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On the Propagation of Smoothness for Semilinear Systems of Maxwell-Klein-Gordon Type (*).

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Abstract. – *We prove a Bony-type result about the propagation of Sobolev smoothness for a particular type of semilinear strictly hyperbolic system with an application to the Maxwell-Klein-Gordon system.*

Introduction.

The problem of propagation of singularities of solutions to semilinear strictly-hyperbolic equations, initiated by RAUCH in [7], for scalar equations of the form

$$P_m(x, D)\, u = f(x, u, \ldots, D^{m-2}u), \qquad m \geq 2,$$

was extended to general scalar equations of the type

$$P_m(x, D)\, u = f(x, u, \ldots, D^{m-1}u), \qquad m \geq 2$$

by BONY in [4], using the machinery of paradifferential calculus.

In this paper we consider a particular type of semilinear system, which is here said to be of Maxwell-Klein-Gordon type. This kind of system is a very special case of a more general one: the two-speed system, introduced by RAUCH and REED in [8]. The problem they study there is the propagation of particular singularities associated to a characteristic foliation of $\boldsymbol{R}^{1+n}$ induced by the two different speeds of the system. More precisely they prove the propagation of the conormality of the solution with respect to a certain family of characteristic hypersurfaces. This means that particular information about the differentiability of the solution with respect to a family of vector field tangent to the characteristic family on the initial hyperplane propagates in the future with respect to the above foliation induced by the two speeds. (It is well

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known that the conormality property is strictly stronger than a mere statement about wavefronts, see [1]).

In this paper we consider instead the problem of propagation of smoothness and prove a Bony-type result. This means that, given a conic neighborhood around a given point of the null bicharacteristic, and given suitable Sobolev regularity of the solution in the above neighborhood, the same regularity holds along the whole null bicharacteristic.

Our approach is along the line of [2]: first a linear theorem is proved about propagation of singularities (using those of BEALS and REED of [2] and HÖRMANDER, plues the Lemma di Rauch (see for instance [3] for a general proof)) and secondly the theorem about propagation of smoothness, by adapting the bootstrap argument of BEALS and REED [2] to the present case.

Finally an application is given to the Maxwell-Klein-Gordon system which, choosing the Lorentz gauge, assumes the form considered here.

We wish to thank MICHAEL BEALS for his advice.

Statement and proof.

DEFINITION. – *A semilinear system is said to be of Maxwell-Klein-Gordon type if it can be written under the form* (P) *below.*

Consider on $\boldsymbol{R}^n = \boldsymbol{R} \times \boldsymbol{R}^{n-1}$ the following system:

$$(P) \qquad \begin{cases} P_m(x, D) A_\mu + Q_{(\mu)}(x, D) A_\mu = \gamma_{\mu, m-2, m-1}(x, \phi, D^\beta \phi, D^\alpha \boldsymbol{A}), \quad \mu = 0, 1, \dots, N-1, \\ P_m(x, D)\phi + \sum\limits_\mu \rho^\mu_{m-2}(x, D^\alpha\phi, D^\alpha \boldsymbol{A}) Q_\mu(x, D)\phi + f_{m-2}(x, D^\alpha\phi, D^\alpha\boldsymbol{A}) = 0, \end{cases}$$

where $\gamma_{\mu, m-2, m-1}$, ρ^μ_{m-2}, $f_{m-2} \in C^\infty$ in the arguments

$$x, \phi, \boldsymbol{A}, D^\beta\phi, D^\alpha\phi, D^\alpha\boldsymbol{A}$$

for $|\alpha| \leqslant m-2$, $|\beta| \leqslant m-1$; $(\boldsymbol{A}, \phi) \in \boldsymbol{R}^{N+1}$; $P_m(x, D)$ is a real smooth-coefficients homogeneous strictly hyperbolic differential operator of order $m \geqslant 2$, $Q_\mu(x, D)$, $Q_{(\mu)}(x, D)$ are smooth-coefficients differential operator of order $m-1$. Here $D = (D_0, D_1, \dots, D_{n-1})$.

