Annali di Matematica pura ed applicata
(IV), Vol. CLV (1989), pp. 243-260

Quenching, Nonquenching, and Beyond Quenching for Solution of Some Parabolic Equations (*).

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Summary. – *In this paper we examine the first initial boundary value problem for* $u_t = u_{xx} + \varepsilon(1-u)^{-\beta}$, $\varepsilon > 0$, $\beta > 0$, *on* $(0, 1)\times(0, \infty)$ *from the point of view of dynamical systems. We construct the set of stationary solutions, determine those which are stable, those which are not and show that there are solutions with initial data arbitrarily close to unstable stationary solutions which quench (reach one in finite time). We also examine the related problem* $u_t = u_{xx}$, $0 < x < 1$, $t > 0$; $u(0, t) = 0$, $u_x(1, t) = \varepsilon(1-u(1, t))^{-\beta}$. *The set of stationary solutions for this problem, and the dynamical behavior of solutions of the time dependent problem are somewhat different.*

1. – Introduction.

In this paper, we present some new results for an old problem first discussed by KAWARADA [8] and later by several authors [1-4, 9-13]. We formulate the problem as in [9] since a simple scaling makes the problem as considered in [1, 3, 8, 9, 12] equivalent to:

$$\text{(A)}\qquad \begin{aligned} &u_t = u_{xx} + \varepsilon(1-u)^{-\beta} && 0 < x < 1\,, \quad 0 < t < T\\ &u(0, t) = u(1, t) = 0 && 0 \leqslant t \leqslant T\\ &u(x, 0) = u_0(x)\,, \quad u_0 < 1 && 0 \leqslant x \leqslant 1\,. \end{aligned}$$

Here $\varepsilon, \beta > 0$ and the interest is in solution of (A) taking values in $[0, 1)$ so that $u_0 \geqslant 0$. In [2, 3, 9, 12] $u_0(x) \equiv 0$ while Kawarada also focused his attention on $\varepsilon > 0$, $\beta = 1$.

Inspired by a result of [1] in which it was shown that for all $\varepsilon > 0$, $\beta \geqslant 1$, it is possible to select initial values so that u quenches (reaches one in finite time) we present here some old and new results that allow us to obtain a more complete understanding of the dynamical problem for all $\varepsilon > 0$, $\beta > 0$ and (smooth) u_0 $(u_0 < 1$ on $[0, 1))$.

(*) Entrata in Redazione il 7 dicembre 1987.

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This research was sponsored by the U.S. Air Force Office of Scientific Research, Air Force Systems Command Grants 84-0252 and 88-0031. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes not withstanding any copyright notation therein.

In [10], we considered a problem closely related to (A), namely

$$\text{(B)} \qquad \begin{array}{lll} u_t = u_{xx} & 0 < x < 1\,, & 0 < t < T \\ u(0, t) = 0 & & 0 \leqslant t < T \\ u_x(1, t) = \varepsilon(1 - u(1, t))^{-\beta} & & 0 < t < T \\ u(x, 0) = u_0(x) \quad (u_0 < 1) & & 0 \leqslant x \leqslant 1 \end{array}$$

in the special case that $u_0 \equiv 0$. We showed that either u exists for all time and

$$\lim_{t \to +\infty} u(x, t) = a\_x$$

where $a\_$ is the smallest root of $a(1-a)^\beta = \varepsilon$ or else $u(1, \cdot)$ reaches one in finite time (quenches) and $u_t(1, \cdot)$ becomes infinite. Higher dimensional results were considered in [11].

In general, problems (A) and (B) are expected to have similar structure. However, there are certain differences in the sets of stationary solutions and consequently in the behavior of the time dependent solutions. We shall also present some new results for (B) here and compare them with those for (A).

Let us summarize our principal results when $\beta = 1$. First, a little history. It has been shown [2, 12], that there is $\varepsilon_0 > 0$ such that if $\varepsilon > \varepsilon_0$ there is no stationary solution of (A) and every solution of (A) must quench. Secondly, if $0 < \varepsilon < \varepsilon_0$, there are exactly two stationary solutions of (A), call them $f_+(x, \varepsilon)$ and $f_-(x, \varepsilon)$ with $0 < f_- < f_+ < 1$ on $(0, 1)$, and these coalesce to a single stationary solution if $\varepsilon = \varepsilon_0$. In this case, it was shown [2, 12] that for $0 < \varepsilon \leqslant \varepsilon_0$, and $0 \leqslant u_0 \leqslant f_-$ on $[0, 1]$ then $u \to f_-$ as $t \to +\infty$. (This was shown for $u_0 \equiv 0$, but it follows by comparison, for other u_0's.)

In this paper, we complete the analysis by showing that there are u_0's in every L_∞ neighborhood of f_+ for which u quenches in a finite time while if $f_- < u_0 < f_+$ on $(0, 1)$ then $u \to f_-$ as $t \to +\infty$.

This analysis is in the spirit of MATANO [14] for strongly orderpreserving (systems of) equations.

Blow up to u_t for (A) at quenching was established for $\beta = 1$ by Kawarada. Acker and Kawohl extended this result to higher dimensional problems in a ball. (See § 4 for a brief discussion of their result.) In [18] we have improved their result (1).

We claimed to have proved blow up for u_t for (B) at quenching in [10]. However, we have found an error in our proof. We shall give a correct (and more general) proof in § 5. In [9], we proposed to consider what happens to solutions of (A)

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(1) See Note added in Proof.

beyond quenching. An appropriate model for such a study was considered by PHILLIPS [17]. His problem in our formulation (and in one dimension) is:

$$(\mathrm{A}_\omega)\qquad \begin{aligned} u_t &= u_{xx} + \varepsilon(1-u)^{-\beta}\chi(u<1) && 0<x<1\,, \quad 0<t<T\\ u(0,t) &= u(1,t) = 0 && \qquad\qquad 0<t<T\\ u(x,0) &= u_0(x) \quad u_0\leqslant 1 && 0<x<1 \end{aligned}$$

with $u_0<1$ at $x=0,1$, $u_0\leqslant 1$, $0<\beta<1$ and where $\chi[A]$ is the indicator function of the set A. PHILLIPS proved global existence of solutions of weak (distribution) solutions of such problems. Unfortunately, he was not able to prove uniqueness of such solutions.

BANDLE and BRAUNER [4] also considered the behavior of (A) beyond quenching. Their results are also incomplete in this regard. The lack of uniqueness makes it difficult to study the problem beyond quenching. We shall make a few observations about (A_ω) and propose a few problems for it.

