Annali di Matematica pura ed applicata
(IV), Vol. CLXV (1993), pp. 109-120

Minima of Some non Convex non Coercive Problems (*).

DANIELA GIACHETTI - ROSANNA SCHIANCHI

Abstract. – *We give here an existence result of minimizers for a class of one dimensional integrals of the Calculus of Variations with non convex, non coercive integrands.*

1. – Introduction and main result.

Let us consider the functional

$$F(u) = \int_0^1 f(x, u(x), u'(x))\, dx \tag{1.1}$$

defined in the class $\mathcal{W}_p = \{u \in W^{1,p}(0, 1): u(0) = 0, u(1) = \lambda, u' \geq 0 \text{ a.e.}\}$ with $\lambda \in R_+$ and $p \geq 1$. The integrand $f = f(x, s, \xi)$ is not assumed to be neither coercive nor convex with respect to ξ. The closure of $\mathcal{W}_p$ in the (either strong or weak) topology of $W^{1,p}_{\mathrm{loc}}(0, 1)$ is given by

$$\overline{\mathcal{W}}_p = \{u \in W^{1,p}_{\mathrm{loc}}(0, 1): u(0) \geq 0, u(1) \leq \lambda, u' \geq 0 \text{ a.e.}\}, \tag{1.2}$$

where the values $u(0)$ and $u(1)$ are defined by

$$u(0) = \inf_{x \in (0,1)} u(x), \qquad u(1) = \sup_{x \in (0,1)} u(x).$$

The extension of F «by lower semicontinuity» from $\mathcal{W}_p$ to $\overline{\mathcal{W}}_p$ is the functional $\overline{F}$ defined for $u \in \overline{\mathcal{W}}_p$ by

$$\overline{F}(u) = \inf_{\{u_k\}} \left\{ \liminf F(u_k): \{u_k\} \subset \mathcal{W}_p,\ u_k \xrightarrow{w - W^{1,p}_{\mathrm{loc}}} u \right\}.$$

Let us precise the hypotheses on the integrand function f:

(*) Entrata in Redazione il 10 dicembre 1990, versione finale ricevuta il 7 gennaio 1991.

Indirizzo degli AA.: Dipartimento di Matematica Pura e Applicata, Facoltà di Scienze, Università dell'Aquila, 67100 L'Aquila, Italia.

This work was performed as part of a national research project supported by M.U.R.S.T. (40%).

Authors are members of G.N.A.F.A., C.N.R.

A_1) f is a Carathéodory function on $[0,1] \times R \times R$;

A_2) there exist $K \geq 0$, a convex function $h = h(\xi)$ and continuous functions $a = a(x, s)$ and $b = b(x, s)$ such that for every $x \in [0,1]$, $s \in R$, $\xi \in R$

i) $a(x, s) h(\xi) - K \leq f(x, s, \xi) \leq a(x, s) h(\xi) + b(x, s)$,

ii) $|\xi| \leq h(\xi) \leq L(1 + |\xi|^p)$, $L \in R_+$,

iii) $a(x, s) \geq 0$.

Then the function f^{**} (which is the greatest function convex with respect to ξ and less than or equal to f) satisfies the same assumptions and the lower semicontinuous extension of

$$G(u) = \int_0^1 f^{**}(x, u(x), u'(x))\, dx, \qquad u \in \mathcal{W}_p$$

to $\overline{\mathcal{W}}_p$, can be represented as

$$(1.3) \qquad \overline{G}(u) = \int_0^1 f^{**}(x, u(x), u'(x))\, dx + \tilde{h}\left[\int_0^{u(0)} a(0, s)\, ds + \int_{u(1)}^{\lambda} a(1, s)\, ds\right],$$

where, for simplicity, we set $\tilde{h} = h_+ = h_-$,

$$h_\pm = \lim_{\xi \to \pm\infty} \frac{h(\xi)}{\xi},$$

(see [B.-M.], Theorem 2.4).

We are interested in the existence of solutions, for the following problem:

$$(1.4) \qquad \min\left\{\int_0^1 f(x, u(x), u'(x))\, dx + \tilde{h}\left[\int_0^{u(0)} a(0, s)\, ds + \int_{u(1)}^{\lambda} a(1, s)\, ds\right], \quad u \in \overline{\mathcal{W}}_p\right\},$$

where $\overline{\mathcal{W}}_p$ is defined by (1.2).

Usually existence for a non convex problem is achieved in two steps: find a minimizer $u_0 \in \overline{\mathcal{W}}_p$ of the relaxed functional (here (1.3)) and then prove that, for such u_0, $f(x, u_0(x), u_0'(x)) = f^{**}(x, u_0(x), u_0'(x))$ a.e. in Ω.

