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Cohesive Categories and Manifolds (*).

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Sunto. – *Le strutture ottenibili per incollamento di « spazi elementari », come le varietà, i fibrati, le varietà fogliettate, possono essere definite da « atlanti di incollamento » e, formalmente, come categorie arricchite su opportune categorie ordinate.*

0. – Introduction.

0.1. Glueing structures, for instance manifolds, fibre bundles, vector bundles or foliations, can be obtained by patching together a family (U_i) of suitable « elementary spaces » by means of partial bijections $u_j^i\colon U_i \to U_j$ expressing the glueing conditions and forming a sort of « glueing atlas », instead of the more usual atlas of charts.

The goal of this paper is to treat these structures as enriched categories over « totally cohesive » categories, that is ordered categories having binary meets and arbitrary joins of pairwise « compatible » morphisms. The morphisms of these « generalized manifolds » are obtained as « compatible » modules between enriched categories, which can be composed precisely because of the existence of compatible joins. The condition of Cauchy-completeness corresponds to the maximality of the glueing atlas; however, since our morphisms are modules, the procedure of Cauchy-completion just produces an isomorphic object.

This approach to glueing structures is clearly related to Ehresmann's one, based on pseudogroups of transformations (e.g. see [E1, E2]).

On the other hand, our setting inscribes in Lawvere's remark that interesting mathematical structures not only organize in categories, but *are* themselves categories, enriched over some suitable base: a *monoidal* category as in Lawvere's original formulation [La], or more generally a *bicategory* as in BETTI [Be]. The bases we actually use are suitable ordered categories (very particular bicategories).

Last, this work is closely related with the notion of « glueing data », considered

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by KASANGIAN and WALTERS [KW] in an involutive ordered category with arbitrary (instead of compatible) suprema of parallel maps.

A short version of some of these results, in a more particular setting, appeared in [G4].

0.2. A *cohesive* category $\mathcal{A}$ is equipped with an order relation $f \leqslant g$ and a *compatibility* (or *linking*) relation $f!g$, both concerning *parallel* morphisms (same domain and codomain), consistent with composition and satisfying some further axioms (2.1). In particular the relation ! is reflexive and symmetrical, generally non transitive; binary *linked* meets $f \wedge g$ have to exist. A *totally cohesive* category, moreover, has arbitrary *linked* joins $\bigvee \varphi$ (of sets φ of parallel, pairwise compatible maps).

The paradigmatic example is the category $\mathcal{S}$ of sets and *partial* mappings, where $f \leqslant g$ means that f is a restriction of g, while $f!g$ means that f and g agree wherever they are both defined.

Analogously for the category $\mathcal{T}$ of topological spaces and continuous partial mappings, defined on open subsets; or the category $\mathcal{C}^r$ of *open euclidean spaces* (i.e. open subspaces of some $\mathbb{R}^n$) and partial mappings of class C^r, defined on open subsets.

This category $\mathcal{C}^r$ contains the elementary spaces we want to glue in order to get C^r-manifolds, together with the morphisms for the glueing. More precisely, the glueing-morphisms will live in the inverse subcategory $\operatorname{Inv} \mathcal{C}^r$ of open euclidean spaces and *partial* C^r-diffeomorphisms (between open subsets of domain and codomain), which in our setting replaces Ehresmann's pseudogroup of (everywhere defined) C^r-diffeomorphisms between open euclidean sets; $\operatorname{Inv} \mathcal{C}^r$ is an *inverse* category, meaning that each morphism u has a unique generalized inverse $\tilde{u}$, with: $u\tilde{u}u = u$ and $\tilde{u}u\tilde{u} = \tilde{u}$. Notice, however, that we need the whole category $\mathcal{C}^r$ to construct the morphisms of manifolds.

0.3. It may be remarked that, in these examples, the linking relation is *determined* by the order: indeed $f!g$ iff f and g have a common upper bound. Such cohesive categories are here called *link-filtered*.

However inverse categories, which form an important class of categories having a canonical cohesion structure, need not be so, and we think useful to keep our present definition of cohesive category, based on independent, if related, order and linking.

0.4. In the previous examples the cohesion structure is also determined by the endomorphisms $\leqslant 1$ (the partial identities), which we call *projections*:

(1) $\quad f \leqslant g \quad$ iff $f = ge \quad$ (for some projection e),

(2) $\quad f!g \quad$ iff $f = fe$, $g = ge'$, $fe' = ge \quad$ (for some projections e, e').

Moreover every morphism f has a *support* $\boldsymbol{e}(f)$: the least projection e, of the domain of f, verifying $f = fe$, and:

$$f \leqslant g \quad \text{iff } f = g \cdot \boldsymbol{e}(f) , \tag{3}$$

$$f \,!\, g \quad \text{iff } f \cdot \boldsymbol{e}(g) = g \cdot \boldsymbol{e}(f) . \tag{4}$$

These facts suggest the more particular notions of *prj-cohesive* and *e-cohesive* category (prj-category and e-category, for short). Notice that these structures are determined by the order, but need *not* be link-filtered: e.g. consider the cohesive subcategory $\mathcal{S}_0$ of $\mathcal{S}$ consisting of those partial mappings whose definition-set has no more than (say) five elements.

Every prj-cohesive category $\boldsymbol{A}$ has a canonical inverse subcategory, Inv $\boldsymbol{A}$ (5.7). Every dominical category, in the sense of Di Paola-Heller [He, Di, DH] and more generally every p-category in the sense of Rosolini [Ro] is e-cohesive (3.8).

0.5. Cohesive categories present two interesting notions of « (co)completion »: the *totally cohesive completion*, concerning linked joins and the *glueing completion*, concerning the « glueing of manifolds ». The first construction is achieved by considering equivalence classes of linked sets of parallel morphisms (cohesive completion theorem, 2.7-2.8).

As to the second, a *manifold* (U_i, u^i_j) in the prj-category $\boldsymbol{A}$ is here an enriched category over $\boldsymbol{A}$ (i.e.: $u^i_i = 1$ and $u^j_k \cdot u^i_j \leqslant u^i_k$, for all i, j, k) satisfying a symmetry condition: $u^i_j \cdot u^j_i \cdot u^i_j = u^i_j$, which forces the glueing morphisms u^i_j into the inverse subcategory Inv $\boldsymbol{A}$. Its *glueing*, if existing, is the lax colimit.

If $\boldsymbol{A}$ is prj-cohesive, with linked joins, the category Mf $\boldsymbol{A}$ of manifolds over $\boldsymbol{A}$ and « linked » modules between them, with the usual matrix composition, is the *glueing completion* of $\boldsymbol{A}$: it is glueing-complete and every prj-functor $\boldsymbol{A} \to \boldsymbol{B}$, preserving linked joins, with values in a glueing-complete prj-category, extends uniquely to Mf $\boldsymbol{A}$ (glueing completion theorem, 7.7).

The e-categories $\mathcal{S}$ and $\mathcal{T}$ are already complete in both regards. Instead the e-category $\mathcal{C}^r$ is just totally cohesive: its glueing completion Mf $\mathcal{C}^r$ yields C^r-manifolds as « glueing atlases ». The topological realization of a manifold is given by the functor Mf $\mathcal{C}^r \to \mathcal{T}$ extending the natural embedding $\mathcal{C}^r \to \mathcal{T}$, by the universal property of the completion itself; it transforms a manifold (U_i, u^i_j) into its glueing in $\mathcal{T}$, i.e. the quotient of the sum-space $\coprod U_i$ modulo the obvious equivalence relation produced by the glueing morphisms.

0.6. Analogously, fibre bundles and vector bundles can be considered as manifolds over the e-cohesive categories $\mathcal{B}$ and $\mathcal{V}$ of trivial fibre or vector bundles, with suitable partial mappings. The topological realization can now be constructed into a (glueing) category whose objects are general fibrations, or also Serre fibrations.

A unified formal treatment of differentiable manifolds and fibre bundles clearly presents advantages. For instance, the trivial tangent bundle functor $T: \mathcal{C}^r \to \mathcal{V}$ ($r \geqslant 1$), transforming the open set U of $\mathbb{R}^n$ into the trivial vector bundle $U \times \mathbb{R}^n$, automatically extends, by the glueing completion theorem, to the tangent bundle functor $\mathrm{Mf}\, \mathcal{C}^r \to \mathrm{Mf}\, \mathcal{V}$ for C^r-manifolds.

0.7. In a different context, the category $L^\infty(\boldsymbol{a}, \boldsymbol{Ban})$ of Banach spaces with spectral measures (on a fixed Boolean σ-algebra $\boldsymbol{a}$) and bounded measurable operators between the former, has a natural prj-cohesive structure which will be sketched here in 1.5 and studied in a subsequent work [G5]. It does not consist of partial mappings and its projections are idempotent operators.

0.8. Chapter 1 contains a more detailed exposition of the examples and motivations recalled above; it also treats *compositive* joins of morphisms in an order category (1.7) and the type of « cardinal bound » ϱ we are going to use to restrict completeness conditions (1.8).

Cohesive, prj-cohesive and e-cohesive categories are introduced and studied in ch. 2-4, together with the ϱ-cohesive completion (with regard to suprema of linked ϱ-sets of parallel morphisms).

