first verify the topological properties of $\operatorname{Spec}_1 G$ announced in the theorem. A formal unit is an (integral) element u in G such that $\{u\}_1 = G$. It is easily seen that the existence of a formal unit in G is equivalent to the quasi-compactness of $\operatorname{Spec}_1 G$ (see p. 16 in [25]). The Hausdorff property is likewise an immediate consequence of the fact that there exists no inclusion relation between two different prime 1-ideals in G. That $\operatorname{Spec}_1 G$ is totally disconnected results from the fact that the basic open sets $U_a = E(a)$ are also closed. In fact, for any a and $a^\perp = e{:}a = \{b \mid |b| \wedge |a| = e\}$ we have the relations

$$E(a) \cup E(a^\perp) = Y \quad \text{and} \quad E(a) \cap E(a^\perp) = \emptyset\,.$$

This follows from the fact that exactly one of the two relations $a \in P$ or $a^\perp \subset P$ holds for each $a \in G$.

We shall next verify that the presheaf $\boldsymbol{L}_Y$ is a sheaf. Hence, let $\{U_a\}$ with $a \in A \subset G$ be a covering of Y by basic open sets and let the family $\{g_a \in G(U_a) \mid a \in A\}$ be selected in such a way that for each pair of elements $a, b \in A$ the presheaf images of g_a and g_b in $G(U_a \cap U_b)$ are equal. We must show that there exists a unique $g \in G = G(Y)$ whose image in $G(U_a)$ is g_a for all $a \in A$.

Since the unicity is obvious let us pass to the existence. Consider the diagram



The two « exterior » maps φ and ψ are ordinary presheaf maps whereas the « inner » maps α, β, γ are canonical maps induced on the factor groups by the inclusions $O_{U_a}, O_{U_b} \subset O_{U_a} + O_{U_b} \subset O_{U_a \cap U_b}$. The crucial point is that the regularity condition in the theorem (every prime 1-ideal is maximal) assures that also $O_{U_a \cap U_b} \subset O_{U_a} + O_{U_b}$ such that γ becomes the identity map. In fact, when this latter inclusion is interpreted in the spectral topology of Y it simply amounts to the inclusion $\overline{U}_a \cap \overline{U}_b \subset \subset \overline{U_a \cap U_b}$ which is trivially true since U_a and U_b are closed sets.

By the compactness of Y we can select a subcovering $\{U_b\}$ of $\{U_a\}$ with $b \in B$ for some finite subset B of A. We now apply Nakano's chinese remainder theorem for 1-ideals [36] to the finite families $\{O_{U_b}\}$ and $\{g_b\}$. Actually, by the initial compatibility condition on the g_a's we have $g_b \equiv g_c (\text{mod } O_{U_b \cap U_c})$ which by the identity $O_{U_b \cap U_c} = O_{U_b} + O_{U_c}$ amounts to

$$g_b \equiv g_c \ (\text{mod } (O_{U_b} + O_{U_c}))$$

for all $b, c \in B$. By Nakano's theorem there exists a $g \in G$ such that

$$g \equiv g_b \ (\text{mod } O_{U_b}) \,. \tag{12.1}$$

This means that g is mapped onto g_b for all $b \in B$ by the given presheaf maps $G(Y) \to G(U_b)$. We now claim that

$$g \equiv g_a \ (\text{mod } O_{U_a}) \tag{12.2}$$

for all $a \in A$. Since $O_{U_b} \subset O_{U_a \cap U_b}$ (12.1) gives

$$g \equiv g_b \ (\text{mod } O_{U_a \cap U_b}) \,. \tag{12.3}$$

Combining (12.3) with the initial condition $g_a \equiv g_b \,(\mathrm{mod}\ O_{U_a \cap U_b})$ we obtain

$$g \equiv g_a \,(\mathrm{mod}\ O_{U_a \cap U_b}) \tag{12.4}$$

for all $b \in B$.

Using (12.4) together with $O_{U_a} = \bigcap_{b \in B} O_{U_a \cap U_b}$ we get (12.2) as desired. This finishes the proof that $\boldsymbol{L}_Y$ is a sheaf and that we hence have an isomorphism of *GCD*-groups $G \simeq \Gamma(Y, \boldsymbol{L}_Y)$.