REMARK 1. – The main result of this paper is still true if (P) is of the form:

$$\begin{cases} \boldsymbol{P}(x, D)\boldsymbol{A} = \gamma_{m-2, m-1}(x, \phi, D^\beta\phi, D^\alpha\boldsymbol{A}), \\ P_m(x, D)\phi + \sum\limits_{\mu=1}^{M} \rho^\mu_{m-2}(x, D^\alpha\phi, D^\alpha\boldsymbol{A}) Q_\mu(x, D)\phi + f_{m-2}(x, D^\alpha\phi, D^\alpha\boldsymbol{A}) = 0, \end{cases}$$

where $\boldsymbol{P}(x, D)$ is an $N \times N$ system such that $Q_{Nm}(x, \xi)$ = principal part of $\det \boldsymbol{P}(x, \xi)$ is of real principal type (i.e. $Q_{Nm}(x, \xi)$ is real and the Hamilton field $H_{Q_{Nm}}$ is non-vanishing and does not have the radial direction when $Q_{Nm} = 0$), and Γ is a null bicharacteristic for both $P_m(x, D)$ and $Q_{Nm}(x, D)$. In fact Hörmander's Theorem is still valid for the system $\boldsymbol{P}(x, D)$, (see [9]). Here $\boldsymbol{A} = (A_0, \ldots, A_{N-1})$ and $\gamma_{m-2, m-1} = (\gamma^0_{m-2, m-1}, \ldots, \gamma^{N-1}_{m-2, m-1})$.

REMARK 2. – The special rôle of ϕ in (P) is the appearance of $D^\beta \phi$, $|\beta| = m - 1$. A nonlinear term of type $\gamma_{\mu, m-2-k, m-1-k}$, $k \geq 1$, would allow the operators $Q_{(\mu)}$ to have coefficients depending on $D^\alpha \phi$, $D^{\alpha'} \boldsymbol{A}$, for $|\alpha|$, $|\alpha'| \leq k - 1$, and the result would remain true, but for the example considered below such coefficients and nonlinear terms do not occur.

Suppose we are given a solution $(\boldsymbol{A}, \phi) = ((A_\mu), \mu = 0, 1, \ldots, N-1; \phi)$ of (P). We want to study the propagation of smoothness of the solution.

Notice that $\operatorname{char}(P_m) \subset \operatorname{char}(\boldsymbol{P})$ and $\operatorname{char}(P_m)$ is foliated by bicharacteristic curves of P_m. Let $\gamma_0 = (x_0, \xi_0) \in \boldsymbol{R}^{2n} \setminus 0$ be a point of Γ, $P_m(\gamma_0) = 0$, Γ the null bicharacteristic of P_m through γ_0. Here $H^t_{\text{loc}} = H^t_{\text{loc}}(\boldsymbol{R}^n)$.

THEOREM A. – *Suppose*

i) $\frac{1}{2}n + m < s + m - 1 \leq r + m - 1 < 2(s + m - 1) - \left(\frac{1}{2}n + m - 1\right)$, $m \geq 2$,

ii) $\forall \mu$, $A_\mu, \phi \in H^{s+m-1}_{\text{loc}} \cap H^{r+m-1}_{\text{ml}}(\Gamma)$,

iii) $\forall \mu$, $A_\mu, \phi \in H^{r+m-1+\varepsilon}_{\text{ml}}(\gamma_0)$, *some* $\varepsilon \leq \min\left\{1, 2s - \frac{1}{2}n - r\right\}$,

iv) $(\boldsymbol{A}, \phi)$ *satisfies* (P).

Then $A_\mu, \phi \in H^{r+m-1+\varepsilon}_{\text{ml}}(\Gamma)$, $\forall \mu = 0, 1, \ldots, N-1$.

THEOREM B. – *Suppose*

i) $\frac{1}{2}n < s \leq \frac{1}{2}n + 1$ and $0 \leq \varepsilon < s - \frac{1}{2}n$ (so $\varepsilon < 1$), $m \geq 2$,

ii) $\forall \mu$, $A_\mu, \phi \in H^{s+m-1}_{\text{loc}}$,

iii) $\forall \mu$, $A_\mu, \phi \in H^{s+m-1+\varepsilon}_{\text{ml}}(\gamma_0)$,

iv) $(\boldsymbol{A}, \phi)$ satisfies (P).

Then $A_\mu, \phi \in H^{s+m-1+\varepsilon}_{\text{ml}}(\Gamma)$, $\forall \mu = 0, 1, \ldots, N-1$.

PROOF OF A. – We are going to use the Propagation Theorems of Beals-Reed ([2]) and Hörmander (see [1], [6] or [9]). Suppose first $m = 2$.

