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Boundary Regularity for Parabolic Quasiminima (*).

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Summary. – *We prove a Wiener-type criterion for parabolic Q-minima.*

1. – Introduction.

Let $D \subset \Re^{n+1}$, $n > 2$, be an open bounded set. In this paper we establish a sufficient condition for a boundary point of D to be regular for a parabolic Q-minimum. Our criterion is analogous to the Wiener-type criterion proved by W. P. ZIEMER [Z-2] for elliptic Q-minima.

The main tools in the proof of our result are the Harnack inequalities. In fact Harnack inequalities for functions in parabolic De Giorgi classes are proved by G. WANG [Wa].

Moreover W. WIESER [Wi] proved that parabolic Q-minima belong to parabolic De Giorgi classes.

The generic point $z \in \Re^{n+1}$ will be denoted by $z = (x, t)$, with $x \in \Re^n$ and $t \in \Re$. u_x or $\nabla_x u$ will denote the gradient on x of a function $u(x, t)$.

Let $H^{0,1}(D)$ be the space $\{u \in L^2(D) \mid \nabla_x u \in L^2(D)\}$.

The existence of the parabolic Q-minima is connected to $V^2(D)$. It is the space of functions u which belong to $H^{0,1}(D)$ and such that $\|u\|_{L^2(D_t)} \in L_\infty(\Re^1)$ where $D_t =: D \cap \{z = (x, \tau) \mid \tau = t\}$. $V^2(D)$ is endowed with the norm

$$\|u\|_{V^2(D)} =: \sup_t \left(\int_{D_t} u(x, t)^2 \, dx \right)^{1/2} + \left(\int_D |\nabla_x u(x, t)|^2 \, dx \, dt \right)^{1/2}.$$

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Let $Z(D)$ be the space $\{u \in \mathcal{D}^1(D) \mid u = \operatorname{div} g,\ g \in L^2(D)\}$ endowed with the norm $\|u\|_{Z(D)} = \operatorname*{Inf}_{\substack{u = \operatorname{div} g \\ g \in L^2(D)}} \|g\|_{L^2(D)}$ and $W^1(D) = \{u \in H^{0,1}(D),\ \partial u/\partial t \in Z(D)\}$.

We denote $B_r(x_0) = \{x \in \mathfrak{R}^n,\ |x - x_0| < r\}$ if $x_0 \in \mathfrak{R}^n$ and

$$\hat{Q}_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2) \quad \text{if } z_0 = (x_0, t_0) \in \mathfrak{R}^{n+1}.$$

If $u \in W^1(D)$, $z_0 \in \partial D$ and $l \in \mathfrak{R}$, we say that

$$u(z_0) \leqslant 1 \text{ weakly} \tag{1.1}$$

if for every $k > 1$ there is $r > 0$ and a sequence $\{u_m\} \subseteq W^{1,\infty}(\mathfrak{R}^{n+1})$ such that $u_m \to u$ in $W^1(D \cap \hat{Q}_r(z_0))$ and $\operatorname{spt} \eta (u_m - k)^+ \subset D \cap \hat{Q}_r(z_0)$ whenever $\eta \in C_0^\infty(\hat{Q}_r(z_0))$. The definition of

$$u(z_0) \geqslant 1 \text{ weakly} \tag{1.2}$$

is analogous, and $u(z_0) = 1$ if both (1.1) and (1.2) hold.

Many different definitions of capacity are introduced to study the regularity problem for parabolic equations. A capacity which is «naturally» associated to the space $V^2(D)$ is

$$\operatorname{cap}_*(E) = \inf\left\{\sup_t \int_D u^2\,dx + \int_D |\nabla_x u|^2\,dx\,dt,\ u \in V^2(\mathfrak{R}^{n+1}),\ E \subset \operatorname{int}[u \geqslant 1]\right\}. \tag{1.3}$$

W. P. ZIEMER [Z-1] proved a sufficient condition for the regularity at the boundary of weak solutions of parabolic equations in term of the capacity cap_*. In analogy with the Newtonian capacity (cfr. [F.Z]), L. C. EVANS and R. GARIEPY [E.G] introduced the thermal capacity, which R. GARIEPY and W. P. ZIEMER [G.Z] proved to be strictly weaker than cap_*. In [G.Z] a Wiener criterion (only sufficient condition) is also proved for bounded weak solutions of parabolic equations, in terms of the thermal capacity.

