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Differential Forms and Resolution on Certain Analytic Spaces.
III. - Spectral Resolution (*).

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Introduction.

This work is the continuation of the two previous works [1], [2]. The general motivations were described in [1] and we briefly recall them. We want to construct a resolution of the sheaf of holomorphic functions on an analytic space S with normal singularities. We have assumed in [1], that the singular locus X of S is smooth and that there exists a desingularization $\varphi: \widetilde{S} \to S$ of S such that the exceptional divisor $\widetilde{X} = \varphi^{-1}(X)$ is irreducible. In this work, we suppose again that X is smooth, but we do not assume that $\widetilde{X}$ is irreducible. We still have to do an hypothesis, namely that for any couple of irreducible components Y_i', Y_j'' of $\widetilde{X}$, φ is surjective from $Y_i \cap Y_j$ to X. This is a rather strong hypothesis (which basically implies that the desingularization φ can be obtained by one blowing up).

Under this hypothesis, we can construct a resolution of the sheaf of holomorphic function. The main tool is the definition of a spectral resolution.

1. - Infinitesimal neighborhood of the exceptional divisor.

1.0. *Notations.*

We consider a compact analytic space S with a singular locus X which is assumed to be a smooth complex manifold. We assume that S has normal singularities.

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Let $\varphi\colon \widetilde{S} \to S$ a desingularization of S and call

$$\widetilde{X} = \varphi^{-1}(X).$$

This is the exceptional divisor which has, in general, many components

$$\widetilde{X} = \bigcup_{i=1}^{N} Y_i, \tag{1.1}$$

where Y_i are smooth hypersurfaces with transversal intersections.

We shall denote I_i the ideal of Y_i. If $\widetilde{U}$ is an open subset of $\widetilde{S}$ touching Y_i, then $\widetilde{U} \cap Y_i$ has an equation $\zeta_i = 0$ and I_i is generated by ζ_i in $\widetilde{U}$.

1.1. *Vanishing of cohomology near* $\widetilde{X}$.

We shall now assume the following hypothesis (A)

(A) *There exists an ideal* $J \subset O_S$ *such that* $J \cdot O_{\widetilde{S}}$ *is generated by one element and such that for all* $O_{\widetilde{S}}$*-module* F *in* $\widetilde{S}$, *there exists an* $r_0 > 0$ *with*

$$H^p(\varphi^{-1}(U), (J \cdot O_{\widetilde{S}})^r \otimes_{O_{\widetilde{S}}} F) = 0\,, \tag{1.2}$$

for all sufficiently small open set $U \subset S$, *all* $p > 0$ *and* $r > r_0$.

This hypothesis is fulfilled in any situation where the desingularization is obtained by successive blowing-up.

We shall apply this hypothesis only in the case where F is a vector bundle on $\widetilde{S}$.

The fact that $J \cdot O_{\widetilde{S}}$ is generated by one element means that there exists $a_1, \ldots, a_N$ positive integers with the property

$$J \cdot O_{\widetilde{S}} = I_1^{a_1} \cdot I_2^{a_2} \ldots I_N^{a_N}. \tag{1.3}$$

We can always assume that for $F = O_{\widetilde{S}}$, $r = 1$ provided we change the definition of the a_j.

Call $J_{\widetilde{X}} = J \cdot O_{\widetilde{S}}$. From the cohomology exact sequence associated to the short exact sequence

$$O \to J_{\widetilde{X}} \to O_{\widetilde{S}} \to O_{\widetilde{S}}/J_{\widetilde{X}} \to 0 \tag{1.4}$$

we deduce the isomorphisms: for all $p > 0$, U small in S,

$$H^p(\varphi^{-1}(U), O_{\widetilde{S}}) \simeq H^p(\varphi^{-1}(U), O_{\widetilde{S}}/J_{\widetilde{X}})\,, \tag{1.5}$$

which is induced by the quotient mapping

$$j_{\widetilde{X}}\colon O_{\widetilde{S}} \to O_{\widetilde{S}}/J_{\widetilde{X}}. \tag{1.6}$$

1.2. *Définitions of various jets.*

a) We consider again

$$J_{\tilde{X}} = \prod_{j=1}^{N} I_j^{a_j}. \tag{1.3}$$

We call

$$\tilde{O}_j = O_{\tilde{S}} / I_j^{a_j} \tag{1.7}$$

and $j_k \colon O_{\tilde{S}} \to O_{\tilde{S}} / I_k^{a_k}$.

If $\tilde{U}$ is a small open set in $\tilde{S}$ such that $\zeta_k^{(\tilde{U})} = 0$ is local equation of $Y_k \cap \tilde{U}$ in $\tilde{U}$, and if f is holomorphic on $\tilde{U}$, then

$$j_k(f) = \sum_{l=0}^{a_k - 1} \frac{\partial^l f}{\partial \zeta_k^{(\tilde{U})l}} \bigg|_{\zeta_k^{(\tilde{U})}=0} \frac{(\zeta^{(\tilde{U})})^l}{l!} \tag{1.8}$$

and we can also say that $\tilde{O}_k$ is the sheaf of holomorphic sections of a vector bundle

$$E_k^{(a_k - 1)} \to Y_k,$$

by the identification

$$j_k(f) \simeq (f_0^{(\tilde{U})}, \dots, f_{a_k - 1}^{(\tilde{U})}) \quad \text{where } f_l^{(\tilde{U})} = \frac{\partial^l f}{(\partial \zeta_k^{(\tilde{U})})^l} \bigg|_{\zeta_k^{(\tilde{U})}=0}.$$

If we change the open set $\tilde{U}$ into another open set $\tilde{V}$, then we have a holomorphic change of coordinates

$$\begin{cases} \zeta_k^{(\tilde{V})} = \zeta_k^{(\tilde{U})} \psi(\zeta_k^{(\tilde{U})}, z^{(\tilde{U})}), \\ z^{(\tilde{V})} = \varphi(z^{(\tilde{U})}), \end{cases}$$

where $z^{(\tilde{U})}$ are holomorphic coordinates along $\tilde{U} \cap Y_k$. When we write $j_k(f)$ in both systems of coordinates $(\operatorname{mod} I_k^{a_k})$ we have the formula for the change of trivialization of $E_k^{(a_k - 1)}$ by

$$\sum_{l=0}^{a_k - 1} f_l^{(\tilde{U})} \frac{(\zeta_k^{(\tilde{U})})^l}{l!} \equiv \sum_{l=0}^{a_k - 1} f_l^{(\tilde{V})} \frac{(\zeta_k^{(\tilde{V})})^l}{l!} \quad (\operatorname{mod} (\zeta_k^{(\tilde{V})})^{a_k}). \tag{1.9}$$

In particular $\tilde{O}_k$ is naturally a O_{Y_k}-module the multiplication being

$$h(f_0^{(\tilde{U})}, \dots, f_{a_k - 1}^{(\tilde{U})}) = (h \cdot f_0^{(\tilde{U})}, \dots h f_{a_k - 1}^{(\tilde{U})})$$

for $h \in \Gamma(\tilde{U} \cap Y_k, O_{Y_k})$.

Using the identification $h \mapsto h \circ \varphi|_{Y_k}$, it also becomes a O_X-module.

b) more generally, we can define

$$\widetilde{O}_{kl} = O_{\widetilde{S}}/(I_{Y_k}^{a_k} + I_{Y_l}^{a_l}), \qquad j_{kl}\colon O_{\widetilde{S}} \to \widetilde{O}_{kl}.$$

If $\widetilde{U}$ is a small open set in $\widetilde{S}$, and $\zeta_k^{(\widetilde{U})} = 0$ and $\zeta_l^{(\widetilde{U})} = 0$ are local equations of $Y_k \cap \widetilde{U}$ and $Y_l \cap \widetilde{U}$, $j_{kl}(f)$ is obtained by taking the Taylor expansion of f in terms of $\zeta_k^{(\widetilde{U})}$ and $\zeta_l^{(\widetilde{U})}$ on $Y_{kl} = Y_k \cap Y_l$ and keeping all powers of orders $< a_k$ in $\zeta_k^{(\widetilde{U})}$ and $< a_l$ in $\zeta_l^{(\widetilde{U})}$ so that

$$j_{kl}(f) \simeq (f_{\alpha\beta}^{(\widetilde{U})})_{\substack{\alpha < a_k \\ \beta < a_l}}$$

$$f_{\alpha\beta}^{(\widetilde{U})} = \left.\frac{\partial^{\alpha+\beta} f}{(\partial\zeta_k^{(\widetilde{U})})^{\alpha}(\partial\zeta_l^{(\widetilde{U})})^{\beta}}\right|_{\zeta_k^{(\widetilde{U})} = \zeta_l^{(\widetilde{U})} = 0}.$$

This is also the sheaf of holomorphic sections of a vectors bundle

$$E_{kl}^{(a_k-1,\, a_l-1)} \to Y_{kl}.$$

It is then naturally a $O_{Y_{kl}}$-module, but also a O_{Y_k}-module, a O_X-module ...

c) We shall also define

$$(1.10) \qquad \begin{cases} \widetilde{O}_{kl\dots m} = O_{\widetilde{S}}/(I_k^{a_k} + I_l^{a_l} + \dots + I_m^{a_m}), \\ j_{kl\dots m}\colon O_{\widetilde{S}} \to \widetilde{O}_{kl\dots m}, \\ E_{kl\dots m}^{(a_k-1,\dots,a_m-1)} \to Y_{kl\dots m}. \end{cases}$$

1.3. *Exact sequence associated to $O_{\widetilde{S}}/J_{\widetilde{X}}$.*

LEMMA 1.1. – *There is an exact sequence of $O_{\widetilde{S}}$-modules*

$$(1.11) \qquad 0 \to O_{\widetilde{S}}/J_{\widetilde{X}} \to \bigoplus_k \widetilde{O}_k \to \bigoplus_{k,l} \widetilde{O}_{kl} \to \bigoplus \widetilde{O}_{klm} \cdots$$

with natural morphisms.

PROOF. – If $f \in O_{\widetilde{S}}$, we can associate

$$(1.12) \qquad f \to (j_k(f))_{k=1\dots N} \in \bigoplus_k \widetilde{O}_k$$

and this induces the first morphism in (1.11).
Then if $(f_k)_{k=1,\dots,N} \in \bigoplus_k \widetilde{O}_k$, we associate

$$(1.13) \qquad (j_l(f_k) - j_k(f_l))_{k,l} \in \bigoplus_{k,l} \widetilde{O}_{k,l}$$

and if $(f_{kl})_{k,\,l=1,\dots,N} \in \bigoplus_{k,\,l} \tilde{O}_{kl}$, we associate the various

$$(j_m(f_{kl}) - j_l(f_{km})) \in \bigoplus \tilde{O}_{klm} \tag{1.14}$$

(the same $\tilde{O}_{klm}$ is repeated several times).