Consider $\Lambda = (1 + |D|^2)^{1/2} \in \Psi^1_{1,0}$ ($\Psi^m_{1,0}$ being the set of pseudo-differential operators of order m and type $(1, 0)$, see [5]. From now on all the pseudodifferential operators used in the following will be supposed to be properly supported.) Then

$$\Lambda P_2(x, D) = P_2(x, D)\Lambda + [\Lambda, P_2(x, D)] = P_2(x, D)\Lambda + ([\Lambda, P_2(x, D)]\Lambda^{-1})\Lambda,$$

$$\Lambda \sum_\mu \rho_0^\mu Q_\mu = \sum_\mu (\rho_0^\mu Q_\mu(x, D)\Lambda + \rho_0^\mu[\Lambda, Q_\mu(x, D)] + [\Lambda, \rho_0^\mu]Q_\mu(x, D)),$$

and, by Rauch's Lemma and our hypothesis, $\Lambda f_0(x, \phi, A) \in H^s_{\rm loc} \cap H^r_{\rm ml}(\Gamma)$. Notice that: $[\Lambda, P_2(x, D)]\Lambda^{-1} + \sum_\mu \rho_0^\mu Q_\mu(x, D)$ is a sum of pseudodifferential operators of order 1, that $\rho_0^\mu \in H^{s+1}_{\rm loc} \cap H^{r+1}_{\rm ml}(\Gamma)$, that $[\Lambda, Q_\mu(x, D)]$ has order 0 and that, by the Commutator Lemma in [2], we get that $[\Lambda, \rho_0^\mu]Q_\mu(x, D)\phi \in H^s_{\rm loc} \cap H^r_{\rm ml}(\Gamma)$.

Therefore the 2nd equation in (P) reduces to an equation in $\Lambda\phi$:

$$P_2(x, D)\Lambda\phi + ([\Lambda, P_2(x, D)]\Lambda^{-1} + \sum_\mu \rho_0^\mu(x, \phi, A)Q_\mu(x, D))\Lambda\phi +$$

$$+ (\sum_\mu \rho_0^\mu(x, \phi, A)p_{0,\mu}(x, D))\Lambda\phi + \sum_\mu G_\mu + \Lambda f_0(x, \phi, A) = 0$$

where $p_{0,\mu}(x, D)$ is in $\Psi^0_{1,0}$; and G_μ, $\Lambda f_0 \in H^s_{\rm loc} \cap H^r_{\rm ml}(\Gamma)$, all of this since $(1/2)n + 1 < s \leqslant r < 2s - (1/2)n$, and $(1/2)n + 1 < s + 1 \leqslant r + 1 < 2(s+1) - ((1/2)n + 1) < 2(s+1) - (1/2)n$ (so that we can use Rauch's Lemma). Notice that now we have the following hypothesis on $\Lambda\phi$:

$$\Lambda\phi \in H^s_{\rm loc} \cap H^r_{\rm ml}(\Gamma) \quad \text{and} \quad \Lambda\phi \in H^{r+\varepsilon}_{\rm ml}(\gamma_0).$$

Then we can apply the theorem of Beals-Reed to get $\Lambda\phi \in H^{r+\varepsilon}_{\rm ml}(\Gamma)$. Now, since $\varepsilon \leqslant 1$, $\phi \in H^{r+\varepsilon}_{\rm ml}(\Gamma)$ and by standard pseudodifferential arguments, $\phi \in H^{r+1+\varepsilon}_{\rm ml}(\Gamma)$.

Consider now the first equation in (P): the overall regularity of the arguments of $\gamma_{\mu, m-2, m-1}$ is $H^s_{\rm loc} \cap H^{r+\varepsilon}_{\rm ml}(\Gamma)$ (again since $\varepsilon \leqslant 1$ and $A_\mu \in H^{s+1}_{\rm loc} \cap H^{r+1}_{\rm ml}(\Gamma)$) since $(1/2)n < s$ and $r + \varepsilon \leqslant r + (2s - (1/2)n - r)$ and Rauch's Lemma applies, and since we already have $D^\beta\phi \in H^s_{\rm loc} \cap H^{r+\varepsilon}_{\rm ml}(\Gamma)$.

By hypothesis we have $A_\mu \in H^{r+1+\varepsilon}_{\rm ml}(\gamma_0)$ and therefore, by Hörmander's Theorem, we get $A_\mu \in H^{r+1+\varepsilon}_{\rm ml}(\Gamma)$.

This concludes the proof in the case $m = 2$.

If $m > 2$ consider $\Lambda^{-(m-2)}\Lambda^{m-2}\phi$; then we get that the second equation in (P) becomes:

$$(P_m(x, D)\Lambda^{-(m-2)})\Lambda^{m-2}\phi +$$

$$+ \sum_\mu \rho_{m-2}^\mu(x, D^\alpha\phi, D^\alpha A)(Q_\mu(x, D)\Lambda^{-(m-2)})\Lambda^{m-2}\phi + f_{m-2}(x, D^\alpha\phi, D^\alpha A) = 0.$$

Then we are back in the case where the principal part has order 2, and we can apply the same procedure as before since the Propagation Theorem of Beals and Reed is valid for strictly hyperbolic pseudodifferential equations.