For the remaining part of the theorem we observe that the very definition of a direct limit gives

$$\varinjlim G/O_U = G/O_P$$

where O_P is the germinal 1-ideal belonging to P and the limit is taken over all spectral (basic) open neighbourhoods of P. Since P is a minimal prime 1-ideal it follows that $O_P = \bigcap_{Q \subset P} Q = P$ and the stalk at P of the sheaf $\boldsymbol{L}_Y$ will hence be isomorphic to the totally ordered group G/P. Since P is a maximal 1-ideal of G this stalk will be order isomorphic to a subgroup of the group of real numbers and this completes the proof of the theorem. ∎

This paper deals with basic arithmetical questions linked to the notion of a t-ideal. With respect to this perspective, one may say that our considerations on germinal 1-ideals and the associated sheaf representation are somewhat marginal. Prime 1-ideals are, however, intimately linked to the prime t-ideals and it is essential to be able to play on both of these types of objects and the duality between them. It should also be noted that the crux of the preceding proof (i.e. the chinese remainder theorem of Nakano) has a distinctly arithmetical origin. Nakano's theorem arose directly out of considerations by KRULL [30] and RIBENBOIM [42] concerning approximation theorems in valuation theory. (For a more general treatment of the relationship between sheaf representations and chinese remainder theorems see CORNISH [14].)

Theorem 9 deals with real *sectional* representation of *GCD*-groups. Let us now turn to real *functional* representation of (partially) ordered groups. The literature on this topic is somewhat confusing and difficult to penetrate. There seems to be a need for a comprehensive exposition which surveys the whole field and which clarifies the interrelations between the different approaches and the different underlying assumptions. A comparison is made difficult by the fact that different authors have different candidates as to the objects which are chosen as the points of the representation space (i.e. the points making up the domain of definition for the representing functions). We shall now show that a neat exposition of the topic of real functional representation of ordered groups is achieved by the use of the Lorenzen t-group and the *GCD*-functor. This is really nothing more than applying the language of the present paper in order to give a more clear exposition of the main content of an interesting but rather cryptic paper by I. FLEISCHER [17].

THEOREM 10. – *A completely integrally closed group G ($\neq \{e\}$) with an archimedian element (strong unit) is order isomorphic to a separating group of continuous real-valued functions on a compact (Hausdorff) space.*

PROOF. – We recall that an archimedian element of G is an element $u > e$ such that for every $g \in G$ there exists $n \geqslant 1$ with $u^n \geqslant g$. Since G is completely integrally closed, it can be embedded (order-isomorphically) in its group of v-ideals G^*. This latter group is a GCD-group such that its t-system has a Lorenzen system x as its trace on G, making G regularly x-closed. G is an ordered subgroup of $\bar{G} = \Lambda_x(G)$ which in turn is an ordered subgroup of G^*. The trace of the t-system in G^* is the t-system in $\bar{G}$ which in turn induces the x-system in G as its trace. (Examples 1° and 2° p. 52 in [21] and Theorem 3 above.) The fact that the group G^* is lattice-complete implies that all its v-ideals are principal and hence form a group isomorphic to G^*. This entails that G^* is completely integrally closed, a property which transfers to its ordered subgroup $\bar{G}$. Thus $\bar{G}$ appears as a completely integrally closed GCD-group with the same archimedian element as G (since any element of $\bar{G}$ is dominated by an element of G).

It is known that for a GCD-group with an archimedian element the condition of complete integral closure amounts to the property that the intersection of its maximal 1-ideals reduces to the identity element (or equivalently to the fact that its monoid of integral elements is equal to the intersection of all the t-valuation monoids arising from localization at minimal prime t-ideals). A short proof of the part of this result which interests us here runs as follows: For any GCD-group H with an archimedian element $u > e$ we can to each prime 1-ideal P_i select a maximal 1-ideal M_i containing P_i (which is itself prime due to the presence of u). This gives rise to obvious homomorphisms of GCD-groups

$$H \xrightarrow{\varphi} \prod_{i\in I} H/P_i \xrightarrow{\psi} \prod_{i\in I} H/M_i$$

