A Universal Branching Set for 4-Dimensional Manifolds (*)(**).

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Abstract. – *In this work, a universal branching set K for orientable 4-manifolds, such that*

$$\Pi_1(S^4 - K) = \langle a, b, c/aca^{-1}c^{-1} = 1 \rangle$$

is proved to exist. This leads to the possibility of representing every closed connected orientable 4-manifold by a suitable transitive set $\{\sigma, \tau, \mu\}$ of permutations, in analogy with known results for dimension three (see [M] and [CV]).

1. – Introduction.

In [M], Montesinos shows the existence of a very simple universal branching set for 3-manifolds (the graph G depicted in figure 3); since $\Pi_1(S^3 - G)$ is a free group on two generators, this leads to the possibility of representing 3-manifolds by means of a transitive pair $\{\sigma, \tau\}$ of permutations in the symmetric group S_h on h elements, for some h . Later on, in [CV], this representation theory is related to the one making use of edge-coloured graphs; in particular, an algorithm is described to pass from a crystallization to a pair of permutations (one of which may be assumed to be of standard type) representing a chosen 3-manifold.

In this paper, we re-obtain the above results for dimension three by means of a standard method—described in [CG]—to yield edge-coloured graphs representing branched covering spaces, starting from the associated monodromy (section 3).

Further, in section 4, the same approach is used to face the analogous problems in dimension four: a universal branching set K is obtained, such that $\Pi_1(S^4 - K)$ admits a presentation $\langle a, b, c/aca^{-1}c^{-1} = 1 \rangle$. Thus, every closed orientable 4-manifold M^4 is proved to be representable by a suitable transitive set $\{\sigma, \tau, \mu\}$ of permutations, such

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that $\sigma\mu\sigma^{-1}\mu^{-1} = 1$ (Corollary 10); moreover, the permutations σ and τ may always be assumed to be of standard type, so that the only permutation μ is sufficient to reconstruct the 4-manifold (Proposition 11).

A glance towards higher dimensions concludes the paper: every closed orientable n -manifold M^n ($n \geq 5$) is shown to be representable by a suitable transitive set $\{\sigma_1, \sigma_2, \dots, \sigma_{2^{n-3}}, \sigma_{2^{n-3}+1}\}$ of permutations (among which only one is essential), constituting the monodromy with respect to a universal branching set (section 5).

2. – Preliminaries.

Throughout this paper, all spaces and maps are piecewise-linear (PL), in the sense of [Gl] or [RS]. Manifolds are always assumed to be connected and orientable. We will often identify a ball complex with its associated polyhedron and every homeomorphic space. For basic graph theory we refer to [W].

An *edge-coloration* on a multigraph $\Gamma = (V(\Gamma), E(\Gamma))$ is a map $\gamma: E(\Gamma) \rightarrow \Delta_n = \{0, 1, \dots, n\}$ such that $\gamma(e) \neq \gamma(f)$, for each pair e, f of adjacent edges. The pair (Γ, γ) , Γ being a regular multigraph of degree $n+1$ and $\gamma: E(\Gamma) \rightarrow \Delta_n$ being an edge-coloration, is said to be an $(n+1)$ -coloured graph. For each $B \subseteq \Delta_n$, we set $\Gamma_B(V(\Gamma), \gamma^{-1}(B))$; each connected component of Γ_B is often called a *B-residue*. An *m-residue* is a *B-residue* such that the cardinality of B is m . The symbol g_B denotes the number of *B-residues*. For each colour $c \in \Delta_n$, we set $\hat{c} = \Delta_n - \{c\}$.

An n -dimensional labelled pseudocomplex $K(\Gamma)$ (see [HW] and [BM]) can be associated with any $(n+1)$ -coloured graph (Γ, γ) , via the following construction:

- i) for each $v \in V(\Gamma)$, take an n -simplex $\sigma(v)$, with its vertices labelled by $0, 1, \dots, n$;
- ii) for each $e \in E(\Gamma)$ with end-points $v, w \in V(\Gamma)$ and colour $\gamma(e) = c$, identify the $(n+1)$ -faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex labelled by c , so that equally labelled vertices coincide.

We say that (Γ, γ) represent $K(\Gamma)$, $|K(\Gamma)|$ and every homeomorphic polyhedron.

Note that every $\{c_1, \dots, c_m\}$ -residue $\mathcal{E}$ of Γ uniquely corresponds to a $(n-m)$ -simplex σ^{n-m} of $K(\Gamma)$, whose vertices are labelled by $\Delta_n - \{c_1, \dots, c_m\}$.

Moreover, it is known (see [FGG]) that:

- A) $K(\Gamma)$ is orientable iff Γ is a bipartite graph;
- B) $K(\Gamma)$ is a (closed) n -manifold iff, for every n -residue $\mathcal{E}$ of Γ , $K(\mathcal{E})$ is homeomorphic to the $(n-1)$ -sphere S^{n-1} .

A *crystallization* of a closed n -manifold M^n is an $(n+1)$ -coloured graph (Γ, γ) representing M^n such that $\Gamma_{\hat{c}}$ is connected for each $c \in \Delta_n$. A theorem of [P] ensures the existence, for every closed n -manifold M^n , of crystallizations of M^n (and hence of

$(n + 1)$ -coloured graphs representing M^n); see also [FGG] and its bibliography for successive extensions of this result.