(We remark that $\text{char}(P_m) = \text{char}(P_m\Lambda^{-(m-2)})$ and that when $P_m(\gamma_0) = 0$ we have $H_{P_m\Lambda^{-(m-2)}} = (1 + |\xi|^2)^{-(m-2)/2}H_{P_m}$, so $H_{P_m\Lambda^{-(m-2)}}$ and H_{P_m} have the same orbits through γ_0.) This concludes the proof of A. ■

PROOF OF B. – As before suppose first that $m = 2$. Then, proceeding the same way as above, we get:

$$\Lambda\phi \in H^s_{\mathrm{loc}}, \quad A_\mu \in H^{s+1}_{\mathrm{loc}}, \quad \mu = 0, 1, \dots, N-1, \quad \rho_0^\mu(x, \phi, A) \in H^{s+1}_{\mathrm{loc}}, \quad \Lambda f_0(x, \phi, A) \in H^s_{\mathrm{loc}},$$

then we can apply the second Propagation Theorem of Beals-Reed getting $\Lambda\phi \in H^{s+\varepsilon}_{\mathrm{ml}}(\Gamma)$, since we are assuming $\Lambda\phi \in H^{s+\varepsilon}_{\mathrm{ml}}(\gamma_0)$.

Again it follows that $\phi \in H^{s+1+\varepsilon}_{\mathrm{ml}}(\Gamma)$. Now

$$\frac{1}{2}n < s \leqslant s + \varepsilon < s + \left(s - \frac{1}{2}n\right),$$

then, by Rauch's Lemma, we have $\gamma_{\mu,0,1} \in H^s_{\mathrm{loc}} \cap H^{s+\varepsilon}_{\mathrm{ml}}(\Gamma)$ (since $s \leqslant (1/2)n + 1$, then $\varepsilon \leqslant 1$, so that $A_\mu \in H^{s+\varepsilon}_{\mathrm{ml}}(\Gamma)$).

Then by Hörmander's Theorem we obtain $A_\mu \in H^{s+1+\varepsilon}_{\mathrm{ml}}(\Gamma)$ $\forall\mu = 0, 1, \dots, N-1$, since by hypothesis $A_\mu \in H^{s+1+\varepsilon}_{\mathrm{ml}}(\gamma_0)$.

This concludes the proof in the case $m = 2$. If $m > 2$ we can apply the same trick as above to get the result. This concludes the proof of B. ■

Now we are ready to state the following Bony-type theorem:

THEOREM C. – *Suppose* $(1/2)n + m - 1 < \sigma \leqslant \tau < 2\sigma - ((1/2)n + m - 1)$ *and* A_μ, $\phi \in H^\sigma_{\mathrm{loc}} \cap H^\tau_{\mathrm{ml}}(\gamma_0)$, $\forall\mu$, *where* $P_m(\gamma_0) = 0$ *and* (A, ϕ) *is a solution to* (P). *If* Γ *is a null bicharacteristic for* $P_m(x, D)$ *through* γ_0, *then* A_μ, $\phi \in H^\sigma_{\mathrm{loc}} \cap H^\tau_{\mathrm{ml}}(\Gamma)$, $\forall\mu = 0, 1, \dots, N-1$, $m \geqslant 2$.

PROOF. – We are going to use the bootstrap ideas as in [2]. Suppose first that A_μ, $\phi \in H^\sigma_{\mathrm{loc}} \cap H^\tau_{\mathrm{ml}}(\gamma_0)$ where $(1/2)n + m < \sigma \leqslant (1/2)n + m + 1$. Then by Theorem A, if $\varepsilon_1 = \min\{1, \tau - \sigma\}$ (notice that in this case

$$\varepsilon_1 \leqslant \min\left\{1, 2\sigma - \left(\frac{1}{2}n + m - 1\right) - \sigma\right\} = \min\left\{1, \sigma - \frac{1}{2}n - m + 1\right\} =$$

$$= \min\left\{1, s - \frac{1}{2}n\right\} = \min\left\{1, 2s - \frac{1}{2}n - r\right\}$$

if $\sigma = s + m - 1$, $r = s$), we have that A_μ, $\phi \in H^\sigma_{\mathrm{loc}} \cap H^{\sigma+\varepsilon_1}_{\mathrm{ml}}(\Gamma)$, $\forall\mu$.

If $\tau - \sigma \leqslant 1$ we are done. Suppose $\tau - \sigma > 1$, then by hypothesis A_μ, $\phi \in H^\sigma_{\mathrm{loc}} \cap H^{\sigma+1+\varepsilon_2}_{\mathrm{ml}}(\gamma_0)$ where now $\varepsilon_1 = 1$, $0 < \varepsilon_2 = \min\{1, \tau - (\sigma + 1)\}$ and $\varepsilon_2 \leqslant \min\{1, 2s - (1/2)n - r\}$; now $\sigma = s + m - 1$, $r = s + 1$. But $\varepsilon_2 = \tau - (\sigma + 1)$ since $\tau - (\sigma + 1) < 1$, for we have

$$\tau - (\sigma + 1) < 2\sigma - (\sigma + 1) - \left(\frac{1}{2}n + m - 1\right) = \sigma - \left(\frac{1}{2}n + m\right) \leqslant 1$$

by assumption.