Ch. 5 is concerned with inverse categories $\boldsymbol{K}$ and their canonical cohesion structure; the inverse ϱ-glueing completion $\varrho\mathrm{IMf}\,\boldsymbol{K}$ of $\boldsymbol{K}$ is constructed in ch. 6. The ϱ-glueing completion $\varrho\mathrm{Mf}\,\boldsymbol{A}$ of an e-cohesive category $\boldsymbol{A}$ is derived from this result in ch. 7.

Fibre bundles, vector bundles and foliations are briefly considered in ch. 8. Finally, ch. 9 contains the proof of some completion theorems.

Capital script letters, like $\mathcal{S}$ or $\mathcal{T}$, usually denote categories of partial mappings.

0.9. Last, some words on the connections of this setting with C. Ehresmann's one. I thank Mrs. A. EHRESMANN for her suggestions on this point.

An ordered category $(\boldsymbol{C}, \prec)$ in Ehresmann's sense (let us say *o-category*, to avoid confusion) abstracts the usual category $\boldsymbol{Set}$ of small sets and (total) mappings, provided with the following order on morphisms: $(f: X \to Y) \prec (f': X' \to Y')$ if $X \subset X'$, $Y \subset Y'$, and f is a restriction of f'. Thus, in an o-category, $f \prec f'$ does not imply that f and f' are parallel; instead, if f, f' are parallel morphisms and $f \prec f'$, it is assumed that $f = f'$.

These o-categories $\boldsymbol{C}$, with suitable regularity conditions, should correspond to e-cohesive categories $\boldsymbol{A}$ with splitting of projections (and possibly some further conditions). Given $\boldsymbol{C}$, construct $\boldsymbol{A} = P(\boldsymbol{C})$ as the category of « partial maps » of $\boldsymbol{C}$, obtained by spans $X \leftarrow \cdot \to Y$ whose first morphism i is an « inclusion » ($i \prec 1_X$). Given $\boldsymbol{A}$, let $\boldsymbol{C}$ be the subcategory of « total maps » u of $\boldsymbol{A}$ ($\boldsymbol{e}(u) = 1$).

Thus, the present glueing completion theorem, restricted to totally cohesive *e-categories*, probably reduces to Ehresmann's « théorème d'élargissement complet d'un foncteur local » [E2]. The connections at the level of cohesive or prj-cohesive categories should be more involved, if possible.

From our viewpoint, ordered categories in the present sense allow to treat manifolds as enriched categories over 2-categories, and their partial mappings as modules between enriched categories. Moreover, this setting seems to be more adapted to applications to measurable operators, using the prj-cohesive Banach categories $L^\infty(\boldsymbol{a}, \boldsymbol{Ban})$, which are not e-cohesive (1.5, [G5]).

1. – Examples and preliminary notions.

1.1. *Cohesion in the category of partial mappings.* Let $\mathcal{S}$ be the category of small sets and *partial mappings* (i.e. univocal correspondences), composed as correspondences. We write Def f the subset of the domain of f on which f is defined.

$\mathcal{S}$ is an ordered category, via:

(1) $f \leqslant g$ if f and g are parallel maps and f coincides with g on Def f,

iff f and g are parallel maps and the graph of f is contained in the graph of g.

Moreover $\mathcal{S}$ is provided with a *proximity relation* (1) which will be called *linking* (or *compatibility*) and written $f!g$:

(2) $f!g$ if f and g are parallel maps and coincide on $\mathrm{Def} f \cap \mathrm{Def}\, g$.

These two relations, order and linking, are closely related. For instance, if $\varphi \subset \mathcal{S}(X, Y)$ is a *linked set* of parallel maps ($f!f'$ for all $f, f' \in \varphi$), the supremum $f_1 = \bigvee \varphi$ and (for $\varphi \neq \emptyset$) the infimum $f_0 = \bigwedge \varphi$ exist: they are given, respectively, by the set-theoretical union and intersection of the graphs; moreover they are compositive, i.e. preserved by composition. It may be noticed that $\bigvee \varphi$ exists iff φ is linked (every set of maps having an upper bound is so), while $\bigwedge \varphi$ always exists for $\varphi \neq \emptyset$; however it is easy to check that the meet is compositive precisely when φ is linked.

1.2. *Cohesion and projections.* A *projection* of X in $\mathcal{S}$ is any « partial identity » $e\colon X \to Y$, i.e. any endomorphism $e \leqslant 1_X$. The projections of X form an ordered set Prj X which is isomorphic to the Boolean algebra $\mathcal{P}X$ of the parts of X, via $e \mapsto \mathrm{Def}\, e$.

The projections of $\mathcal{S}$ are determined by the order; conversely, they determine

(1) We mean: a binary relation between parallel maps, reflexive, symmetrical and consistent with composition.

both the order and the linking relation:

(1) $f \leqslant g$ iff there is some projection e such that $f = ge$,

(2) $f \,!\, g$ iff there are projections e, e' such that $f = fe$, $g = ge'$, $fe' = ge$,

in the latter case the pair (e, e') will be called a *resolution* of f and g, and we have: $f \wedge g = fe' = ge$.

Last, each partial mappings $f\colon X \to Y$ has a least projection $\boldsymbol{e}(f) \in \operatorname{Prj} X$ such that $f = fe$, namely the partial identity on $\operatorname{Def} f$; it will be written $\boldsymbol{e}(f)$ and called the *support* of f. Clearly:

(3) $$f \leqslant g \quad \text{iff } f = g \cdot \boldsymbol{e}(f) ,$$

(4) $$f \,!\, g \quad \text{iff } f \cdot \boldsymbol{e}(g) = g \cdot \boldsymbol{e}(f) .$$

In the following (ch. 2, 3) we shall introduce the notion of cohesive category $(\boldsymbol{A}, \leqslant, !)$, of prj-cohesive category $(\boldsymbol{A}, \operatorname{Prj})$, of e-cohesive category $(\boldsymbol{A}, \boldsymbol{e})$. Every prj-cohesive category $\boldsymbol{A}$ has an associated cohesion structure defined as in (1)-(2), or more simply as in (3)-(4) if $\boldsymbol{A}$ is also e-cohesive.

1.3. *Some categories of continuous partial mappings.* Consider the category $\mathcal{C}$ of small topological sets and continuous partial mappings, defined on open subsets. Consider also the subcategory $\mathcal{C}^r$ of $\mathcal{C}$ whose objects are the open subspaces of all $\mathbb{R}^n$ $(n \in \mathbb{N})$, with partial mappings of class C^r defined on open subsets; here and in the following, $r \in \mathbb{N} \cup \{\infty, \omega\}$ and class C^ω means analytic.

If $\boldsymbol{A}$ is any of these categories, the (faithful) forgetful functor $U\colon \boldsymbol{A} \to \mathcal{S}$ creates an e-cohesive structure on $\boldsymbol{A}$, provided with *arbitrary linked joins* and *binary linked meets* (1.1), distributive with respect to the former. The projections of the object X form an ordered set $\operatorname{Prj} X$, isomorphic to the locale (2) $\mathcal{O}(X)$ of the open sets of X.

Other examples, related to fibre bundles, vector bundles and foliations, will be considered in ch. 8.

1.4. *Cohesion for measurable functions.* Let X be a measurable space and Y a normed one. The following very simple cohesion structure on the *set* Y:

(1.1) $$y \leqslant y' \Leftrightarrow (y = 0 \text{ or } y = y') , \qquad y \,!\, y' \Leftrightarrow (y \leqslant y' \text{ or } y' \leqslant y) ,$$

yields, by the usual «pointwise» argument, a cohesion structure on the normed space $L^\infty(X, Y)$ of bounded measurable mappings from X into Y:

(2) $$f \leqslant f' \Leftrightarrow (\forall x \in X\colon fx \leqslant gx) \Leftrightarrow (\forall x \in X\colon fx \neq 0 \Rightarrow fx = gx) ,$$

(3) $$f \,!\, f' \Leftrightarrow (\forall x \in X\colon fx \,!\, gx) \Leftrightarrow (\forall x \in X\colon fx \neq 0 \neq gx \Rightarrow fx = gx) ,$$

(2) I.e., a complete lattice in which binary meets distribute over arbitrary joins.

which is finitely cohesive, i.e. provided with finite linked joints. It is easy to guess that the universal completion of $L^\infty(X, Y)$ with respect to σ-joins of linked sets is the space $M(X, Y)$ of all measurable mappings from X to Y: indeed any such map $f\colon X \to Y$ is the linked join of the increasing sequence of bounded measurable mappings $f_n = e_n \circ f$, where $e_n\colon Y \to Y$ is the following measurable (non linear) mapping: $e_n(y) = y$ if $\|y\| \leqslant n$, $e_n(y) = 0$ otherwise.

It may also be noticed that the category $\mathcal{S}$ considered in 1.1 is equivalent to the category $\mathcal{S}'$ of pointed sets and pointed (everywhere defined) mappings; writing 0 the base point, the cohesion structure of the hom-sets $\mathcal{S}(X, Y)$ may be described as above.

1.5. *Cohesion for operators.* The category $L^\infty(\boldsymbol{a}, \boldsymbol{Ban})$ of *bounded measurable operators* in the category $\boldsymbol{Ban}$ of Banach spaces, on the Boolean σ-algebra $\boldsymbol{a}$, has for objects all the pairs (X, E) where X is a Banach space and $E\colon \boldsymbol{a} \to \boldsymbol{Ban}\,(X)$ is a (bounded) σ-additive spectral measure with values in X (see [DS], XV.2.3-4). A morphism $S\colon (X, E) \to (Y, F)$ is a bounded linear mapping $S\colon X \to Y$ commuting with the measures E, F: $S\cdot E(a) = F(a)\cdot S$, for all $a \in \boldsymbol{a}$.