This is an exact sequence, because the Y_k are all smooth transversal hypersurfaces.

REMARK 1. – Although each of these modules in (1.11) except $O_{\tilde{S}}/J_{\tilde{X}}$ is naturally a $O_{\tilde{X}}$-module, this is not a sequence of $O_{\tilde{X}}$-modules because the morphisms (1.13), (1.14) ... are note $O_{\tilde{X}}$-morphisms.

REMARK 2. – If there are only two components Y_1, Y_2 of $\tilde{X}$, (1.11) is a short exact sequence

$$0 \to O_{\tilde{S}}/J_{\tilde{X}} \to \tilde{O}_1 \oplus \tilde{O}_2 \to \tilde{O}_{12} \to 0\,. \tag{1.15}$$

1.4. *The infinitesimal neighborhood as an O_X-module.*

We shall now assume the following hypothesis (S)

(S) *For any $i, j = 1, \dots, N$, the morphisms $\varphi|_{Y_{ij}}\colon Y_{ij} \to X$ are surjective.*

We want to prove the following theorem.

THEOREM 1.2. – *Let us assume hypothesis* (S). *With the natural structures of O_X-modules on the sheaves $\tilde{O}_k$ and $\tilde{O}_{kl}$, the morphism of the sequence* (1.11)

$$\bigoplus_k \tilde{O}_k \to \bigoplus_{k,\,l} \tilde{O}_{kl} \tag{1.16}$$

is a O_X-morphism and in particular $O_{\tilde{S}}/J_{\tilde{X}}$ is a O_X-module.

To prove this result, we shall state several lemmas:

LEMMA 1.3. – *Let T_i be the set of points $\tilde{m}$ in Y_i such that the rank of $d(\varphi|_{Y_i})$ at $\tilde{m}$ is less than* $\dim X$. *Then $T_k \cap T_l$ is different from Y_{kl}.*

PROOF. – Because T_k is an analytic set in Y_k, $\varphi(T_k)$ is an analytic subset in X which is of codimension ≥ 1 in X because X is smooth and because of Sard's theorem. Then $\varphi(T_k \cup T_l)$ is also different from X and because $\varphi(Y_{kl}) = X$ we have $Y_{kl} \neq T_k \cup T_l$.

LEMMA 1.4. – *The set of points $\tilde{m} \in Y_{kl}$ such that the fiber $(\varphi|Y_k)^{-1}(\varphi(\tilde{m}))$ or the fiber $(\varphi|Y_l)^{-1}(\varphi(\tilde{m}))$ have a singular point at $\tilde{m}$ is a proper analytic subset of Y_{kl}.*

PROOF. – At a point $\widetilde{m} \in Y_k$ such that the fiber $(\varphi | Y_k)^{-1}(\varphi(\widetilde{m}))$ has a singular point, the rank of the differential $d(\varphi|_{Y_k})$ at $\widetilde{m}$ is not maximal, so that $\widetilde{m}$ is in T_k and Lemma 1.4 follows from Lemma 1.3.

PROOF OF THEOREM 1.2. – We have already seen that $\widetilde{O}_k$ and $\widetilde{O}_{kl}$ are O_X-modules. We consider a point $\widetilde{m} \in Y_{kl}$ and the morphism

$$\widetilde{O}_k \oplus \widetilde{O}_l \to \widetilde{O}_{kl}$$

around that point defined by

$$f_k \oplus f_l \to (j_l(f_k) - j_k(f_l)) . \tag{1.17}$$

Let us first assume that $\widetilde{m}$ is not in $T_k \cup T_l$. The picture around $\widetilde{m}$ is as follows: the fibers $(\varphi|_{Y_k})^{-1}(m')$ determine a local analytic foliation of Y_k around $\widetilde{m}$ for m' near $m = \varphi(\widetilde{m})$ and the fibers $(\varphi | Y_l)^{-1}(m')$ determine a local analytic foliation of Y_l around $\widetilde{m}$ for m' near m.

Let us choose local coordinates in $\widetilde{S}$ around $\widetilde{m}$, (ζ_k, ζ_l, z) where $\zeta_k = 0$ (resp. $\zeta_l = 0$) are local equation for Y_k (resp. Y_l) near $\widetilde{m}$ and z are local coordinates on Y_{kl} around $\widetilde{m}$ containing local coordinates around $m = \varphi(\widetilde{m})$ in X and maybe other coordinates.

Then if $h \in O_{X,m}$, it is clear that $h \circ (\varphi|_{Y_k})$ is constant on the fibers of $\varphi|_{Y_k}$ and

$$\frac{\partial^r (h \circ (\varphi|_{Y_k}))}{\partial \zeta_l^r} = 0 \qquad \text{for } r > 0 .$$

Then it is clear that

$$j_l((h \circ (\varphi|_{Y_k})) f_k) = (h \circ \varphi|_{Y_{kl}}) j_l(f_k) \tag{1.18}$$

and (1.17) is a morphism of $O_{X,m}$-modules near $\widetilde{m}$.

Now let us suppose that $\widetilde{m}$ is in $T_k \cup T_l$ and let us choose coordinates around $\widetilde{m}$ in $\widetilde{S}$ of the form $(\widetilde{\zeta}_k, \widetilde{\zeta}_l, \widetilde{z}')$ so that $\widetilde{\zeta}_k = 0$ is a local equation for Y_k (resp. $\widetilde{\zeta}_l = 0$ is a local equation for Y_l) and $(\widetilde{\zeta}_l, \widetilde{z}')$ (resp. $(\widetilde{\zeta}_k, z')$) are local coordinates for Y_k (resp. for Y_l). There exist points $\widetilde{m}' \in Y_{kl} - (T_k \cup T_l)$ tending to $\widetilde{m}$, and we can assume that there exists local coordinates of $\widetilde{S}$ around $\widetilde{m}'$ of the form (ζ'_k, ζ'_l, z') with $z' = \widetilde{z}'$ and z containing the coordinates of X near $\varphi(\widetilde{m})$ and with $\zeta'_k = 0$ (resp. $\zeta'_l = 0$) local equation for Y_k (resp. Y_l) near $\widetilde{m}'$. Moreover

$$\zeta'_k = \widetilde{\zeta}_k \psi_k(\widetilde{\zeta}_k, \widetilde{\zeta}_l, \widetilde{z}'), \qquad \zeta'_l = \widetilde{\zeta}_l \psi_l(\widetilde{\zeta}_k, \widetilde{\zeta}_l, \widetilde{z}') .$$

Now

$$\frac{\partial(h\circ\varphi|_{Y_k})}{\partial\tilde{\zeta}_l}=\frac{\partial(h\circ\varphi|_{Y_k})}{\partial\zeta'_l}\,\frac{\partial\zeta'_l\,|_{Y_k}}{\partial\tilde{\zeta}_l}. \tag{1.19}$$

But around $\tilde{m}'$, this is 0, so

$$\frac{\partial(h\circ\varphi|_{Y_k})}{\partial\tilde{\zeta}_l}(\tilde{m}')=0\,.$$

But $\partial(h\circ\varphi|_{Y_k})/\partial\tilde{\zeta}_l$ is continuous, so that

$$\frac{\partial(h\circ\varphi|_{Y_k})}{\partial\tilde{\zeta}_l}(\tilde{m})=0\,. \tag{1.20}$$

All higher derivatives of $(h\circ\varphi)|_{Y_k}$ with respect to $\tilde{\zeta}_l$ can be treated similarly by derivations of (1.19) so that (1.18) is still true at points $\tilde{m}\in Y_{kl}\cap(T_k\cap T_l)$. This proves that the morphism (1.16) is a morphism of O_X-modules. Now, $O_{\tilde{S}}/J_{\tilde{X}}$ is the kernel of this morphism and so it is naturally a O_X-module.

DEFINITION. – *We define the sheaf $G_{\tilde{X}}$ to be the image of the natural morphism*

$$\bigoplus_k \tilde{O}_k\to\bigoplus_{k,l}\tilde{O}_{kl}\,.$$

It is then a O_X-module and we deduce

COROLLARY 1.5. – *Under hypothesis (S), we have a short exact sequence of O_X-modules:*

$$0\to O_{\tilde{S}}/J_{\tilde{X}}\to\bigoplus_k\tilde{O}_k\to G_{\tilde{X}}\to 0 \tag{1.21}$$

where $G_{\tilde{X}}$ is the image of the morphism

$$\bigoplus_k \tilde{O}_k\to\bigoplus_{k,l}\tilde{O}_{kl}\,.$$

NOTATION. – The formal neighborhood is denoted $\hat{O}_{\tilde{S}}$.
By definition,

$$\hat{O}_S=\varprojlim_k\,(O_{\tilde{S}}/J^k_{\tilde{X}})\,. \tag{1.22}$$

We can take the direct image $\varphi_*(\hat{O}_{\tilde{S}})$. This is in fact $\hat{O}_S$, the formal neighborhood of X in S, by a result of Grothendieck [3], but we shall not need this fact below.

On the other hand, each $\varphi_*(O_{\tilde{S}}/J_{\tilde{X}}^k)$ is a O_X-module and the natural morphisms $O_{\tilde{S}}/J_{\tilde{X}}^k \to O_{\tilde{S}}/J_{\tilde{X}}^{k-1}$ induce morphisms of O_X-modules. As a consequence, we have

COROLLARY 1.6. – *Under hypothesis (S), the infinitesimal neighborhood $\varphi_*(\hat{O}_{\tilde{S}})$ is a O_X-module.*

2. – Spectral resolutions.