If (Γ, γ) is an $(n + 1)$ -coloured graph representing a closed n -manifold M^n , and Γ_c is connected, for some $c \in \Delta_n$, it is known [Gr] that a presentation $\langle X/R \rangle$ of the fundamental group $\Pi_1(M^n)$, called *c-edge presentation*, can be obtained in the following way:

- *) the generators of X are the c -coloured edges, arbitrarily oriented;
- **) the relators of R are obtained by walking along the (arbitrarily oriented) 2-residues of $\Gamma_{\{c, d\}}$, for each $d \in \Delta_n - \{c\}$, giving the exponent $+1$ or -1 to each generator whether the orientation of the 2-residue is coherent or not with the orientation of the generator.

Note that, if Γ_c is not connected, the c -edge presentation of $\Pi_1(M)$ can be obtained in a similar way: it is sufficient to complete the relators of R with a minimal set of generators such that the corresponding c -coloured edges connect Γ_c .

Let $(\Gamma, \gamma), (\Gamma', \gamma')$ be $(n + 1)$ -coloured graphs. For a positive integer m ($1 \leq m \leq n$), a map $f: V(\Gamma') \rightarrow V(\Gamma)$ is said to be an *m-covering* [V] if f preserves c -adjacency for all $c \in \Delta_n$ and is bijective when restricted to m -residues. The branching $(m + 1)$ -residues are the $(m + 1)$ -residues of (Γ, γ) covered by at least one $(m + 1)$ -residue of (Γ', γ') on which f is not injective.

The covering f naturally induces a topological map $|f|: K(\Gamma') \rightarrow K(\Gamma)$. An n -covering induces a topological covering between the underlying topological space, while a 1-covering induces a topological covering branched over the $(n - 2)$ -subcomplex of $K(\Gamma)$ whose $(n - 2)$ -simplexes are represented by the branching 2-residues of (Γ, γ) .

Let (Γ, γ) be an $(n + 1)$ -coloured graph representing a closed (orientable) n -manifold $K(\Gamma) = M^n$. Suppose Γ_c connected and let L be the $(n - 2)$ -subcomplex of $K(\Gamma)$ represented by a given set $\{C_1, C_2, \dots, C_p\}$ of 2-residues containing the colour c . In this hypothesis we say that $(K(\Gamma), L)$ is a *branched $(n + 1)$ -coloured pair (with respect to the colour c)*.

If $P = \langle X = \{x_1, \dots, x_s\} / R = \{r_1, \dots, r_t\} \rangle$ is a c -edge presentation of the fundamental group $\Pi_1(M^n)$ and if $r_{t-p+1}, \dots, r_t$ are the relators obtained by reading the 2-residues $C_1, \dots, C_p$, then it is easy to check that $P' = \langle X / R' = \{r_1, \dots, r_{t-p}\} \rangle$ is a presentation (called again *c-edge presentation*) of $\Pi_1(M^n - L)$.

The existence of a one-to-one correspondence Φ between transitive representations ω of $\Pi_1(M^n - L)$ in S_d (the symmetric group on the set $N_d = \{1, 2, \dots, d\}$) and d -fold covering space of M^n branched over L , is well-known (see [F]). In [CG], the following method is described for constructing an $(n + 1)$ -coloured graph $(\tilde{\Gamma}, \tilde{\gamma})$ such that $K(\tilde{\Gamma}) = \Phi(\omega)$:

- set $V(\tilde{\Gamma}) = V(\Gamma) \times N_d$;
- for each $k \in \Delta_n - \{c\}$ and $i \in N_d$, join (v, i) with (w, i) by a k -coloured edge if v, w are k -adjacent in (Γ, γ) ;

— join (v, i) with (w, j) by a c -coloured edge if in (Γ, γ) there is an oriented c -coloured edge x_i from v to w and $w(x_i)(i) = j$.

Roughly speaking, $(\tilde{\Gamma}, \tilde{\gamma})$ is obtained by taking d copies of the (connected) n -coloured graph $\Gamma_{\tilde{c}}$ and by drawing the c -coloured edges in the way induced by the permutations of $\omega(X)$.

3. – Re-obtaining known results in dimension three.

Let (Λ, λ) be a crystallization of a closed connected orientable 3-manifold M^3 , and let $V'(\Lambda), V''(\Lambda)$ be the two bipartition classes of its vertex set. Further, let $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$ be an arbitrarily fixed permutation of the colour set; in the following, for sake of notational simplicity, we shall always write i instead of ν_i , for every $i \in \Delta_3$.

Let $(\tilde{\Lambda}, \tilde{\lambda})$ be the 4-coloured graph representing M^3 such that the pseudocomplex $K(\tilde{\Lambda})$ is obtained from $K(\Lambda)$ by baricentrically subdividing the 2-simplex opposite to the (unique) 3-labelled vertex v_3 , and then by starring from v_3 itself. It is easy to check that $(\tilde{\Lambda}, \tilde{\lambda})$ can directly be obtained from (Λ, λ) by means of the following rules:

— for every $v \in V(\Lambda)$, construct a copy of the 2-coloured cycle (H, h) of figure 1, and label its vertices by the elements of the set $N_6 \times \{v\}$, so that the vertex (i, v) is the copy of the vertex i of (H, h) ;

— for every $i \in \{1, 2\}$ (resp. $i \in \{3, 4\}$) (resp. $i \in \{5, 6\}$), connect (i, v) with (i, w) by a 2-coloured edge iff v and w are connected in (Λ, λ) by a 0-coloured (resp. 1-coloured) (resp. 2-coloured) edge;

Fig. 1.

Fig. 2.

— for every $i \in N_6$, connect (i, v) with (i, w) by a 3-coloured edge iff v and w are connected in (Λ, λ) by a 3-coloured edge.

Let now $(\bar{\Lambda}, \bar{\lambda})$ denote the standard 4-coloured graph of figure 2, which evidently represents S^3 . An easy direct computation proves the following result.