Since we have $\varepsilon_1 = 1$ and A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+1}_{\rm ml}(\Gamma)$, then by Theorem A, we get A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+1+\varepsilon_2}_{\rm ml}(\Gamma) = H^\sigma_{\rm loc} \cap H^\tau_{\rm ml}(\Gamma)$.

This concludes the proof when $(1/2)n + m < \sigma \leq (1/2)n + m + 1$.

Suppose now $(1/2)n + m + 1 < \sigma \leq (1/2)n + m + 2$.

As before: A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+\varepsilon_1}_{\rm ml}(\Gamma)$, where $\varepsilon_1 = \min\{1, \tau - \sigma\} > 0$; if $\tau - \sigma \leq 1$ we are done; if $\tau - \sigma > 1$, we have A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+1+\varepsilon_2}_{\rm ml}(\gamma_0)$, $\varepsilon_2 = \min\{1, \tau - (\sigma + 1)\} > 0$, thus A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+1+\varepsilon_1}_{\rm ml}(\Gamma)$ (notice that $\varepsilon_1 = 1$). If $\tau - (\sigma + 1) \leq 1$ we are done, so suppose $\tau - (\sigma + 1) > 1$ so that $\sigma + 2 < \tau < 2\sigma - ((1/2)n + m - 1)$, ($\exists \tau$, since $\sigma > (1/2)n + m + 1$), and $\tau - (\sigma + 2) > 0$; now we have that A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+2+\varepsilon_3}_{\rm ml}(\gamma_0)$, where $\varepsilon_3 = \min\{1, \tau - (\sigma + 2)\} \leq \min\{1, 2s - (1/2)n - r\}$ with $\sigma = s + m - 1$, $r = s + 2$ and $\varepsilon_3 = \tau - (\sigma + 2)$ because $\tau - (\sigma + 2) < 1$, for we have

$$\tau - (\sigma + 2) < 2\sigma - \left(\frac{1}{2}n + m - 1\right) - (\sigma + 2) = \sigma - \left(\frac{1}{2}n + m + 1\right) \leq 1,$$

by assumption.

Therefore A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+2+\varepsilon_3}_{\rm ml}(\Gamma) = H^\sigma_{\rm loc} \cap H^\tau_{\rm ml}(\Gamma)$.

Similarly, in the case $(1/2)n + m - 1 + k < \sigma \leq (1/2)n + m - 1 + k + 1$, $k \in \boldsymbol{N}$, we get up to A_μ, $\phi \in H^\sigma_{\rm loc} \cap H^{\sigma+k+\varepsilon_{k+1}}_{\rm ml}(\gamma_0)$, where $\varepsilon_{k+1} = \min\{1, \tau - (\sigma + k)\} \leq \min\{1, 2s - (1/2)n - r\}$ with $\sigma = s + m - 1$ and $r = s + k$ and $\varepsilon_{k+1} = \tau - (\sigma + k)$ since

$$\tau - (\sigma + k) < 2\sigma - \left(\frac{1}{2}n + m - 1\right) - (\sigma + k) = \sigma - \left(\frac{1}{2}n + m + k - 1\right) \leq 1,$$

by assumption.

Again by Theorem A we can conclude

$$A_\mu, \qquad \phi \in H^\sigma_{\rm loc} \cap H^{\sigma+k+\varepsilon_{k+1}}_{\rm ml}(\Gamma), \qquad \forall \mu = 0, 1, \ldots, n-1.$$

It is left to treat the case $(1/2)n + m - 1 < \sigma \leq (1/2)n + m$: set $\sigma = s + m - 1$ and $\varepsilon = \tau - \sigma$, then $\varepsilon = \tau - \sigma < \sigma - ((1/2)n + m - 1) = s - (1/2)n < 1$ so that we have

$$A_\mu, \qquad \phi \in H^{s+m-1}_{\rm loc} \cap H^{s+m-1+\varepsilon}_{\rm ml}(\gamma_0) \text{ with } \varepsilon < s - \frac{1}{2}n.$$

Then, by Theorem B, we get A_μ, $\phi \in H^{s+m-1}_{\rm loc} \cap H^{s+m-1+\varepsilon}_{\rm ml}(\Gamma)$, i.e.

$$A_\mu, \qquad \phi \in H^\sigma_{\rm loc} \cap H^\tau_{\rm ml}(\Gamma), \qquad \forall \mu = 0, 1, \ldots, N-1.$$

This concludes the proof of Theorem C. ■

COROLLARY 1. – *Suppose* $\phi: \boldsymbol{R}^n \to \boldsymbol{C}$ *with* $\phi = \phi^1 + i\phi^2$ *and suppose* A_μ, ϕ^i ($\mu = 0, \ldots, n-1$; $i = 1, 2$) *satisfy the hypotheses of Theorems* A, B, C. *Then the conclusions of* A, B, C *are still true.*

Applications.