Therefore we need assumptions on f^{**} in order to prove existence of minima of the functional (1.3) in the class (1.2):

B_1) f^{**} admits continuous partial derivatives

$$f^{**}_{sx}, \; f^{**}_{\xi s}, \; f^{**}_{\xi xx}, \; f^{**}_{\xi xs}, \; f^{**}_{\xi ss};$$

B_2) there exist an exponent $p \geq 1$ and a function $M: R_+ \times R_+ \to R_+$ such that for $\delta, r > 0$,

$$|f_s^{**}(x, s, \xi)| \leq M(\delta, r)(1 + |\xi|^p), \qquad \forall (x, s, \xi) \in [\delta, 1 - \delta] \times [-r, r] \times R.$$

REMARK 1. – The assumptions B_1) implies that the functions defined by

$$\varphi = \varphi(x, s, \xi) = f_s^{**} - f_{\xi x}^{**} - \xi f_{\xi s}^{**}, \tag{1.5}$$

$$\psi = \psi(x, s, \xi) = \varphi_x + \varphi_s \xi, \tag{1.6}$$

are continuous.

Existence results for problem (1.4), when the integrand f is convex, are proved in [B.-M.]. Here the authors consider both the cases where $\varphi = f_s - f_{\xi x} - \xi f_{\xi s}$ has a definite sign or it changes its sign.

The non convex case is considered in [M.2] under the assumption that the function φ given by (1.5) has a definite sign.

Our aim in this paper is to prove (see the theorem below) an existence result for (1.4) in the non convex case when φ changes its sign.

This framework could be a general approach to prove existence of a minimizer for the following non convex functional related to the problem of cavitation in non linear elasticity:

$$\text{Min}\left\{\int_0^1 r^{n-1}\Phi\left[\frac{v}{r}, v'\right]dr + \tilde{h}\frac{[v(0)]^n}{n}: v \in W^{1,p}_{\text{loc}}(0,1): v \geq 0\, v(1) = \lambda\, v' \geq 0 \text{ a.e.}\right\} \tag{1.7}$$

where the energy $\Phi = \Phi(\eta, \xi)$, satisfies assumptions of type A_1), A_2), B_2) and $\tilde{h}$ is defined as above. The functional to minimize in problem (1.7) has first been considered by P. MARCELLINI in [M.1].

Up to now, no existence result seems to be applicable to problem (1.7), when Φ is not convex with respect to ξ.

The problem of cavitation has been first studied by J. BALL in [B.1] and [B.2].

More exhaustive references on the subject can be found in [M.2].

We now state our main theorem.

THEOREM. – *Assume that* $f(x, s, \xi)$ *satisfies* A_1), A_2), B_1) *and* B_2) *and that the functions* φ *and* ψ *defined in* (1.5) *and* (1.6) *satisfy the following assumption*

$$\text{C)} \quad \begin{cases} \text{i) } \varphi(x, s, 0) \neq 0 \quad \forall x, s, \\ \text{ii) } \varphi(x, s, \xi) = 0, \;\; \xi \neq 0 \Rightarrow \xi\psi(x, s, \xi) > 0. \end{cases}$$

Then the variational problem (1.4) *has a solution* u_0 *which belongs to* $W^{1,\infty}_{\text{loc}}(0,1)$ *and satisfies the following estimate*

$$|u_0'(x)| \leq 4\frac{\lambda}{\delta} \quad \forall x \in [\delta, 1-\delta], \;\; \delta \in \left]0, \frac{1}{2}\right[. \tag{1.7}$$

REMARK 2. – Let us point out that, if $\tilde{h} = +\infty$ and $a(0, s)$, $a(1, s)$ are almost everywhere positive, the minimum u_0 satisfies $u_0(0) = 0$, $u_0(1) = \lambda$ and therefore u_0 is also a minimum of the functional F defined by (1.1) in the class $\mathcal{W}_p$.

Moreover our theorem also looks at the case where $f = f(x, s, \xi)$ grows at most linearly when $|\xi| \to \infty$.

REMARK 3. – The existence result in [B.-M.] is related to a convex integrand f such that $\varphi = f_s - f_{\xi x} - f_{\xi s}\xi$ changes its sign according to the following assumption:

$\forall \delta \in]0, 1/2[$ and $r > 0$ there exists $k_0 = k_0(\delta, r) > 0$ such that for every (x, s, ξ) belonging to $[\delta, 1-\delta] \times [-r, r] \times R$ with $|\xi| > k_0$, if $\varphi(x, s, \xi) = 0$ then $\xi\psi(x, s, \xi) > 0$.

The plan of the paper is the following: in Section 2 we define approximating problems which are convex and coercive and we prove some properties of their solutions.

In Section 3 we prove some geometrical properties (concavity-convexity properties) of the approximating solutions defined in Section 2 and a priori estimates.