The plan of the paper is as follows: In the next section we characterize the sets of stationary solutions for (A) and (B). In the third section we define and present the stability and quenching results for these problems. In the fourth section, we discuss the blow up of u_t: Finally, we propose some problems for (A_ω).

We shall not discuss the question of local existence or continuation of (L^∞) solutions of either (A) or (B) here. This was done in [10] for (B) and follows for (A) by the same type of argument (or from more general considerations to be found in standard treatises).

A word about notation. A solution of (A) with initial values $u_0(\cdot)$, will be written variously as $u(x,t;\varepsilon,0_0)$, $u(x,t;u_0)$, $u(x,t;\varepsilon)$ or, where no confusion can occur, as $u(x,t)$. Similarly, if (A) has only one positive stationary solution, we will write it as $f(x;\varepsilon)$. If there are more than one we will label them by the order of their maximum values.

2. – Stationary solutions.

We begin with the study of classical stationary solutions. Some of the results here are probably well known. (See [7] where the n dimensional version was studied for $n>1$.) We include them for completeness. Let

$$\varphi(u) \equiv (1-u)^{-\beta} \qquad -\infty<u<1\,.$$

Let $f(x)(<1)$ be a stationary solution of (A), C^2 on $(0,1)$. Then f solves

$$(\mathrm{SA})\qquad \begin{aligned} 0 &= f''(x) + \varepsilon\varphi(f(x)) \qquad 0<x<1\\ f(0) &= f(1) = 0\,. \end{aligned}$$

Then, since $\varphi > 0$, we have $f'' < 0$, $f > 0$ on $(0, 1)$ and f has exactly one maximum at $\bar{x} \in (0, 1)$. Let $\Phi'(u) = \varphi(u)$, $\Phi(0) = 0$. Then f also solves

$$\tfrac{1}{2}(f'(x))^2 + \varepsilon\Phi(f(x)) = \varepsilon\Phi(M) \tag{2.1}$$

where $M = f(\bar{x})$. Since $g(x) = f(1 - x)$ is also a solution of (SA) and of (2.1), we have

$$\bar{x} = 1 - \bar{x} = \frac{1}{\sqrt{2\varepsilon}} \int_0^M [\Phi(M) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta = \frac{1}{2} \tag{2.2}$$

and therefore, $f(x) \equiv f(1 - x)$. (This also follows from [5].) Consequently, there is exactly one solution of (SA) with $f(\frac{1}{2}) = M$. Thus, for $0 \leqslant x \leqslant \frac{1}{2}$, $f(x)$ is given by

$$\sqrt{2\varepsilon}\, x = \int_0^{f(x)} [\Phi(M) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta \tag{2.3}$$

and by $f(x) = f(1 - x)$ if $\frac{1}{2} < x < 1$ where M satisfies (2.2). It remains only to count the solutions of (2.2), Define, for $0 < M < 1$,

$$\theta = \theta(M) = \begin{cases} \big(\ln(1/(1 - M))\big)^{\frac{1}{2}} & \beta = 1\,, \\ (1 - (1 - M)^{\beta-1})^{\frac{1}{2}} & \beta > 1\,, \\ ((1 - M)^{\beta-1} - 1)^{\frac{1}{2}} & \beta < 1\,. \end{cases} \tag{2.4}$$

Then, solution of (2.2) is equivalent to the solution of the following (since $\theta(M)$ is strictly increasing on $(0, 1)$):

$$e^{-\theta^2} \int_0^\theta e^{\sigma^2}\, d\sigma = \sqrt{\varepsilon/8} \qquad \theta \in (0, \infty) \tag{2.2$_1$}$$

when $\beta = 1$;

$$(1 - \theta^2)^{\frac{1}{2}((\beta+1)/(\beta-1))} \int_0^\theta (1 - \sigma^2)^{\beta/(1-\beta)}\, d\sigma = \sqrt{\frac{\varepsilon}{8}(\beta - 1)} \qquad \theta \in (0, 1) \tag{2.2$_\beta$}$$

when $\beta > 1$; and

$$(1 + \theta^2)^{\frac{1}{2}((\beta+1)/(\beta-1))} \int_0^\theta (1 + \sigma^2)^{\beta/(1-\beta)}\, d\sigma = \sqrt{\frac{\varepsilon}{8}(1 - \beta)} \qquad \theta \in (0, \infty) \tag{2.2$_\beta$}$$

for $0 < \beta < 1$.

Define intervals

$$I_\beta = \begin{cases} (0, \infty) & 0 < \beta \leqslant 1 \\ (0, 1) & \beta > 1\,, \end{cases}$$

constants

$$(2.5)\qquad C(\varepsilon, \beta) = \begin{cases} \sqrt{\dfrac{\varepsilon}{8}} & \beta = 1 \\ \sqrt{\dfrac{\varepsilon|\beta - 1|}{8}} & \beta \neq 1 \end{cases}$$

and functions on I_β

$$(2.6)\qquad G(\theta, \beta) = \begin{cases} e^{-\theta^2}\int_0^\theta e^{\sigma^2}\, d\sigma & \beta = 1, \quad \theta \in I_1 \\ (1-\theta^2)^{\frac{1}{2}((\beta+1)/(\beta-1))}\int_0^\theta (1-\sigma^2)^{\beta/(1-\beta)}\, d\sigma & \beta > 1, \quad \theta \in I_\beta \\ (1+\theta^2)^{\frac{1}{2}((\beta+1)/(\beta-1))}\int_0^\theta (1+\sigma^2)^{\beta/(1-\beta)}\, d\sigma & 0 < \beta < 1, \quad \theta \in I_\beta . \end{cases}$$

Then the set of solutions of $G(\theta, \beta) = C(\varepsilon, \beta)$ on I_β is exactly the same as the set of zeros of function

$$H(\theta, \beta) = \begin{cases} e^{\theta^2}(G(\theta, 1) - C(\varepsilon, 1)) & \beta = 1, \quad \theta \in I_1 \\ (1-\theta^2)^{-\frac{1}{2}((\beta+1)/(\beta-1))}(G(\theta, \beta) - C(\varepsilon, \beta)) & \beta > 1, \quad \theta \in I_\beta \\ (1+\theta^2)^{-\frac{1}{2}((\beta+1)/(\beta-1))}(G(\theta, \beta) - C(\varepsilon, \beta)) & 0 < \beta < 1, \quad \theta \in I_\beta \end{cases}$$

in I_β.