This result improved of [Z-1].

In this paper we consider another type of parabolic capacity, which we will denote $\operatorname{cap}(E)$:

$$\operatorname{cap}(E) = \inf\left\{\int_D |\nabla_x u|^2,\ u \in C_0^\infty(\mathfrak{R}^{n+1}),\ E \subset \operatorname{int}[u \geqslant 1]\right\}. \tag{1.4}$$

Here E was a compact set.

BIROLI and MOSCO [B.M-2] showed that $\operatorname{cap}(E)$ is a Choquet capacity and, for any bounded Borel set E,

$$\operatorname{cap}(E) = \int N - \operatorname{cap}(E_t)\,dt \tag{1.5}$$

where $E_t =: E \cap \{z = (x, \tau) \mid \tau = t\}$ and N-cap denote the usual Newtonian capacity of E_t with respect to $\mathfrak{R}^n$. It is known (cf. [M]) that $N\text{-cap}(E_t) =$

$= \sup \{ \|R_1 * \mu\|_2^{-1} \,|\, \mathrm{spt}\,\mu \subset E_t,\ \mu(E_t) = 1\}$, where μ is a positive Borel measure in $\Re^n$ and $R_1(x) = |x|^{1-n}$.

If $z_0 = (x_0, t_0) \in \partial D$ we will denote, for α, $r > 0$,

$$R_\alpha(r) = B_r(x_0) \times \left(t_0 - \frac{3}{4}\alpha r^2,\ t_0 + \frac{1}{4}\alpha r^2\right),$$

$$R_\alpha^*(r) = B_r(x_0) \times \left(t_0 - \frac{9}{4}\alpha r^2,\ t_0 - \frac{5}{4}\alpha r^2\right),$$

$$\delta_\alpha(r) = \frac{\mathrm{cap}\,(R_\alpha^*(r) - D)}{\mathrm{cap}\,(R_\alpha^*(r))}.$$

We will prove that

$$\int_0^1 \exp\left(-[C\delta_\alpha(r)^{-1}]\right)\frac{dr}{r} = +\infty$$

is a sufficient condition for the regularity of a parabolic Q-minimum in z_0, where C depends on $\|u\|_{L^2(D)}$ and the structure conditions of the Q-minimum.

Obviously $\mathrm{cap}\,(E) \leqslant \mathrm{cap}_*(E)$, and, a fortiori, thermal capacity is weaker than cap.

Then our result would be refined using cap_* or thermal capacity.

We don't know if it is possible to employ these capacities.

2. – Q-minima and De Giorgi classes.

Consider a Caratheodory function

$$F = F(z, u, p): D \times \Re \times \Re^n \to \Re \tag{2.1}$$

satisfying the growth condition

$$\lambda|p|^2 - b|u|^2 - g(z) \leqslant F(z, u, p) \leqslant \mu|p|^2 + b|u|^2 + g(z) \tag{2.2}$$

where g is a nonnegative function, $g \in L^q(D)$, $q > n$ and b, λ, μ are positive constants.

DEFINITION 1. – *For $Q \geqslant 1$ we define a function $u: D \to \Re$ with $u, u_x \in L^2(D)$, to be a parabolic Q-minimum if for every $\phi \in C_0^\infty(D)$ the following inequality*

$$-\int_K u\phi'\,dz + E(u, K) \leqslant QE(u - \phi, K) \tag{2.3}$$

holds, where $K = \operatorname{spt}\phi$ *and*

$$E(w, K) = \int_K F(z, w, \nabla_x w)\, dz .$$

We can suppose that ϕ is a Lipschitz function with $\operatorname{spt}\phi \subset D$. A result of WEISER [Wi] is

PROPOSITION 1. – *Let* F *satisfy* (2.8) *and* $u: D \to \mathfrak{R}$ *be a* Q*-minimum such that* $u, u_x \in L^2(D)$. *Then* $u \in V^2(D)$.