2.1. *The cone construction.*

LEMMA 2.1. – *Let us consider a commutative diagram:*

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
0 \to A \to & & B & \longrightarrow & C & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & B^0 & \xrightarrow{u^0} & C^0 & & \\
 & & {\scriptstyle d^0}\downarrow & & \downarrow{\scriptstyle \delta^0} & & \\
 & & B^1 & \xrightarrow{u^1} & C^1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \\
 & & {\scriptstyle d^{n-1}}\downarrow & & \downarrow{\scriptstyle \delta^{n-1}} & & \\
 & & B^n & \xrightarrow{u^n} & C^n & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

with exact columns and a top short exact sequence. Let us define the complex $Q^\bullet$

$$Q^k = B^k \oplus C^{k-1} \qquad (C^{-1} \equiv 0)$$

with a differential given by the matrix

$$D^k = \begin{pmatrix} d^k & 0 \\ u^k & -\delta^{k-1} \end{pmatrix}$$

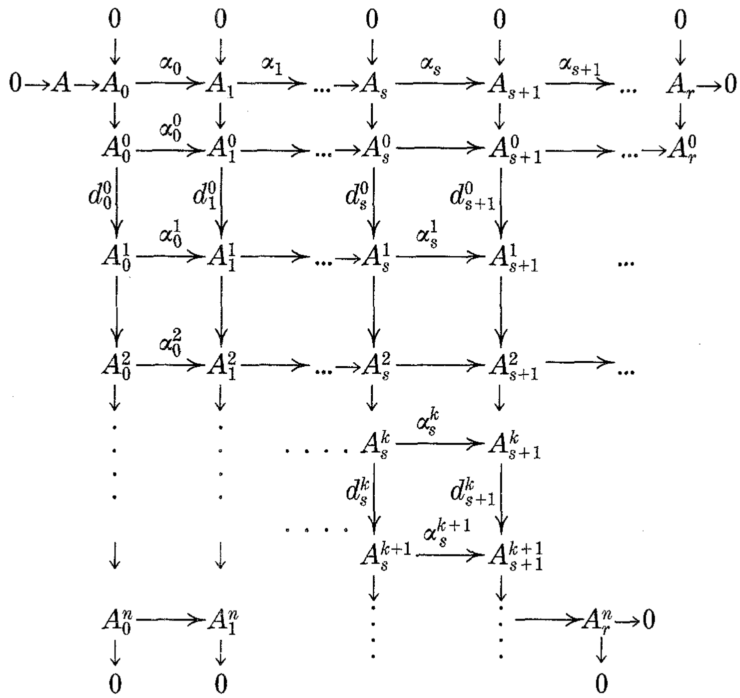
and the natural injection $A \to Q^0 \equiv B^0$ then $(Q^\bullet, D^\bullet)$ is a resolution of A.

PROOF. – This is an exercise.

2.2. *Spectral resolution.*

Let us now consider a commutative diagram

(2.1)



and assume that:

(i) the top line is exact,

(ii) the columns are exact,

(iii) the lines are complexes.

Let us now define for a fixed s a complex

(2.2) $$Z_s^k = A_s^k \oplus A_{s+1}^{k-1} \oplus \ldots \oplus A_{s+k}^0,$$

(we define $A_l^q = 0$ if $q < 0$, $q > n$, $l < 0$ or $l > r$) with the differential

$$D_s^k : Z_s^k \to Z_s^{k+1}$$

given by the matrix

$$D_s^k = \begin{pmatrix} d_s^k & 0 & \cdots\cdots & & 0 \\ \alpha_s^k & -d_{s+1}^{k-1} & & & \\ 0 & -\alpha_{s+1}^{k-1} & d_{s+2}^{k-2} & \ddots & \\ \vdots & & & \ddots & \\ 0 & & & (-1)^k \alpha_{s+k}^0 & 0 \end{pmatrix}. \tag{2.3}$$

We check easily that

$$D_s^{k+1} D_s^k = 0$$

so that $Z_s^\bullet$ is a complex.

LEMMA 2.2. – *Let* $H_s = \operatorname{Ker} \alpha_s$ *with the natural injection* $H_s \hookrightarrow A_s \hookrightarrow Z_s^0 \equiv A_s^0$. *Then* $(Z_s^\bullet, D_s^\bullet)$ *is a resolution of* H_s.

PROOF OF LEMMA 2.2. – The proof is by a descending recursion on s. For $s = r - 1$ we have the short exact sequence

$$0 \to H_{r-1} \to A_{r-1} \to A_r \to 0$$

and we apply Lemma 2.1 to get the result.

Let us now assume that H_{s+1} has the resolution $(Z_{s+1}^\bullet, D_{s+1})$. We have a short exact sequence

$$0 \to H_s \to A_s \to H_{s+1} \to 0$$

(because the top line is exact).

Moreover we have a morphism of complexes

$$A_s^\bullet \to Z_{s+1}^\bullet$$

given by

$$u_s^k \colon a_s^k \in A_s^k \to (\alpha_s^k(a_s^k), 0, \ldots, 0) \in A_{s+1}^k \oplus \ldots \equiv Z_{s+1}^k$$

(it is easy to check that it is a morphism of complexes because

$$D_{s+1}^k u_s^k(a_s^k) = D_{s+1}^k(\alpha_s^k a_s^k), 0, \ldots, 0) = (d_{s+1}^k \alpha_s^k(a_s^k), 0, \ldots, 0),$$

$$= (\alpha_s^{k+1} d_s^{k+1}(a_s^k), 0, \ldots, 0)$$

$$= u_s^{k+1} d_s^{k+1}(a_s^k)).$$

We can apply the Lemma 2.1 to the situation:

$$\begin{array}{ccccccccc} 0 & \to & H_s & \to & A_s & \to & H_{s+1} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & A_s^\bullet & \to & Z_{s+1}^\bullet . & & \end{array}$$

Then:

$$Q^k = A_s^k \oplus Z_{s+1}^{k-1} = Z_s^k .$$

Moreover the morphisms $Q^k \to Q^{k+1}$ given in Lemma 2.1 are identical to the morphisms $Z_s^k \to Z_s^{k+1}$: in fact in the notations of Lemma 2.1, we have

$$d^k = d_s^k$$

$$u^k = (\alpha_s^k, 0, \dots, 0)$$

$$\delta^{k-1} = \begin{pmatrix} -d_{s+1}^{k-1} & 0 & \dots\dots\dots & 0 & \\ -\alpha_{s+1}^{k-1} & d_{s+2}^{k-2} & & & \\ & \alpha_{s+2}^{k-2} & -d_{s+3}^{k-3} & \dots\dots\dots & \\ 0 & & \ddots & & \ddots \end{pmatrix}.$$

So that the block matrix

$$\begin{pmatrix} d^k & 0 \\ u^k & -\delta^{k-1} \end{pmatrix} \equiv D_s^k .$$

As a consequence, we obtain by taking $s = 0$:

THEOREM 2.1. – *The complex* $(Z_0^\bullet, D_0^\bullet)$ *is a resolution of* A *(with the natural injection* $A \to Z_0^0$*).*

DEFINITION. – *We shall call* $(Z_0^\bullet, D_0^\bullet)$ *the spectral resolution of* A *associated to the array of resolutions* (2.1).

We shall denote more precisely

$$Z_0^\bullet = \Delta^\bullet (A_0^\bullet \to A_1^\bullet \to \dots)$$

if we want to emphasize the functorial dependence of the spectral resolution of the resolutions $A_s^\bullet$.

3. – Reducible exceptional divisor: resolution of infinitesimal neighborhoods.

3.1. *Resolution of the infinitesimal neighborhood of the exceptional divisor.*

We now consider the general situation where the eceptional divisor $\widetilde{X}$ is reducible. We write as in Sections 1.0, 1.1, 1.2

$$\widetilde{X} = \bigcup_{i=1}^{N} Y_i$$

and we consider the ideals $J_{\widetilde{X}}, I_j^{a_j}, \ldots$ and the rings $O_{\widetilde{S}}/J_{\widetilde{X}}$, $\widetilde{O}_j = O_{\widetilde{S}}/I_j^{a_j}$, $\widetilde{O}_{jk}, \ldots$ as defined in Sections 1.2. We also consider the long exact sequence of Lemma 1.1

$$0 \to O_{\widetilde{S}}/J_{\widetilde{X}} \to \bigoplus_{k=1}^{N} \widetilde{O}_k \to \bigoplus_{k<l} \widetilde{O}_{kl} \to \bigoplus_{k,l,m} \widetilde{O}_{klm} \to \ldots \tag{3.1}$$

with the natural morphisms defined by jets in Section 1.3. Now, I_k is the ideal of definition of Y_k, and $\widetilde{O}_k$ is the sheaf of holomorphic sections of a vector bundle over Y_k and more generally, $\widetilde{O}_{kl\ldots m}$ is the sheaf of holomorphic sections of a vector bundle over $Y_{kl\ldots m}$. In particular, we have a resolution of the sheaf $\widetilde{O}_{k_1 k_2 \ldots k_r}$ by taking the tensor product of $\widetilde{O}_{k_1\ldots k_r}$ with the Dolbeault resolution $\Lambda^{\bullet}_{Y_{k_1\ldots k_r}}$ of $Y_{k_1\ldots k_r}$.

NOTATIONS. – We denote $\Lambda^{\bullet}_M$ the Dolbeault resolution of the complex manifold M, so that Λ^p_M is the sheaf of C^∞ forms of type $(0,p)$ on M with the $\bar\partial$ operator

$$\bar\partial_M, \Lambda^p_M \to \Lambda^{p+1}_M$$

as differential. For any sheaf F of O_M-modules on M, $R^{\bullet}_M(F) = F \otimes_{O_M} \Lambda^{\bullet}_M$ is a resolution of F by fine sheaves. Here we shall denote simply:

$$\boldsymbol{R}^{\bullet}(k_1 \ldots k_r) = \widetilde{O}_{k_1\ldots k_r} \otimes_{O_{Y_{k_1\ldots k_2}}} \Lambda^{\bullet}_{Y_{k_1\ldots k_r}} \tag{3.2}$$

with the differential $Id \otimes \bar\partial$ which we simply denote $\bar\partial$. We also define the jet

$$j^{\bullet}_{k_1\ldots k_r}\colon \Lambda^{\bullet}_{\widetilde{S}} \to \boldsymbol{R}^{\bullet}(k_1, \ldots, k_r) \tag{3.3}$$

as follows: call $\zeta_{k_1} = \ldots = \zeta_{k_r}$ local holomorphic equations for $Y_{k_1\ldots k_r}$ and call $(z_1, \ldots, z_{n-r})$ holomorphic coordinates along $Y_{k_r\ldots k_r}$. Let π be a $(0,p)$ form on $\widetilde{S}$. We express it locally in the coordinate systems $(z_1, \ldots, z_{n-r}, \zeta_{k_1}, \ldots, \zeta_{k_r})$. To compute $j^{\bullet}_{k_1\ldots k_r}(\pi)$ we first suppress all the components of the form π containing one or several of the $d\bar\zeta_{k_1}, \ldots, d\bar\zeta_{k_r}$, then we reduce the coefficients modulo $\bar\zeta_{k_1}, \ldots, \bar\zeta_{k_r}$ and modulo $I_{k_1\ldots k_r} \equiv I_{k_1}^{a_{k_1}} + \ldots + I_{k_r}^{a_{k_r}}$.

It is then clear that

$$j^{\bullet+1}_{k_1\ldots k_r} \bar\partial_{\widetilde{S}} = \bar\partial j^{\bullet}_{k_1\ldots k_r} \tag{3.4}$$

so that $j^{\bullet}_{k_1\ldots k_r}$ *is a morphism of resolutions.*

We also have natural jet morphisms

$$j^{\bullet}: \boldsymbol{R}^{\bullet}(k_1, \dots, k_r) \to \boldsymbol{R}^{\bullet}(k_1, \dots, k_r, k_{r+1}) \tag{3.5}$$

such that the diagram is commutative

$$\begin{array}{ccc} \Lambda_S^{\bullet} & \xrightarrow{\quad Id \quad} & \Lambda_S^{\bullet} \\ {\scriptstyle j^{\bullet}_{k_1 \dots k_r}} \downarrow & & \downarrow {\scriptstyle j^{\bullet}_{k_1}} \\ \boldsymbol{R}^{\bullet}(k_1 \dots k_r) & \xrightarrow{\quad j \quad} & \boldsymbol{R}^{\bullet}(k_1 \dots k_{r+1}) \end{array}$$

$j^{\bullet}$ is also a morphism of resolutions.