As an application of Theorem C we study the propagation of smoothness of solutions to the coupled Maxwell-Klein-Gordon (MKG) system in the Lorentz gauge (see [10] for the notations):

$$\text{(MKG)}\qquad \begin{cases} F_{\mu\nu}{}^{,\nu} = \gamma_\mu\,, \\ {}^*F_{\mu\nu}{}^{,\nu} = 0\,, \\ (\partial^\mu + iA^\mu)(\partial_\mu + iA_\mu)\phi = 0\,, \end{cases}$$

where $\gamma_\mu = (i/2)(\phi\partial_\mu\bar{\phi} - \bar{\phi}\partial_\mu\phi) + A_\mu|\phi|^2$. (Here $n = 4$).

Notice that the equation ${}^*F_{\mu\nu}{}^{,\nu} = 0$ says that there exists a potential A_μ such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Moreover (MKG) is invariant under the transformations

$$T\colon ((A_\mu), \phi) \mapsto ((A_\mu + \partial_\mu\varphi), e^{-i\varphi}\phi)\,, \qquad \varphi\colon \boldsymbol{R}^n \to \boldsymbol{R}\,.$$

Under the Lorentz-gauge condition $\partial^\mu A_\mu = 0$ (we are using Einstein's summation convention with respect to $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$) we get from (MKG) to

$$\text{(MKGL)}\qquad \begin{cases} \Box A_\mu = \gamma_\mu\,, \qquad \mu = 0, 1, 2, 3\,, \\ \Box\phi - 2iA^\mu\partial_\mu\phi + A^\mu A_\mu\phi = 0 \end{cases}$$

and $T\colon ((A_\mu), \phi) \mapsto ((A_\mu + \partial_\mu\varphi), e^{-i\varphi}\phi)$ preserves this system provided that $\Box\varphi = 0$.

We now split the system (MKGL) into real and imaginary part so that we are in position to use Theorem C. Writing $\boldsymbol{A}^1 = \mathrm{Re}(\boldsymbol{A})$, $\boldsymbol{A}^2 = \mathrm{Im}(\boldsymbol{A})$, $\phi^1 = \mathrm{Re}(\phi)$, $\phi^2 = \mathrm{Im}(\phi)$, since $\Box$ is a real operator, we get the following system (here $N = 2n$):

$$\text{(MKGL1)}\qquad \begin{cases} \Box A_\mu^1 = \phi^1\partial_\mu\phi^2 - \phi^2\partial_\mu\phi^1 + A_\mu^1|\phi|^2, \qquad \mu = 0, 1, 2, 3\,, \\ \Box A_\mu^2 = A_\mu^2|\phi|^2, \qquad \mu = 0, 1, 2, 3\,, \\ \Box\phi^1 + 2(A^{1\mu}\partial_\mu\phi^2 + A^{2\mu}\partial_\mu\phi^1) + ((A^{1\mu}A_\mu^1 - A^{2\mu}A_\mu^2)\phi^1 - 2A^{1\mu}A_\mu^2\phi^2) = 0\,, \\ \Box\phi^2 - 2(A^{1\mu}\partial_\mu\phi^1 - A^{2\mu}\partial_\mu\phi^2) + ((A^{1\mu}A_\mu^1 - A^{2\mu}A_\mu^2)\phi^2 + 2A^{1\mu}A_\mu^2\phi^1) = 0\,. \end{cases}$$

Now, splitting the action of the gauge-transformation T into real and imaginary part, we see that T is of the form

$$T_R\colon ((A_\mu^1, A_\mu^2); \phi^1, \phi^2) \mapsto (A_\mu^1 + \partial_\mu\varphi, A_\mu^2; \phi^1\cos\varphi + \phi^2\sin\varphi, \phi^2\cos\varphi - \phi^1\sin\varphi)$$

so that, since $\Box\varphi = 0$, T_R preserves (MKGL1) if and only if T preserves (MKGL).

From now on we will think of the action of T on (MKGL) as the equivalent action of T_R on (MKGL1).

Hence, in order for the notion of smoothness of the solution to (MKGL) to make sense, T must preserve the regularity.