This category has a natural prj-cohesive structure, defined as in 1.2.1-2, the projections of the object (X, E) being the endomorphisms $E(a)$, for $a \in \boldsymbol{a}$. The structure is not complete with regard to linked joins: its σ-cohesive completion may be concretely described as the category $M(\boldsymbol{a}, \boldsymbol{Ban})$ of closed densely defined, measurable operators, as it will be shown in [G5].

1.6. *Cohesion for inverse categories.* A category $\boldsymbol{K}$ is inverse if every morphism $a\colon A \to A'$ has a unique *generalized inverse* $\tilde{a}\colon A' \to A$, with $a\tilde{a}a = a$ and $\tilde{a}a\tilde{a} = \tilde{a}$. For example: the category $\mathcal{I} = \text{Inv}\,\mathcal{S}$ of sets and partial bijections, or the category $\text{Inv}\,\mathcal{T}$ of topological spaces and partial homeomorphisms between open subspaces (every prj-cohesive category $\boldsymbol{A}$ has an associated invese subcategory, $\text{Inv}\,\boldsymbol{A}$, as shown in 5.7).

The inverse category $\boldsymbol{K}$ has a *canonical cohesion structure*:

$$(1) \qquad a \leqslant b \quad \text{iff } a = b\cdot\tilde{a}a\,, \text{ iff } a = a\tilde{a}\cdot b\,, \text{ iff } a = a\tilde{b}a\,, \ \dots,$$

$$(2) \qquad a\,!\,b \quad \text{if } (a\cdot\tilde{b}b = b\cdot\tilde{a}a \text{ and } b\tilde{b}\cdot a = a\tilde{a}\cdot b)\,,$$

which is not prj-cohesive, at least in the present sense: the linking relation has to be described by *double resolutions*, on domain and codomain (5.4), or equivalently by supports (on domain) and cosupports (on codomain): $\boldsymbol{e}(a) = \tilde{a}a$, $\boldsymbol{e}^*(a) = a\tilde{a}$. This structure will be studied in ch. 5, and its glueing completion in ch. 6.

1.7. *Compositive joins and meets* (3). Let $\boldsymbol{A}$ be an *ordered* category: $\boldsymbol{A}$ is provided with an order relation $\leqslant$ on parallel maps, which is assumed to be reflexive, transi-

(3) The two last sections of ch. 1 contain some preliminary tools.

tive, antisymmetrical and compositive. We say that a set $\alpha \subset \boldsymbol{A}(A, B)$ of parallel maps has *compositive* join (or union) $\hat{a} = \bigvee \alpha$ if:

(1) for all the morphisms $x: A' \to A$, $y: B \to B'$ we have: $y\hat{a}x = \bigvee_{a \in \alpha} yax$, in the ordered set $\boldsymbol{A}(A', B')$;

in particular, $\hat{a}$ is the supremum of α in the ordered set $\boldsymbol{A}(A, B)$. Compositive joins have the following elementary properties:

a) *associativity*: if $a = \bigvee a_i$ $(i \in I)$, and for every i, $a_i = \bigvee a_{ij}$ $(j \in J_i)$ are compositive joins, then $a = \bigvee a_{ij}$ $(i \in I,\ j \in J_i)$ is so;

b) *composition*: if $a = \bigvee a_i$ is compositive, $yax = \bigvee (ya_ix)$ is so;

c) if $a = \bigvee a_i$ is compositive and for every i, $a_i \leqslant a_i' \leqslant a$, then $a = \bigvee a_i'$ is a compositive join.

Dually one defines compositive meets (or intersections), enjoying dual properties. A category provided with binary compositive meets (of parallel pairs of maps):

$$y(a \wedge a')x = yax \wedge ya'x\ , \tag{2}$$

is « the same » as a category enriched over the closed category of $\wedge$-semilattices; it will be called a *semilatticed* category.

The stronger *cartesian compositive property*:

$$(b \wedge b') \cdot (a \wedge a') = ba \wedge b'a'\ , \tag{3}$$

will appear in prj-cohesive categories, with respect to linked meets (3.3.3); a category provided with binary meets, compositive in this stronger sense, is the same as a category enriched over the category of $\wedge$-semilattices, provided with the *monoidal* structure of cartesian product (instead of the *closed* structure considered above).

1.8. *Smallness and cardinal bounds.* A universe $\mathcal{U}$ is fixed throughout; a *small* set is any set belonging to $\mathcal{U}$. A $\mathcal{U}$-*category* $\boldsymbol{A}$ is assumed to have each object and each hom set $\boldsymbol{A}(A, B)$ belonging to $\mathcal{U}$: e.g. the category of small sets, of small groups and so on; it is *small* if also its object-set belongs to $\mathcal{U}$. All the categories we explicitly use are assumed to be $\mathcal{U}$-categories, except of course some « very large » 2-category of categories, like the 2-category $\varrho\mathbf{CH}$ of ϱ-cohesive $\mathcal{U}$-categories mentioned in 2.6.

A *section* of cardinals will be a set ϱ of small cardinals verifying:

a) $1 \in \varrho$, if $x, y \in \varrho$ then $x \cdot y \in \varrho$,

b) if $x \in \varrho$ and $0 \neq y \leqslant x$, then $y \in \varrho$.

Thus ϱ is either $\{1\}$, or $\{0, 1\}$, or an interval $[0, x[$ or $[1, x[$ where x is any small infinite cardinal, or the set Ω (Ω') of all small (non-null) cardinals. If ϱ is infinite, it is also closed with respect to the sum. In particular, we write $f = [0, \aleph_0[$, the set of finite cardinals, and $\sigma = [0, \aleph_1[= [0, \aleph_0]$. We also write ϱ' the section of non-null cardinals of ϱ.

A *ϱ-set* is a small set whose cardinal belongs to ϱ; the section $\{0, 1\}$ will be shortened to 0 in prefixes.

A ϱ-lattice will be a (small) ordered set having join and meet of all its ϱ-subsets; thus 0-lattices are ordered sets with supremum and infimum, f-lattices are lattices with supremum and infimum, Ω-lattices are the complete lattices. Ordinary lattices coincide with f'-lattices and ordered sets with $\{1\}$-lattices.

Analogously one can consider ϱ-distributive lattices, Boolean ϱ-algebras, ϱ-locales and so on. An ordinary locale is the same as an Ω-locale.

A section ϱ is fixed throughout this paper.

2. – Cohesive categories.

2.1. DEFINITION. – A *cohesive category* will be a category $\boldsymbol{A}$ provided with two binary relations, the *order* $\leqslant$ and the *linking* (or *compatibility*) relation !, both on parallel morphisms, verifying:

(CH.1) $\leqslant$ is an order of categories (reflexive, transitive, antisymmetrical and consistent with composition);

(CH.2) ! is reflexive, symmetrical and consistent with composition in the strong sense (4): if $a!a'$ and $b!b'$ are consecutive, then $ba!b'a'$;

(CH.3) if $a \leqslant a'$, $b \leqslant b'$ and $a'!b'$ then $a!b$;

(CH.4) if $a!b$, the (*linked*) meet $a \wedge b$ exists and is compositive in $\boldsymbol{A}$.

The notion of cohesive category is selfdual.

Clearly, if $a, b \leqslant c$ then $a!b$ (CH.2, 3); we say that the cohesive category $\boldsymbol{A}$ is *link-filtered* if the converse holds too:

(1) $\qquad a!b \quad$ iff a and b have a common upper bound,

in which case the linking relation is determined by the order. A link-filtered cohesive category is clearly the same as an ordered category provided with binary

(4) As ! is not transitive, this consistency is stronger than « left and right consistency with composition with one map ».

filtered meets, consistent with composition. The cohesive category $\mathcal{S}_0$ considered in 0.4 is not link-filtered.

Every category has a *discrete* cohesive structure, with $a \leqslant b$ iff $a!b$ iff $a = b$. On the other hand, a cohesive category with *trivial* linking ($a!a'$ iff a and a' are parallel) is the same as a semilatticed category, i.e. a category enriched over the closed category of semilattices (1.7).

In this chapter, $\boldsymbol{A}$ will always be a cohesive category.

2.2. *Linked joins of morphisms.* A *linked* (or *compatible*) *set* α of $\boldsymbol{A}$ is any set of parallel morphisms such that $a!a'$ for all $a, a' \in \alpha$; if also β is so, $\alpha!\beta$ will mean that α and β are parallel and $a!b$ for all $a \in \alpha$, $b \in \beta$; or equivalently, that $a \cup \beta$ is linked. Any subset which has an upper bound is linked.

Say that the set α (of parallel morphisms) has *linked join* if:

a) α has a compositive join $\bigvee \alpha$ (in particular, it is a linked set),

b) for each linked morphism b ($b!a$, for all a in α), $(\bigvee \alpha)!b$ and $(\bigvee \alpha) \wedge b$ is the compositive join of $\{a \wedge b | a \in \alpha\}$ (which is linked, by (CH.3)).

It is easy to see that linked joins verify properties similar to those considered in 1.7 *a*)-*c*) for compositive joins.

2.3. DEFINITION. – A *ϱ-localic cohesive* category (or *ϱ-cohesive* category, for short) will be a cohesive category $\boldsymbol{A}$ such that every linked ϱ-set of parallel morphisms has linked join.