Clearly, $G(0, 1) = 0$. From L'Hopital's rule

$$G(0, 1) = \lim_{\theta \to +\infty} G(\theta, 1) = 0$$

and for $\beta > 1$, $G(0, \beta) = 0$ and

$$\lim_{\theta \leftarrow 1^-} G(\theta, \beta) = 0 .$$

From a routine claculation, one easily sees that $\partial H/\partial\theta(\theta, \beta)$ changes sign exactly once in I_β and $H_\theta(0+, \beta) > 0$ for $\beta \geqslant 1$. Thus, for $\beta \geqslant 1$, $(2.2)_\beta$ has zero, one or two solutions accordingly as

$$(2.7.1)\qquad C(\varepsilon, \beta) > \max_{\theta \in I_\beta} G(\theta, \beta)$$

$$(2.7.2)\qquad C(\varepsilon, \beta) = \max_{\theta \in I_\beta} G(\theta, \beta)$$

$$(2.7.3)\qquad C(\varepsilon, \beta) < \max_{\theta \in I_\beta} G(\theta, \beta) .$$

For the case $\beta \in (0, 1)$, the situation is somewhat more complicated. Here we have

$$G(0, \beta) = 0$$

while

$$\lim_{\theta \leftarrow \infty} G(\theta, \beta) = (1 - \beta)/(1 + \beta) .$$

In this case $H(0, \beta) < 0$ and

$$H_\theta(\theta, \beta) = \left(\frac{1+\beta}{1-\beta}\right)(1 + \theta^2)^{\beta/(1-\beta)}\left[\frac{1-\beta}{1+\beta} - C(\varepsilon, \beta)\frac{\theta}{\sqrt{1+\theta^2}}\right].$$

If

$$C(\varepsilon, \beta) \leqslant \left(\frac{1-\beta}{1+\beta}\right)$$

then $H_\theta > 0$ on I_β, $H(\theta, \beta) \to +\infty$ as $\theta \to +\infty$. Therefore $(2.2)_\beta$ has exactly one solution in this case.

If

$$C(\varepsilon, \beta) > \left(\frac{1-\beta}{1+\beta}\right), \tag{2.8}$$

then H can have at most two zeros on I_β.

To show that $H(\cdot, \beta)$ can sometimes have at least two zeros (which will be the case if (2.7.3) holds), it suffices to show that $G_\theta(\theta, \beta) < 0$ for all $\theta \gg 1$. We have

$$G_\theta(\theta, \beta) = \theta(1 + \theta^2)^{-\frac{1}{2}((\beta+1)/(\beta-1))} L(\theta)$$

where, with $m = 2\beta/(1 - \beta)$

$$L(\theta) = \theta^{-1}(1 + \theta^2)^{(m/2+1)} - (m + 1)\int_0^\theta (1 + \sigma^2)^{m/2}\, d\sigma .$$

A single integration by parts yields

$$L(\theta) = \theta^{-1}(1 + \theta^2)^{m/2} - m\int_0^\theta (1 + \sigma^2)^{m/2-1}\, d\sigma .$$

We see that for $m = 1$,

$$L(\theta) \approx -\ln\theta \qquad (\theta \to +\infty) .$$

For $m \geqslant 2$, we have that

$$L(\theta) \leqslant \theta^{-1}(1 + \theta^2)^{m/2} - \frac{m}{m-1}\,\theta^{m-1} \approx -\frac{\theta^{m-1}}{(m-1)} \qquad (\theta \to \infty) .$$

A second integration by parts yields

$$L(\theta) = \theta^{-1}(1+\theta^2)^{m/2} - \frac{m}{m-1}\,\theta(1+\theta^2)^{(m-2)/2} - \frac{m(m-2)}{(m-1)}\int_0^\theta (1+\sigma^2)^{(m-4)/2}\,d\sigma\,.$$

For $1 < m \leqslant 2$, we have

$$L(\theta) \approx -\frac{\theta^{m-1}}{m-1} + constant \;\; (\theta \to \infty)\,.$$

Finally, if $0 < m < 1$,

$$L(\theta) \to -\frac{m(m-2)}{(m-1)}\int_0^\infty (1+\sigma^2)^{(m-4)/2}\,d\sigma < 0\,.$$

Thus, in all cases, G_θ is eventually negative if $0 < \beta < 1$.

We have, for (2.2_β), one solution if

(2.8.1) $$0 < C(\varepsilon, \beta) \leqslant (1-\beta)/(1+\beta)$$

or

(2.8.2) $$C(\varepsilon, \beta) = \max_{\theta\in I_\beta} G(\theta, \beta);$$

two solutions if

(2.8.3) $$(1-\beta)/(1+\beta) < C(\varepsilon, \beta) < \max_{\theta\in I_\beta} G(\mathrm{p}, \beta);$$

and no solutions if

(2.8.4) $$C(\varepsilon, \beta) > \max_{\theta\in I_\beta} G(\theta, \beta)\,.$$

THEOREM 2.1 A.

(i) *If* $\beta \geqslant 1$, *there is* $\varepsilon(\beta) > 0$ *such that* (SA) *has two positive stationary solutions* $f_+(\cdot, \varepsilon)$, $f_-(\cdot, \varepsilon)$ *for* $0 < \varepsilon < \varepsilon(\beta)$, *one solution if* $\varepsilon = \varepsilon(\beta)$ *and none for* $\varepsilon > \varepsilon(\beta)$. *When* $0 < \beta < 1$, *there are two positive numbers* $\varepsilon_0(\beta), \varepsilon(\beta)$ *with* $0 < \varepsilon_0 < \varepsilon(\beta)$ *such that* (SA) *has exactly one solution if* $0 < \varepsilon \leqslant \varepsilon_0$ *or* $\varepsilon = \varepsilon(\beta)$, *two solutions if* $\varepsilon_0(\beta) < \varepsilon < \varepsilon(\beta)$ *and none if* $\varepsilon > \varepsilon(\beta) \cdot (\varepsilon_0(\beta) = 8(1-\beta)/(1+\beta)^2)$.

(ii) *On any interval* $I = (\varepsilon_a, \varepsilon_b)$ *where* $M(\cdot) \equiv f(\frac{1}{2}, \cdot)$ *is a continuous function of* ε *which satisfies* (2.2), $f(x, \varepsilon)$ *is a continuous function of* ε *in the uniform norm.*

(iii) *On any interval* I *where* $M(\cdot)$ *is strictly increasing,* $f(x, \cdot)$ *is strictly increasing for all* $x \in (0, 1)$.