Wieser proved also that any Q-minimum belongs to a De Giorgi class. To expound a general definition of De Giorgi class we will need some new notations: for any $\rho, \tau > 0$, $\bar{z} = (\bar{x}, \bar{t}) \in \mathfrak{R}^{n+1}$, and $k \in \mathfrak{R}$ we will denote

$$B_\rho = B_\rho(\bar{x}), \qquad Q_{\rho,\tau} = Q_{\rho,\tau}(\bar{z}) = B(\bar{x}, \rho) \times (\bar{t}, \bar{t} + \tau),$$

$$(u - k)^{\pm} = \max(\pm u \mp k, 0) \quad \text{and} \quad A^{\pm}_{k,\rho,\tau} = Q_{\rho,\tau} \cap [(u - k)^{\pm} > 0].$$

DEFINITION 2. – *A function* $u \in V^2(D)$ *belongs to a De Giorgi class* $DG_2^{\pm}(D) = DG_2^{\pm}(D, L, q)$ *if there exist some constants* $L > 0$ *and* $q > 1$ *such that, for any* $Q_{\rho,\tau} = Q_{\rho,\tau}(\bar{z}) \subset D$ *and any* $\sigma_1, \sigma_2 \in (0, 1)$, $k \geqslant 0$ *the following inequalities hold:*

$$\max_{t \in [\bar{t}, \bar{t} + \tau]} \int_{B_{(1-\sigma_1)\rho}} |(u - k)^{\pm}(t)|^2 dx \leqslant \int_{B_\rho} |(u - k)^{\pm}(\bar{t})|^2 dx + L\left\{(\sigma_1 \rho)^{-2} \int_{Q_{\rho,\tau}} |(u - k)^{\pm}|^2 dz + |A^{\pm}_{k,\rho,\tau}|^{1-1/q}\right\}, \tag{2.4}$$

$$\max_{t \in [\bar{t}, \bar{t} + (1-\sigma_2)\tau]} \int_{B_{(1-\sigma_1)\rho}} |(u - k)^{\pm}(t)|^2 dx + \int_{Q_{(1-\sigma_1)\rho,(1-\sigma_2)\tau}} |\nabla(u - k)^{\pm}|^2 dz + L\left\{[(\sigma_1\rho)^{-2} + (\sigma_2\tau)^{-1}] \int_{Q_{\rho,\tau}} |(u - k)^{\pm}|^2 dz + |A^{\pm}_{k,\rho,\tau}|^{1-1/q}\right\}. \tag{2.5}$$

The De Giorgi classes $DG_2(D)$ are defined as

$$DG_2(D) = DG_2(D, L, q) =: DG_2^+(D, L, q) \cap DG_2^-(D, L, q) \quad \text{(with the same } D, L, q).$$

The Harnack inequalities for functions in De Giorgi classes obtained by WANG [Wa] are the following:

THEOREM 1. – *Let* $u(x, t) \in DG_2(D)$, $\theta > 0$, $R > 0$, $B_R(\bar{x}) \times (\bar{t}, \bar{t} + \theta R^2) \subset D$, $\sigma \in (0,1)$. *Then for any* $p > 0$ *there exists a constant* $C > 0$ *such that*

$$\sup_{B_{\sigma R}(\bar{x}) \times (\bar{t} + (1-\sigma^2)\theta R^2, \bar{t} + \theta R^2)} u(x, t) \leq C \left\{ \fint_{B_{\sigma R}(\bar{x}) \times (\bar{t}, \bar{t} + \theta R^2)} |u^+|^p \, dx \, dt \right\}^{1/p}$$

where C *depends on* p, n, L, σ, θ.

We supposed $\fint_S v \, dx \, dt = \frac{1}{|S|} \int_S v \, dx \, dt$ *where* $|S|$ *denote the* $(n+1)$*-dimensional measure of* S.

THEOREM 2. – *Let* $u(x, t) \in DG_2(D)$, $\theta > 0$, $R > 0$, $B_R(\bar{x}) \times (\bar{t}, \bar{t} + \theta R^2) \subset D$. *Then for any* $\sigma_1, \sigma_2 \in (0, 1)$, $0 < \theta_1 < \theta_1 < \theta$ *there exist positive constants* p_0 *and* C *such that*

$$\inf_{B_{\sigma_1 R}(\bar{x}) \times (\bar{t} + \theta_2 R^2, \bar{t} + \theta R^2)} u(x, t) \geq C \left\{ \fint_{B_{\sigma_2 R}(\bar{x}) \times (\bar{t}, \bar{t} + \theta_1 R^2)} |u|^{p_0} \, dx \, dt \right\}^{1/p_0}$$

where C *depends only* n, L, θ_1, θ_2, θ, σ_1 *and* σ_2.