Equivalently, $\boldsymbol{A}$ has to satisfy:

(CH.5ϱ) every linked ϱ-set $\alpha \subset \boldsymbol{A}(A, B)$ has join $\bigvee \alpha$, compositive in $\boldsymbol{A}$; linked binary meets distribute over joins of linked ϱ-sets:

$$(\bigvee \alpha) \wedge b = \bigvee_{a \in \alpha} (a \wedge b)\,, \quad \text{if } \alpha!b\,. \tag{1}$$

The necessity of (CH.5ϱ) being obvious, assume that it holds. $(\bigvee \alpha)!b$ is trivial for $\varrho \subset \{0, 1\}$; otherwise the set $\beta = \alpha \cup \{b\}$ is a linked ϱ-set and $\bigvee \alpha$, $b \leqslant \bigvee \beta$, hence $\bigvee \alpha!b$. Moreover the meets $a \wedge b$ ($a \in \alpha$) form a linked ϱ-set (by (CH.3) or by (1) itself), hence their join has to be compositive.

In particular we have *cohesive*, 0-*cohesive*, *f-cohesive* (or *finitely cohesive*), *σ-cohesive*, *totally cohesive categories* when, respectively: $\varrho = \{1\}, \{0, 1\}, f, \sigma, \Omega$ (1.8). The categories $\mathcal{S}$, $\mathcal{C}$, $\mathcal{C}^r$ are totally cohesive (1.1-3); $L^\infty(\boldsymbol{a}, \boldsymbol{Ban})$ is just finitely cohesive (1.5).

2.4. *Elementary properties.* Let $\boldsymbol{A}$ be ϱ-cohesive. A non-empty ϱ-set $\alpha \subset \boldsymbol{A}(A, B)$ of parallel morphisms is linked iff it has some upper bound (e.g. $\bigvee \alpha$).

If α and β are parallel linked ϱ-sets of morphisms and $\alpha!\beta$, then $\bigvee\alpha!\bigvee\beta$ and:

$$(\bigvee\alpha)\wedge(\bigvee\beta)=\bigvee a\wedge b \qquad (a\in\alpha,\ b\in\beta)\ ; \tag{1}$$

further, if α and γ are consecutive linked ϱ-sets of morphisms then $\gamma\alpha=\{ca|a\in\alpha, c\in\gamma\}$ is again a linked ϱ-set (CH.2) and:

$$\bigvee(\gamma\alpha)=\bigvee\gamma\cdot\bigvee\alpha\,. \tag{2}$$

2.5. *Characterizations.* A cohesive category $\boldsymbol{A}$ is 0-cohesive (resp. f-cohesive, σ-cohesive) iff it satisfies the first (resp. the first two, the following three) conditions:

(CH.5a) for all objects A, B the set $\boldsymbol{A}(A, B)$ has a minimum 0^A_B (the *zero* morphism from A to B), compositive in $\boldsymbol{A}$: the composition of a zero morphism with any other is a zero morphism;

(CH.5b) every pair $a, b\in\boldsymbol{A}(A, B)$ of linked morphisms ($a!b$) has join $a\vee b$, compositive in $\boldsymbol{A}$; linked binary meets distribute over joins of linked pairs;

(CH.5c) every increasing sequence (a_n) in $\boldsymbol{A}(A, B)$, obviously linked, has join $\bigvee a_n$, compositive in $\boldsymbol{A}$; linked binary meets distribute over increasing countable joins.

The proof reduces to calculate the join of a countable linked set $\alpha=\{a_n\colon n\in\mathbb{N}\}$ in $\boldsymbol{A}(A, B)$ by means of an increasing sequence of finite suprema $b_n=\bigvee\{a_k\colon k\leqslant n\}$.

Moreover, if $2\in\varrho$ (i.e. $f'\subset\varrho$), every ϱ-cohesive category is link-filtered (2.1). Thus an ordered category $\boldsymbol{X}$ is ϱ-cohesive, with linking relation expressed by 2.1.1, iff:

(C.1) $\boldsymbol{X}$ has compositive filtered binary meets,

(C.2ϱ) ϱ-sets of parallel maps, filtered in $\boldsymbol{X}$, have compositive join; filtered binary meets distribute over these ϱ-joins.

2.6. *Cohesive functors and transformations.* A *ϱ-cohesive functor* $F\colon \boldsymbol{A}\to\boldsymbol{B}$ will be a functor between ϱ-cohesive categories which preserves order, linking, linked binary meets and linked ϱ-joins. For $\varrho\subset\sigma$ there are characterizations of such functors, similar to those in 2.5.

A *ϱ-cohesive transformation* $\varphi\colon F\to G\colon \boldsymbol{A}\to\boldsymbol{B}$ will be a natural transformation between ϱ-cohesive functors.

A *ϱ-cohesive subcategory* of the ϱ-cohesive category $\boldsymbol{A}$ is any subcategory $\boldsymbol{A}'$ which is closed under linked binary meets and linked ϱ-joins; then $\boldsymbol{A}'$, provided with the induced order and linking relation, is ϱ-cohesive as well as the inclusion $\boldsymbol{A}'\to\boldsymbol{A}$.

A *ϱ-cohesive embedding* F: $\boldsymbol{A} \to \boldsymbol{B}$ will be a ϱ-cohesive functor, injective on the objects and reflecting the order and linking relations. Then F is also faithful and $F(\boldsymbol{A})$ is a ϱ-cohesive subcategory of $\boldsymbol{B}$, isomorphic to $\boldsymbol{A}$.

The concrete 2-category $\varrho\mathbf{CH}$ of ϱ-cohesive $\mathfrak{U}$-categories (1.8), ϱ-cohesive functors and natural transformations is easily seen to be *2-complete* (i.e. to have all small indexed 2-limits). The cohesion structure on a cartesian product $\Pi \boldsymbol{A}_i$ of ϱ-cohesive categories is quite obvious.

2.7. THEOREM (*the ϱ-cohesive completion*). – Every cohesive category has a universal cohesive embedding η: $\boldsymbol{A} \to \varrho c\boldsymbol{A}$ in a ϱ-cohesive category, preserving the existing linked ϱ-joins: the *ϱ-cohesive completion* of $\boldsymbol{A}$.

The universality of η means that: for each cohesive functor F: $\boldsymbol{A} \to \boldsymbol{B}$ preserving the existing linked ϱ-joins, with values in a ϱ-cohesive category, there exists precisely one ϱ-cohesive functor G: $\varrho c\boldsymbol{A} \to \boldsymbol{B}$ extensing F ($F = G\eta$).

PROOF. – See 9.1-2.

2.8. *A description of the ϱ-cohesive completion.* The ϱ-cohesive completion $\varrho c\boldsymbol{A}$ may be constructed in the following way.

First form the category $\mathcal{T}_\varrho \boldsymbol{A}$ having the same objects as $\boldsymbol{A}$ and morphisms α: $A \to B$ given by the linked ϱ-sets $\alpha \subset \boldsymbol{A}(A, B)$, with composition:

(1) $$\beta\alpha = \{ba \mid a \in \alpha,\ b \in \beta\}\ .$$

Consider on the category $\mathcal{T}_\varrho \boldsymbol{A}$ the preorder $\prec$:

(2) $$\alpha \prec \beta \quad \text{iff } \alpha ! \beta \text{ and } \forall a \in \alpha,\ a = \bigvee_{b \in \beta} (a \wedge b) \text{ (linked join)}\,,$$

and the quotient category:

(3) $$\varrho c\boldsymbol{A} = \mathcal{T}_\varrho \boldsymbol{A}/\sim\,,$$

where $\sim$ is the congruence associated to $\prec$.

The order and the linking relation in $\varrho c\boldsymbol{A}$ are given by:

(4) $$[\alpha] \leqslant [\beta] \quad \text{iff } \alpha \prec \beta\,, \qquad [\alpha]\,!\,[\beta] \quad \text{iff } a ! \beta \text{ as linked sets of } \boldsymbol{A}\,,$$

independently from the choice of representatives.

Linked meets and linked ϱ-joins are calculated in $\varrho c\boldsymbol{A}$ by the following formulas:

(5) $$[\alpha] \wedge [\beta] = [\{a \wedge b \mid a \in \alpha,\ b \in \beta\}]\,,$$

(6) $$\bigvee \Sigma' = [\cup \Sigma]\,,$$

where $[\alpha]![\beta]$, Σ is any linked ϱ-set of ϱ-sets of $\boldsymbol{A}$ ($\alpha!\alpha'$, for all $\alpha, \alpha' \in \Sigma$) and

$$\Sigma' = \{[\alpha] | \alpha \in \Sigma\} .$$

In particular:

(7) $\bigvee \alpha = [\alpha]$ (in $\varrho c\boldsymbol{A}$, for any linked ϱ-set α of $\boldsymbol{A}$).

The universal embedding $\eta\colon \boldsymbol{A} \to \varrho c\boldsymbol{A}$ takes the object A into itself and the morphism a into the equivalence class of $\{a\}$.

The σ-cohesive completion of a *finitely cohesive* category $\boldsymbol{A}$ may be given a simpler description, since for each morphism a in $\sigma c\boldsymbol{A}$ there is an *increasing* sequence of parallel morphisms $(a_n)_{n\in\mathbb{N}}$ of $\boldsymbol{A}$ such that $a = [\{a_n : n \in \mathbb{N}\}]$ (see 2.5). This case will be considered in [G5].