Now we apply to (3.1) the spectral construction. We have a complex of resolutions deduced from (3.1)

$$\bigoplus_{k=1}^{N} \boldsymbol{R}^{\bullet}(k) \to \bigoplus_{k<l} \boldsymbol{R}^{\bullet}(k, l) \to \bigoplus_{k, l, m} \boldsymbol{R}^{\bullet}(k, l, m) \to \dots \tag{3.6}$$

The horizontal arrows of this complex of resolutions are induced by difference of various jets (3.5). From Theorem 2.3 of Section 2.2, we deduce the following theorem

THEOREM 3.1. – *The diagonal complex associated to the complex of resolutions* (3.6) *is a resolution of $O_{\tilde{S}}/J_{\tilde{X}}$ by acyclic sheaves* (*in fact fine sheaves*).

NOTATION. – We shall denote

$$\boldsymbol{R}^{\bullet}(O_{\tilde{S}}/J_{\tilde{X}}) = \Delta^{\bullet}\{\oplus \boldsymbol{R}^{\bullet}(k) \to \oplus \boldsymbol{R}^{\bullet}(k, l) \to \dots\} \tag{3.7}$$

so that

$$\begin{cases} \boldsymbol{R}^0(O_{\tilde{S}}/J_{\tilde{X}}) = \bigoplus\limits_{k=1}^{N} \boldsymbol{R}^0(k)\,, \\ \boldsymbol{R}^1(O_{\tilde{S}}/J_{\tilde{X}}) = \left(\bigoplus\limits_{k=1}^{N} \boldsymbol{R}^1(k)\right) \oplus \left(\bigoplus\limits_{k<l} \boldsymbol{R}^0(k, l)\right), \\ \dots \\ \boldsymbol{R}^p(O_{\tilde{S}}/J_{\tilde{X}}) = \left(\bigoplus\limits_{k=1}^{N} \boldsymbol{R}^p(k)\right) \oplus \dots \oplus \left(\bigoplus\limits_{|K|=r} \boldsymbol{R}^{p-r+1}(K)\right) \oplus \dots. \end{cases} \tag{3.8}$$

3.2. *Associated resolution of $O_{\tilde{S}}$.*

Let us now replace in (3.1) $O_{\tilde{S}}/J_{\tilde{X}}$ (resp. each $\tilde{O}_{k_1 \dots k_r}$) by $O_{\tilde{S}}$ and each difference of jets by the corresponding difference of elements.

We have then a corresponding exact sequence and a morphism of exact sequences given by the jets morphisms

$$(3.9)\qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & O_{\tilde S} & \longrightarrow & \bigoplus\limits_k O_{\tilde S} & \longrightarrow & \bigoplus\limits_{k<l} O_{\tilde S} & \longrightarrow & \bigoplus\limits_{k,l,m} O_{\tilde S} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O_{\tilde S}/J_{\tilde X} & \longrightarrow & \bigoplus\limits_k \tilde O_k & \longrightarrow & \bigoplus\limits_{k<l} \tilde O_{kl} & \longrightarrow & \bigoplus\limits_{k,l,m} \tilde O_{k,l,m} & \longrightarrow & \dots \end{array}$$

Let $G_{\tilde S}$ denote the image of the morphism

$$\bigoplus_k O_{\tilde S} \to \bigoplus_{k<l} O_{\tilde S}.$$

We have a morphism of exact sequences of sheaves

$$(3.10)\qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & O_{\tilde S} & \longrightarrow & \bigoplus\limits_k O_{\tilde S} & \longrightarrow & G_{\tilde S} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O_{\tilde S}/J_{\tilde X} & \longrightarrow & \bigoplus\limits_k \tilde O_k & \longrightarrow & \bigoplus\limits_k \tilde O_{kl} & \longrightarrow & \bigoplus \tilde O_{k,l,m} & \longrightarrow & \dots \end{array}$$

In the top line we replace the $O_{\tilde S}$ by their Dolbeault resolutions $\Lambda^\bullet_{\tilde S}$ and $G_{\tilde S}$ by its Dolbeault resolution $G_{\tilde S} \otimes_{O_{\tilde S}} \Lambda^\bullet_{\tilde S}$. We deduce from (3.10) a morphism of complexes

$$(3.11)\qquad \begin{array}{ccccccc} \bigoplus\limits_k \Lambda^\bullet_{\tilde S} & \longrightarrow & G_{\tilde S} \otimes_{O_{\tilde S}} \Lambda^\bullet_{\tilde S} & \longrightarrow & 0 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus\limits_k \boldsymbol{R}^\bullet(k) & \longrightarrow & \bigoplus\limits_{k,l} \boldsymbol{R}^\bullet(k,l) & \longrightarrow & \bigoplus\limits_{k,l,m} \boldsymbol{R}^\bullet(k,l,m) & \longrightarrow & \dots \end{array}$$

where the vertical arrows are defined by the jets (3.3). We take now, the diagonal complex associated with the complex of resolutions of the top line of (3.10). More precisely, we define

$$(3.12)\qquad \boldsymbol{R}^\bullet(O_{\tilde S}) = \Delta^\bullet\{\bigoplus_k \Lambda^\bullet_{\tilde S} \to G_{\tilde S} \otimes_{O_{\tilde S}} \Lambda^\bullet_{\tilde S} \to 0 \dots\}.$$

By Theorem 2.3 of Section 2.2, this is a resolution of $O_{\tilde S}$ by fine sheaves and we have a morphism of resolutions

$$(3.13)\qquad J^\bullet : \boldsymbol{R}^\bullet(O_{\tilde S}) \to \boldsymbol{R}^\bullet(O_{\tilde S}/J_{\tilde X})$$

induced by the vertical morphism of (3.11).

In fact if $(\alpha_k^{(p)})_k \in \bigoplus\limits_k \Lambda^p_{\tilde S}$ and $(\beta_{kl}^{(p-1)})_{k,l} \in G_{\tilde S} \otimes_{O_{\tilde S}} \Lambda^p_{\tilde S}$, we have

$$(3.14)\qquad J^{(p)}((\alpha_k^{(p)}) \oplus (\beta_{kl}^{(p-1)})) = (j_k(\alpha_k^{(p)}))_k \oplus (j_{kl}(\beta_{kl}^{(p-1)}))_{kl} \oplus 0 \oplus 0 \dots \in \boldsymbol{R}^p(O_{\tilde S}/J_{\tilde X}).$$

The morphism $J^\bullet$ is compatible with the jet morphism

$$j_{\tilde X} : O_{\tilde S} \to O_{\tilde S}/J_{\tilde X}.$$

Moreover if we now assume the vanishing of cohomology (hyporthesis (A) of Section 1.1), we know that for sufficiently small U in S and all $p > 0$

$$H^p(\varphi^{-1}(U), O_{\bar{S}}) \simeq H^p(\varphi^{-1}(U), O_{\bar{S}}/J_{\bar{X}}).$$

We apply Theorem 1.1 of comparison of cohomologies of [1] to the morphism of resolutions $J^\bullet$ given by (3.14) and we deduce that $J^\bullet$ induces an isomorphism between the De Rham-Dolbeault cohomology groups of $O_{\bar{S}}$ and $O_{\bar{S}}/J_{\bar{X}}$ computed using the resolutions $\boldsymbol{R}^\bullet(O_{\bar{S}})$ and $\boldsymbol{R}^\bullet(O_{\bar{S}}/J_{\bar{X}})$.

This is the content of the Theorem 3.2.

THEOREM 3.2. – *The morphism of the diagonal resolutions* $J^\bullet: \boldsymbol{R}^\bullet(O_{\bar{S}}) \to \to \boldsymbol{R}^\bullet(O_{\bar{S}}/J_{\bar{X}})$ *is compatibile with the jet* $j_{\bar{X}}: O_{\bar{X}} \to O_{\bar{X}}/J_{\bar{X}}$.

Under hypothesis (A), *it induces an isomorphism between the De Rham-Dolbeault cohomology groups of* $O_{\bar{S}}$ *and* $O_{\bar{S}}/J_{\bar{X}}$ *computed with these resolutions.*

3.3. *Long exact sequence of cohomology sheaves on X.*

We shall henceforth assume hypothesis (S)

(S) *for any* $k < l$, $\varphi|_{Y_{kl}}: Y_{kl} \to X$ *is surjective.*

From theorem 1.2, we know that the short exact sequence of Corollary 1.5

$$0 \to O_{\bar{S}}/J_{\bar{X}} \to \bigoplus_{k=1}^{N} \tilde{O}_k \to G_{\bar{X}} \to 0 \tag{3.15}$$

is a short exact sequence of O_X-morphisms. Here, we recall that $G_{\bar{X}}$ is the image of the natural morphism

$$\bigoplus_{k=1}^{N} \tilde{O}_k \to \bigoplus_{k<l} \tilde{O}_{kl}.$$

In particular, the direct image sheaves $R^0_{\varphi_*}(\tilde{O}_k)$ and $R^0_{\varphi_*}(G_{\bar{X}})$ are naturally O_X-modules. The cohomology sheaves $R^p_{\varphi_*}(\tilde{O}_k)$ and $R^p_{\varphi_*}(G_{\bar{X}})$ are also O_X-modules (for example using their description in Cech cohomology). From (3.15), we deduce a long exact sequence of cohomology modules over O_X

$$0 \to R^0_{\varphi_*}(O_{\bar{S}}/J_{\bar{X}}) \to \bigoplus R^0_{\varphi_*}(\tilde{O}_k) \to R^0_{\varphi_*}(G_{\bar{X}}) \xrightarrow{\delta} R^1_{\varphi_*}(O_{\bar{S}}/J_{\bar{X}}) \to \bigoplus R^1_{\varphi_*}(\tilde{O}_k) \to \dots \to \tag{3.16}$$

$$\to R^p_{\varphi_*}(O_{\bar{S}}/J_{\bar{X}}) \to \bigoplus R^p_{\varphi_*}(\tilde{O}_k) \to R^p_{\varphi_*}(G_{\bar{X}}) \xrightarrow{\delta} R^{p+1}_{\varphi_*}(O_{\bar{S}}/J_{\bar{X}}) \to \dots.$$

The δ is the coboundary operator in cohomology, the other morphisms are obvious. Now, all the modules in this long exact sequence are O_X-modules and they have a Dolbeault resolution obtained by taking the tensor product of the module by the Dolbeault resolution $\Lambda^\bullet_X$ of X (this is where we use the fact that X is smooth so that $\Lambda^\bullet_X$

is a flat O_X-module). If F is anyone of these sheaves, we denote

$$R_X^\bullet(F) = F \otimes_{O_X} \Lambda_X^\bullet. \tag{3.17}$$

We abreviate the notations as follows:

$$R^p_{\varphi_*}(O_k) \equiv R^p(k)$$

and also we suppress the index φ_*.