3. – The Wiener condition.

If $\bar{x} \in \mathfrak{R}^n$, $0 < r < R$, we will say that $\eta(x)$ is a cut-off function for the pair $(B_r(\bar{x}), B_R(\bar{x}))$ if $\eta(x) \in C_0^\infty(\mathfrak{R}^n)$, $\eta(x) = 1$ on $B_r(\bar{x})$, $\eta(x) = 0$ on $\mathfrak{R}^n - B_R(\bar{x})$, $0 \leq \eta(x) \leq 1$, $|\nabla \eta| \leq C/(R - r)$.

If $\bar{z} = (\bar{x}, \bar{t}) \in \mathfrak{R}^{n+1}$, $0 < r < R$, $0 < s < S$, we will say that $\eta(x, t)$ is a cut-off function for the pair $(Q_{r,s}(\bar{z}), Q_{R,S}(\bar{z}))$ if $\eta(x, t)$ is a Lipschitz function

$$\eta(x, t) = 1 \ \text{on} \ Q_{r,s},$$

$$\eta(x, t) = 0 \ \text{on} \ (\mathfrak{R}^{n+1} - Q_{R,S})_{t > \bar{t}} = \{(x, t) \in Q_{R,S} \,|\, t > \bar{t}\}, \qquad 0 \leq \eta(x, t) \leq 1,$$

$$|\eta'| + |\nabla \eta|^2 \leq C\left[\frac{1}{(R - r)^2} + \frac{1}{S - s}\right].$$

We observe that no conditions are imposed on η for $t \leq \bar{t}$. In fact we will always multiply η for a function depending on the only variable t and having the support inclosed in $\{t > \bar{t}\}$.

Let $z_0 = (x_0, t_0) \in \partial D$ be. If $u(z_0) \leq 1$ weakly, and $l < k$ denote $u_k = (u - k)^+$ prolonged by zero out of D and let $r > 0$ and $\{u_m\} \subset W^{1,\infty}(\mathfrak{R}^{n+1})$ such that $u_m \to u$ in $W^1(D \cap \hat{Q}_{3r}(z_0))$, $\operatorname{spt} \eta(u_m - k)^+ \subset D \cap \hat{Q}_{3r}(z_0)$ wheanever $\eta \in C_0^\infty(\hat{Q}_{3r}(z_0))$.

We observe that the function u_k is bounded on any compact set of D because of the

Harnack inequality (Theorem 1). Moreover if u_k is prolonged out of D by zero, it remains bounded also near the boundary of D, the bound depending on $\|u\|_{L^2(D)}$ and the structure conditions (2.1). Therefore for any fixed $\alpha > 0$, $u_k(2r) = \operatorname{Sup}\{u_k(z),\, z \in R_\alpha(2r)\}$ is finite for all small r. This is applied in the proof of the following theorem.

THEOREM 3. – *There exist positive constants C and p_0 depending on $\|u\|_{L^2(D)}$ and the structure conditions* (2.1) *such that, for any* $\alpha > 0$,

$$u_k(2r) - u_k(r) \geqslant C\left(\int_{R_\alpha^*(r)} [u_k(2r) - u_k(z)]^{p_0}\, dz\right)^{1/p_0}$$

PROOF. – Let $\bar{x} = x_0$, $\bar{t} = t_0 - (9/4)\alpha r^2$, $\bar{z} = (\bar{x}, \bar{t})$. Denote $v(x) = u_k(2r) - u_k(z)$ prolonged by $u_k(2r)$ out of D. We observe that $v,\, v_x \in L^2(D \cup Q_{2r,(5/4)\alpha r^2}(\bar{z}))$.