Before proving this, let us state the corresponding result for stationary solutions of (B):

$$\text{(SB)}\qquad \begin{aligned} 0 &= f''(x) \qquad 0 < x < 1 \\ f(0) &= 0 \\ f'(1) &= \varepsilon(1 - f(1))^{-\beta} . \end{aligned}$$

The equation that replaces (2.2) is clearly $M(1 - M)^{\beta} = \varepsilon$.

THEOREM 2.1 B.

(i) *For every* $\beta > 0$, *there is* $\varepsilon(\beta) > 0$ *such that* (SB) *has exactly two solutions if* $0 < \varepsilon < \varepsilon(\beta)$, *one solution if* $\varepsilon = \varepsilon(\beta)$ *and no solutions if* $\varepsilon > \varepsilon(\beta)$.

(ii) *This is the same as in the previous theorem.*

(iii) *On any interval* $I \subset (0, \varepsilon(\beta)($, *if* $M(\cdot)$ *is strictly monotone increasing (decreasing)* $f(x, \cdot)$ *is also strictly monotone increasing (decreasing).*

Theorem 2.1B is a simple consequence of the linearity of solutions of (SB).

To prove Theorem 2.1A we first establish (ii), (iii). The solutions satisfy (2.3) on $[0, \frac{1}{2}]$ and $f(x, \varepsilon) = f(1 - x, \varepsilon)$ on $[\frac{1}{2}, 1]$. It follows from (2.1) that

$$f'(0) = \big(2\varepsilon\Phi(M(\varepsilon))\big)^{\frac{1}{2}} := f'(0, \varepsilon)$$

and consequently, from standard arguments in ordinary differential equations that (ii) holds. For (iii) one observes that if $M(\varepsilon') < M(\varepsilon'')$, then, on $[0, \frac{1}{2}]$,

$$\int_0^{f(x,\varepsilon'')} [\Phi(M(\varepsilon')) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta = \sqrt{\varepsilon'} \cdot C(\beta) \cdot x < \sqrt{\varepsilon''}\, C(\beta) x = \int_0^{f(x,\varepsilon'')} [\Phi(M(\varepsilon'')) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta .$$

However,

$$\int_0^{f(x,\varepsilon')} [\Phi(M(\varepsilon')) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta > \int_0^{f(x,\varepsilon')} [\Phi(M(\varepsilon'')) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta$$

and consequently

$$\int_{f(x,\varepsilon')}^{f(x,\varepsilon'')} [\Phi(M(\varepsilon'')) - \Phi(\eta)]^{-\frac{1}{2}}\, d\eta > 0$$

and the statement follows.

Finally, by similar reasoning, if $M_-(\varepsilon) < M_+(\varepsilon)$, if follows that $f_-(x, \varepsilon) < f_+(x, \varepsilon)$ on $(0, 1)$.

The bifurcation diagrams are given below.

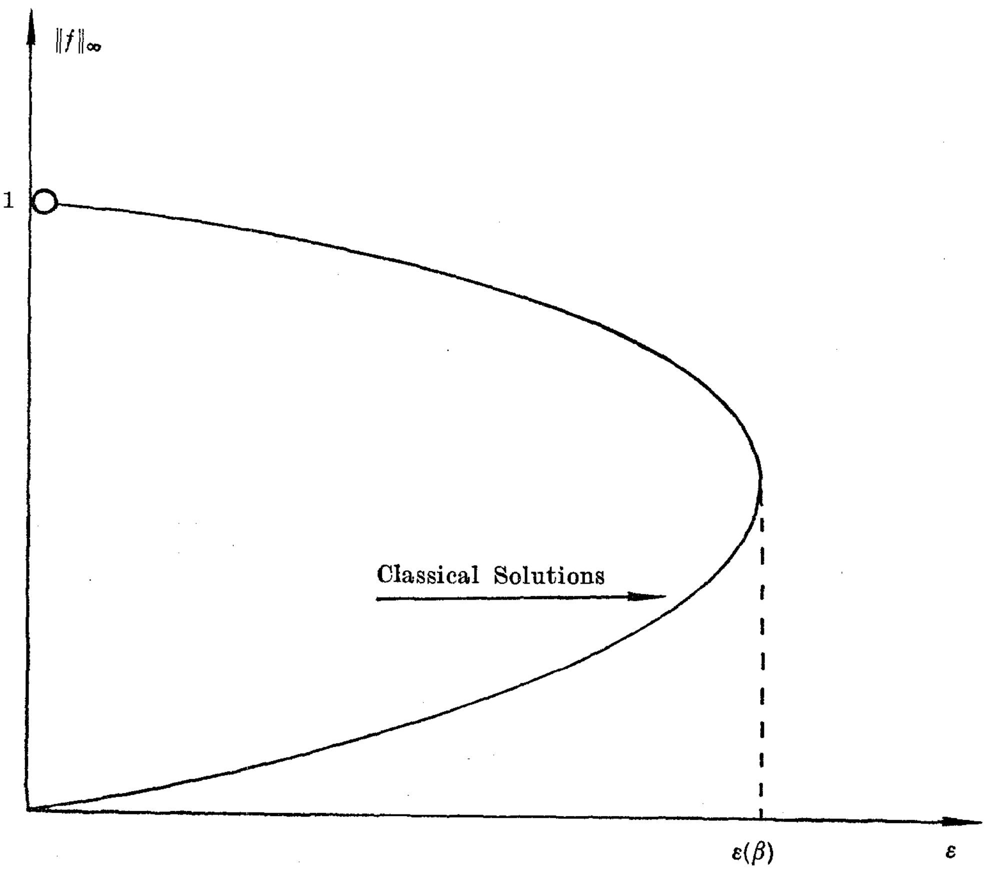

Fig. 1. – Bifurcation diagram for (SA) $\beta \geqslant 1$ and for (SB) $0 < \beta < \infty$.