Finally, in Section 4, we prove the main theorem.

2. – Approximating solutions, monotonicity properties.

In this section a double approximating scheme is introduced in order to obtain smooth convex and coercive integrand functions. We consider

$$g^{\varepsilon k}(x, s, \xi) = \alpha_k * f^{**}(x, s, \xi) + \varepsilon(1 + |\xi|^2)^{q/2} + k(\xi^-)^q \tag{2.1}$$

where $q \geq \max\{p, 4\}$, $\alpha = \alpha(\xi)$ is a positive mollifier with compact support in $[-1, 1]$, $\alpha_k(\xi) = k\alpha(k\xi)$ and $\xi^- = -\min\{\xi, 0\}$.

The variational problem

$$\operatorname{Min}\left\{G^{\varepsilon k}(u) = \int_0^1 g^{\varepsilon k}(x, u(x), u'(x))\,dx : u \in W^{1,q}(0, 1),\ u(0) = 0,\ u(1) = \lambda\right\} \tag{2.2}$$

related to the convex and coercive integral $G^{\varepsilon k}(u)$ admits a solution $u_{\varepsilon k}(x)$ which satisfies the properties stated in the following lemma.

LEMMA 2.1. – *For* $\varepsilon \in]0, 1[$ *and* $k > 0$, $u_{\varepsilon k} \in C^3[0, 1]$ *and satisfies*

$$\frac{d}{dx}[g_\xi^{\varepsilon k}(x, u_{\varepsilon k}, u'_{\varepsilon k})] = g_s^{\varepsilon k}(x, u_{\varepsilon k}, u'_{\varepsilon k}). \tag{2.3}$$

Moreover, for fixed $\varepsilon \in]0, 1]$, $\|u'_{\varepsilon k}\|_{L^\infty(0,1)}$ *is bounded uniformly with respect to* k.

PROOF. – A classical argument due to Morrey (see Th. 1.10.1 in [Mo.]) provides solutions $u_{\varepsilon k} \in C^3[0, 1]$ of the Euler's equation (2.3). The uniform C^1 bound of $u_{\varepsilon k}$ is obtained following the outline of the proof of Lemma 5.7 in [M.2].

LEMMA 2.2. – *The sequence $\{u_{\varepsilon k}\}_{k \in N}$, for fixed ε, is relatively compact in the weak topology of $W^{1,q}(0, 1)$: up to a subsequence, $\{u_{\varepsilon k}\}_{k \in N}$ weakly converges to a solution u_ε of the following minimum problem:*

$$(2.4)\qquad \operatorname{Min}\left\{\int_0^1 g^\varepsilon(x, u(x), u'(x))\,dx\colon u \in W^{1,q}(0, 1),\ u(0) = 0,\ u(1) = \lambda,\ u' \geqslant 0 \text{ a.e.}\right\}$$

where $g^\varepsilon(x, s, \xi) = f^{**}(x, s, \xi) + \varepsilon(1 + |\xi|^2)^{q/2}$.

PROOF. – Let us begin by proving that the sequence $\{u_{\varepsilon k})_{k \in N}$ is bounded in the $W^{1,q}(0,1)$-norm uniformly with respect to k.

Since $u_{\varepsilon k}$ solves the problem (2.2), for $v = \lambda x$, $\forall x \in (0, 1)$, we get

$$G^{\varepsilon k}(u_{\varepsilon k}) \leqslant G^{\varepsilon k}(v) \leqslant C_1$$

where C_1 is a positive constant independent of $\varepsilon \in]0, 1]$ and $k \in N$.

By the growth condition on f (see i) in A_2)) and the definition of f^{**}, we get

$$(2.5)\qquad \varepsilon\|u'_{\varepsilon k}\|^q_{L^q(0,1)} + k\|(u'_{\varepsilon k})^-\|^q_{L^q(0,1)} - K \leqslant G^{\varepsilon k}(u_{\varepsilon k}) \leqslant C_1 .$$

This proves the boundedness of the sequence $\{u_{\varepsilon k}\}_{k \in N}$ in $W^{1,q}(0, 1)$. Then there exists $u_\varepsilon \in W^{1,q}(0, 1)$ which is the weak limit in $W^{1,q}(0, 1)$ of $\{u_{\varepsilon k}\}_{k \in N}$ (up to a subsequence).

By (2.5), since $k\|(u'_{\varepsilon k})^-\|^q_{L^q(0,1)}$ is bounded for each $k \in N$, then the negative part of $u'_{\varepsilon k}$ converges strongly to zero in $L^q(0, 1)$ and thus $u'_\varepsilon \geqslant 0$ a.e. in $[0, 1]$.

We show now that u_ε solves problem (2.4).