2.9. *Density.* If $\boldsymbol{A}$ is a cohesive subcategory of a ϱ-cohesive category $\boldsymbol{B}$, with the same objects, the embedding $F\colon \boldsymbol{A} \to \boldsymbol{B}$ is the ϱ-cohesive completion of $\boldsymbol{A}$ iff:

i) F preserves the existing linked ϱ-joins,

ii) $\boldsymbol{A}$ is *ϱ-dense* in $\boldsymbol{B}$: for every morphism b in $\boldsymbol{B}$ there is a linked ϱ-set α in $\boldsymbol{A}$ whose join in $\boldsymbol{B}$ is b.

Indeed, the necessity of these conditions being obvious, assume that they hold: we must show that the ϱ-cohesive functor $G\colon \varrho c\boldsymbol{A} \to \boldsymbol{B}$ extending F is an isomorphism of cohesive categories. Since it is surjective, by ii), it suffices to show that it reflects the order (hence it is injective) and the linking relation.

Let α and β be parallel, linked ϱ-sets in $\boldsymbol{A}$. If $G[\alpha] \leqslant G[\beta]$ in $\boldsymbol{B}$, for every $a \in \alpha$: $a = Ga \leqslant G[\beta] = \bigvee_b b$, hence $a = \bigvee_b a \wedge b$, linked join in $\boldsymbol{B}$. Since a and all $a \wedge b$ are in $\boldsymbol{A}$, the linked join holds in $\boldsymbol{A}$, which proves that $[\alpha] \leqslant [\beta]$ in $\varrho c\boldsymbol{A}$.

Last, if $G[\alpha]!G[\beta]$ in $\boldsymbol{B}$: $\bigvee \alpha = \bigvee \beta$ in $\boldsymbol{B}$, whence $a!b$ in $\boldsymbol{B}$ for every $a \in \alpha$ and $b \in \beta$, and the same holds in the cohesive subcategory $\boldsymbol{A}$; in other words, $[\alpha]![\beta]$ in $\varrho c\boldsymbol{A}$.

3. – Prj-cohesive and e-cohesive categories.

As we have seen in ch. 1, cohesion structures are often defined by assigning for each object a set of commuting idempotent endomorphisms, which will be called « projections ». This yields the notions of prj-cohesive and e-cohesive category, the latter being stronger than the former.

3.1. DEFINITION. – A *prj-cohesive* category (or prj-category for short) will be a category $\boldsymbol{A}$ provided, for every object A, with a set $\mathrm{Prj}\, A \subset \boldsymbol{A}(A)$ of endomorphisms of A (the *projections* of A) so that:

(PCH.1) every identity is a projection; if e is a projection, $ee = e$; if e and f are parallel projections, $ef = fe$ is a projection;

(PCH.2) if $a\colon A \to B$ is in $\boldsymbol{A}$ and $f \in \operatorname{Prj} B$, there exists some $e \in \operatorname{Prj} A$ such that: $fa = ae$.

Thus $\operatorname{Prj} A$ is a commutative idempotent unitary subsemigroup of $\boldsymbol{A}(A)$ and a 1-semilattice in its own right, with $e \wedge f = ef = fe$, $e \leqslant f$ iff $e = ef$ $(= fe)$ and maximum 1_A.

A *prj-cohesive* functor $F\colon \boldsymbol{A} \to \boldsymbol{B}$ is a functor between prj-cohesive categories which preserves projections.

3.2. *The cohesion structure.* The prj-category $\boldsymbol{A}$ has the following *associated* order and linking relations (which make $\boldsymbol{A}$ into a cohesive category, as it is proved n the following section):

(1) $a \leqslant b$ if there is a projection e such that $a = be$ (note: $ae = a$),

2) $a!b$ if there are projections e, f such that: $a = ae$, $b = bf$, $af = be$; in this case we say that (e, f) is a *resolution* of the linked pair (a, b).

This order extends the canonical order of projections: if $e = f\cdot g$ in $\operatorname{Prj} A$, then $ef = fgf = fg = e$. An endomorphism $a \in \boldsymbol{A}(A)$ is a projection iff $a \leqslant 1_A$: thus all (the existing) joins and *non-empty* meets [5] of projections are the same in $\operatorname{Prj} A$ or in $\boldsymbol{A}(A)$. The identity 1_A is maximal in $\boldsymbol{A}(A)$: if $a \geqslant 1_A$ then $1 = ae$ hence,

$$e = ae\cdot e = ae = 1 \quad \text{and} \quad a = 1\,.$$

It will also be useful to remark that the projection e in (1) and (2) may be replaced with each projection e_0 such that $e_0 \leqslant e$ and $a\cdot e_0 = a$.

3.3. PROPOSITION. – The prj-category $\boldsymbol{A}$ with the associated order and linking relations is a cohesive category (2.1). If (e, e') is a resolution of the linked pair (a, b), the meet of the latter is:

(1) $$a \wedge b = ae' = be\,;$$

---

[5] Warning: the empty set has infimum 1 in $\operatorname{Prj} A$, but generally (e.g. in the examples of ch. 1) no infimum in $\boldsymbol{A}(A)$: the latter has no greatest element.

moreover, if in the diagram (2) $a!b$ and $c!d$:

$$A \underset{b}{\overset{a}{\rightrightarrows}} B \underset{d}{\overset{c}{\rightrightarrows}} C\,, \tag{2}$$

then the cartesian compositive property (3) holds (see 1.7):

$$ca \wedge db = (c\wedge d)\cdot(a\wedge b)\,. \tag{3}$$

Every set ε of parallel projections is linked; the linked meet of two parallel projections is their meet in $\mathrm{Prj}\, A$: $e\wedge f = ef = fe$, which is therefore compositive in $\boldsymbol{A}$.

A functor between prj-cohesive categories is cohesive iff it is prj-cohesive, iff it preserves the order.

PROOF. – The letters e, f, e', f' ... always denote projections.

For the first two axioms (CH.1-2) the only non-trivial checkings concern the composition. Let be given the diagram (2).

If $a\leqslant b$ and $c\leqslant d$, let: $a = be$, $c = df$; by (PCH.2) there is a projection e' such that $fa = ae'$, and: $db\cdot ee' = d\cdot be\cdot e' = dae' = dfa = ca$.

Instead, if $a!b$ and $c!d$, let: $a = ae$, $b = be'$, $ae' = be$, $c = cf$, $d = df'$, $cf' = df$. By (PCH.2) there are projections $\hat{e}$, $\hat{e}'$ such that: $fa = a\hat{e}$, $f'b = b\hat{e}'$; we want to show that $(e\hat{e}, e'\hat{e}')$ is a resolution of the pair (ca, db). Indeed: $ca\cdot e\hat{e} = c\cdot a\hat{e} = cfa = ca$, and analogously: $db\cdot e'\hat{e}' = db$; last:

$$ca\cdot e'\hat{e}' = cbe\hat{e}' = cb\hat{e}'e = cf'be = dfbe = dfae' = da\hat{e}e' = dae'\hat{e} = db\cdot e\hat{e}\,. \tag{4}$$

As to (CH.3): if $a\leqslant a'$, $b\leqslant b'$ and $a'!b'$, let $a = a'e$, $b = b'f$ and (e', f') be a resolution of (a', b'). It is sufficient to check that (ee', ff') is a resolution of (a, b): $a\cdot ee' = a$, $b\cdot ff' = b$, $a\cdot ff' = a'eff' = b'efe' = b\cdot ee'$.

Now we prove (CH.4) and the properties (1), (3). Let $a!b$, with resolution (e, e'): $a = ae$, $b = be'$, $ae' = be$; we must show that $h = ae' = be$ is the meet of a and b; clearly $h\leqslant a, b$, while if $k\leqslant a, b$ then $k = af = bf'$ and $k = ae\cdot f = bf'\cdot e\leqslant h$. It is now easy to deduce (3), hence the compositive property of meets: with the hypothesis $a!b$, $c!d$ and the notations above (proof of (CH.2)), we have:

$$ca\wedge db = ca\cdot e'\hat{e}' = df\cdot be = (c\wedge d)\cdot(a\wedge b)\,. \tag{5}$$

The last remarks are now trivial; in particular a functor between prj-cohesive categories preserves the order iff it preserves the projections, in which case it also preserves resolutions, hence the linking relation and also binary linked meets, because of (1).

3.4. REMARK. – A cohesive category $(\boldsymbol{A}, \leqslant, !)$ may be determined by at most one prj-cohesive structure on $\boldsymbol{A}$, given by:

$$\mathrm{Prj}\, A = \{a\in \boldsymbol{A}(A) \mid a\leqslant 1_A\}\,, \tag{1}$$

which happens iff $a \leqslant 1$ implies $aa = a$ and moreover the characterizations 3.2.1-2, concerning the order and linking relations, hold.

Indeed, if this is the case, define the projections by (1). (PCH.1): if $e, f \in \mathrm{Prj}\, A$ then $ef \leqslant 1$ is again a projection, hence an idempotent: it follows that $ef = ef \cdot ef \leqslant f \cdot e$; analogously: $fe \leqslant ef$. (PCH.2): from $fa \leqslant a$ and the condition 3.2.1 it follows the existence of a projection e such that $fa = a \cdot e$.

Thus a cohesive category will be said to be prj-cohesive when these facts hold.