There remains the question of the existence of nonclassical solutions of (SA). These must satisfy

$$(*) \qquad 0 = \int_0^1 [\psi'' f + \varepsilon\varphi(f)\psi]\, dx$$

for all $\psi \in C_0^\infty(0, 1)$. This definition precludes solutions which are one on a set of positive measure. For these we must consider stationary solutions of (A_∞). By taking a suitable sequence of ψ's, we see that $(*)$ holds if and only if

$$f(x) = \varepsilon \int_0^1 G(x, y)\varphi\big(f(y)\big)\, dy \tag{2.9}$$

where

$$G(x, y) = \begin{cases} x(1-y) & 0 \leqslant x \leqslant y \leqslant 1 \\ y(1-x) & 0 \leqslant y \leqslant x \leqslant 1 \end{cases} \tag{2.10}$$

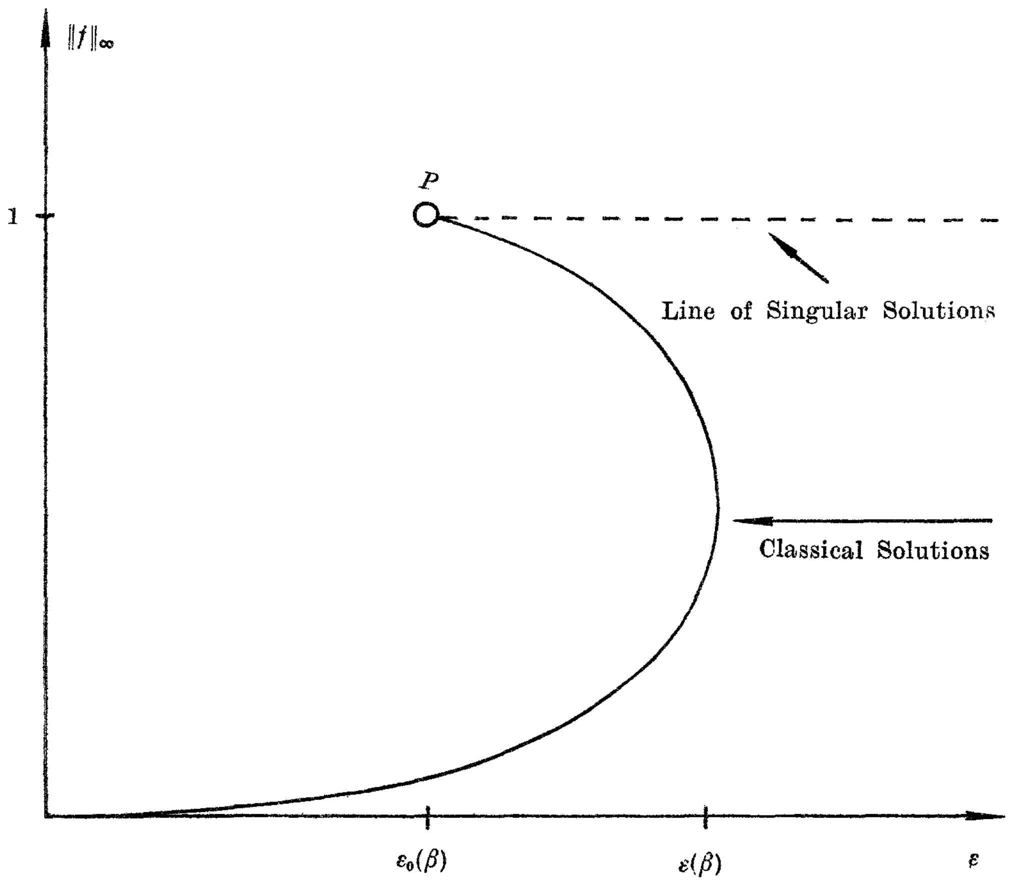

Fig. 2. – Bifurcation diagram for (SA) with $0 < \beta < 1$. $\lim_{\beta \to {}^+0} \varepsilon_0(\beta) = \lim_{\beta \leftarrow 0^+} \varepsilon(\beta) = 8$.

is the Green's function for $-d^2/dx^2$ with Dirichlet boundary conditions. From the integral equation, $y\varphi(f(y))$ and $(1-y)\varphi(f(y))$ are in $L^1_{\text{loc}}(0, 1)$. Therefore f is absolutely continuous and, whenever $f(x) < 1$,

$$f'(x) = \varepsilon\int_x^1 (1-\varphi)\varphi(f(y))\,dy - \varepsilon\int_0^x y\varphi(f(y))\,dy\,. \tag{2.11}$$

Therefore, f' is also absolutely continuous and wherever $f(x) < 1$

$$f''(x) = -\varepsilon\varphi(f(x))$$

when $f < 1$. Thus f is concave down and $[f = 1] = \emptyset$ or $[f = 1] = \{a\}$ where $a \in (0, 1)$. In the case $[f = 1] = \emptyset$, $f < 1$ on $[0, 1]$ and the solution of $(*)$ is classical. If $[f = 1] = [a]$, we see from (2.1) that $\beta \geqslant 1$ is not possible. A routine computation shows that

$$\int_0^{f(x)} [(1-u)^{1-\beta} + P_a^2]^{-\frac{1}{2}}\,du = \frac{4C(\varepsilon,\beta)}{1-\beta}\,x \qquad 0 \leqslant x \leqslant a \tag{2.12$_a$}$$

and

$$\int_0^{f(x)} [(1-u)^{1-\beta} + N_a^2]^{-\frac{1}{2}}\,du = \frac{4C(\varepsilon,\beta)}{1-\beta}\,(1-x) \qquad a \leqslant x \leqslant 1\,. \tag{2.13$_a$}$$

where, with

$$P_a = \sqrt{\frac{1-\beta}{2\varepsilon}}\,\lim_{x\to a^-} f'(x) \tag{2.14$_a$}$$

$$N_a = \sqrt{\frac{1-\beta}{2\varepsilon}}\,\lim_{x\to a^+} f'(x) \tag{2.15$_a$}$$

are the unique nonnegative and nonpositive roots of

$$H(P_a^2) = \frac{4C(\varepsilon,\beta)}{1-\beta}\,a \tag{2.16$_a$}$$

$$H(N_a^2) = \frac{4C(\varepsilon,\beta)}{1-\beta}\,(1-a) \tag{2.17$_a$}$$

where

$$H(z^2) = \int_0^1 [(1-u)^{1-\beta} + z^2]^{-\frac{1}{2}}\,du\,. \tag{2.18}$$

Since f' is continuous, $P_a = N_a = 0$, $a = 1 - \alpha = \frac{1}{2}$ and no solution is possible unless

$$C(\varepsilon,\beta) = \frac{1-\beta}{1+\beta}$$

i.e. $\varepsilon = \varepsilon_0(\beta)$ of Theorem 2.1A.