Then we have a long exact sequence of resolutions obtained by taking the tensor product of (4.14) with $\Lambda_X^\bullet$

$$\bigoplus_k \boldsymbol{R}_X^\bullet(R^0(k)) \to \boldsymbol{R}_X^\bullet(R^0(G_{\bar X})) \to \boldsymbol{R}_X^\bullet(R^1(O_{\bar S}/J_{\bar X})) \to \bigoplus_k \boldsymbol{R}_X^\bullet(R^1(k)) \to \ldots \to \tag{3.18}$$

$$\to \boldsymbol{R}_X^\bullet(R^p(O_{\bar S}/J_{\bar X})) \to \bigoplus_k \boldsymbol{R}_X^\bullet(R^p(k)) \to \boldsymbol{R}_X^\bullet(R^p(G_{\bar X})) \to \boldsymbol{R}_X^\bullet(R^{p+1}(O_{\bar S}/J_{\bar X}) \to \ldots .$$

Then we take the diagonal complex. We define

$$\boldsymbol{R}^\bullet(\varphi_*(O_{\bar S}/J_{\bar X})) = \boldsymbol{\Delta}^\bullet\{\bigoplus_k \boldsymbol{R}_X^\bullet(R^0(k)) \to \boldsymbol{R}_X^\bullet(R^0(G_{\bar X})) \to \ldots\}$$

and deduce from Theorem 2.3:

THEOREM 3.3. – *$\boldsymbol{R}^\bullet(\varphi_*(O_{\bar S}/J_{\bar X}))$ is a resolution of $\varphi_*(O_{\bar S}/J_{\bar X})$ by fine sheaves on S.*

We shall now investigate more closely the structure of the resolution $\boldsymbol{R}^\bullet(\varphi_*(O_{\bar S}/J_{\bar X}))$ and prepare notations for the next chapter. We have

$$\boldsymbol{R}^0(\varphi_*(O_{\bar S}/J_{\bar X})) = \bigoplus_k \boldsymbol{R}_X^0(R^0(k)),$$

$$\boldsymbol{R}^1(\varphi_*(O_{\bar S}/J_{\bar X})) = (\bigoplus_k \boldsymbol{R}_X^1(R^0(k))) \oplus \boldsymbol{R}_X^0(R^0(G_{\bar X})), \tag{3.19}$$

$$\boldsymbol{R}^2(\varphi_*(O_{\bar S}/J_{\bar X})) = (\bigoplus_k \boldsymbol{R}_X^2(R^0(k))) \oplus \boldsymbol{R}_X^1(R^0(G_{\bar X})) \oplus \boldsymbol{R}_X^0(R^1(O_{\bar S}/J_{\bar X}))$$

and more generally we shall split $\boldsymbol{R}^p(\varphi_*(O_{\bar S}/J_{\bar X}))$ into three pieces

$$\boldsymbol{R}^p(\varphi_*(O_{\bar S}/J_{\bar X})) = (\bigoplus_k \boldsymbol{R}_X^p(R^0(k))) \oplus \boldsymbol{R}_X^{p-1}(R^0(G_{\bar X})) \oplus C^p, \tag{3.20}$$

where C^p is defined as

$$C^p = \boldsymbol{R}_X^{p-2}(R^1(O_{\bar S}/J_{\bar X})) \oplus (\bigoplus_k \boldsymbol{R}_X^{p-3}(R^1(k))) \oplus (\boldsymbol{R}_X^{p-4}(R^1(G_{\bar X}))) \oplus \tag{3.21}$$

$$\oplus \boldsymbol{R}_X^{p-5}(R^2(O_{\bar S}/J_{\bar X})) \oplus (\bigoplus_k \boldsymbol{R}_X^{p-6}(R^2(k))) \oplus (\boldsymbol{R}_X^{p-7}(R^2(G_{\bar X}))) \oplus \ldots \oplus \boldsymbol{R}_X^0(\Gamma^{p-1})$$

where Γ^{p-1} is defined as

$$\begin{aligned}\Gamma^{p-1} &= R^{p-1}(G_{\bar{X}}) && \text{if } p \equiv 1 \bmod 3\,,\\ &= R^{p-1}(O_{\bar{S}}/J_{\bar{X}}) && \textit{if } p \equiv 2 \bmod 3\,,\\ &= \bigoplus_k R^{p-1}(k) && \textit{if } p \equiv 0 \bmod 3\,.\end{aligned}$$

Moreover in (3.21) what is inside the symbols $R_X^\bullet$ mimics the long exact sequence (3.16).

We shall also denote $\hat{\bar{D}}^\bullet$ the differential of $\boldsymbol{R}^\bullet(\varphi_*(O_{\bar{S}}/J_{\bar{X}}))$.

We denote typical elements of the modules as:

$$\alpha^{(p,q)} = (\alpha_k^{(p,q)})_k \in \bigoplus_k \boldsymbol{R}_X^q(R^p(k))$$

$$\beta^{(p,q)} = (\beta_{kl}^{(p,q)})_{kl} \in \boldsymbol{R}_X^q(R^p(G_{\bar{X}}))$$

$$\gamma^{(p,q)} \in \boldsymbol{R}_X^q(R^p(O_{\bar{S}}/J_{\bar{X}}))\,.$$

We now see that, in general the morphism $\hat{\bar{D}}^p$ splits in the following manner

$$(3.22)\qquad \hat{\bar{D}}^p\colon (\bigoplus_k \boldsymbol{R}_X^p(R^0(k))) \oplus \boldsymbol{R}_X^{p-1}(R^0(G_{\bar{X}})) \oplus C^p \to \to (\bigoplus_k \boldsymbol{R}_X^{p+1}(R^0(k))) \oplus \boldsymbol{R}_X^p(R^0(G_{\bar{X}})) \oplus C^{p+1}$$

where the splitting has a triangular structure coming naturally from the diagonal complex

$$(3.23)\qquad \hat{\bar{D}}^p(\alpha^{(p,q)} \oplus \beta^{(0,p-1)} \oplus C^{(p)}) = = (\bigoplus_k \bar{\partial}_X \alpha_k^{(0,p)}) \oplus [\bigoplus_{k<l} (j_l(\alpha_k^{(0,p)}) - j_k(\alpha_l^{(0,p)}) - \bar{\partial}_X \beta_{kl}^{(0,p-1)}] \oplus C^{(p+1)}.$$

We denote simply

$$(3.24)\qquad C^{(p+1)} \equiv \partial^{(p)}(\beta^{(0,p-1)} \oplus C^{(p)})$$

and call it the coboundary part of $\hat{\bar{D}}^p$: it depends only of $\beta^{(0,p-1)}$ and $C^{(p)}$.

We also denote by $\hat{\bar{\partial}}^{(p)}$ the morphism

$$(3.25)\qquad \hat{\bar{\partial}}^{(p)}(\alpha^{(0,p)} \oplus \beta^{(0,p-1)}) = (\bigoplus_k \bar{\partial}_X \alpha_k^{(0,p)}) \oplus \bigoplus_{k<l} (j_l(\alpha_k^{(0,p)}) - j_k(\alpha_l^{(0,p)}) - \bar{\partial}_X \beta_{kl}^{(0,p-1)})$$

with the natural splitting

$$\widehat{\widehat{D}}^p(\alpha^{(0,p)}\oplus\beta^{(0,p-1)}\oplus C^{(p)}) = \widehat{\widehat{\partial}}^{(p)}(\alpha^{(0,p)}\oplus\beta^{(0,p-1)})\oplus\delta^{(p)}(\beta^{(0,p-1)}\oplus C^{(p)}). \tag{3.26}$$

4. – Reducible exceptional divisor: spectral resolution of O_S.

We are now ready to construct a canonical resolution of O_S. In this chapter, we shall assume that the hypothesis (A) (of Section 1.1) and (S) (of Section 1.4) hold. Moreover, we shall refer freely to the notations of the preceding Chapter 3.

4.1. *Construction of differential forms on* S *and* $\bar\partial$ *operator.*

a) Differential forms at a regular point of S. Let m be a regular point of S. We define the germs of differential forms of type $(0,p)$ at m by the formula

$$\Lambda^p_{S,m} = \left(\bigoplus_{k=1}^N \Lambda^p_{\tilde S,\tilde m}\right)\oplus(G_{\tilde S,\tilde m}\otimes_{O_{\tilde S,\tilde m}}\Lambda^{p-1}_{\tilde S,\tilde m})$$

(with the obvious convention $\Lambda^{-1}_{\tilde S}=0$) and $\tilde m = \varphi^{-1}(m)$). In other word

$$\Lambda^p_{S,m} = \boldsymbol{R}^p(O_{\tilde S})_{\varphi^{-1}(m)} \tag{4.1}$$

b) The space $\Gamma(X\cap U, \boldsymbol{R}^p_X(R^0(k)))$ *and its lifting to* Y_k. Let m be a point in X and U an open neighborhood of m in S. For a given k, we can define the space $\Gamma(X\cap U, \boldsymbol{R}^p_X(R^0(k)))$ which are the sections on $X\cap U$ of the sheaves

$$\boldsymbol{R}^p_X(R^0(k)) = \varphi_*(\tilde O_k)\otimes_{O_X}\Lambda^p_X.$$

If $\alpha^{(p)}_k$ is in $\Gamma(X\cap U, \boldsymbol{R}^p_X(R^0(k)))$, we can write

$$\alpha^{(p)}_k = \sum_i u_i\otimes_{O_X}\omega^{(p)}_i, \tag{4.2}$$

where $\omega^{(p)}_i$ are C^∞ sections on $X\cap U$ of Λ^p_X and u_i are global holomorphic sections on $\varphi^{-1}(U)$ of $\tilde O_k$.