Analogously, an ordered category $(\boldsymbol{A}, \leqslant)$ is prj-cohesive, with projections defined by (1), iff $a \leqslant 1$ implies $aa = a$ and 3.2.1 holds.

3.5. DEFINITION. – An *e-cohesive* category (or *e-category* for short) will be a category $\boldsymbol{A}$ provided, for every object A, with a projection-set $\mathrm{Prj}\, A \subset \boldsymbol{A}(A)$, verifying (PCH.1) and:

(ECH.1) for each $a\colon A \to B$ in $\boldsymbol{A}$, the set of projections e of A such that $ae = a$ has a least element $\boldsymbol{e}(a)$: the *support* of a,

(ECH.2) for every $a\colon A \to B$, $b\colon B \to C$ in $\boldsymbol{A}$: $\boldsymbol{e}(b) \cdot a = a \cdot \boldsymbol{e}(ba)$.

Elementary properties, for $a, a'\colon A \to B$, $b\colon B \to C$, $e \in \mathrm{Prj}\, A$, $f \in \mathrm{Prj}\, B$:

(1) $$\boldsymbol{e}(e) = e\,,$$

(2) $$\boldsymbol{e}(ba) \leqslant \boldsymbol{e}(a)\,,$$

(3) $$fa = a \cdot \boldsymbol{e}(fa)\,,$$

(4) $$\boldsymbol{e}(a \cdot e) = e \cdot \boldsymbol{e}(ae) = \boldsymbol{e}(a) \cdot e\,,$$

(5) $$a \leqslant a' \Rightarrow \boldsymbol{e}(a) \leqslant \boldsymbol{e}(a')\,,$$

(6) if a is monic, then $\boldsymbol{e}(a) = 1$.

In particular, (3) shows that the axiom (PCH.2) is satisfied: $\boldsymbol{A}$ is prj-cohesive, hence cohesive.

An *e-cohesive* functor will be a functor between e-cohesive categories which preserves supports; by (1) it also preserves projections, hence it is prj-cohesive and (3.3) cohesive.

3.6. *The cohesion structure.* By the last remark in 3.2, if $\boldsymbol{A}$ is e-cohesive the associated order and linking relations are characterized by:

(1) $a \leqslant b$ iff $a = b \cdot \boldsymbol{e}(a)$,

(2) $a\,!\,b$ iff $a \cdot \boldsymbol{e}(b) = b \cdot \boldsymbol{e}(a)$, iff $(\boldsymbol{e}(a), \boldsymbol{e}(b))$ is a *resolution* of (a, b),
iff $a \cdot \boldsymbol{e}(b) \leqslant b$ and $b \cdot \boldsymbol{e}(a) \leqslant a$.

Further, if $a!b$, by 3.3 and 3.5.4:

$$a \wedge b = a \cdot \boldsymbol{e}(b) = b \cdot \boldsymbol{e}(a) \,, \tag{3}$$

$$\boldsymbol{e}(a \wedge b) = \boldsymbol{e}(a) \wedge \boldsymbol{e}(b) \,. \tag{4}$$

Similarly, if $\boldsymbol{A}$ is ϱ-cohesive, it is easy to check that:

$$\boldsymbol{e}(\bigvee \alpha) = \bigvee_{a \in \alpha} \boldsymbol{e}(a) \,, \qquad \text{(for every linked } \varrho\text{-set } \alpha) \,. \tag{5}$$

3.7. *Counterimages of projections.* If $\boldsymbol{A}$ is e-cohesive, every morphism $a\colon A \to B$ in $\boldsymbol{A}$ determines a mapping:

$$a^P\colon \operatorname{Prj} B \to \operatorname{Prj} A \,, \qquad a^P(f) = \boldsymbol{e}(fa) \,. \tag{1}$$

Thus Prj becomes a contravariant functor from $\boldsymbol{A}$ into the category of semilattices:

$$1^P = 1 \,, \qquad (ba)^P = a^P b^P \,, \qquad a^P(f \wedge g) = a^P(f) \wedge a^P(g) \,. \tag{2}$$

Indeed:

$$a^P b^P(g) = a^P(\boldsymbol{e}(gb)) = \boldsymbol{e}(\boldsymbol{e}(gb) \cdot a) = \boldsymbol{e}(a \cdot \boldsymbol{e}(gba)) = \boldsymbol{e}(a) \cdot \boldsymbol{e}(gba) = \boldsymbol{e}(g \cdot ba) = (ba)^P(g) \,.$$

Further:

$$a^P(fg) = a^P(f^P(g)) = (fa)^P(g) = (a \cdot a^P(f))^P(g) = $$
$$= (a^P(f))^P(a^P(g)) = a^P(f) \cdot a^P(g) = a^P(f) \wedge a^P(g) \,.$$

Other properties, for $a, a'\colon A \to B$, $b\colon B \to C$, $e, e' \in \operatorname{Prj} A$, $f \in \operatorname{Prj} B$:

$$e^P(1) = e \,, \tag{3}$$

$$fa = a \cdot a^P(f) \,, \tag{4}$$

$$e^P(e') = ee' \,, \tag{5}$$

$$\boldsymbol{e}(ba) = \boldsymbol{e}(\boldsymbol{e}(b) \cdot a) \leqslant \boldsymbol{e}(a) \,, \tag{6}$$

for (6), write $f = b^P(1)$, so that:

$$\boldsymbol{e}(ba) = (ba)^P(1) = a^P b^P(1) = a^P(f) = a^P f^P(1) = (fa)^P(1) = \boldsymbol{e}(fa) = \boldsymbol{e}(\boldsymbol{e}(b) \cdot a) \,.$$

Conversely, if $\boldsymbol{A}$ verifies (ECH.1) and the mappings (1) are given, satisfying (2)-(4), hence (5), then $\boldsymbol{A}$ is e-cohesive, with $\boldsymbol{e}(a) = a^P(1)$. Indeed: $a \cdot a^P(1) = a$; if $ae = a$ then $a^P(1) = (ae)^P(1) = e^P a^P(1) = e \cdot a^P(1)$, i.e. $a^P(1) \leqslant e$; further $a \cdot (ba)^P(1) = a \cdot a^P b^P(1) = a \cdot a^P(f) = fa = b^P(1) \cdot a$, where $f = b^P(1)$.

3.8. EXAMPLES. – *a*) The cohesive categories $\mathcal{S}$, $\mathcal{T}$, $\mathcal{C}^r$ described in ch. 1 are *e*-cohesive, with projections given by the partial identities.

b) An *e*-cohesive category need not be link-filtered: e.g. the subcategory of $\mathcal{S}$ considered in 0.4.

c) Every *dominical* category ([He, Di, DH]), more generally every *p-category* [Ro] $\boldsymbol{A}$, is *e*-cohesive, with:

$$\text{(1)} \qquad \text{Prj}\, A = \{\text{dom}\, x | x \in \boldsymbol{A}(A)\} = \{e \in \boldsymbol{A}(A) | \exists a \text{ in } \boldsymbol{A} \text{ such that } e = \text{dom}\, a\} = \\ = \{e \in \boldsymbol{A}(A) | e = \text{dom}\, e\}\,,$$

$$\text{(2)} \qquad \boldsymbol{e}(a) = \text{dom}\, a\,.$$

This follows from the following properties of *domains* proved in [Ro], 2.1.4-5, for morphisms $a: A \to B$, $b: B \to C$, $a': A \to B'$:

i) $\text{dom}\, 1_A = 1_A$, ii) $\text{dom}\,(ba) = \text{dom}\,((\text{dom}\, b)\cdot a)$,

iii) $(\text{dom}\, b)\cdot a = a \cdot \text{dom}\,(ba)$, iv) $(\text{dom}\, a)\cdot(\text{dom}\, a') = (\text{dom}\, a')\cdot(\text{dom}\, a)$,

v) $a \cdot \text{dom}\, a = a$, vi) $\text{dom}\,(\text{dom}\, a) = \text{dom}\, a$,

vii) $(\text{dom}\, a)\cdot(\text{dom}\, a) = (\text{dom}\, a)$, viii) $\text{dom}\,((\text{dom}\, a)\cdot(\text{dom}\, a')) = (\text{dom}\, a)\cdot(\text{dom}\, a')$.

Indeed the second and third equalities in (1) come from the property vi). The axiom (PCH.1) follows from i), vii), iv) and viii), while (ECH.2) coincides with iii). As to (ECH.1): if $a: A \to B$, then $a = a \cdot \text{dom}\, a$, by v); on the other hand, if $a = ae$ and $e \in \text{Prj}\, A$, then $e \leqslant \text{dom}\, a$, as (by ii) and viii)):

$$\text{(3)} \qquad \text{dom}\, a = \text{dom}\,(ae) = \text{dom}\,((\text{dom}\, a)\, e) = \text{dom}\,((\text{dom}\, a)(\text{dom}\, e)) = \\ = (\text{dom}\, a)\cdot(\text{dom}\, e) = (\text{dom}\, a)\cdot e\,.$$

d) The category $L^\infty(\boldsymbol{a}, \boldsymbol{Ban})$ described in 1.5 is prj-cohesive [G5].

3.9. *Cartesian products and duality.* The cartesian product $\boldsymbol{A} = \Pi\, \boldsymbol{A}_i$ of a family of prj-cohesive categories $(\boldsymbol{A}_i)_{i\in I}$ is prj-cohesive, with $\text{Prj}_{\boldsymbol{A}}\,(A_i) = \Pi\,(\text{Prj}\, A_i)$. If the factors $\boldsymbol{A}_i$ are *e*-cohesive, so is the product $\boldsymbol{A}$ with: $\boldsymbol{e}((a_i)_{i\in I}) = (\boldsymbol{e}(a_i))_{i\in I}$.