THEOREM 2.2 A. – *If* $\beta \geqslant 1$ (A) *cannot have any singular stationary solutions (in the sense of distributions). If* $0 < \beta < 1$, (A) *has exactly one such solution when* $\varepsilon = \varepsilon_0(\beta)$ *given by*

$$f_s(x,\varepsilon_0) = 1 - (1-2x)^{2/(1+\beta)} \qquad 0 < x \leqslant \tfrac{1}{2}$$

and by $f_s(x,\varepsilon_0) = f_s(1-x,\varepsilon_0)$ *if* $\frac{1}{2} < x \leqslant 1$.

For singular stationary solutions of (A_∞), we must have

$$f(x) = \varepsilon \int_0^1 G(x, y)\varphi(f(y))\,\chi[f < 1]\,dy\,.$$

Since f is classical wherever $f < 1$ [4, 16], it follows that there is, for every $\varepsilon \geqslant \varepsilon_0$, a unique stationary solution given by

$$(2.19) \qquad f(x) = \begin{cases} 1 - (1 - x/a)^{2/(1+\beta)} & 0 \leqslant x \leqslant a \\ 1 & a \leqslant x \leqslant \frac{1}{2} \end{cases}$$

on $[0, \frac{1}{2}]$ and $f(x) = f(1 - x)$ for $x \in [\frac{1}{2}, 1]$ where

$$(2.20) \qquad a = \frac{1}{2C(\varepsilon, \beta)} \cdot \frac{(1-\beta)}{(1+\beta)} = \sqrt{\frac{2(1-\beta)}{\varepsilon(1+\beta)^2}}\,.$$

THEOREM 2.3 A. – *If* $0 < \beta < 1$ *and* $\varepsilon \geqslant \varepsilon_0$, *then there is exactly one singular stationary solution of* (A_∞) *which is given by* (2.19), (2.20).

We designate this solution by $f_s(x; \varepsilon)$.

3. – Quenching and nonquenching.

Suppose we have a solution of (A) which does not quench. Then

$$(3.1) \qquad F(x, t) = \int_0^1 G(x, y)\,u(y, t)\,dy$$

is bounded in $[0, 1] \times [0, \infty)$ and

$$(3.2) \qquad F_t(x, t) = \int_0^1 G(x, y)\,u_t(y, t)\,dy = -\,u(x, t) + \varepsilon \int_0^1 G(x, y)\varphi(u(y, t))\,dy\,.$$

If $u_t \geqslant 0$ also, then $F_t \geqslant 0$ and u increases to a solution of (*) of Section 2. (For each $x \in (0, 1)$ there is a sequence $\{t_n(x)\}_{n=1}^{\infty}$ with $t_n \to +\infty$ such that $F_t(x, t_n(x)) \to 0$, otherwise there would be x_0 for which $F(x_0, t)$ is not bounded. Since u increases in t, $\lim_{t\to\infty} u(x, t) = f(x)$ exists. Taking $t = t_n(x)$ and taking note of the monotone convergence theorem, we see that f satisfies (2.9).)

THEOREM 3.1 A. – *Suppose* $\beta \geqslant 1$.

(i) *If* $\varepsilon > \varepsilon(\beta)$, *then every solution of* (A) *with* $0 \leqslant u_0 < 1$ *quenches.*

(ii) *If* $\varepsilon = \varepsilon(\beta)$ *we have the following*: (*a*) *If* $u_0(x) \leqslant f(x, \varepsilon(\beta))$ *then* u *is a global solution and* $\lim_{t\to+\infty} u(x, t) = f(x, \varepsilon(\beta))$. (*b*) *For every* $\delta > 0$, *there exists* $u_0 \geqslant 0$ *with* $u_0 < 1$ *and* $\|u_0(\cdot) - f(\cdot, \varepsilon(\beta))\|_{L^\infty} < \delta$ *such that* u *quenches.*

(iii) *If* $\varepsilon < \varepsilon(\beta)$, *we have the following*: (*a*) *If* $u_0(x) < f_+(x, \varepsilon)$ *and* $u_0 \in C^1[0, 1]$ *then* u *is global and* $\lim_{t\to+\infty} u(x, t) = f_-(x, \varepsilon)$. (*b*) *For every* $\delta > 0$, *there is* $u_0(< 1)$ *with* $\|u_0(\cdot) - f_+(\cdot, \varepsilon)\|_{L^\infty} < \delta$ *such that* u *quenches.*

PROOF OF THEOREM 3.1 A. – Recall that in this case all stationary solutions are classical.

In order to prove (i), we need only consider $u_0 \equiv 0$, since by comparison $u(x, t; u_0) \geqslant u(x, t; 0)$ if $u_0 \geqslant 0$. In this case however, $v = u_t(x, t; 0)$ satisfies $v = 0$ for $x = 0, 1$, $0 < t < T$ and $v > 0$ for $t = 0$, $0 < x < 1$ and a linear parabolic equation and therefore $v \geqslant 0$. Now suppose u does not quench. Then $T = +\infty$ and u increases for each x, to a function $w(x)$ as $t \to +\infty$. Therefore, w must, by the opening remarks of this section, be a stationary solution of (A) for $\varepsilon > \varepsilon(\beta)$. Since there are none, we are done in this case.

We next prove (iii). (The proof of (ii) is easier and is omitted.) To prove (iii)(*a*), we observe that since $0 \leqslant u_0(x) < f_+(x, \varepsilon)$, there is a number $\sigma > 0$ such that $0 \leqslant u_0(x) < f_+(x, \varepsilon + \sigma)$ and $f_+(\frac{1}{2}, \varepsilon + \sigma) < f_+(\frac{1}{2}, \varepsilon)$. Let $v_0(x) = f_+(x, \varepsilon + \sigma)$. Then, for as long as both solutions exist, $u(x, t; u_0) \leqslant u(x, t; v_0)$. However, $u_t(x, 0; v_0) < 0$ on $(0, 1)$. Therefore, by standard arguments $u_t(x, t; v_0) \leqslant 0$ and consequently $u(x, t; v_0) \leqslant v_0$ and $u(x, t; v_0)$ global. Therefore so is $u(x, t; u_0)$. Also $\lim_{t\to\infty} u(x, t; v_0) = \psi(x)$ exists and $\psi(\frac{1}{2}) < f_+(\frac{1}{2}, \varepsilon)$. By an argument similar to that used to prove part (i), $\psi(x)$ is a stationary solution. But then $\psi(x) = f_-(x, \varepsilon)$. Also by the argument in part (i), $u(x, t; 0)$ exists globally and $\lim_{t\to\infty} u(x, t; 0) = F(x)$, which must also be a stationary solution. Since

$$u(x, t; 0) \leqslant u(x, t; u_0) \leqslant u(x, t; v_0)$$

it follows that $F(x) = f_-(x, \varepsilon)$ and $\lim_{t\to\infty} u(x, t, u_0) = f_-(x, \varepsilon)$ pointwise as claimed.