We want to prove that v belongs to a De Giorgi class $DG_2^-(D \cup Q_{2r,(5/4)\alpha r^2}(\bar{z}))$. We prove that v verifies (2.4) and (2.5) for any $Q_{\rho,\tau}(\bar{z})$ with $\rho < 2r$, $\tau < (5/4)\alpha r^2$. For $0 < h < u_k(2r)$ consider $(v-h)^-$ prolonged out of D by zero. Let $\varphi_{m,\varepsilon,h}(x,t) = -(\eta^2(v_m - h)^-_{} \aleph^\alpha_{[t_1,\bar{t}+\tau]})_\varepsilon$ where η is a cut-off function for the pair $(B_{2s_1r}(x_0), B_{2s_2r}(x_0))$, $0 < s_1 < s_2 < 1$, $\bar{t} < t_1 < \bar{t} + \tau$, $0 < \tau < \alpha(tr)^2$ and, for any $l > 0$, $\tau_1 < \tau_2$

$$\aleph^l_{\tau_1,\tau_2}(t) = \begin{cases} 0 & \text{if } t \leqslant \tau_1 - l \text{ or } t \geqslant \tau_2 + l\,, \\ 1 & \text{if } \tau_1 \leqslant t \leqslant \tau_2\,, \\ 1 + \dfrac{t-\tau_1}{l} & \text{if } \tau_1 - l \leqslant t \leqslant \tau_1\,, \\ 1 - \dfrac{t-\tau_2}{l} & \text{if } \tau_2 \leqslant t \leqslant \tau_2 + l\,, \end{cases}$$

denote the piecewise linear approximation of the characteristic function $\aleph_{[\tau_1,\tau_2]}$. ψ_ε denote a «time»-mollification of ψ. So $\varphi \in W_0^{1,2}(D \cap R_\alpha(2r))$. Inserting φ as test function in (2.3) and letting to the limit $m \to +\infty$ and $\alpha, \varepsilon \to 0$, we obtain that v satisfies the «$-$»-part of (2.4) with $\rho = 2s_2r$, $\sigma_1 = 1 - s_1/s_2$. We avoid the calculus which allows us to this issue. It can be obtained following the same line of reasoning of Theorem 3.1 of [Z-2] and Theorem 4.1 of [Wi]. Let now $\eta = \eta(x,t)$ in the test function φ be a cut-off function for the pair $(Q_{2s_1r,\tau_1}(\bar{z}), Q_{2s_2r,\tau_2}(\bar{z}))$, $0 < \tau_1 < (5/2)\alpha(s_1r)^2$, $0 < \tau_2 < (5/2)\alpha(s_2r)^2$, $\tau_2 < \tau_1$. Inserting φ as test function in (2.3), we obtain in this case that v satisfies the «$-$»-part of (2.5).

We can so affirm that v belongs to a De Giorgi class $DG_2^-(D \cup Q_{2r,(5/4)\alpha r^2}(\bar{z}))$. Therefore, applying the Harnack inequality (Theorem 2) to the function v, with $R = r$, $\theta_1 = 1$, $\theta_2 = (3/2)\alpha$, $\theta = (5/2)\alpha$, $\sigma_1 = \sigma_2 = 1$, Theorem 3 is proved.

We recall the following Lemma:

LEMMA 1 (see Lemma 3.2 of [Z-2]). – *Let $u \in W^{1,2}(B)$ where B is an open ball of $\Re^n$ of radius r, and let μ be a probability measure supported in $B \cap [u=0]$. Then, if $k>0$,*

$$k|[|u| \geq k]| \leq Cr\int_B |\nabla u| + Cr^n \int_B (R_1 * \mu)|\nabla u|. \tag{3.1}$$

Let $u_k(2r)$ be defined as at the beginning of Theorem 3 and let $k_j = u(2r) - 2^{-j}u_k(2r)$, $j=0,1,\dots$ where $u(2r) = \sup_{R_\alpha(2r)} u$. Denote $A^{\pm}(k_j, r) = R_\alpha^*(r) \cap [(u - k_j)^{\pm} > 0]$.

LEMMA 2. – *There is a constant $C>0$, depending on $\|u\|_{L^2(D)}$ and the structure conditions* (2.1) *such that for any positive integer s,*

$$sr^n \operatorname{cap}(R_\alpha^* - D) \leq Csr^{n-2}|A^-(k_s, r)| + Cr^{n+1/2}\left[1 + \frac{r^a}{u(2r)-k_s}\right] \tag{3.2}$$

where $a = 1-(n+1)/2q$.