PROPOSITION 1. — Let $g: (\tilde{\Lambda}, \tilde{\lambda}) \rightarrow (\bar{\Lambda}, \bar{\lambda})$ be the map defined by

$$g(i, v) = \begin{cases} A & \text{if } i \text{ even and } v \in V'(\Lambda), \\ B & \text{if } i \text{ odd and } v \in V'(\Lambda), \\ A' & \text{if } i \text{ even and } v \in V''(\Lambda), \\ B' & \text{if } i \text{ odd and } v \in V''(\Lambda). \end{cases}$$

Then, g is a 1-covering, branched over the 2-residues of colours $\{0, 1\}$, $\{1, 2\}$, $\{2, 3\}$.

The 1-covering g naturally induces a topological covering $|g|: M^3 \rightarrow S^3$, branched over the subcomplex G of $K(\bar{\Lambda})$ represented by the branching 2-residues of $(\bar{\Lambda}, \bar{\lambda})$. It is now easy to check, by analyzing the pseudosimplicial structure of $K(\bar{\Lambda})$, that the above defined subcomplex G is exactly the graph depicted in figure 3 (already proved to be a universal branching set for 3-manifolds in [M]).

COROLLARY 2. — Every closed connected orientable 3-manifold M^3 is a covering of S^3 branched over the universal set G .

Since branched covering spaces are uniquely determined by the monodromy of the covering map [F], we are now going to determine the group of G , i.e. $\Pi_1(S^3 - G)$.

For, we consider the (unique) bisection $\bar{\Lambda}'$ of type $(0, 2)$ of $\bar{\Lambda}$ (see [G]); $\bar{\Lambda}'$ (which is drawn in figure 4) represents a pseudocomplex $K(\bar{\Lambda}')$ triangulating S^3 , such that

Fig. 3.

Fig. 4.

$(K(\bar{\Lambda}'), G)$ is a branched 4-coloured pair (with respect to the colour 0). Thus, it is possible to obtain $\Pi_1(S^3 - G)$ from the 0-edge presentation of $\Pi_1(S^3)$ associated to $\bar{\Lambda}'$.

PROPOSITION 3. - $\Pi_1(S^3 - G) = \langle a, b \rangle$.

PROOF. - As pointed out in section 2, $\Pi_1(S^3 - G)$ is obtained from the 0-edge presentation of $\Pi_1(S^3)$ associated to $\bar{\Lambda}'$ by deleting all the relators corresponding to

branching 2-residues (i.e. to 1-simplices of $K(\bar{\Lambda}')$ belonging to the subcomplex G). Hence,

$$\begin{aligned} \Pi_1(S^3 - G) &= \langle x_1, x_2, x_3, x_4 / x_1 x_2 x_3 x_4 = 1, x_1 x_3 = 1 \rangle = \\ &= \langle x_1, x_2, x_4 / x_1 x_2 x_1^{-1} x_4 = 1 \rangle = \langle x_1, x_2 / x_1 x_2 x_1^{-1} x_1 x_2^{-1} x_1^{-1} = 1 \rangle = \langle x_1, x_2 / \rangle. \quad \blacksquare \end{aligned}$$

COROLLARY 4. – *Every closed connected orientable 3-manifold M^3 is represented by a suitable transitive set $\{\sigma, \tau\}$ of permutations.*

PROOF. – If $\omega: \Pi_1(S^3 - G) \rightarrow S_d$ is the monodromy associated to the covering map from M^3 to S^3 branched over G , then it is sufficient to set $\sigma = \omega(a)$, $\tau = \omega(b)$, a, b being the generators of the presentation of $\Pi_1(S^3 - G)$ obtained in Prop. 3. $\blacksquare$

We want now to analyze the algebraic features of the set $\{\sigma, \tau\}$ of permutations associated to a given 3-manifold M^3 , by computing the monodromy of the topological covering $|q|$ induced by the 1-covering g of Prop. 1. For, we consider the (global) bi-section $\tilde{\Lambda}'$ of type $(0, 2)$ of $\tilde{\Lambda}$, so that the 1-covering g between $\tilde{\Lambda}$ and $\bar{\Lambda}$ naturally induces a 1-covering g' between the graph $\tilde{\Lambda}'$ (representing M^3) and the graph $\bar{\Lambda}'$ (representing S^3). It is now easy to check that the 1-covering $g': \tilde{\Lambda}' \rightarrow \bar{\Lambda}'$ can be obtained as an application of a standard method—due to [CG] and briefly described in section 2—to yield coloured graphs representing branched coverings of the sphere, starting from the associated monodromy. Since $\tilde{\Lambda}'$ contains three copies of $\bar{\Lambda}'_2$ for every pair $\{v, w\}$ of 3-adjacent vertices of (Λ, λ) , g' induces a topological $(3 \cdot q/2)$ -fold covering ($q = \# V(\Lambda)$), whose monodromy is described by the 0-coloured edges connecting the various sheets.

Thus, σ, τ may be thought of as permutations on the set $T = N_3 \times N_{q/2}$, by denoting $V'(\Lambda) = \{1, 2, \dots, q/2\}$ and $V''(\Lambda) = \{1', 2', \dots, q'/2\}$, with i 3-adjacent to i' for every $i \in N_{q/2}$; moreover, the combinatorial properties of the graph $\tilde{\Lambda}'$ ensure that σ may always be assumed to be of standard type (only depending on $q = \# V(\Lambda)$), while τ directly depends on the combinatorial properties of the starting crystallization (Λ, λ) of M^3 :

$$\begin{aligned} \sigma &= \prod_{i=1}^{q/2} ((1, i)(2, i)(3, i)), \\ (*) \quad \tau((x, i)) &= (x, \tau_{j(x)}(i)), \quad \text{where } j(x) = \begin{cases} 0 & x \in \{1, 2\}, \\ 1 & x \in \{3, 4\}, \\ 2 & x \in \{5, 6\}, \end{cases} \end{aligned}$$

and $\tau_j(i)$ ($j \in \{0, 1, 2\}$), for every vertex $i \in V'(\Lambda)$, is the vertex $r \in V'(\Lambda)$ which is 3-adjacent to the vertex $r' \in V''(\Lambda)$ j -adjacent to i .