To prove (iii) (*b*), we invoke Theorem 2.1A. Given $\delta > 0$ we choose $\sigma > 0$ so small that

$$\|f_+(\cdot, \varepsilon - \sigma) - f_+(\cdot, \varepsilon)\|_{L^\infty} < \delta$$

and note that $f_+(\frac{1}{2}, \varepsilon - \sigma) > f_+(\frac{1}{2}, \varepsilon)$.

We now set $u_0(x) = f_+(x, \varepsilon - \sigma)$ and observe that for $u(x, t, u_0)$ we have $u_t(x, 0; u_0) > 0$ on $(0, 1)$. Again we find that $u_t(x, t; u_0) > 0$ on the existence interval wherever u_t exists. Thus, if this u does not quench, $T = +\infty$ and $u(x, t) \to w(x)$, a solution of (2.1) which must be a stationary solution of (A) with $w(\frac{1}{2}) > f_+(\frac{1}{2}, \varepsilon)$. However, this is not possible unless w is a weak stationary solution. Since for $\beta \geqslant 1$, there are none, u must quench.

THEREFORE 3.2 A. – *Let* $0 < \beta < 1$.

(i) *If* $\varepsilon > \varepsilon_0$, *then statements* (i), (ii) *and* (iii) *of the preceding hold in this case accordingly as* $\varepsilon > \varepsilon(\beta)$, $\varepsilon = \varepsilon(\beta)$ *and* $\varepsilon_0 < \varepsilon < \varepsilon(\beta)$.

(ii) *If* $\varepsilon = \varepsilon_0$, *we have the following*: (*a*) *If* $u_0 < f_s$ *and* $u_0 \in C^1$, *then* u *is global and* $\lim_{t\to\infty} u(x, t) = f_-(x, \varepsilon_0)$. (*b*) *There exist* (*smooth*) *initial values for which* u *quenques.*

(iii) *If* $0 < \varepsilon < \varepsilon_0$, *we have the following*: (*a*) *If* $0 \leqslant u_0 < f_s(x, \varepsilon_0)$, *then* u *canot quench, even in infinite time and* $\lim_{t\to+\infty} u(x, t) = f(x, \varepsilon)$. (*b*) *There exist* (*smooth*) *initial values* $u_0 < 1$ *for which* u *quenches.*

PROOF OF THEOREM 3.2 A. – (i) Suppose $\varepsilon > \varepsilon_0$. If $\varepsilon < \varepsilon(\beta)$ and $u_0 \equiv 0$, then $u = u(x, t; 0)$ must quench. To see this, since $u_t \geqslant 0$, we note that if u did not quench, even in infinite time, then u would approach a classical stationary solution of (A) which is impossible. If u reached one in infinite time, it would have to approach a stationary solution of the form $f_s(\cdot, \varepsilon_0)$ as $t \to \infty$ (i.e. a stationary solution taking the value 1 at a single point) since $(x - \frac{1}{2}) u_x(x, t) > 0$ for $x \neq \frac{1}{2}$ and all $t > 0$. However, no such weak stationary solutions exist when $\varepsilon > \varepsilon_0$. For $0 < u_0 < 1$, the finite time quenching follows by comparison. This proves the statement (i) of Theorem 3.1A for this case.

Next we prove (iii) of Theorem 3.1A for this case. (The proof of (ii) is similar and is omitted.) The proof of (*a*) is exactly the same if $0 < \beta < 1$ as for $\beta \geqslant 1$. The proof of (*b*) follows as before, except we must rule out the possibility of infinite time quenching. This we do as above, since the choice of $u_0 (= f_+(x, \varepsilon - \sigma))$ again yields $(x - \frac{1}{2}) u_x > 0$ for $x \neq \frac{1}{2}$.

The proof of the theorem when $\varepsilon = \varepsilon_0$ is exactly like the case for which $\varepsilon < \varepsilon_0$ and is omitted.

To prove (iii) (*a*), we choose $v_0 = f_+(x, \gamma)$ where $\varepsilon_0 < \gamma < \varepsilon(\beta)$ and γ is so close to ε_0 that $u_0 \leqslant v_0$ on $(0, 1)$. Therefore, by comparison,

$$u(x, t; \varepsilon, 0) \leqslant u(x, t; \varepsilon, u_0) \leqslant u(x, t; \varepsilon, v_0) .$$

However, with $v(x, t) = u(x, t; \varepsilon, v_0)$, and $w = v_t$, we have $w(0, t) = w(1, t) = 0$, $w(x, 0) < 0$ and thus $w(x, t) < 0$. Therefore $v(x, t)$ cannot quench, even in infinite time, and $v(x, t) \to f(x, \varepsilon)$, the only stationary solution of (A) in this case. Also $u(x, t; \varepsilon, 0)$ increases to $f(x, \varepsilon)$.

To prove (*b*) we let

$$F(t) = \int_0^1 u(x, t)\, \psi(x)\, dx$$

where ψ is the first (Dirichlet) eigenfunction for $-d^2/dx^2$. Then

$$F'(t) \geqslant -\pi^2 F + \varepsilon (1 - F)^{-\beta} \equiv Q(F) .$$

We choose $u_0(x)$ so close to one that $F(0) > r_1$ where r_1 is the largest root of Q in $(-\infty, 1)$. Then $F'(t) > 0$, $Q(F(t))$ cannot change sign and

$$\infty > \int_{F(0)}^{1} \frac{d\sigma}{Q(\sigma)} \geqslant t$$

for all t in the existence interval. Therefore u quenches.

For problem (B) we have

THEOREM 3.2 B. – *If* $0 < \beta < \infty$, *then* (i), (ii) *and* (iii) *of Theorem* 3.1A *also hold for solutions of* (SB) *where now* $\varepsilon(\beta)$ *is as in Theorem* 2.1B.