Indeed, by Lemma 2.1, $u'_{\varepsilon k}$ is bounded in $L^\infty(0, 1)$ uniformly with respect to k and since $\alpha_k * f^{**}$ converges uniformly on bounded sets of $[0, 1] \times R \times R$, then we have, for $\delta \in]0, 1/2[$,

$$\lim_{k \to +\infty} \int_\delta^{1-\delta} \{\alpha_k * f^{**}(x, u_{\varepsilon k}, u'_{\varepsilon k}) - f^{**}(x, u_{\varepsilon k}, u'_{\varepsilon k})\}\,dx = 0 .$$

Therefore, using lower semicontinuity arguments, for $v \in W^{1,q}(0, 1)$ such that

$v(0) = 0$, $v(1) = \lambda$, $v' \geqslant 0$ a.e. in $[0,1]$, we get

$$\int_{\delta}^{1-\delta} g^{\varepsilon}(x, u_{\varepsilon}, u_{\varepsilon}')\,dx \leqslant \liminf_{k\to+\infty} \int_{\delta}^{1-\delta} g^{\varepsilon}(x, u_{\varepsilon k}, u_{\varepsilon k}')\,dx =$$

$$= \liminf_{k\to+\infty} \int_{\delta}^{1-\delta} \{\alpha_k * f^{**}(x, u_{\varepsilon k}, u_{\varepsilon k}') + \varepsilon(1 + |u_{\varepsilon k}'|^2)^{q/2}\}\,dx \leqslant$$

$$\liminf_{k\to+\infty} G^{\varepsilon k}(u_{\varepsilon k}) \leqslant \liminf_{k\to+\infty} G^{\varepsilon k}(v) = \int_0^1 g^{\varepsilon}(x, v, v')\,dx .$$

By the monotone convergence theorem, as $\delta \to 0$ we get the result.

A strict monotonicity property of u_{ε} is stated in the following lemma.

LEMMA 2.3. – *For fixed* ε, *the functions* u_{ε} *are strictly increasing in* $(0,1)$.

PROOF. – First of all, let us prove that there not exists any interval $I \subseteq [0,1]$ such that $u_{\varepsilon}'(x) = 0$ $\forall x \in I$, where u_{ε} is defined in the previous lemma.

Indeed, if such an interval I exists, set $I = (x_1, x_2) \subseteq [0,1]$, u_{ε} solves Euler's equation in weak form and also in the form

$$g_{\xi}^{\varepsilon}(x, u_{\varepsilon}(x), u_{\varepsilon}'(x)) = \text{const} + \int_{x_1}^{x} g_{s}^{\varepsilon}(t, u_{\varepsilon}(t), u_{\varepsilon}'(t))\,dt, \qquad \forall x \in (x_1, x_2)$$

Differentiation with respect to x, taking into account that $u_{\varepsilon}'(x) = 0$ $\forall x \in (x_1, x_2)$, gives

$$f_{\xi x}^{**}(x, u_{\varepsilon}(x), 0) = f_{s}^{**}(x, u_{\varepsilon}(x), 0) \qquad \forall x \in (x_1, x_2),$$

which contradicts the assumption i) in C).

Since, by Lemma 2.2, we know that $u_{\varepsilon}' \geqslant 0$ a.e. in $[0,1]$, u_{ε} is an increasing function in $[0,1]$. Indeed the first part of the proof implies that it is strictly increasing.

As a consequence, we get

$$0 = u_{\varepsilon}(0) \leqslant u_{\varepsilon}(x) \leqslant u_{\varepsilon}(1) = \lambda \qquad \forall x \in [0,1]. \tag{2.6}$$

3. – Geometrical properties and a priori estimates for approximating solutions.

This section is devoted to the study of concavity-convexity properties of the approximating solutions $u_{\varepsilon k}$ and to the related a priori estimates. Both of them will hold true for the limit function u_{ε}.

Let $\delta \in]0, 1/2[$ and $\varepsilon \in]0, 1]$ be fixed. For $k \in N$, define the following subsets of $]\delta, 1-\delta[$

$$Y_k = \{x \in]\delta, 1-\delta[: u'_{\varepsilon k}(x) \neq 0\}, \tag{3.1}$$

$$Z_k = \{x \in Y_k : u''_{\varepsilon k}(x) = 0\}. \tag{3.2}$$

By Lemma 2.3, $\{k \in N: Y_k \neq \emptyset\}$ is infinite.

In order to prove the stated properties of $u_{\varepsilon k}$, we need the following lemma.