Note that $\{\sigma, \tau\}$ is obviously a transitive set, since $\tilde{\Lambda}'$ is a connected graph.

Further, we point out that the only permutation τ on T is sufficient to reconstruct

(Λ, λ) . In fact, (Λ, λ) is isomorphic to the 4-coloured graph $\Lambda(\tau)$ obtained by the following rules:

- the vertex set of $\Lambda(\tau)$ is $\{1, 2, \dots, q/2\} \cup \{1', 2', \dots, q'/2\}$;
- every vertex $i \in \{1, 2, \dots, q/2\}$ is connected with the homonymous vertex $i' \in \{1', 2', \dots, q'/2\}$ by a 3-coloured edge;
- every vertex $i \in \{1, 2, \dots, q/2\}$ is connected with the vertex $(\tau_j(i))' \in \{1', 2', \dots, q'/2\}$ by a j -coloured edge, for every $j \in \{0, 1, 2\}$.

Thus, we have described an algorithm for obtaining, from any 3-manifold M^3 —given by a crystallization (Λ, λ) —, a permutation τ which completely represents it; though essentially different, this algorithm gives the same result as the one described in [CV].

PROPOSITION 5. — *Every closed connected orientable 3-manifold M^3 is represented by a suitable permutation τ on $T = N_3 \times N_{q/2}$ such that*

$$\pi_1 \circ \tau = 1 \quad \text{and} \quad (\pi_2 \circ \tau)((x, i)) = \begin{cases} \tau_0(i) & x \in \{1, 2\}, \\ \tau_1(i) & x \in \{3, 4\}, \\ \tau_2(i) & x \in \{5, 6\}, \end{cases}$$

(or, equivalently, by a suitable set $\{\tau_0, \tau_1, \tau_2\}$ of permutations on $N_{q/2}$).

REMARK. — The three permutations τ_0, τ_1, τ_2 on $N_{q/2}$ correspond to the rows of the «code» for 3-manifolds, defined in [LD]. Thus, the conditions obtained in [CV] for deciding whether a given set $\{\tau_0, \tau_1, \tau_2\}$ of permutations represents a 3-manifold M^3 or not, may be used to solve the analogous problem for «codes».

If $\mathcal{T} = \{\tau_0, \tau_1, \tau_2\}$ is a set of permutations representing a closed 3-manifold M^3 , then the crystallization $\Lambda(\tau)$ of M^3 , τ being obtained from $\{\tau_0, \tau_1, \tau_2\}$ by formula (*), is said to be *canonically associated* to $\mathcal{T}$.

The following result shows how easily $\mathcal{T} = \{\tau_0, \tau_1, \tau_2\}$ may produce the fundamental group of the represented 3-manifold, via the 3-edge presentation of the canonically associated crystallization.

PROPOSITION 6. — *Let $\mathcal{T} = \{\tau_0, \tau_1, \tau_2\}$ be a set of permutations on $N_{q/2}$ ($q \in N$), representing a closed 3-manifold M^3 . Then, a (positive) presentation of $\Pi_1(M^3)$ is*

$$\langle X/R_0 \cup R_1 \cup R_2 \rangle,$$

where the set $X = \{x_1, x_2, \dots, x_{q/2}\}$ of generators is in bijection with $N_{q/2}$, and the set $R_i = \{r_i^1, r_i^2, \dots, r_i^{\sigma_i}\}$ of relators ($i \in \Delta_2$) is in bijection with the set $\{c_i^1, c_i^2, \dots, c_i^{\sigma_i}\}$ of disjoint cycles of the permutation τ_i , r_i^s ($s = 1, 2, \dots, \sigma_i$) being obtained by juxtaposing the generators corresponding to the ordered elements of c_i^s .

PROOF. – Let $\Lambda = \Lambda(\tau)$ be the crystallization of M^3 canonically associated to the set $\mathcal{T}$ of permutations. By construction, the set of 3-coloured edges of Λ is $\{x_1, x_2, \dots, x_{q/2}\}$, $i \in \{1, 2, \dots, q/2\}$ and $i' \in \{1', 2', \dots, q/2\}$ being the end-points of x_i , for every $i \in \{1, 2, \dots, q/2\}$. If every edge x_i ($i \in \{1, 2, \dots, q/2\}$) is oriented from i to i' , then $\langle X/R_0 \cup R_1 \cup R_2 \rangle$ is exactly the 3-edge presentation of $\Pi_1(M^3)$ associated to Λ , since τ_0, τ_1, τ_2 are directly related with the 2-residues of Λ containing the colour 3. ■

4. – Main results (dimension four).

Let (Γ, γ) be a crystallization of a closed connected orientable 4-manifold M^4 , and let $V'(\Gamma), V''(\Gamma)$ be the two bipartition classes of its vertex set. Further, let $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ be an arbitrarily fixed permutation of the colour set. As in the previous section, for sake of notational simplicity, we shall always write c instead of ε_c , for every $c \in \Delta_4$.