A *prj*-cohesive* category will be a pair $\boldsymbol{A} = (\boldsymbol{A}, \mathit{Prj})$ verifying (PCH.1) and (PCH.2*): for all a and e there is some f such that $fa = ae$; the associated cohesion structure has: $a \leqslant b$ iff there is some projection f such that $a = fb$, and analogously for the linking (determined by *coresolutions* of pairs of morphisms). Then $\boldsymbol{A}$ is an *e*-cohesive* category if it is provided with *cosupports* $\boldsymbol{e}^*(a)$ verifying (ECH.1*, ECH.2*).

4. – Adequate prj-cohesive categories.

$\boldsymbol{A}$ is always a prj-category; we examine conditions ensuring that the ϱ-cohesive completion of $\boldsymbol{A}$ is again a prj-category.

4.1. *Resolution of sets.* It is easy to show that a set $\alpha \subset \boldsymbol{A}(A, B)$ of parallel morphisms is linked iff there is a family of projections $e_{ab} \in \operatorname{Prj} A$ $(a, b \in \alpha)$ such that:

(1) $$a = a \cdot e_{ab}\,, \qquad a \cdot e_{ba} = b \cdot e_{ab}\,, \qquad \text{for all } a, b \in \alpha\,.$$

More particularly, a *resolution* of α will be a family $(e_a)_{a\in\alpha}$ of projections of A such that:

(2) $$a = a \cdot e_a\,, \qquad a \cdot e_b = b \cdot e_a\,, \qquad \text{for all } a, b \in \alpha\,;$$

the second condition may also be written: $a \cdot e_b \leqslant b$. A set admitting a resolution is clearly linked, but these two facts are indeed equivalent in most cases we are interested in, as we shall soon see (4.3-4).

Any prj-cohesive functor preserves resolution of sets.

4.2. *Transfer of resolutions.* A resolution (e_a) of α may be *transferred* by composition in the following way. Given the morphisms x, y:

(1) $$A' \xrightarrow{x} A \xrightarrow{a} B \xrightarrow{y} B'\,, \qquad (a \in \alpha)\,,$$

choose, for each $a \in \alpha$, a projection $e'_a \in \operatorname{Prj} A'$ such that $e_a \cdot x = x \cdot e'_a$: then, a trivial checking shows that:

(2) $$(e'_a) \text{ is a resolution of } y\alpha x = \{yax \mid a \in \alpha\}\,.$$

4.3. *Existence of resolutions.* Let $\boldsymbol{A}$ be prj-cohesive.

a) Every set ε of parallel projections has a canonical resolution: $(e)_{e\in\varepsilon}$.

b) More generally, every set α which has an upper bound $\hat{a}$ has a resolution. Indeed, let $a = \hat{a} \cdot e_a$ $(a \in \alpha)$:

(1) $$ae_a = \hat{a}e_a \cdot e_a = a\,, \qquad a \cdot e_b = \hat{a}e_a \cdot e_b = \hat{a}e_b \cdot e_a = b \cdot e_a\,.$$

Thus: if $\boldsymbol{A}$ is ϱ-cohesive, each linked ϱ-set has a resolution.

c) If $\boldsymbol{A}$ has ϱ'-meets of projections (in $\operatorname{Prj} A$ or equivalently in $\boldsymbol{A}(A)$, by 3.2), compositive in $\boldsymbol{A}$, we are going to show that each linked ϱ-set α has a resolution (e_a) and also (for $\alpha \neq \emptyset$) compositive meet:

(2) $$\bigwedge \alpha = b \cdot \bigwedge_a e_a\,, \qquad \text{for any } b \in \alpha\,.$$

Indeed, with the notations of 4.1.1, the family $e_a = \bigwedge_b e_{ab}$ $(a \in \alpha)$ is a resolution of α:

$$a \cdot e_a = a \cdot \bigwedge_b e_{ab} = \bigwedge_b a \cdot e_{ab} = \bigwedge_b a = a \, , \tag{3}$$

$$a \cdot e_b = a e_a \cdot e_b = (a \cdot e_{ba} e_b) e_a = (b e_{ab} e_b) e_a = b e_b \cdot e_{ab} e_a = b e_a \, , \tag{4}$$

as to (2): $b \cdot \bigwedge e_a \leqslant b \cdot e_a = a \cdot e_b \leqslant a$ for all $a \in \alpha$; if $x \leqslant a$ for all $a \in \alpha$, then $x \leqslant a \wedge b = b \cdot e_a$, hence $x \leqslant b \cdot \bigwedge e_a = \bigwedge b \cdot e_a$; last, the compositive property of the meet is a straightforward consequence of the transfer of resolutions (4.2).

d) In particular, as any prj-category has linked f'-meets, it follows that *each finite linked set has a resolution.*

e) If $\boldsymbol{A}$ is *e-cohesive*, a set α of parallel morphisms is linked iff (3.6.2):

$$a \cdot \boldsymbol{e}(b) \leqslant b \, , \qquad \text{for all } a, b \in \alpha \, , \tag{5}$$

iff the family $e_a = \boldsymbol{e}(a)$ of their supports is a resolution of α (the *e-resolution*).

4.4. *Adequate prj-cohesive categories.* We shall say that the prj-coherent category $\boldsymbol{A}$ is *ϱ-adequate* if it satisfies:

(PCH.3_ϱ) each linked ϱ-set of $\boldsymbol{A}$ has a resolution,

(PCH.4_ϱ) $\boldsymbol{A}$ has ϱ-joins of projections, compositive in $\boldsymbol{A}$.

A prj-category which is ϱ-cohesive is also ϱ-adequate (4.3 *b*); trivially, it is 0-cohesive iff it is 0-adequate. The category $L^\infty(\boldsymbol{a}, \boldsymbol{Ban})$ (1.5) is σ-adequate, because of 4.3 *c*), whereas it is not σ-cohesive.

A *ϱ-adequate functor* will be a prj-cohesive functor between ϱ-adequate (prj-cohesive) categories, which preserves ϱ-joins of projections.

4.5. PROPOSITION. – If $\boldsymbol{A}$ is a ϱ-adequate prj-category, a linked ϱ-set of parallel morphisms has linked join (2.2) iff it has an upper bound. Every existing ϱ-join of morphisms is linked.

A ϱ-adequate functor preserves all the existing ϱ-joins.

PROOF. – The thesis being trivial for $\varrho \subset \{0, 1\}$, assume that ϱ is infinite.

First, every ϱ-set α with an upper bound has compositive join: if $a = c \cdot e_a$, for $a \in \alpha$, set $\hat{e} = \bigvee e_a$ (compositive join) and $\hat{a} = c \cdot \hat{e}$; then $\hat{a} = c \cdot \bigvee e_a = \bigvee_a c e_a = \bigvee_a a = \bigvee \alpha$ is a *compositive* join.

Take now a parallel map b linked with α: we have to show that $\hat{a} ! b$ and $\hat{a} \wedge b = \bigvee_a (a \wedge b)$. By (PCH.$3\varrho$), the linked ϱ-set $\gamma = \alpha \cup \{b\}$ has a resolution $(e_c)_{c \in \gamma}$;

the compositive join $\hat{e} = \bigvee e_a$ $(a \in \alpha)$ yields a resolution $(\hat{e}, e_b)$ of the pair $(\hat{a}, b)$, proving that it is linked:

$$(1) \qquad \hat{a}\cdot\hat{e} = \bigvee_{a,a'} ae_{a'} = \bigvee_a a = \hat{a}\,, \qquad \hat{a}\cdot e_b = \bigvee_a (a\cdot e_b) = \bigvee_a (b\cdot e_a) = b\cdot\bigvee_a e_a = b\cdot\hat{e}\,.$$

The distributivity follows, calculating the meets by the resolutions (3.3.1):

$$(2) \qquad (\bigvee \alpha)\wedge b = \hat{a}\cdot e_b = (\bigvee_a a)\cdot e_b = \bigvee_a (a\cdot e_b) = \bigvee_a (a\wedge b)\,.$$

Thus, every existing ϱ-join in $\boldsymbol{A}$ is linked. If $F\colon \boldsymbol{A}\to\boldsymbol{B}$ is a ϱ-adequate functor, $\hat{a} = \bigvee\alpha$ is a ϱ-join, $a = \hat{a}\cdot e_a$ $(a\in\alpha)$ and $\hat{e} = \bigvee e_a$ as above:

$$(3) \qquad F(\bigvee a) = F(\hat{a}) = F(\hat{a}\cdot\hat{e}) = F(\hat{a})\cdot F(\hat{e}) = F(\hat{a})\cdot F(\bigvee e_a) =$$
$$= F(\hat{a})\cdot(\bigvee Fe_a) = \bigvee (F\hat{a}\cdot Fe_a) = \bigvee F(\hat{a}\cdot e_a) = \bigvee Fa\,.$$

4.6. COROLLARY. – A prj-category is ϱ-cohesive iff it is ϱ-adequate and every linked ϱ-set of parallel morphisms has an upper bound. A functor between ϱ-cohesive prj-categories is ϱ-cohesive iff it is ϱ-adequate.