LEMMA 3.1. – *If the set $\{k \in N: Z_k \neq \emptyset\}$ is infinite, up to a subsequence, the functions $u'_{\varepsilon k}$ have a unique global minimum point x_k with $u'_{\varepsilon k}(x_k) > 0$.*

PROOF. – Since $u_{\varepsilon k} \in C^3$, the Euler's equation (2.3) can be differentiated obtaining:

$$g^{\varepsilon k}_{\xi\xi} u''_{\varepsilon k} = \alpha_k * f^{**}_s - \{\alpha_k * f^{**}_{\xi x} + u'_{\varepsilon k} \cdot \alpha_k * f^{**}_{\xi s}\} \tag{3.3}$$

and in the set Z_k:

$$\begin{aligned} g^{\varepsilon k}_{\xi\xi} u'''_{\varepsilon k} = {} & \alpha_k * (f^{**}_{sx} + f^{**}_{ss} u'_{\varepsilon k}) - \alpha_k * f^{**}_{\xi x x} - \\ & - u'_{\varepsilon k}(\alpha_k * f^{**}_{\xi s x}) - \alpha_k * (f^{**}_{\xi x s} u'_{\varepsilon k}) - u'_{\varepsilon k}[\alpha_k * (f^{**}_{\xi s s} u'_{\varepsilon k})]. \end{aligned} \tag{3.4}$$

If we set

$$L_1(r) = \sup\{|f^{**}_{\xi s x}(x, s, \xi)| : x \in [0, 1], |s| \leq r, |\xi| \leq r\}, \tag{3.5}$$

$$L_2(r) = \sup\{|f^{**}_{\xi s s}(x, s, \xi)| : x \in [0, 1], |s| \leq r, |\xi| \leq r\}, \tag{3.6}$$

then for such values x, s, ξ, we have

$$\begin{aligned} & |\xi \alpha_k * f^{**}_{\xi s x} - \alpha_k * \xi f^{**}_{\xi s x}| = \\ & = \left| \xi \int_R \alpha_k(t) f^{**}_{\xi s x}(x, s, \xi - t)\, dt - \int_R \alpha_k(t)(\xi - t) f^{**}_{\xi s x}(x, s, \xi - t)\, dt \right| \leq \\ & \leq L_1(r+1) \int_R \alpha_k(t)|t|\, dt \leq \frac{L_1(r+1)}{k} \int_R \alpha(t)|t|\, dt \leq \\ & \leq \frac{L_1(r+1)}{k} \int_R \alpha(t)\, dt = \frac{L_1(r+1)}{k}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} & |\xi \alpha_k * (f^{**}_{\xi s s} \xi) - \alpha_k * f^{**}_{\xi s s} \xi^2| = \\ & = \left| \xi \int_R \alpha_k(t) f^{**}_{\xi s s}(x, s, \xi - t)(\xi - t)\, dt - \int_R \alpha_k(t) f^{**}_{\xi s s}(x, s, \xi - t)(\xi - t)^2\, dt \right| \leq \end{aligned} \tag{3.8}$$

$$\leq L_2(r+1)\left|\int_R [\xi\alpha_k(t)(\xi-t)-\alpha_k(t)(\xi-t)^2]\,dt\right| =$$

$$= L_2(r+1)\left|\int_R [\alpha_k(t)\xi t-\alpha_k(t)t^2]\,dt\right| \leq L_2(r+1)\int_R \alpha_k(t)|\xi-t|\,|t|\,dt \leq$$

$$\leq (r+1)L_2(r+1)\int_R |t|\alpha_k(t)\,dt \leq \frac{L_2(r+1)}{k}(r+1).$$

By (3.4)-(3.8), for $r \geq \sup_{k>0}\|u'_{\varepsilon k}\|_{L^\infty(0,1)}$, taking into account the definition (1.6), we have, for $x \in Z_k$,

$$|g^{\varepsilon k}_{\xi\xi}u'''_{\varepsilon k} - \alpha_k * \psi| \leq \frac{L_1(r+1)}{k} + \frac{L_2(r+1)}{k}(r+1). \tag{3.9}$$

In a similar way, we can prove (see also (5.18) in [M.2])

$$|g^{\varepsilon k}_{\xi\xi}u''_{\varepsilon k} - \alpha_k * \varphi| \leq \frac{L(r+1)}{k}, \tag{3.10}$$

where φ is defined in (1.5) and $L(r)$ is defined by

$$L(r) = \sup\{|f^{**}_{\xi s}(x, s, \xi)|: x \in [0,1],\ |s| \leq r,\ |\xi| \leq r\}.$$

Consider now the infinite set $\{k \in N: Z_k \neq \phi\}$. We can assume, possibly extracting a subsequence, that for each k, $Z_k \neq \phi$. Let be $x_k \in Z_k$, then $u'_{\varepsilon k}(x_k) \neq 0$, $u''_{\varepsilon k}(x_k) = 0$ and $\{(x_k, u_{\varepsilon k}(x_k), u'_{\varepsilon k}(x_k))\}_{k \in N}$ converges to some point $(x, s, \xi) \in [0,1] \times [-r, r] \times \times [-r, r]$.