Let $(\tilde{\Gamma}, \tilde{\gamma})$ be the 5-coloured graph representing M^4 such that the pseudocomplex $K(\tilde{\Gamma})$ is obtained from $K(\Gamma)$ by baricentrically subdividing the tetrahedron opposite to the (unique) 4-labelled vertex v_4 , and then by starring from v_4 itself. It is easy to check that $(\tilde{\Gamma}, \tilde{\gamma})$ can be directly obtained from (Γ, γ) by means of the following rules:

— for every $v \in V(\Gamma)$, construct a copy of the 3-coloured graph (G, g) of figure 5, and label its vertices by the elements of the set $N_{24} \times \{v\}$, so that the vertex (i, v) is the copy of the vertex i of (G, g) ;

— for every $i \in \{1, \dots, 6\}$ (resp. $i \in \{7, \dots, 12\}$) (resp. $i \in \{13, \dots, 18\}$) (resp. $i \in \{19, \dots, 24\}$), connect (i, v) with (i, w) by a 3-coloured edge iff v and w are connected in (Γ, γ) by a 0-coloured (resp. 1-coloured) (resp. 2-coloured) (resp. 3-coloured) edge;

— for every $i \in N_{24}$, connect (i, v) with (i, w) by a 4-coloured edge iff v and w are connected in (Γ, γ) by a 4-coloured edge.

Let now $(\bar{\Gamma}, \bar{\gamma})$ denote the standard 5-coloured graph of figure 6, which evidently represents S^4 . An easy direct computation proves the following result.

PROPOSITION 7. – Let $f: (\tilde{\Gamma}, \tilde{\gamma}) \rightarrow (\bar{\Gamma}, \bar{\gamma})$ be the map defined by

$$f(i, v) = \begin{cases} A & \text{if } i \text{ even and } v \in V'(\Gamma), \\ B & \text{if } i \text{ odd and } v \in V'(\Gamma), \\ A' & \text{if } i \text{ even and } v \in V''(\Gamma), \\ B' & \text{if } i \text{ odd and } v \in V''(\Gamma). \end{cases}$$

Fig. 5.

Then, f is a 1-covering, branched over the 2-residues of colours $\{0, 1\}$, $\{1, 2\}$, $\{0, 2\}$, $\{2, 3\}$, $\{3, 4\}$. ■

The 1-covering f naturally induces a topological covering $|f|: M^4 \rightarrow S^4$, branched over the subcomplex K of $K(\bar{T})$ represented by the branching 2-residues of $(\bar{T}, \bar{\gamma})$. It is easy to check, by analyzing the pseudosimplicial structure of $K(\bar{T})$, that the branching set K is the bidimensional pseudocomplex depicted in figure 7 (where equally labelled vertices equally signed edges have to be identified).

Note that, in analogy with the branching set G described in the previous section, K contains exactly two 2-spheres «connected» by one 2-simplex, but it further contains an additional cone from a point of one sphere to a maximum circle of the other one.

Fig. 6.

COROLLARY 8. – *Every closed connected orientable 4-manifold M^4 is a covering of S^4 branched over the universal set K .*

Since branched covering spaces are uniquely determined by the monodromy of the covering map $[F]$, we are now going to determine the structure of the group of K , i.e. $\Pi_1(S^4 - K)$. For, we consider the (unique) bisection $\bar{\Gamma}'$ of type $(2, 0)$ of $\bar{\Gamma}$ (see [G]) and then the (unique) bisection of type $(2, 3)$ of $\bar{\Gamma}'$; the resulting graph $\bar{\Gamma}''$ (which is drawn in figure 8) represents a pseudocomplex $K(\bar{\Gamma}'')$ triangulating S^4 , such that $(K(\bar{\Gamma}''), K)$ is a branched 5-coloured pair (with respect to the colour 2). Thus, it is possible to yield $\Pi_1(S^4 - K)$ from the 2-edge presentation of $\Pi_1(S^4)$ associated to $\bar{\Gamma}''$.

PROPOSITION 9. – $\Pi_1(S^4 - K) = \langle a, b, c/aca^{-1}c^{-1} = 1 \rangle$.

PROOF. – As pointed out in section 2, $\Pi_1(S^4 - K)$ is obtained from the 2-edge presentation of $\Pi_1(S^4)$ associated to $\bar{\Gamma}''$ by deleting all the relators corresponding to branching 2-residues (i.e. to 2-simplices of $K(\bar{\Gamma}'')$ belonging to the subcomplex K). Hence,

$$\Pi_1(S^4 - K) = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8/x_5 = x_6, x_7 = x_8, x_5 = x_8, \quad$$

$$x_6 = x_7, x_1x_6x_4^{-1}x_7^{-1} = 1, x_1 = x_4, x_2 = x_3 \rangle = \langle x_1, x_2, x_5/x_1x_5x_1^{-1}x_5^{-1} = 1 \rangle. \quad \blacksquare$$

COROLLARY 10. – *Every closed connected orientable 4-manifold M^4 is represented by a suitable transitive set $\{\sigma, \tau, \mu\}$ of permutations, such that $\sigma\mu\sigma^{-1}\mu^{-1} = 1$.*

Fig. 7.

PROOF. – If $\omega: \Pi_1(S^4 - K) \rightarrow S_d$ is the monodromy associated to the covering map from M^4 to S^4 branched over K , it is sufficient to set $\sigma = \omega(a)$, $\tau = \omega(b)$, $\mu = \omega(c)$,

Fig. 8.

a, b, c being the generators of the presentation of $\Pi_1(S^4 - K)$ obtained in Prop. 9. ■

Given a transitive set $\{\sigma, \tau, \mu\}$ of permutations on N_d such that $\sigma\mu\sigma^{-1}\mu^{-1} = 1$, it is univocally determined the d -fold covering space $\Phi(\omega)$ of S^4 branched over K , $\omega: \Pi_1(S^4 - K) \rightarrow S_d$ being the (unique) transitive representation extending the map $\psi: \{a, b, c\} \rightarrow S_d$ defined by $\psi(a) = \sigma$, $\psi(b) = \tau$, $\psi(c) = \mu$. However, it is easy to check that the 4-dimensional polyhedron $\Phi(\omega)$ is not necessarily a manifold.