We can lift $\alpha^{(p)}_k$ on $\varphi^{-1}(U)\cap Y_k$ by $(\varphi|_{Y_k})^*$ as in [1] Section 2.3

$$(\varphi|_{Y_k})^*(\alpha^{(p)}_k) = \sum_i u_i(\varphi|_{Y_k})^*(\omega^{(p)}_i). \tag{4.3}$$

This is a $(0,p)$-form on Y_k with coefficient in the jet sheaf $\tilde O_k$ (which is the sheaf of sections of a vector bundle on Y_k). We can prove exactly as in [1] Section 2.3, Theorem 2.2:

LEMMA 4.1. – *The mapping* $(\varphi|_{Y_k})^*$ *defined by* (4.3) *is a well defined and injective mapping*

$$(\varphi|_{Y_k})^*\colon \Gamma(X\cap U, \boldsymbol{R}^p_X(R^0(k)))\to\Gamma(\varphi^{-1}(U)\cap Y_k, \tilde O_k\otimes_{O_{Y_k}}\Lambda^p_{Y_k}).$$

Moreover

$$(\varphi|_{Y_k})^*(\bar{\partial}_X\alpha_k^{(p)}) = \bar{\partial}_{Y_k}((\varphi|_{Y_k})^*(\alpha_k^{(p)})) \tag{4.4}$$

c) The space $\Gamma(X\cap U, \boldsymbol{R}_X^p(R^0(G_{\bar{X}})))$ *and its lifting to* $\bigcup_{k<l} Y_{kl}$. Let $(\beta_{kl}^{(p)})_{k<l}$ an element in $\Gamma(X\cap U, \boldsymbol{R}_X^p(R^0(G_{\bar{X}})))$. By definition we write

$$(\beta_{kl}^{(p)})_{kl} = \sum_i (u_{kl}^{(i)})_{(k<l)} \otimes_{O_X} \omega_i^{(p)} \tag{4.5}$$

where $\omega_i^{(p)}$ are again C^∞ form of type $(0,p)$ on $X\cap U$ and $(u_{kl}^{(i)})_{k<l}$ are holomorphic sections on $\varphi^{-1}(U)$ of the sheaf $G_{\bar{X}}$. For each $k<l$, we can take

$$(\varphi|_{Y_{kl}})^*(\beta_{kl}^{(p)}) = \sum_i u_{kl}^{(i)}(\varphi|_{Y_{kl}})^*(\omega_i^{(p)}) \tag{4.6}$$

which is a C^∞ form on $\varphi^{-1}(U)\cap Y_{kl}$ of type $(0,p)$ with coefficients in the sheaf $\tilde{O}_{kl}$. As in Lemma 4.1 this mapping is well defined and injective and the collection of these mappings induces a mapping

$$(\varphi|_{\bigcup_{k<l} Y_{kl}})^*((\beta_{kl}^{(p)})) \equiv ((\varphi|_{Y_{kl}})^*(\beta_{kl}^{(p)}))_{k<l} \tag{4.7}$$

so that we obtain an element of

$$\bigoplus_{k<l}\Gamma(\varphi^{-1}(U)\cap Y_{k,l}, \boldsymbol{R}^p(k,l))$$

and we have

LEMMA 4.2. – *The mapping* $(\varphi|_{\bigcup_{k<l} Y_{kl}})^*$ *defined by* (4.7) *induces a well defined and injective mapping*

$$(\varphi|_{\bigcup_{k<l} Y_{kl}})^*: \Gamma(X\cap U, \boldsymbol{R}_X^p(R^0(G_{\bar{X}}))) \to \bigoplus_{k<l}\Gamma(\varphi^{-1}(U)\cap Y_{kl}, \boldsymbol{R}^p(k,l)).$$

Moreover

$$(\varphi|_{\cup Y_{kl}})^* \circ \bar{\partial}_X = \bigoplus_{k<l} \bar{\partial}_{Y_{kl}} \circ (\varphi|_{Y_{kl}})^*. \tag{4.8}$$

Finally the image

$$(\varphi|_{\cup Y_{kl}})^*\left(\Gamma(X\cap U, \boldsymbol{R}_X^p(R^0(G_{\bar{X}})))\right)$$

is contained in the kernel of the natural jet difference mapping

$$\bigoplus_{k<l}\boldsymbol{R}^p(k,l) \to \bigoplus_{k,l,m}\boldsymbol{R}^p(k,l,m)$$

of Section 3.1 *formula* (3.6).

The last statement of this Lemma 4.2 results from two facts: the definition of the jet difference morphism and the fact that if

$$(\beta_{kl}^{(p)})_{k<l} \in \Gamma(X\cap U, \boldsymbol{R}_X^p(R^0(G_{\bar{X}}))$$

is has the form (4.5) where the various $(u_{kl}^{(i)})_{k<l} \in \Gamma(\varphi^{-1}(U), G_{\bar{X}})$ and are killed by the jet difference morphism

$$\bigoplus_{k<l} \widetilde{O}_{kl} \to \bigoplus_{k,l,m} \widetilde{O}_{klm}$$

by definition of $G_{\bar{X}}$.

d) Definition of a germ of differential form on S at a point in X. We are now ready to define a germ of a C^∞ differential form of type $(0,p)$ at a point $m \in X$. Let U be a small open neighborhood of m in S.

DEFINITION. – *We call π a C^∞ form of type $(0,p)$ on U to be a collection*

$$\pi = \left(\bigoplus_{k=1}^{N} \pi_k\right) \oplus \overline{\omega}$$

which is a section on $\varphi^{-1}(U)$ of $\boldsymbol{R}^p(O_{\bar{S}})$ with the following property:

(P) *if we take $J^{(p)}(\pi) \in \boldsymbol{R}^p(O_{\bar{S}}/J_{\bar{X}})$, then*

$$(4.9)\qquad \begin{aligned} (J^{(p)}(\pi))_k &= j_k(\pi_k) \in \Gamma(X \cap U, \boldsymbol{R}_X^p(R^0(k))) \\ (J^{(p)}(\pi))_{k<l} &= (j_{kl}(\overline{\omega}_k)) \in \Gamma(X \cap U, \boldsymbol{R}_X^{p-1}(R^0(G_{\bar{X}}))) \end{aligned}$$

(Here $\boldsymbol{R}^\bullet(O_{\bar{S}})$ and $J^\bullet$ are defined by (3.12) and (3.13) respectively).

This definition makes sense, because of Lemmas 4.1 and 4.2 which give a meaning to the fact that a jet on Y_k of a form (resp. a collection of jets on Y_{kl} of form) belong to the space $\Gamma(X \cap U, \boldsymbol{R}_X^p(R^0(k))$ (resp. $\Gamma(X \cap U, \boldsymbol{R}_X^{p-1}(R^0(G_{\bar{X}}))))$.

In other words, a $C^\infty(0,p)$ form at a point $m \in X$, is basically a C^∞ section on $\varphi^{-1}(U)$ of $R^p(O_{\bar{S}})$ such that the corresponding jet $J^{(p)}$ in $\boldsymbol{R}^p(O_{\bar{S}}/J_{\bar{X}})$ is a tensor product of global holomorphic sections on $\varphi^{-1}(U)$ of the sheaves $\widetilde{O}_k$ and $G_{\bar{X}}$ with C^∞ forms of type $(0,p)$ on X.

NOTATION. – *We denote $\Lambda_{S,m}^p$ the germs of $C^\infty(0,p)$ forms on S at m.*

Then it is clear that the collection of all $\Lambda_{S,m}^p$ is a sheaf on S and moreover using the same kind of partition of unity as in [1] Section 3.1 we have:

LEMMA 4.3. – *The sheaves Λ_S^p are fine sheaves on S.*

Finally we also have

LEMMA 4.4. – *We have a natural jet morphism*

$$(4.10)\qquad J_X^\bullet : \Lambda_S^\bullet \to \bigoplus_{k=1}^{N} \boldsymbol{R}_X^\bullet(R^0(k)) \oplus \boldsymbol{R}_X^{\bullet-1}(R^0(G_{\bar{X}}))\,.$$

PROOF. – This statement is exactly the basic property (4.9) of the definition of $\Lambda_S^\bullet$ at a singular point and the fact that the liftings are well defined and injective.

Moreover, we have

LEMMA 4.5. – *Let m be a singular point on X. We have a commutative diagramm:*

$$(4.11)\qquad \begin{array}{ccc} \Lambda_{S,m}^\bullet & \xrightarrow{J_X^\bullet} & \bigoplus_{k=1}^N \boldsymbol{R}_X^\bullet(R^0(k))_m \oplus \boldsymbol{R}_X^{\bullet-1}(R^0(G_{\tilde X}))_m \\ \downarrow & & \\ \varphi_*(\boldsymbol{R}^\bullet(O_{\tilde S}))_m & \xrightarrow{J^\bullet} & \varphi_*(\boldsymbol{R}^\bullet(O_{\tilde S}/J_{\tilde X}))_m . \end{array}$$

Here $J^\bullet$ is defined as in formula (3.13), $J_X^\bullet$ is defined as in (4.10). The left vertical morphism is the inclusion. The right vertical morphism is given by Lemmas 4.1 and 4.2.

e) Definition of the $\bar\partial^{(p)}$. Call $\bar\partial^\bullet : \boldsymbol{R}^\bullet(O_{\tilde S}) \to \boldsymbol{R}^{\bullet+1}(O_{\tilde S})$ the differential of the resolution $\boldsymbol{R}^\bullet(O_{\tilde S})$ (induced by the diagonal construction (3.12)).

LEMMA 4.6. – $\bar\partial^\bullet$ *induces a differential*

$$\bar\partial^\bullet : \Lambda_S^\bullet \to \Lambda_S^{\bullet+1}$$

and with this differential $J_X^\bullet$ *becomes a morphism of complexes*

$$(4.12)\qquad \begin{array}{ccc} \Lambda_S^\bullet & \xrightarrow{J_X^\bullet} & \bigoplus_{k=1}^N \boldsymbol{R}_X^\bullet(R^0(k)) \oplus \boldsymbol{R}_X^{\bullet-1}(R^0(G_{\tilde X})) \\ \bar\partial^\bullet \downarrow & & \downarrow \hat{\bar\partial}^\bullet \\ \Lambda_S^{\bullet+1} & \xrightarrow{J^{\bullet+1}} & \bigoplus_{k=1}^N \boldsymbol{R}_X^{\bullet+1}(R^0(k)) \oplus \boldsymbol{R}_X^\bullet(R^0(G_{\tilde X})) \end{array}$$

where the vertical right $\hat{\bar\partial}^\bullet$ *has been defined in* (3.23).

PROOF. – To prove this fact, we first remark that the lifting of Lemmas 4.1 and 4.2 induce an injective morphism of complexes

$$(4.13)\qquad \bigoplus_{k=1}^N \boldsymbol{R}_X^\bullet(R^0(k)) \oplus \boldsymbol{R}_X^{\bullet-1}(R^0(G_{\tilde X})) \to \varphi_*(\boldsymbol{R}^\bullet(O_{\tilde S}/J_{\tilde X}))$$

obtained by associating to $(\alpha_k^\bullet) \oplus (\beta_{kl}^{\bullet-1})$ the

$$(4.14)\qquad ((\varphi|_{Y_k})^*(\alpha_k^\bullet)) \oplus (\varphi|_{UY_{kl}}(\beta_{kl}^{\bullet-1})) \oplus 0 \oplus 0 \oplus 0 \ldots$$

where the O denotes the 0 section in the $\varphi_*(\bigoplus_{|K|=r} \boldsymbol{R}^\bullet(K))$ for $r \geqslant 3$. The fact that it is a morphism of complexes is also deduced from the fact that $(\varphi|_{UY_{kl}}(\beta_{kl}^{\bullet-1}))$ is an element in the image of the natural morphism

$$\bigoplus_k \boldsymbol{R}^\bullet(k) \to \bigoplus_{k<l} \boldsymbol{R}^\bullet(k, l)$$

and thus, it gives O in $\bigoplus_{k,l,m} \boldsymbol{R}^\bullet(k, l, m)$ (Lemma 4.2).