On the other hand, by the continuity of φ and (3.10) used for $x = x_k$,

$$\lim_{k\to\infty} \varphi(x_k, u_{\varepsilon k}(x_k), u'_{\varepsilon k}(x_k)) = \varphi(x, s, \xi) = 0$$

therefore, by assumption i) in C), ξ must be different from zero, and by ii) in C), $\xi\psi(x, s, \xi) > 0$ which implies definitively that $u'_{\varepsilon k}(x_k)\cdot\psi(x_k, u_{\varepsilon k}(x_k), u'_{\varepsilon k}(x_k)) > 0$. Now we use (3.9) and, taking into account that $g^{\varepsilon k}_{\xi\xi}$ and α_k are positive, we conclude that definitively $u'_{\varepsilon k}(x_k)$ and $u'''_{\varepsilon k}(x_k)$ have the same sign.

It follows that definitively x_k is a local minimum for $u'_{\varepsilon k}(x)$ with $u'_{\varepsilon k}(x_k) > 0$ if $\xi > 0$ or, definitively, x_k is a local maximum with $u'_{\varepsilon k}(x_k) < 0$ if $\xi < 0$.

Indeed x_k is a strict global minimum for the function $|u'_{\varepsilon k}|$, because if it was strict local but not global, it would imply the existence elsewhere of a local positive maximum, which is excluded by the previous argument. For the same reason it is unique. The lemma follows now from the strong L^∞-convergence of $u_{\varepsilon k}$ to u_ε and Lemma 2.3.

REMARK 4. – From the above proof it follows also that $u'_{\varepsilon k}$ cannot have a positive local maximum.

Now we can state the lemma which exhibits the mentioned geometrical properties of the approximating solutions $u_{\varepsilon k}$.

LEMMA 3.2. – *Let be $\delta \in]0, 1/2[$. There exists a subsequence of $\{u_{\varepsilon k}\}_{k \in N}$, still denoted by $\{u_{\varepsilon k}\}_{k \in N}$ and two sequences $\{x_k^1\}$ and $\{x_k^2\}$, $\delta \leq x_k^1 \leq x_k^2 \leq 1 - \delta$ such that*

i) $u'_{\varepsilon k}(x) = 0 \ \forall x \in]x_k^1, x_k^2[$;

ii) *if $u'_{\varepsilon k}(x) > 0$ (resp. $u'_{\varepsilon k}(x) < 0$) in $]\delta, x_k^1[$, then $u_{\varepsilon k}$ is concave (resp. convex) in $]\delta, x_k^1[$;*

if $u'_{\varepsilon k}(x) > 0$ (resp. $u'_{\varepsilon k}(x) < 0$) in $]x_k^2, 1 - \delta[$, then $u_{\varepsilon k}$ is convex (resp. concave) in $]x_k^2, 1 - \delta[$.

PROOF. – Assume first that the set $\{k \in N: Y_k =]\delta, 1 - \delta[\}$ is infinite (the set Y_k is defined by (3.1)); up to a subsequence, we can assume that $Y_k =]\delta, 1 - \delta[$ $\forall k \in N$.

If the set $\{k \in N: Z_k \neq \emptyset)$ is finite then definitively $Z_k = \emptyset$ and $u_{\varepsilon k}$ are convex in $]\delta, 1 - \delta[$ or concave in $]\delta, 1 - \delta[$ and the lemma is proved by choosing $x_k^1 = x_k^2 = \delta$ or $x^1 = x_k^2 = 1 - \delta$.

If the set $\{k \in N: Z_k \neq \emptyset\}$ is infinite, by Lemma 3.1, up to subsequence, $u'_{\varepsilon k}(x)$ is decreasing for $x < x_k$ and increasing for $x > x_k$. We can conclude also in this case that the lemma is true, by choosing $x_k^1 = x_k^2 = x_k$.

Assume now that the set $\{k \in N: Y_k =]\delta, 1 - \delta[\}$ is finite. Therefore, definitively $Y_k \neq]\delta, 1 - \delta[$, i.e. there exists $\bar{k} \in N$ such that, for $k > \bar{k}$, there exists at least one point $x_k \in]\delta, 1 - \delta[$ satisfying $u'_{\varepsilon k}(x_k) = 0$. Moreover for large values of k, $Z_k = \emptyset$ because if not, the set $\{k \in N: Z_k \neq \emptyset\}$ would be infinite and, by Lemma 3.1, it would exists a positive local minimum for $u'_{\varepsilon k}$ in $\bar{x}_k \in Y_k$. On the other hand we have that $u'_{\varepsilon k}(x_k) = 0$, which implies the presence of a local maximum point for $u'_{\varepsilon k}$ in the interval with end points x_k and $\bar{x}_k$ and this contradicts Remark 4.