PROOF. – Inserting in (2.3) the test function $\varphi_{m,\varepsilon,h} = -(\eta^2(u_m - k_j)^+ \aleph^h_{[t_1,\bar t+\tau_1]})_\varepsilon$ where $\{u_m\} \subset W^{1,\infty}(\Re^{n+1})$, $u_m \to u$ in $W^{1,\infty}(D)$ and $\operatorname{spt}(u_m - h)^- \subset D$, η is a cut-off function for the pair $(Q_{r,\tau_1}(\bar z), Q_{2r,\tau_2}(\bar z))$, $\bar z = (\bar x, \bar t)$, $\bar x = x_0$, $\bar t = t_0 - (9/4)\alpha r^2$, $0<\tau_1<\alpha r^2$, $0<\tau_2<(13/4)\alpha r^2$, $\tau_1<\tau_2$, we obtain

$$\int_{A^+(k_j,r)} |\nabla u|^2 \leq C[r^{n-1}(u(2r)-k_j)^2 + |A^+(k_j,2r)|^{1-1/q}] \leq \tag{3.3}$$

$$\leq Cr^{n-1}(k_{j+1}-k_j)^2\left[1+\frac{r^a}{u(2r)-k_j}\right]^2 \quad \text{where } s \geq j+1,\ a = 1-\frac{n+1}{2q}.$$

For any $t \in \Re$ let now μ_t be a positive measure supported in $(R_\alpha^*(r) - A^+(k,r))_t$ such that

$$\int |R_1 * \mu_t|^2 \leq (4\cdot N - \operatorname{cap}(R_\alpha^*(r) - D))_t^{-2}. \tag{3.4}$$

Denote $N - \operatorname{cap}(R_\alpha^*(r) - D)_t = \gamma_{r,t}$. Apply Lemma 1 to the function $v(x) = (u-k_j)^+ + k_j$ with $k = k_j$ for the level t (we observe that $v(x) > k_j \Leftrightarrow u(x) > k_j$). We obtain

$$k_j|A^+(k_j,r)_t| \leq C\left(\int_{A^+(k_j,r)_t} |\nabla u|^2\right)^{1/2}\left(r|A^+(k_j,r)_t|^{1/2} + Cr^n\left(\int_{A^+(k_j,r)_t} |(R_1*\mu_t)|^2\right)^{1/2}\right). \tag{3.5}$$

We will suppose $s \geq j$, so $A^+(k_s, r) \subset A^+(k_j, r)$. Then

$$k_j |A^+(k_s, r)_t| \leq C\left(\int_{A^+(k_j, r)_t} |\nabla u|^2\right)^{1/2} \cdot \left(r^{1+n/2} + r^n \left(\int_{A^+(k_j, r)_t} (R_1 * \mu_t)^2\right)^{1/2}\right) \leq$$

$$\leq C\left(\int_{A^+(k_j, r)_t} |\nabla u|^2\right)^{1/2} \cdot (r^{1+n/2} + r^n \gamma_{r,t}^{-1}) \leq C\left(\int_{A^+(k_j, r)_t} |\nabla u|^2\right)^{1/2} \cdot r^{n+1/2} \cdot \gamma_{r,t}^{-1}.$$

Hence

$$k_j |A^+(k_s, r)_t| \gamma_{r,t} \leq C\left(\int_{A^+(k_j, r)_t} |\nabla u|^2\right)^{1/2} \cdot r^{1+n/2}.$$

Integrating in t and taking into account (3.3) we have

$$k_j \int |A^+(k_s, r)_t| \gamma_{r,t}\, dt \leq C r^{n+1/2} (k_{j+1} - k_j)\left[1 + \frac{r^a}{u(2r) - k_s}\right].$$

Summing over j from 0 to $s+1$ implies

$$\left(\sum_{j=0}^{s+1} k_j\right) \int |A^+(k_s, r)_t| \gamma_{r,t}\, dt \leq C r^{n+1/2} \left(\sum_{j=0}^{s+1} k_{j+1} - k_j\right)\left[1 + \frac{r^a}{u(2r) - k_s}\right].$$