We want now to analyze the algebraic features of the set $\{\sigma, \tau, \mu\}$ of permutations associated to a given 4-manifold M^4 , by computing the monodromy of the topological covering $|f|$ induced by the 1-covering f of Prop. 7. In order to do this, we consider the (global) bisection $\tilde{\Gamma}'$ of type $(2, 0)$ of $\tilde{\Gamma}$, and then the (global) bisection of type $(2, 3)$ of $\tilde{\Gamma}'$, so that the 1-covering f between $\tilde{\Gamma}$ and $\tilde{\Gamma}$ naturally induces a 1-covering f'' between the resulting graph $\tilde{\Gamma}''$ (representing M^4) and the graph $\tilde{\Gamma}$ (representing S^4). It is now easy to check that the 1-covering $f'': \tilde{\Gamma}'' \rightarrow \tilde{\Gamma}$ can be obtained as an application of a standard method—due to [CG] and briefly described in section 2—to

yield coloured graphs representing branched coverings of the sphere, starting from the associated monodromy. Since $\tilde{\Gamma}''$ contains twelve copies of $\tilde{\Gamma}''_2$ for every pair $\{v, w\}$ of 4-adjacent vertices of (Γ, γ) , f'' induces a topological $(12 \cdot p/2)$ -fold covering ($p = \# V(\Gamma)$) whose monodromy is described by the 2-coloured edges connecting the various sheets.

Thus, σ, τ, μ may be thought of as permutations on the set $S = N_{12} \times N_{p/2}$, by denoting $V'(\Gamma) = \{1, 2, \dots, p/2\}$ and $V''(\Gamma) = \{1', 2', \dots, p'/2\}$, with i 4-adjacent to i' for every $i \in N_{p/2}$; moreover, the combinatorial properties of the graph $\tilde{\Gamma}''$ ensure that σ and τ may always be assumed to be of standard type (only depending on $p = \# V(\Gamma)$), while μ directly depends on the combinatorial properties of the starting crystallization (Γ, γ) of M^4 :

$$\sigma = \prod_{i=1}^{p/2} [((1, i)(2, i)(3, i))((4, i)(5, i)(6, i))((7, i)(8, i)(9, i))((10, i)(11, i)(12, i))],$$

$$\tau = \prod_{i=1}^{p/2} [((1, i)(8, i)(10, i))((2, i)(4, i)(7, i))((3, i)(11, i)(6, i))((5, i)(12, i)(9, i))],$$

$$(\#) \quad \mu((x, i)) = (x, \mu_{j(x)}(i)), \quad \text{where } j(x) = \begin{cases} 0 & x \in \{1, 2, 3\}, \\ 1 & x \in \{4, 5, 6\}, \\ 2 & x \in \{7, 8, 9\}, \\ 3 & x \in \{10, 11, 12\}, \end{cases}$$

and $\mu_j(i)$ ($j \in \{0, 1, 2, 3\}$), for every vertex $i \in V'(\Gamma)$ is the vertex $r \in V'(\Gamma)$ which is 4-adjacent to the vertex $r' \in V''(\Gamma)$ j -adjacent to i .

Note that the above defined permutations σ, τ, μ obviously verify condition $\sigma\mu\sigma^{-1}\mu^{-1} = 1$; moreover, $\{\sigma, \tau, \mu\}$ is a transitive set, since $\tilde{\Gamma}''$ is a connected graph.

Further, we point out that the only permutation μ on S is sufficient to reconstruct (Γ, γ) . In fact, (Γ, γ) is isomorphic to the 5-coloured graph $\Gamma(\mu)$ obtained by the following rules:

- the vertex set of $\Gamma(\mu)$ is $\{1, 2, \dots, p/2\} \cup \{1', 2', \dots, p'/2\}$;
- every vertex $i \in \{1, 2, \dots, p/2\}$ is connected with the homonymous vertex $i' \in \{1', 2', \dots, p'/2\}$ by a 4-coloured edge;
- every vertex $i \in \{1, 2, \dots, p/2\}$ is connected with the vertex $(\mu_j(i))' \in \{1', 2', \dots, p'/2\}$ by a j -coloured edge, for every $j \in \{0, 1, 2, 3\}$.

Thus, we have described an algorithm for obtaining, from any 4-manifold M^4 —given by a crystallization (Γ, γ) —, a permutation μ which completely represents it; this leads to the following result.

PROPOSITION 11. – *Every closed connected orientable 4-manifold M^4 is represented by a suitable permutation μ on $S = N_{12} \times N_{p/2}$ such that*

$$\pi_1 \circ \mu = 1 \quad \text{and} \quad (\pi_2 \circ \mu)((x, i)) = \begin{cases} \mu_0(i) & x \in \{1, 2, 3\}, \\ \mu_1(i) & x \in \{4, 5, 6\}, \\ \mu_2(i) & x \in \{7, 8, 9\}, \\ \mu_3(i) & x \in \{10, 11, 12\}, \end{cases}$$

(or, equivalently, by a suitable set $\{\mu_0, \mu_1, \mu_2, \mu_3\}$ of permutations on $N_{p/2}$).

REMARK. – The four permutations $\mu_0, \mu_1, \mu_2, \mu_3$ on $N_{p/2}$ may be thought of as the rows of a «code» for 4-manifolds, extending the one defined in [LD] for dimension three.