Now we prove that, for k large enough, the set $\{x \in]\delta, 1 - \delta[: u'_{\varepsilon k}(x) = 0\}$ is an interval. In fact, let be x, y such that $u'_{\varepsilon k}(x) = u'_{\varepsilon k}(y) = 0$; if $u'_{\varepsilon k}(\bar{x})$ is different from zero in some point $\bar{x}$ between x and y, the function $u'_{\varepsilon k}$ must have an extremum between x and y in contradiction with the fact that $Z_{k'} = \emptyset$ definitively.

Setting

$$x_k^1 = \inf\{x \in [\delta, 1 - \delta]: u'_{\varepsilon k}(x) = 0\},$$

$$x_k^2 = \sup\{x \in [\delta, 1 - \delta]: u'_{\varepsilon k}(x) = 0\},$$

then assertion i) in the statement of the lemma is proved.

Since $Z_k = \emptyset$ for k large enough, $|u'_{\varepsilon k}|$ is decreasing in $]\delta, x_k^1[$ and increasing in $]x_k^2, 1 - \delta[$ which proves assertion ii).

Finally we are able to prove the a priori local estimate on $u'_{\varepsilon k}$.

LEMMA 3.3. – *Let* $\{u_{\varepsilon k}\}_{k \in N}$ *be the subsequence satisfying the statement in the Lemma* 3.2, *then the following estimate holds*:

$$\forall \delta \in \left]0, \frac{1}{2}\right[\quad \|u'_{\varepsilon k}\|_{L^\infty(\delta, 1-\delta)} \leqslant \frac{4}{\delta}\|u_{\varepsilon k}\|_{L^\infty(0,1)}. \tag{3.11}$$

PROOF. – Let us apply Lemma 3.2 with δ replaced by $\delta/2$. Different situations are possible, but in any case we get the following estimate:

$$|u'_{\varepsilon k}(x) \leqslant \frac{|u_{\varepsilon k}(x) - u_{\varepsilon k}(\delta/2)}{|x - \delta/2|} \qquad \forall x \in]\delta, x_k^1[.$$

Then, $\forall x \in]\delta, x_k^1[$

$$|u'_{\varepsilon k}(x)| \leqslant \frac{4}{\delta}\|u_{\varepsilon k}\|_{L^\infty(0,1)}. \tag{3.12}$$

In a similar way we proceed to prove estimate (3.12) for $x \in]x_k^2, 1-\delta[$. By (3.12) and i) in Lemma 3.2 we get the estimate (3.11).

Let us observe that estimate (3.11) holds true passing to the limit for $k \to \infty$. In fact the boundedness in $L^\infty(\delta, 1-\delta)$ of $\{u'_{\varepsilon k}\}$ implies that this sequence converges in the weak* topology to u'_ε and by lower semicontinuity of the norm, we get

$$\|u'_\varepsilon\|_{L^\infty(\delta, 1-\delta)} \leqslant \liminf_{k\to\infty} \|u'_{\varepsilon k}\|_{L^\infty(\delta, 1-\delta)} \leqslant \frac{4\lambda}{\delta}. \tag{3.13}$$

4. – Proof of the main theorem.

Here we follow the outline of the proof of Theorem 5.4 of [M.2]. Let us consider for each ε the function $u_\varepsilon(x)$ obtained as limit, for $k \to \infty$, of $u_{\varepsilon k}$. By inequality (3.13), $\{u_\varepsilon\}$ is relatively compact in the weak* topology of $W^{1,\infty}_{\text{loc}}(0,1)$ and there exists a function $u_0 \in W^{1,\infty}_{\text{loc}}(0,1)$ such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{w-*} u_0 \quad \text{in } W^{1,\infty}_{\text{loc}}(0,1) \text{ for } \varepsilon \to 0. \tag{4.1}$$

By the definition of $\overline{G}$ (see 1.3), recalling that u_ε is a solution of problem (2.4) (see Lemma 2.2), $\forall v \in \mathcal{W}_q = \mathcal{W}_p \cap W^{1,q}(0,1)$, we get

$$\overline{G}(u_0) \leqslant \liminf_{\varepsilon\to 0} \int_0^1 f^{**}(x, u_\varepsilon, u'_\varepsilon)\,dx \leqslant \liminf_{\varepsilon\to 0} \int_0^1 g^\varepsilon(x, u_\varepsilon, u'_\varepsilon)\,dx \leqslant$$

$$\leqslant \liminf_{\varepsilon\to 0} \int_0^1 g^\varepsilon(x, v, v')\,dx = \int_0^1 f^{**}(x, v, v')\,dx = G(v)$$