We have $\sum_{j=0}^{s+1} k_j > s \cdot u_k(2r)$ and $\sum_{j=0}^{s+1} (k_{j+1} - k_j) < u_k(2r)/2$, then

$$s \int |A^+(k_s, r)_t| \gamma_{r,t}\, dt \leq C r^{n+1/2}\left[1 + \frac{r^a}{u(2r) - k_s}\right]. \tag{3.6}$$

We have $|A^+(k_s, r)_t| \geq C(|B_r| - |A^-(k_s, r)_t|)$, then

$$s \int |A^+(k_s, r)_t| \gamma_{r,t}\, dt \geq sC \int r^n \gamma_{r,t}\, dt - sC \int |A^-(k_s, r)_t| \gamma_{r,t}\, dt.$$

Hence, taking into account (3.6) and that $\gamma_{r,t} \leq C r^{n-2}$ with C independent of t and r, Lemma 2 is proved.

THEOREM 4. – *Let $u \in W^{1,2}(D)$ be a Q-minimum and let $z_0 \in \partial D$. Then there is a constant $C > 0$, depending on $\|u\|_{L^2(D)}$ and the structure conditions* (2.1) *such that, if*

$$\int_0^1 \exp(-[C \delta_z(r)^{-1}]) \frac{dr}{r} = +\infty \tag{3.7}$$

for some $\alpha > 0$, *and* $u(z_0) \leqslant 1$ *weakly, then*

$$\limsup_{\substack{z \to z_0 \\ z \in D}} u(z) \leqslant 1 .$$

PROOF. – The proof is analogous to that of Theorem 3.4 of [Z-2].

Let $\limsup_{\substack{z \to z_0 \\ z \in D}} u(z) \geqslant 1$. Then there are numbers $k > 1$ and $\theta > 0$ such that

$$u_k(2r) \geqslant \theta > 0 \quad \text{for all small } r > 0 . \tag{3.8}$$

Let $\omega(r) = u_k(2r) - u_k(r)$ and observe that $\int_0^1 \omega(r)(dr/r) < +\infty$.

From Theorem 3 follows that there exists $p_0 > 0$ such that

$$\omega(r)^{p_0} \geqslant (2^{-j} u_k(2r))^{p_0} \frac{|A^-(k_j, r)|}{|R_\alpha^*(r)|} \quad \text{for any integer } j \geqslant 1 .$$

Thus, from (3.8), $|A^-(k_j, r)|/|R_\alpha^*(r)| \leqslant (\theta^{-1} 2^j \omega(r))^{p_0}$.

Taking into account Lemma 2 and that $\operatorname{cap}(R_\alpha^*(r)) \approx Cr^n$,

$$s\delta(r) \leqslant Cr^{1/2}\left[1 + \frac{r^a}{u(2r) - k_s}\right] + Csr^n(\theta^{-1} 2^s \omega(r))^{p_0} \leqslant \tag{3.9}$$

$$\leqslant Cr^{1/2}\left[1 + \frac{2^s r^a}{u_k(2r)}\right] + Csr^n(\theta^{-1} 2^s \omega(r))^{p_0} .$$

Let $s = s(r)$ be the largest integer such that

$$s^1(\theta^{-1} 2^s \omega(r))^{p_0} \leqslant \frac{1}{2} . \tag{3.10}$$

Then $2^s(\theta^{-1} 2^s \omega(r))^{p_0} \geqslant 1/2^{p_0+1}$ and therefore

$$s(r)(p_0 + 1) \log 2 + p_0 \log[\omega(r)] \geqslant \log \frac{1}{2}\left(\frac{\theta}{2}\right)^{p_0}$$

hence, dividing by $p_0 + 1$,

$$s(r) + \log[\omega(r)] \geqslant \log M , \qquad M = M(\theta, p_0) . \tag{3.11}$$

So, dividing (3.9) by s and taking into account of (3.10), we have

$$\delta(r) \leqslant Cs(r)^{-1} \tag{3.12}$$

with $C = C(\theta, p_0)$. From (3.7), (3.10) and (3.12) we obtain $\int_0^1 \omega(r)\, dr/r = +\infty$. This yields a contradiction, and Theorem 4 is proved.

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