Obviously, Prop. 11 can not be reversed: not every set $\{\mu_0, \mu_1, \mu_2, \mu_3\}$ of permutations on $N_{p/2}$ ($p \in N$) is the result of the described algorithm applied to a p -vertices crystallization (Γ, γ) of a 4-manifold M^4 . Nay, we know no direct method to decide whether this happens or not. A partial answer is based on the combinatorial properties of the 5-coloured graph $I(\mu)$, μ being obtained from $\{\mu_0, \mu_1, \mu_2, \mu_3\}$ by formula (#).

PROPOSITION 12. – *Let $\mathcal{P} = \{\mu_0, \mu_1, \mu_2, \mu_3\}$ be a set of permutations on $N_{p/2}$ ($p \in N$). $\mathcal{P}$ represents a singular 4-manifold M^4 if and only if $I(\mu)$ satisfies (1) $g_{\bar{c}} = 1$ and (2)*

$$\sum_{j, k \in \Delta_4 - \{c\}} g_{jk} - \sum_{j \in \Delta_4 - \{c\}} g_{\bar{c}j} = p \quad \text{for every } c \in \Delta_4.$$

PROOF. – It is evident that $\mathcal{P}$ represents a singular 4-manifold M^4 if and only if $I(\mu)$ does, i.e. if its (unique) $\bar{c}$ -residue $\Gamma_{\bar{c}}$ represents a closed 3-manifold, for every $c \in \Delta_4$. Condition (2) now easily follows by computing the Euler characteristic of $K(\Gamma_{\bar{c}})$:

$$X(K(\Gamma_{\bar{c}})) =$$

$$= \sum_{j \in \Delta_4 - \{c\}} g_{\bar{c}j} - \sum_{j, k \in \Delta_4 - \{c\}} g_{jk} + \#E(\Gamma_{\bar{c}}) - \#V(\Gamma_{\bar{c}}) = \sum_{j \in \Delta_4 - \{c\}} g_{\bar{c}j} - \sum_{j, k \in \Delta_4 - \{c\}} g_{jk} + p. \quad \blacksquare$$

If $\mathcal{P} = \{\mu_0, \mu_1, \mu_2, \mu_3\}$ is a set of permutations representing a closed 4-manifold M^4 , then the crystallization $I(\mu)$ of M^4 , μ being obtained from $\{\mu_0, \mu_1, \mu_2, \mu_3\}$ by formula (#), is said to be *canonically associated* to $\mathcal{P}$.

The following result shows how easily $\mathcal{P} = \{\mu_0, \mu_1, \mu_2, \mu_3\}$ may produce the fundamental group of the represented 4-manifold, via the 4-edge presentation of the canonically associated crystallization.

PROPOSITION 13. – Let $\mathcal{P} = \{\mu_0, \mu_1, \mu_2, \mu_3\}$ be a set of permutations on $N_{p/2}$ ($p \in N$), representing a closed 4-manifold M^4 . Then, a (positive) presentation of $\Pi_1(M^4)$ is

$$\langle X/R_0 \cup R_1 \cup R_2 \cup R_3 \rangle,$$

where the set $X = \{x_1, x_2, \dots, x_{p/2}\}$ of generators is in bijection with $N_{p/2}$, and the set $R_i = \{r_i^1, r_i^2, \dots, r_i^{\sigma_i}\}$ of relators ($i \in \Delta_3$) is in bijection with the set $\{c_i^1, c_i^2, \dots, c_i^{\sigma_i}\}$ of disjoint cycles of the permutation μ_i , r_i^s ($s = 1, 2, \dots, \sigma_i$) being obtained by juxtaposing the generators corresponding to the ordered elements of c_i^s .

PROOF. – Let $\Gamma = \Gamma(\mu)$ be the crystallization of M^4 canonically associated to the set $\mathcal{P}$ of permutations. By construction, the set of 4-coloured edges of Γ is $\{x_1, x_2, \dots, x_{p/2}\}$, $i \in \{1, 2, \dots, p/2\}$ and $i' \in \{1', 2', \dots, p'/2\}$ being the end-points of x_i , for every $i \in \{1, 2, \dots, p/2\}$. If every edge x_i ($i \in \{1, 2, \dots, p/2\}$) is oriented from i to i' , then $\langle X/R_0 \cup R_1 \cup R_2 \cup R_3 \rangle$ is exactly the 4-edge presentation of $\Pi_1(M^4)$ associated to Γ , since $\mu_0, \mu_1, \mu_2, \mu_3$ are directly related with 2-residues of Γ containing colour 4. ■

5. – A glance towards higher dimensions.

Without going into depth about detailed proofs, we devote the present last section to show how to extend the procedures developed in the previous section to arbitrary dimension.

Let (Ω, ω) be a crystallization of a closed connected orientable n -manifold M^n ; w.l.o.g., we may fix the permutation $\varepsilon = (0, 1, \dots, n)$ of the colour set. Let $(\tilde{\Omega}, \tilde{\omega})$ be the $(n+1)$ -coloured graph representing M^n such that the pseudocomplex $K(\tilde{\Omega})$ is obtained from $K(\Omega)$ by baricentrically subdividing the $(n-1)$ -simplex opposite to the (unique) n -labelled vertex v_n , and then by starring from v_n itself. The particular geometrical structure of $K(\tilde{\Omega})$ induces particular combinatorial properties on the $(n+1)$ -coloured graph $(\tilde{\Omega}, \tilde{\omega})$:

- a) for $i \in \{0, 1, \dots, n-3\}$, every $\{i, i+1\}$ -residue of $(\tilde{\Omega}, \tilde{\omega})$ has valence six;
- b) for $i, j \in \{0, 1, \dots, n-2\}$, $j \neq i \pm 1$, every $\{i, j\}$ -residue of $(\tilde{\Omega}, \tilde{\omega})$ has valence four;
- c) for $i \in \{0, 1, \dots, n-3\}$ (resp. $i \in \{0, 1, \dots, n-2\}$) every $\{i, n-1\}$ -residue (resp. $\{i, n\}$ -residue) of $(\tilde{\Omega}, \tilde{\omega})$ has valence four.