then

$$\overline{G}(u_0) \leq G(v) \quad \forall v \in \mathcal{W}_q. \tag{4.2}$$

Let now be $w \in \mathcal{W}_p$, because of the density of $W^{1,q}$ in $W^{1,p}$, there exists a sequence $\{v_k\} \subset \mathcal{W}_q$ such that $v_k \to w$ in $W^{1,p}(0, 1)$. Moreover since, by A_2), G is strongly continuous in $W^{1,p}$, inequality (4.2) applied to $v = v_k$, to the limit, gives

$$\overline{G}(u_0) \leq G(w) \quad \forall w \in \mathcal{W}_p. \tag{4.3}$$

Finally let $v \in \overline{\mathcal{W}}_p$, by the definition of $\overline{G}$, for a sequence $\{v_k\} \subset \mathcal{W}_p$ such that $v_k \rightharpoonup v$ in the weak topology of $W^{1,p}_{\text{loc}}(0, 1)$, $\lim G(v_k) = \overline{G}(v)$. By replacing w with v_k in the previous inequality (4.3) and passing to the limit, we see that

$$\overline{G}(u_0) \leq \overline{G}(v) \quad \forall v \in \overline{\mathcal{W}}_p$$

and u_0 solves the minimum problem related to the functional (1.3) in $\overline{\mathcal{W}}_p$.

To conclude our proof we must only prove that

$$f(x, u_0(x), u_0'(x)) = f^{**}(x, u_0(x), u_0'(x)) \quad \text{a.e. in } (0, 1) \tag{4.4}$$

since from (3.13) immediatly follows the analogous estimate for u_0', by semicontinuity arguments.

Let us point out that u_0' is a piecewise monotone function because of the geometrical properties of $u_{\varepsilon\kappa}$ stated in the Lemma 3.2. Then u_0' is almost everywhere continuous. Let be $A = \{x \in (0, 1): u_0' \text{ is continuous in } x\}$ and choose $x \in A$ such that $f(x, u_0(x), u_0'(x)) \neq f^{**}(x, u_0(x), u_0'(x))$. We recall that f^{**} is a linear function with respect to $\xi = u_0'(x)$ and therefore, taking the derivative at x of the Euler's equation in the weak form,

$$f_\xi^{**}(x, u_0(x), u_0'(x)) = c + \int_0^x f_s^{**}(t, u_0(t), u_0'(t))\, dt,$$

we get

$$\varphi(x, u_0(x), u_0'(x)) = f_s^{**} - f_{\xi x}^{**} - f_{\xi s}^{**} u_0'(x) = 0.$$

By i) in the assumption C), it follows that $u_0'(x) \neq 0$.

On the other hand $\tilde{\varphi}(x) = \varphi(x, u_0(x), u_0'(x))$ is strictly increasing in this point x because of assumption ii) in C). Then there exists a neighbourhood $I(x)$ such that, for each $y \in I(x) - \{x\}$, $\tilde{\varphi}(y) \neq 0$. It follows that, for each $y \in I(x) - \{x\}$, either $y \notin A$ or $f(y, u_0(y), u_0'(y)) = f^{**}(y, u_0(y), u_0'(y))$ otherwise, by the previous arguments, $\tilde{\varphi}(y)$ would be equal to zero.

Since u_0' is almost everywhere continuous, then $(f - f^{**})(y, u_0(y), u_0'(y)) = 0$ a.e. in $I(x) - \{x\}$. This contradicts the fact that $(f - f^{**})(x, u_0(x), u_0'(x))$ is different from zero in x which is a continuity point for u_0'. We conclude that (4.4) holds true.

Acknowledgment. We would like to thank Prof. P. MARCELLINI for the useful discussions on the subject.

REFERENCES

[B.1] J. M. BALL, *Convexity conditions and existence theorems in non linear elasticity,* Arch. Rational Mech. Anal., **63** (1977), pp. 337-403.

[B.2] J. M. BALL, *Discontinuous equilibrium solutions and cavitation in non linear elasticity,* Phil. Trans. R. Soc. London, **306** (1982), pp. 557-611.

[B.-M.] B. BOTTERON - P. MARCELLINI, *A general approach to the existence of minimizers of one-dimensional non coercive integrals of the calculus of variations,* Ann. I.H.P. Anal. Nonlin., to appear.

[M.1] P. MARCELLINI, *The stored energy for some discontinuous deformations in non linear elasticity,* in *Essays in Honor of E. De Giorgi,* Vol. II, edited by F. Colombini et al., Birkhäuser (1989), pp. 767-786.

[M.2] P. MARCELLINI, *Non Convex Integrals of the Calculus of Variations,* Lectures Notes in Math., **1446** (1990), pp. 16-57.

[Mo.] G. B. MORREY, *Multiple Integrals in the Calculus of Variations,* D. Grundl. Math. Wiss, **130**, Springer-Verlag (1966).