4. – Blow up of u_t at quenching.

In his original paper, Kawarada showed that when u quenched (in our sense) then u_t became unbounded at the quenching point. He considered only the case $\beta = 1$ and $u_0 = 0$. In [9], we suggested that his result (and proof) should extend to higher dimensional problems, Indeed, recently, in [1], it has been shown that when $(0, 1)$ is replaced by the n ball and the discussion is restricted to radial solutions with radially decreasing initial values such that $u_t(r, 0) \geqslant 0$, then for solutions which quench at $r = 0$, u_t blows up at $r = 0$. Their proof did not follow the lines of Kawarada's. Rather, they examined the differential equation satisfied by $v = (1 - u)^\beta u_t$. They show that $v_t \geqslant \Delta_r v$ where Δ_r is the radial Laplacian. Their arguments seem to require that the quenching point be isolated.

Although their result provides a partial answer to the problem we proposed, the last word in this problem has yet to be said. For example, it is easy to write down simple problems for which quenching takes place on a continuum. (Consider $u_t = \Delta u + \varepsilon(1 - u)^{-\beta}$ in the annulus $1 < r < 2$, with $u = 0$ on $r = 1$, $r = 2$ and $u(r, \theta, 0) = 0$ for ε sufficiently large.) Nor is the question answered in an n ball when the data is not radial (or when it is, when it is not radially decreasing). In [18] we have recently obtained an improvement of the blow up result for u_t of [1] (1).

For problem (B), the blow up of u_t at $x = 1$ was claimed by us in [10]. However, the proof of Corollary 2.7, where this claim was made, is not correct. The equation at the top of page 1145 should read (with $L = \varepsilon$ and $u_0 \equiv 0$),

$$u_t(x, t) = \varepsilon G(x, 1, t)\varphi(u(1, 0)) + \int_0^t G(x, 1; t - \eta)\varphi'(u(1, \eta))\, u_n \, d\eta$$

where

$$G(x, y; t) = 2\sum_{n=1}^{\infty} \exp(-\lambda_n^2 t) \sin(\lambda_n x) \sin(\lambda_n y)$$

and $\lambda_n = \frac{1}{2}(2n - 1)\pi$.

We prove

THEOREM 4.1. – *Let* $u_0'(x)\geqslant 0$, $u_0''(x)\geqslant 0$ *on* $[0,1]$ *and suppose* u, *a solution of* (B) *with* $u(x,0)=u_0(x)$ *quenches at time* T. *Then the quenching occurs at* $x=1$ *and* $u_t(1,t)$ *blows up as* t *approaches* T *from below.*

PROOF. – By the arguments of [10] it follows that $u_x>0$, $u_t>0$ in $(0,1)\times(0,T)$ so that quenching cannot occur on $[0,1)\times(0,T)$.

From the representation formula for the solution, we have, for $t<T$,

$$(4.1)\qquad u(x,t)=\int_0^1 G(x,y;t)\,u_0(y)\,dy+\varepsilon\int_0^0 G(x,1;t-\eta)\,\varphi(u(1,\eta))\,d\eta\,,$$

and consequently, after differentiating and integrating by parts we find that for $0<t<T$, $0<x<1$,

$$(4.2)\qquad u_t(x,t)=\int_0^1 G_t(x,y;t)\,u_0(y)\,dy-\varepsilon\int_0^1\frac{\partial}{\partial\eta}[G(x,1;t-\eta)]\varphi(u(1,\eta))\,d\eta=$$

$$=\int_0^1 G_t(x,y;t)\,u_0(y)\,dy+\varepsilon G(x,1;t)\varphi(u(1,0))+\varepsilon\int_0^t G(x,1;t-\eta)\varphi'(u(1,\eta))\,u_\eta(1,\eta)\,d\eta\,.$$

We let $x\to 1^-$ in this last expression and use the (assumed) continuity of u_t on $[0,1]\times(0,T)$. Since $u_\eta\geqslant 0$, $\varphi'\geqslant 0$, we have, for $\delta\leqslant t<T$, $\delta>0$ and fixed

$$u_t(1,t)\geqslant C_1+\varepsilon C_2[\varphi(u(1,t))-\varphi(u(1,0))]$$

where

$$C_1=\inf_{\delta\leqslant t\leqslant T}\left[\int_0^1 G^t(1,y;t)\,u_0(y)\,dy\right]+\varepsilon\varphi(u(1,0))\inf_{\delta\leqslant t\leqslant T}[G(1,1;t)]$$

and

$$C_2=\inf_{\delta\leqslant t\leqslant T}G(1,1,t)$$

are positive constants. The result now follows from (4.3).

5. – Beyond quenching.

As we remarked in the introduction, the study of (A_∞) is inhibited by the lack of a uniqueness theorem. However, we conjecture the following: Suppose $u_t\geqslant 0$

in the region where $u<1$. If the solutions of (A_ω) with these properties quench and $\beta \geqslant 1$, or $0<\beta<1$ and $\varepsilon<\varepsilon_0$, then

$$\lim_{t\to\infty} u(x,t)=1 \qquad x\in(0,1);$$

or else if $0<\beta<1$ and $\varepsilon \geqslant \varepsilon_0$, then

$$\lim_{t\to\infty} u(x,t)=f_s(x;\varepsilon)$$

where f_s is given in Theorem 2.3A. If $\beta>1$, the first limit is obtained in finite time (complete quenching).

Acknowledgement. – The author thanks the referee for a number of valuable comments.

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Note added in Proof.

Recently DENG and LEVINE, by using Lemma 4.1 of FRIEDMAN and MCLEOD (Indiana J. Math., **24** (1985), pp. 425-447) and [5], have been able to show that u_t blows up when u quenches for a much wider class of initial data than considered in [1] as well as for convex regions (in one or more dimensions).

They also show (in one or more dimensions) that if the special domain is convex, the set of quenching points is a compact subset of that domain.

Recently GUO [19], has shown that if $\beta > 0$ than (A) can have at most a finite number of quenching points. In view of our remark in § 4, this result fails in more than one dimension.

He has also shown that if $\beta \geqslant 3$, then near a quenching point $a \in (0, 1)$,

$$\lim_{t \to T^-} [1 - u(x, t)]\,(T - t)^{-\gamma} = (\varepsilon/\gamma)^{\gamma}$$

uniformly for $|x - a|^2 \leqslant c(T - t)$ for any $c > 0$ where $\gamma = 1/(\beta + 1)$.