So we have the following

DEFINITION. – *T is acceptable if, given A^i, $\phi^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(\gamma)$, $i = 1, 2$ then A'^i_μ, $\phi'^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(\gamma)$, $i = 1, 2$ where $((A'_\mu), \phi') = T(((A_\mu), \phi))$. ($\gamma$ can be either (x_0, ξ_0) or Γ, a null bicharacteristic of $\Box$).*

Then

LEMMA 1. – *Suppose A^i_μ, $\phi^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(\gamma)$, $i = 1, 2$ and define, for $(x_0, \xi_0) \in \gamma$,*

$$\mathcal{S} = \{\varphi \in H^{\sigma+1}_{\text{loc}} \cap H^{\tau+1}_{\text{ml}}(x_0, \xi_0);\ \Box\varphi = 0,\ \varphi \text{ real valued}\},$$

where $(1/2)n < \sigma \leqslant \tau < 2\sigma - (1/2)n$. Then

$$T_\varphi \colon ((A_\mu), \phi) \mapsto ((A_\mu + \partial_\mu \varphi), e^{-i\varphi}\phi)$$

is acceptable $\forall \varphi \in \mathcal{S}$.

PROOF. – Since $(1/2)n < \sigma + 1 \leqslant \tau + 1 < 2\sigma - (1/2)n + 1 < 2(\sigma + 1) - (1/2)n$, we have

$$(e^{-i\varphi})^i \in H^{\sigma+1}_{\text{loc}} \cap H^{\tau+1}_{\text{ml}}(x_0, \xi_0) \subset H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(x_0, \xi_0), \qquad i = 1, 2$$

so that

$$(A_\mu + \partial_\mu \varphi)^i, \qquad (e^{-i\varphi}\phi)^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(x_0, \xi_0) \qquad \forall \mu = 0, 1, \dots, n-1;\ \ i = 1, 2.$$

This concludes the proof of the Lemma if $\gamma = (x_0, \xi_0)$. If $\gamma = \Gamma$ it suffices to notice that, by Hörmander's Theorem and by $\varphi \in \mathcal{S}$, $\varphi \in H^{\sigma+1}_{\text{loc}} \cap H^{\tau+1}_{\text{ml}}(\Gamma)$. This concludes the proof of Lemma 1. ∎

REMARK 3. – The hypothesis A^i_μ, $\phi^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(x_0, \xi_0)$, $i = 1, 2$ implies that A_μ, $\phi \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(x_0, \xi_0)$ if and only if $\overline{A}_\mu$, $\overline{\phi} \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(x_0, -\xi_0)$ if and only if

$$A^i_\mu, \qquad \phi^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(x_0, \xi_0) \cap H^\tau_{\text{ml}}(x_0, -\xi_0), \qquad i = 1, 2.$$

Hence the conclusion of the foregoing Lemma 1 is still true in the case where γ_0 is replaced by $\gamma_0^\mp = (x_0, \pm\xi_0)$ and Γ replaced by $\Gamma_\pm$, for any $\varphi \in \mathcal{S}_\pm = \{\varphi \in H^{\sigma+1}_{\text{loc}} \cap H^{\tau+1}_{\text{ml}}(x_0, \xi_0) \cap H^{\tau+1}_{\text{ml}}(x_0, -\xi_0);\ \Box\varphi = 0,\ \varphi \text{ real valued}\} = \mathcal{S}$, φ being real valued.

REMARK 4. – Suppose

$$\gamma_\mu = \frac{i}{2}(\phi\partial_\mu\overline{\phi} - \overline{\phi}\partial_\mu\phi) + \frac{1}{2}(A_\mu + \overline{A}_\mu)|\phi|^2$$

(i.e. allow A_μ to be complex). Then a nontrivial solution to (MKGL) is given by

$$A_\mu(x) = \xi_\mu(e^{i\langle x,\xi\rangle} - e^{-i\langle x,\xi\rangle}), \qquad \phi(x) = ze^{i\langle x,\xi\rangle} + we^{-i\langle x,\xi\rangle}$$

where $\xi^\mu \xi_\mu = 0$, $|z|^2 = |w|^2$; $z, w \in \boldsymbol{C}$, $\xi \in \boldsymbol{R}^n$. (Notice that $\partial^\mu A_\mu = 0$).