On the other hand, the valences of the $\{n-1, n-1\}$ - and $\{n-2, n\}$ -residues of $(\tilde{\Omega}, \tilde{\omega})$ directly depend upon the incidence structure of $K(\Omega)$.

It is now easy to check the existence of a canonical 1-covering f between $(\tilde{\Omega}, \tilde{\omega})$ and the standard $(n+1)$ -coloured graph $(\bar{\Omega}, \bar{\omega})$ representing S^n of figure 9; moreover.

Fig. 9.

the branching 2-residues of $(\bar{\Omega}, \bar{\omega})$ are the $\{i, i+1\}$ -residues, for $i \in \{0, 1, \dots, n-3\}$ (with branching index three), the $\{i, j\}$ -residues, for $i, j \in \{0, 1, \dots, n-2\}$, $j \neq i \pm 1$ (with branching index two), the $\{n-2, n-1\}$ - and $\{n-1, n\}$ -residues. Thus, the subcomplex $X = X^{n-2}$ of $K(\bar{\Omega})$ represented by the branching 2-residues of $(\bar{\Omega}, \bar{\omega})$ constitutes a universal branching set for n -manifolds.

PROPOSITION 14. – *Every closed connected orientable n -manifold M^n is a covering of S^n branched over X^{n-2} .*

By performing on $(\bar{\Omega}, \bar{\omega})$ $n-2$ successive bisections (of type $(n-2, i)$, for $i = 0, 1, \dots, n-4$ and finally for $i = n-1$), a new graph $\bar{\Omega}^*$ representing S^n is obtained, such that $(K(\bar{\Omega}^*), X)$ is a branched $(n+1)$ -coloured pair (with respect to the colour $n-2$); thus, it is possible to yield $\Pi_1(S^n - X)$ from the $(n-2)$ -edge presentation of $\Pi_1(S^n)$ associated to $\bar{\Omega}^*$:

PROPOSITION 15. – $\Pi_1(S^n - X) = \langle x_1, x_2, \dots, x_{2^{n-3}}, x_{2^{n-3}+1} = \bar{x}/x_j \bar{x} = \bar{x}x_j \forall j \in \{1, \dots, 2^{n-3} - 1\} \rangle$.

HINT OF PROOF. – The presentation is obtained by considering, at first, all $(n-2)$ -coloured edges of $\bar{\Omega}^*$ as generators, and then by reading the relations only from the non-branching 2-residues of $\bar{\Omega}^*$, i.e. from the $\{i, n-2\}$ -residues of $\bar{\Omega}^*$, for $i \in \Delta_n -$

$-\{n-1, n-2\}$, which are obtained by subdivision from the $\{i, n-1\}$ -residues, for $i \in \{0, \dots, n-3\}$, and the $\{j, n\}$ -residues, for $j \in \{0, \dots, n-2\}$, of $\bar{\Omega}$. ■

The following result is a direct consequence of the fact that branched covering spaces are uniquely determined by the monodromy of the covering map [F].

COROLLARY 16. – *Every closed connected orientable n -manifold M^n is represented by a suitable transitive set $\{\sigma_1, \sigma_2, \dots, \sigma_{2^{n-3}}, \sigma_{2^{n-3}+1} = \bar{\sigma}\}$ of permutations, such that $\sigma_j \bar{\sigma} = \bar{\sigma} \sigma_j$, $\forall j \in \{1, \dots, 2^{n-3} - 1\}$.*

REMARK. – Corollary 16 holds for dimensions three and four, as well: in fact, for $n = 3$ (resp. $n = 4$) it reduces to Corollary 4 (resp. Corollary 10).

It is now possible to analyze the algebraic features of the set $\{\sigma_1, \sigma_2, \dots, \sigma_{2^{n-3}}, \sigma_{2^{n-3}+1} = \bar{\sigma}\}$ of permutations associated to a given n -manifold $M^n = K(\Omega)$, by means of the same techniques applied in sections three and four: by performing on $(\bar{\Omega}, \bar{\omega})$ $n-2$ successive bisections, a new $(n+1)$ -coloured graph $(\bar{\Omega}^*, \bar{\omega}^*)$ is obtained, so that the induced 1-covering between $(\bar{\Omega}^*, \bar{\omega}^*)$ and $(\bar{\Omega}, \bar{\omega})$ can be seen as an application of the construction described in [CG]. Thus, $\{\sigma_1, \sigma_2, \dots, \sigma_{2^{n-3}}, \sigma_{2^{n-3}+1} = \bar{\sigma}\}$ may be thought of as the permutations defining the monodromy of a topological $(p \cdot n!/4)$ -fold covering ($p = \# V(\Omega)$); moreover, the combinatorial properties of the graph $\bar{\Omega}^*$ ensure that, for every $i \in \{1, \dots, 2^{n-3}\}$, σ_i may be assumed to be of standard type (only depending on $p = \# V(\Omega)$ and on the structure of any $\bar{n}$ -residue of $(\bar{\Omega}, \bar{\omega})$, representing an $(n-1)$ -simplex baricentrically subdivided), while $\bar{\sigma}$ directly depends on the combinatorial properties of the starting crystallization (Ω, ω) of M^n .

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