where $\varphi = \{\varphi_i\}_{i\in I}$ is known to be injective. Assume now that H is completely integrally closed and that $\operatorname{Ker}(\psi\circ\varphi) \neq \{e\}$. Since $\operatorname{Ker}(\psi\circ\varphi)$ is an 1-ideal we can assume $a \in \operatorname{Ker}(\psi\circ\varphi)$ with $a > e$ and we must then have $a^n \nleqslant u$ for a certain $n \geqslant 1$, because of the complete integral closure. In view of the fact that φ is an isomorphism and H/P_i is totally ordered, this entails $\varphi_i(a)^n > \varphi_i(u)$ for some i. Since $\varphi_i(u)$ is an archimedian element in H/P_i it follows that $\varphi_i(u) \notin M_i$ and hence that $(\psi_i\circ\varphi_i(a))^n$ is strictly positive in H/M_i. From this we infer that $\psi_i\circ\varphi_i(a)$ is different from the identity element in H/M_i, contradicting that $a \in \operatorname{Ker}(\psi\circ\varphi)$.

Once the « strong 1-semisimplicity » has been proved, the functional representation of $\bar{G}$ over the *set* $\operatorname{Spec}\min_t \bar{G}$ (or equivalently over the *set* $\operatorname{Spec}\max_1 \bar{G}$), results immediately since $\bar{G}/P_1$ is a real group for any maximal 1-ideal P_1. Endowing the set $\operatorname{Spec}\min_t \bar{G}$ with the coarsest topology making all the representing functions continuous, we clearly obtain a representation of $\bar{G}$ which has the properties announced in the theorem.

It remains to be seen how the representation of $\bar{G}$ induces the desired representation of G via inclusion and how the representation space may be described in terms of entities in G. It is convenient to do the latter part first: We know already that the maps α and β in the diagram (11.3) are bijections (α is a bijection because of Lorenzens theorem and β is a bijection since a *GCD*-group is a t-Prüfer group). These two bijections induce the bijections

$$\text{(12.5)} \qquad \operatorname{Spec}\max \operatorname{val}_t G \to \operatorname{Spec}\max \operatorname{val}_t \bar{G} \to \operatorname{Spec}\min_t \bar{G}$$

where the left hand side denotes the set of all maximal t-valuation monoids of G—or equivalently the set of all real-valued t-valuations of G. We thus only transport the above-mentioned weak topology of the right-hand side of (12.5) to the left-hand side, which indeed consists of a family of objects directly attached to G.

We must finally show that the restriction of the representation from $\bar{G}$ to G retains the property of point-separation. Assume hence that $\hat{g}(v_1) = \hat{g}(v_2)$ for $v_1, v_2 \in \operatorname{Spec}\max \operatorname{val}_t G$ and all $g \in G$. This means that $v_1(g) = v_2(g)$ for all $g \in G$. By the Lorenzen theorem, v_1 and v_2 are uniquely extendible to $v_1' = \alpha^{-1}(v_1)$ and $v_2' = \alpha^{-1}(v_2) \in \operatorname{Spec}\max \operatorname{val}_t \bar{G}$ (using the notation of (11.3)). Thus $v_1'(h) = v_2'(h)$ or $\hat{h}(v_1') = \hat{h}(v_2')$ for all $h \in \bar{G}$. This means that $v_1' = v_2'$ and hence $v_1 = v_2$ as desired. ∎

At first sight, the reader will probably have some difficulty in recognizing the above proof as a precision of Fleischers proof, which hardly contains more than hints. But if one observes that his group $\bar{G}$ occurring at the bottom of page 261 of [17] is nothing but our group $\bar{G} = \Lambda_x(G)$ and that the « maximal closed semigroups » in the second paragraph of page 262 coincide with our maximal t-valuation monoids, one sees that the spirit of our proof is in fact quite close to Fleischers proof-suggestions—although we make a much more explicit use of our heritage from Lorenzen. Another exposition of Fleischers work has been given by P. RIBENBOIM in [41]. As to the origin of Theorem 10, it goes back to more analytical work of YOSIDA and STONE and a later paper by KY FAN [31]. The present neat formulation seems to be due to Fleischer. RIBENBOIM [41] (Theorem 11, p. 75) gives reference to JAFFARD [22] for a similar result, but this reference does not seem to be quite accurate. Theorem 10 occurs also, essentially, as a corollary of a more complicated and more general representation theory given in [38].

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