REMARK 5. – Consider the following equation (satisfied by the imaginary part of $\boldsymbol{A}$)

$$(*) \qquad \Box u = |\phi|^2 u$$

on $\boldsymbol{R}^n$, suppose $\phi \in H^s(\boldsymbol{R}^n)$ where $s > n/2 + 1$ and that there exists a solution $u \in H^s(\boldsymbol{R}^n)$ to $(*)$ with compactly supported initial data. We have

$$\phi \in C_{(0)}(\boldsymbol{R}, H^{s-1/2}(\boldsymbol{R}^{n-1})) \subset C_{(0)}(\boldsymbol{R}, L^2(\boldsymbol{R}^{n-1})) \cap C_{(0)}(\boldsymbol{R}, L^\infty(\boldsymbol{R}^{n-1}))$$

($C_{(0)}$ is the space of continuous functions vanishing at infinity) since $s - 1/2 > ((n - 1)/2) + 1$. Take q such that $1/2 = 1/q + 1/(n-1)$ i.e. $q = (2n-2)/(n-3)$. (Recall that $n - 1 \geq 3$ is the interesting case for (MKGL).)

By Sobolev's Lemma we have:

$$\frac{1}{2}\partial_t \|u'(t,\cdot)\|^2 = \int_{\boldsymbol{R}^{n-1}} \partial_t u(t,x)\Box u(t,x)\,dx =$$

$$= \int u_t |\phi|^2 u\,dx \leq \|u_t\| \left(\int |\phi|^{2n-2}dx\right)^{1/(n-1)} \left(\int |u|^{(2n-2)/(n-3)}dx\right)^{(n-3)/(2n-2)} \leq$$

$$\leq C_n \|u'\| \, \|\phi\|^2_{L^{2n-2}} \|\nabla u\| \leq C_n \|\phi\|^2_{L^{2n-2}} \|u'\|^2$$

since $\phi(t,\cdot) \in L^2 \cap L^\infty$ implies $\phi(t,\cdot) \in L^{2n-2}$. (Here $u' = (\partial_t u, \nabla u)$ and $\|\cdot\| = \|\cdot\|_{L^2(\boldsymbol{R}^{n-1})}$.) Hence, by Gronwall's inequality, we conclude that

$$\|u'(t,\cdot)\| \leq \|u'(0,\cdot)\| \exp\left(\frac{C_n}{2}\int_0^t \|\phi(\tau,\cdot)\|^2_{L^{2n-2}} d\tau\right).$$

Therefore we can say that if in (MKGL1) $\boldsymbol{A}$ is satisfying the same hypotheses as the above u, and $\boldsymbol{A}$ is real initially ($A_\mu^2 = 0$, $\partial_t A_\mu^2 = 0$, say at time $t = 0$), then it must be real locally in later times (by the previous argument and the fact that A_μ^2 satisfies an equation of the form $(*)$).

We can now state the following

THEOREM. – *Suppose* $(\boldsymbol{A}, \phi)$ *is a solution to* (MKGL) *such that, for* $i = 1, 2$, A_μ^i, $\phi^i \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(\gamma_0)$, *where* $(1/2)n + 1 < \sigma < \tau < 2\sigma - ((1/2)n + 1)$.

Then A_μ, $\phi \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(\Gamma_+) \cap H^\tau_{\text{ml}}(\Gamma_-)$, *and*

$$T_\varphi(\boldsymbol{A}, \phi) \in H^\sigma_{\text{loc}} \cap H^\tau_{\text{ml}}(\Gamma_+) \cap H^\tau_{\text{ml}}(\Gamma_-), \qquad \forall \varphi \in \mathcal{S}.$$

PROOF. – It is just an application of Theorem C, Corollary 1 and Lemma 1 to (MKGL1) followed by Remark 3. ■

REFERENCES

[1] M. BEALS, *Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems*, Birkhäuser, Boston (1989).

[2] M. BEALS - M. REED, *Propagation of singularities for hyperbolic pseudodifferential operators and applications to nonlinear problems*, Comm. Pure Appl. Math., **35** (1982), pp. 169-184.

[3] M. BEALS - M. REED, *Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems*, Trans. Am. Math. Soc., **285** (1984), pp. 159-184.

[4] J. M. BONY, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non lineaires*, Ann. Sc. Ec. Norm. Sup., **14** (1981), pp. 209-246.

[5] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I, II, III, IV*, Springer-Verlag, Berlin (1983-1985).

[6] L. NIRENBERG, *Lectures on linear partial differential equations*, C.B.M.S. Reg. Conf. Ser. in Math., **17**, Am. Math. Soc. (1973).

[7] J. RAUCH, *Singularities of solutions to semilinear wave equations*, J. Math. Pures Appl., **58** (1979), pp. 299-308.

[8] J. RAUCH - M. REED, *Striated solutions of semilinear two-speed wave equations*, Indiana Univ. Math. J., **34** (1985), pp. 337-353.

[9] M. TAYLOR, *Pseudodifferential Operators*, Princeton University Press, Princeton (1981).

[10] R. WALD, *General Relativity*, University of Chicago Press, Chicago (1984).