Then the proof of lemma (4.6) is deduced from the following facts

(i) $J^\bullet : \boldsymbol{R}^\bullet(O_{\bar{S}}) \to \boldsymbol{R}^\bullet(O_{\bar{S}}/J_{\bar{X}})$ is a morphism of complexes (Theorem 3.2);

(ii) we have the injective morphism of complexes (4.13);

(iii) we have the commutative diagram (4.11).

4.2. *Definition of the spectral resolution* $\boldsymbol{R}^\bullet(O_S)$.

a) Definition of $\boldsymbol{R}^\bullet(O_S)$. We come back to the sheaves C^p defined as in (3.19). These sheaves are supported on X. We define

$$\boldsymbol{R}^\bullet(O_S) = \Lambda_S^\bullet \oplus C^\bullet \tag{4.15}$$

and we define also a differential $\overline{D}^\bullet$ as follows

$$\overline{D}^p : \Lambda_S^p \oplus C^p \to \Lambda_S^{p+1} \oplus C^{p+1}$$

by saying that we associate to

$$(\pi_k) \oplus (\overline{\omega}_{kl}) \oplus C^{(p)} \in \Lambda_S^p \oplus C^p$$

the element

$$\overline{D}^{(p)}((\pi_k) \oplus (\overline{\omega}_{kl}) \oplus C^{(p)}) = \bar{\partial}^{(p)}((\pi_k)_k \oplus (\overline{\omega}_{kl})_{k<l}) \oplus \delta^{(p)}((j_{kl}(\overline{\omega}_{kl}))_{k<l} \oplus C^{(p)}) \tag{4.16}$$

where $\delta^{(q)}$ is the coboundary part of $\hat{\overline{D}}^{(p)}$ in (3.24) and by (4.9), $(j_{kl}(\overline{\omega}_{kl}))_{k<l}$ is in $\boldsymbol{R}_X^{p-1}(R^0(G_{\bar{X}}))$.

b) The jet morphism.

LEMMA 4.7. – *We have a morphism of complexes*

$$J^\bullet : \boldsymbol{R}^\bullet(O_S) \to \boldsymbol{R}^\bullet(\varphi_*(O_{\bar{X}}/J_{\bar{X}})) \tag{4.17}$$

defined as follows: the morphism:

$$J^\bullet : \Lambda_S^\bullet \oplus C^\bullet \to \bigoplus_{k=1}^N \boldsymbol{R}_X^\bullet(R^0(k)) \oplus \boldsymbol{R}_X^\bullet(R^0(G_{\bar{X}})) \oplus C^\bullet$$

is defined to be

$$J^\bullet(\pi^\bullet \oplus C^\bullet) = J_X^\bullet(\pi^\bullet) \oplus C^\bullet \tag{4.18}$$

where $J_X^\bullet$ *has been defined by* (4.10) *in Lemma* 4.4.

It is clear by construction and by Lemma 4.6 and the definition of $\overline{D}^\bullet$ that we have a morphism of complexes.

c) Finally, it is clear that the $\boldsymbol{R}^\bullet(O_S)$ *are fine sheaves.*

DEFINITION. – *We call* $\boldsymbol{R}^\bullet(O_S)$ *the spectral complex of* O_S.

4.3. *Relations between the various resolutions.*

LEMMA 4.8. – *There is a natural morphism of complexes* $\mu^\bullet$ *obtained by forgetting the* $C^\bullet$ *part of* $\boldsymbol{R}^\bullet(O_S)$

$$\mu^\bullet : \boldsymbol{R}^\bullet(O_S) \to \varphi_*(\boldsymbol{R}^\bullet(O_{\tilde{S}}))\,. \tag{4.19}$$

PROOF. – Let $\alpha \in \boldsymbol{R}^\bullet(O_S)$ of the type

$$\alpha = \pi \oplus C \in \Lambda_S^\bullet \oplus C^\bullet\,.$$

By definition π is in $\varphi_*(R^\bullet(O_{\tilde{S}}))$ and we define

$$\mu^\bullet(\pi \oplus C) = \pi\,.$$

It is clear that μ commute with the $\bar{\partial}$, i.e.

$$\mu^\bullet(\overline{D}^\bullet(\pi \oplus C)) = \mu^\bullet(\bar{\partial}^\bullet(\pi) \oplus \delta^\bullet(\bar{\omega} \oplus C)) = \bar{\partial}^\bullet(\pi)$$

$$= \bar{\partial}^\bullet(\mu^\bullet(\pi \oplus C))\,.$$

Now we can relate in a clear way, the relations between the various complexes $\boldsymbol{R}^\bullet(\ldots)$. This is the content of the following theorem.

THEOREM 4.1. – *We have a commutation diagram of morphisms of complexes*

$$\begin{array}{ccc} \boldsymbol{R}^\bullet(O_S) & \xrightarrow{\mu^\bullet} & \varphi_*(\boldsymbol{R}^\bullet(O_{\tilde{S}})) \\ {\scriptstyle J^\bullet}\downarrow & & {\scriptstyle J^\bullet}\downarrow \\ \boldsymbol{R}^\bullet(\varphi_*(O_{\tilde{S}}/J_{\tilde{X}})) & \xrightarrow{\nu^\bullet} & \varphi_*(\boldsymbol{R}^\bullet(O_{\tilde{S}}/J_{\tilde{X}})) \end{array} \tag{4.20}$$

PROOF. – $\mu^\bullet$ has been explained in Lemma 4.8 and the left vertical $J^\bullet$ in Lemma 4.7. The right vertical $J^\bullet$ comes from (3.13) (Theorem 3.2) and $\nu^\bullet$ is constructed as follows. We start from

$$(\alpha_k^{(p)}) \oplus (\beta_{kl}^{(p-1)}) \oplus C^{(p)} \in \bigoplus_k \boldsymbol{R}_X^p(R^0(k)) \oplus \boldsymbol{R}_X^{p-1}(R^0(G_{\bar{X}})) \oplus C^p$$

we forget the component $C^{(p)}$ and lift the $(\alpha_k^{(p)}) \oplus (\beta_{kl}^{(p-1)})$ through the injection morphism (4.13) (i.e. through $(\varphi|_{Y_k})^*$ and $(\varphi|_{Y_{kl}})^*$) to an element of $\varphi_*(\boldsymbol{R}^p(O_{\bar{S}}/J_{\bar{X}}))$ (using also the embedding (4.14)).

We clearly get morphisms of complexes.

4.4. *$\boldsymbol{R}^\bullet(O_S)$ is a resolution of O_S (cases $p = 0, 1$).*

The fundamental theorem is the following

THEOREM 4.2. – *The complex $\boldsymbol{R}^\bullet(O_S)$ is a resolution of O_S by fine sheaves on S.*

In this section we prove the cases $p = 0$ and 1 which are slightly special. We shall work around $m \in X$

a) Case $p = 0$.

Let $\alpha \in \boldsymbol{R}^0(O_S)$, so that $\alpha = (\alpha_k^{(0)})$ is simply an element of $\Gamma(\varphi^{-1}(U), \bigoplus_k \Lambda_{\bar{S}}^0)$. Because α is $\overline{D}_0$ closed this implies that each $\alpha_k^{(0)}$ is $\bar{\partial}_{\bar{S}}$-closed and that all the $\alpha_k^{(0)}$ coincide. Then $(\alpha_k^{(0)})_k$ define a unique C^∞ function on $\varphi^{-1}(U)$, say $\alpha^{(0)}$, which is holomorphic. Because the analytic space S is normal, this function is in $\Gamma(U, O_S)$.

b) Case $p = 1$.

We start with $\alpha^1 \in \boldsymbol{R}^1(O_S)$ at m, which is $\hat{\overline{D}}_1$-closed and we want to prove that it is $\overline{D}_0$ closed.

1st step: image of α^1 in $\boldsymbol{R}^1(\varphi_*(O_{\bar{S}}/J_{\bar{X}}))$.

This image is $J^1(\alpha^1)$ (see diagram (4.20)) and it is $\hat{\overline{D}}_1$-closed. But $\boldsymbol{R}^\bullet(\varphi_*(O_{\bar{S}}/J_{\bar{X}}))$ is a resolution, so $J^1(\alpha^1)$ is $\hat{\overline{D}}_0$-exact or

$$J^1(\alpha^1) = \hat{\overline{D}}_0 \hat{\alpha}^0 \tag{4.21}$$

where $\hat{\alpha}^0 \in \boldsymbol{R}^0(\varphi_*(O_{\bar{S}}/J_{\bar{X}}))$.

2nd step: the image of α^1 in $\varphi_*(\boldsymbol{R}^1(O_{\bar{S}}))$ is $\bar{\partial}^0$-exact on all of $\varphi^{-1}(U)$

$$\mu^1(\alpha^1) \text{ is in } \Gamma(\varphi^{-1}(U), \boldsymbol{R}^1(O_{\bar{S}}));$$

let us take $J^1(\mu^1(\alpha^1)) \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}^1(O_{\bar{S}}/J_{\bar{X}}))$.

Because of the commutative diagram (5.20) this is also $\nu^1(J^1\alpha^1)$ and because of

(5.21) and the fct that all these morphisms are morphisms of resolutions, this is also

$$J^1(\mu^1(\alpha^1)) = \bar{\partial}^0 \nu^0(\widetilde{\alpha}^0) \qquad \nu^0(\widetilde{\alpha}^0) \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}^0(O_{\bar{S}}/J_{\bar{X}})) .$$

Now the morphism $J^\bullet: \boldsymbol{R}^\bullet(O_{\bar{S}}) \to \boldsymbol{R}^\bullet(O_{\bar{S}}/J_{\bar{X}})$ induces an isomorphism of the Dolbeault cohomology groups on $\varphi^{-1}(U)$ (Theorem 3.2 of comparison of cohomologies) so that there exists some f^0 in $\Gamma(\varphi^{-1}(U), \boldsymbol{R}^0(O_{\bar{S}}))$ with

$$\mu^1(\alpha^1) = \bar{\partial} f^0 . \tag{4.22}$$

3rd step: *α^1 is $\overline{D}^0$-exact in $\boldsymbol{R}^\bullet(O_S)$.*

Now, because we are in rank $p = 1$, by definition $\mu^1 = Id$, so that (4.22) is really

$$\alpha^1 = \bar{\partial} f^0 . \tag{4.23}$$

Take the J^1 of (4.23) and compare with (4.21), to obtain

$$\bar{\partial}^{(0)}(J^0 f^0 - \widetilde{\alpha}^0) = 0$$

which means that $J^0 f^0 - \widetilde{\alpha}^0$ is an holomorphic section of $\bigoplus_{k=1}^{N} \widetilde{O}_k$ on $\varphi^{-1}(U)$ and in particular is in $\bigoplus_{k=1}^{N} R^0(k)$. This implies, because

$$\widetilde{\alpha}^0 \in \boldsymbol{R}^0\left(\bigoplus_{k=1}^{N} R^0(k)\right) \equiv \bigoplus_{k=1}^{N} (R^0(k) \otimes_{O_X} \Lambda_X^0)$$

that

$$J^0 f^0 \in \boldsymbol{R}^0\left(\bigoplus_{k=1}^{N} R^0(k)\right).$$

This means by definition that $f^0 \in \Lambda_S^0$ in U and the definition of $\overline{D}^0 \equiv \bar{\partial}$ in degree 0, implies

$$\alpha^1 = \overline{D}^0 f^0 \qquad \text{from (4.23)} .$$

4.5. *$\boldsymbol{R}^\bullet(O_S)$ is a resolution of O_S (case $p > 1$).*

This is a little bit more intricate. We start from $\alpha^p \in \boldsymbol{R}^p(O_S)$ with $\overline{D}^p \alpha^p = 0$ on some open subset U of S which is a neighborhood of a point m in X.

1st step: *decomposition of $\alpha^{(p)}$.*

According to the definition of $\boldsymbol{R}^p(O_S)$, we can decompose α^p as follows

$$\alpha^p = [(\bigoplus_k \alpha_k^p) \oplus (\bigoplus_{k<l} \alpha_{kl}^{p-1})] \oplus a^p \in \Lambda_S^p \oplus C^p . \tag{4.24}$$

2nd step: *$J^p(\alpha^p)$ in $\boldsymbol{R}^p(\varphi_*(O_{\bar{S}}/J_{\bar{X}}))$.*

We consider by definition

$$J^p(\alpha^p) = J_X^p[(\bigoplus \alpha_k^p) \oplus (\bigoplus_{k<l} \alpha_{kl}^{p-1})] \oplus a^p$$

and because J^p is a morphism of complexes $\hat{\bar{D}}^p J^p(\alpha^p) = 0$, so that there exists a $\beta^{(p-1)} \in \boldsymbol{R}^{p-1}(\varphi_*(O_{\tilde{X}}/J_{\tilde{X}}))$ with

$$\hat{\bar{D}}^{p-1}\beta^{(p-1)} = J^p(\alpha^p). \tag{4.25}$$

We decompose

$$\beta^{(p-1)} = [\bigoplus_k \beta_k^{(p-1)} \oplus \bigoplus_{k<l} \beta_{kl}^{(p-2)}] \oplus b^{p-1} \in \bigoplus_k \boldsymbol{R}^{p-1}(R^0(k)) \oplus \bigoplus \boldsymbol{R}^{p-2}(R^0(G_{\tilde{X}})) \oplus C^{p-1}. \tag{4.26}$$

The definition of $\hat{\bar{D}}^{p-1}$ tells us that

$$\hat{\bar{\partial}}^{p-1}[\bigoplus_k \beta_{kl}^{(p-1)} \oplus \bigoplus_{k<l} \beta_{kl}^{(p-2)}] = J_X^p[(\bigoplus_k \alpha_k^p) \oplus (\bigoplus_{k<l} \alpha_{kl}^{p-1})]. \tag{4.27}$$

3rd step: image of α^p in $\boldsymbol{R}^p(O_{\tilde{S}})$.

The image by μ^p of α^p in $\boldsymbol{R}^p(O_{\tilde{S}})$ is exactly $(\bigoplus_k \alpha_k^p) \oplus (\bigoplus_{k<l} \alpha_{kl}^{p-1})$ and this is $\bar{\partial}^p$-closed (because α^p is $\bar{D}^p$-closed and $\bar{D}^p$ coincide with $\bar{\partial}^p$ on $\mu^p(\alpha^p)$). To see that $\mu^p(\alpha^p)$ is $\bar{\partial}^{p-1}$-exact in $\boldsymbol{R}^\bullet(O_{\tilde{S}})$, it is sufficient to prove that its Dolbeault cohomology class on $\varphi^{-1}(U)$ for $O_{\tilde{S}}$ is O, so that by Theorem 1.2 of comparison of cohomology of [1], it is sufficient to prove that $J^p(\mu^p(\alpha^p))$ has a Dolbeault cohomology class O on $\varphi^{-1}(U)$ for $O_{\tilde{S}}/J_{\tilde{X}}$. But because the diagram (4.20) is commutative, because of the definition of ν and of (4.27), $J^p(\mu^p\alpha^p)$ is $\bar{\partial}^p$-exact on $\varphi^{-1}(U)$ in the complex $\boldsymbol{R}^\bullet(O_{\tilde{S}}/J_{\tilde{X}})$. This means that there exists

$$\gamma^{(p-1)} \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}^{p-1}(O_{\tilde{S}}))$$

with

$$\mu^p(\alpha^p) = \bigoplus_k \alpha_k^p \oplus \bigoplus_{k<l} \alpha_{kl}^{p-1} = \bar{\partial}^{p-1}(\bigoplus_k \gamma_k^{p-1} \oplus \bigoplus_{k<l} \gamma_{kl}^{p-2}) \tag{4.28}$$

with

$$\gamma_k^{p-1} \in \Gamma(\varphi^{-1}(U), \Lambda_{\tilde{S}}^{p-1})$$

$$\bigoplus_{k<l} \gamma_{k,l}^{p-2} \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}_{\tilde{S}}^{p-2}(G_{\tilde{S}})).$$

4th step: comparison of $\gamma^{(p-1)}$ and $\beta^{(p-1)}$.

If we take the jet J^p of equation (4.28) and substract from (4.27) (skipping the identification through ν to abreviate the notations), we deduce that

$$\bar{\partial}^{(p-1)}(\nu\beta^{(p-1)} - J^{p-1}(\gamma^{p-1})) = 0$$

which implies that $\nu\beta^{p-1} - J^{p-1}(\gamma^{p-1})$ defines a cohomology class in $H^{p-1}(\varphi^{-1}(U), O_{\tilde{S}}/J_{\tilde{X}})$. Again the comparison Theorem 1.2 of [1], tells us that this class comes from a class $H^{p-1}(\varphi^{-1}(U), O_{\tilde{S}})$ so that there exists an $\omega^{p-1} \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}^{p-1}(O_{\tilde{S}}))$ which is $\bar{\partial}^{(p-1)}$-closed in $\boldsymbol{R}^{p-1}(O_{\tilde{S}})$ and a $\psi^{(p-2)} \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}^{p-2}(O_{\tilde{S}}/J_{\tilde{X}}))$ so that

$$\text{(4.29)} \qquad \begin{cases} J^{p-1}(\gamma^{p-1}) - \nu\beta^{(p-1)} = J^{p-1}(\omega^{p-1}) + \bar{\partial}^{p-2}\psi^{p-2} \\ \bar{\partial}^{(p-1)}\omega^{p-1} = 0\,. \end{cases}$$

5th step: modification of γ^{p-1}.

Let us chose $\tilde{\psi}^{(p-2)} \in \Gamma(\varphi^{-1}(U), \boldsymbol{R}^{p-2}(O_{\tilde{S}}))$ so that $J^{p-2}(\tilde{\psi}^{p-2}) = \psi^{p-2}$ and let us define

$$\rho^{p-1} = \gamma^{p-1} - \omega^{p-1} - \bar{\partial}^{p-2}\Big(\bigoplus_k \tilde{\psi}_k^{p-2} \oplus \bigoplus_{k<l} \tilde{\psi}_{kl}^{p-2}\Big).$$

It is clear that from (4.29)

$$\text{(4.30)} \qquad \begin{cases} \bar{\partial}^{p-1}\rho^{p-1} = \mu^p(\alpha^p) \\ J^{p-1}(\rho^{p-1}) = \nu\beta^{(p-1)} \\ \beta^{(p-1)} \in \bigoplus_k \boldsymbol{R}_X^{p-1}(R^0(k)) \oplus \boldsymbol{R}_X^{p-2}(R^0(G_{\tilde{X}}))\,. \end{cases}$$

6th step: resolution of the $\overline{D}^{p-1}$ equation.

Let us now define $\pi^{p-1} \in \boldsymbol{R}^{p-1}(O_S)$ as

$$\text{(4.31)} \qquad \pi^{p-1} = \rho^{p-1} \oplus b^{p-1}$$

(where ρ^{p-1} was defined in (4.30) and b^{p-1} was defined in (4.26)).

It is then clear from (4.30) that

$$\text{(4.32)} \qquad \rho^{p-1} \in \Lambda_S^{p-1}.$$

Let us now compute $\overline{D}^{p-1}\pi^{p-1}$: this has a part on Λ_S^p, which is exactly $\bar{\partial}^{p-1}(\pi^{p-1}) = \mu^p(\alpha^p)$ which is exactly what we want.

Now, to compute the other part of $\overline{D}^{p-1}(\pi^{p-1})$, we take the image of π^{p-1} in $\boldsymbol{R}^{p-1}(\varphi_*(O_{\tilde{S}}/J_{\tilde{X}}))$ by J^{p-1}: this is

$$J^{p-1}(\pi^{p-1}) = J_X^{p-1}(\rho^{p-1}) \oplus b^{p-1} = \nu\beta^{(p-1)} \oplus b^{p-1}$$

but $\nu\beta^{(p-1)} \oplus b^{p-1}$ is exactly β^{p-1} by definition of ν, so

$$J^{p-1}(\pi^{p-1}) = \beta^{p-1}.$$

Then we take the $\hat{\overline{D}}^{p-1}$ of $J^{p-1}(\pi^{p-1})$ and take the C^p part of $\hat{\overline{D}}^{p-1}J^{p-1}(\pi^{p-1}) = \hat{\overline{D}}^{p-1}\beta^{p-1} = J^p(\alpha^p)$ (see (4.25)). But the C^p part of $J^p(\alpha^p)$ is exactly a^p because J^p is the identity on the C^p part of $\boldsymbol{R}^p(O_S)$ by definition. These computations prove that

$$\overline{D}^{p-1}(\pi^{p-1}) = \alpha^p$$

where π^{p-1} is defined by (4.31) with ρ^{p-1} in Λ_S^{p-1}.

DEFINITION. – *We shall call* $\boldsymbol{R}^\bullet(O_S)$ *with the differential* $\overline{D}^\bullet$ *the spectral resolution of* O_S.

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