4.7. THEOREM (*the ϱ-completion for ϱ-adequate prj-categories*). – If $\boldsymbol{A}$ is a ϱ-adequate prj-category, the ϱ-cohesive completion $\varrho c\boldsymbol{A}$ (2.8) is prj-cohesive, with the same projections. The embedding $\boldsymbol{A}\to\varrho c\boldsymbol{A}$ is ϱ-adequate, and may also be considered as the universal ϱ-adequate functor from $\boldsymbol{A}$ into a ϱ-cohesive prj-category.

The linking and order relations in $\varrho c\boldsymbol{A}$ can also be described as follows, for α and β parallel linked ϱ-sets of $\boldsymbol{A}$-morphisms:

(1) $\quad [\alpha]\,!\,[\beta]$ $\quad$ iff there is a resolution $(e_x)_{x\in\alpha\cup\beta}$ of $\alpha\cup\beta$ in $\boldsymbol{A}$,

(2) $\quad [\alpha]\leqslant[\beta]$ $\quad$ iff there is such $(e_x)_{x\in\alpha\cup\beta}$ with: $a = a\cdot(\bigvee_{b\in\beta} e_b)$, for $a\in\alpha$,

iff there is such $(e_x)_{x\in\alpha\cup\beta}$ with: $\bigvee_{a\in\alpha} e_a \leqslant \bigvee_{b\in\beta} e_b$.

PROOF. – If ε is a ϱ-set of parallel projections of $\boldsymbol{A}$, $e = \bigvee\varepsilon$ is a linked join in $\boldsymbol{A}$, by the previous proposition (4.5); as linked joins are preserved by the embedding in $\varrho c\boldsymbol{A}$, $e = [\varepsilon]$ in $\varrho c\boldsymbol{A}$. It follows that the projections of $\boldsymbol{A}$ coincide with the endomaps $[\alpha]\leqslant 1$ of $\varrho c\boldsymbol{A}$: indeed, the relation $e\leqslant 1$ in $\boldsymbol{A}$ is preserved by the embedding, while if $[\alpha]\leqslant 1$ in $\varrho c\tilde{\boldsymbol{A}}$, each morphism $a\in\alpha$ is a projection ($a\leqslant[\alpha]\leqslant 1$ in $\varrho c\tilde{\boldsymbol{A}}$, hence $a\leqslant 1$ in $\boldsymbol{A}$) and $[\alpha] = \bigvee\alpha\leqslant 1$ in $\boldsymbol{A}$.

We have to prove that the ϱ-cohesive category $\varrho c\boldsymbol{A}$ is prj-cohesive, with the same projections as $\boldsymbol{A}$. Because of 3.4, this reduces to check the characterizations 3.2.1-2 for the order and the linking relation in $\varrho c\boldsymbol{A}$; in the same time, we shall also verify the characterizations (1) and (2) of these relations.

First, consider the linking. If $\alpha\,!\,\beta$ in $\mathcal{F}_\varrho\boldsymbol{A}$, then $\alpha\cup\beta$ is a linked ϱ-set, with resolution $(e_x)_{x\in\alpha\cup\beta}$. Given such a resolution, the subsets $\varepsilon = \{e_a | a\in\alpha\}$, $\eta = \{e_b | b\in\beta\}$

and their joins $e = \bigvee \varepsilon = [\varepsilon]$, $f = \bigvee \eta = [\eta]$ yield a resolution of $[\alpha]$ and $[\beta]$ in $\varrho c\boldsymbol{A}$, in agreement with 3.2.2:

$$\alpha \cdot e = \{a \cdot e | a \in \alpha\} = \alpha\,, \qquad \beta \cdot f = \beta\,, \tag{3}$$

$$[\alpha]\cdot f = [\alpha]\cdot[\eta] = [\{a \cdot e_b | a \in \alpha,\, b \in \beta\}] = [\{b \cdot e_a | a \in \alpha,\, b \in \beta\}] = [\beta]\cdot[\varepsilon] = [\beta]\cdot e\,. \tag{4}$$

Last, assume we have such a resolution: $[\alpha] = [\alpha]\cdot e$, $[\beta] = [\beta]\cdot f$, $[\alpha]\cdot f = [\beta]\cdot e$. By (PCH.3$\varrho$) there are resolutions (e_a) of α and (f_b) of β; thus: $a = [\alpha]\cdot e_a = [\alpha]\cdot ee_a = ae$, and we may assume that $e_a \leqslant e$, for all $a \in \alpha$; similarly $f_b \leqslant f$, for $b \in \beta$. Then, in $\varrho c\boldsymbol{A}$:

$$a \cdot f_b = [\alpha]\cdot e_a \cdot f_b = [\alpha] f \cdot e_a f_b = [\beta] e \cdot e_a f_b = [\beta]\cdot e_a f_b = b \cdot e_a\,, \tag{5}$$

whence $a!b$ in $\boldsymbol{A}$ (for all a and b) and $[\alpha]![\beta]$ in $\varrho c\boldsymbol{A}$.

Now, consider the order. If $[\alpha] \leqslant [\beta]$ in $\varrho c\boldsymbol{A}$, every resolution (e_x) of $\alpha \cup \beta$ (with e and f as above) yields: $a = \bigvee_b (a \wedge b) = \bigvee_b ae_b = af$. Given such a resolution, replace each e_a with $e_a \cdot f$: this gives a new resolution of $\alpha \cup \beta$ verifying $e \leqslant f$. If this property holds, by (3) and (4): $[\alpha] = [\alpha]\cdot e = [\alpha]\cdot f = [\beta]\cdot e$, as required by 3.2.1. Last, if $[\alpha] = [\beta]\cdot h$ for some projection h, the relation $h \leqslant 1$ in $\varrho c\boldsymbol{A}$ implies $[\alpha] \leqslant [\beta]$.

Finally the embedding $\boldsymbol{A} \to \varrho c\boldsymbol{A}$ preserves the projections by the above remarks, and their ϱ-joins (as all the existing linked ϱ-joins) by definition; the new universal property is a particular case of the known one (2.7).

4.8. THEOREM (*the ϱ-completion for ϱ-adequate e-categories*). – If $\boldsymbol{A}$ is a ϱ-adequate e-category, then $\varrho c\boldsymbol{A}$ is e-cohesive, with supports:

$$\boldsymbol{e}[\alpha] = [\{\boldsymbol{e}(a) | a \in \alpha\}] = \bigvee_{a \in \alpha} \boldsymbol{e}(a) \qquad (\alpha\text{: linked } \varrho\text{-set of } \boldsymbol{A})\,. \tag{1}$$

The embedding η: $\boldsymbol{A} \to \varrho c\boldsymbol{A}$ is a ϱ-adequate e-functor; it is the universal ϱ-adequate e-functor from $\boldsymbol{A}$ into a ϱ-cohesive e-category.

PROOF. – Let us consider the ϱ-set of projections $\varepsilon = \{\boldsymbol{e}(a) | a \in \alpha\}$: it is an endomap in $\mathcal{S}_\varrho \boldsymbol{A}$. Clearly $\alpha \cdot \varepsilon = \{a \cdot \boldsymbol{e}(a') | a, a' \in \alpha\} \sim \alpha$; on the other hand, if $[\alpha]\cdot e = [\alpha]$ then (as in the proof of 4.7) $ae = a$ for all $a \in \alpha$: i.e. $\boldsymbol{e}(a) \leqslant e$, for all a, and $[\varepsilon] \leqslant e$. Hence, in $\varrho c\boldsymbol{A}$, $[\alpha]$ has support $[\varepsilon] = \bigvee \boldsymbol{e}(a)$; this proves also that the embedding η preserves supports.

As to (ECH.2), if β is a linked ϱ-set of $\boldsymbol{A}$, composable with α, for all $a, a' \in \alpha$ and $b \in \beta$:

$$a \cdot \boldsymbol{e}(ba') \leqslant a \cdot \boldsymbol{e}(ba')\, \boldsymbol{e}(a) = a \cdot \boldsymbol{e}(ba' \cdot \boldsymbol{e}(a)) = a \cdot \boldsymbol{e}(ba \cdot \boldsymbol{e}(a')) \leqslant a \cdot \boldsymbol{e}(ba)\,, \tag{2}$$

$$\begin{aligned}\boldsymbol{e}[\beta]\cdot[\alpha] &= [\{\boldsymbol{e}(b)\cdot a | a \in \alpha,\, b \in \beta\}] = [\{a \cdot \boldsymbol{e}(ba) | a \in \alpha,\, b \in \beta\}] = \\ &= [\{a \cdot \boldsymbol{e}(ba') | a, a' \in \alpha,\, b \in \beta\}] = [\alpha]\cdot \boldsymbol{e}[\beta\alpha]\,.\end{aligned} \tag{3}$$

4.9. REMARK. – Let $\boldsymbol{A}$ be a prj-category. It can be shown that its ϱ-cohesive completion $\varrho c\boldsymbol{A}$ is prj-cohesive provided that $\boldsymbol{A}$ satisfies (PCH.3ϱ) and the following condition, weaker than (PCH.4ϱ):

a) for every morphism $a\colon A \to B$, every $e_0 \in \operatorname{Prj} A$ and every ϱ-set ε of projections of A, if $ae_0 = a = \bigvee_{e\in\varepsilon} ae$ is a linked join (i.e. $ae_0 = a \sim a\varepsilon$ in $\mathcal{F}_\varrho \boldsymbol{A}$) then there exists a projection e_1 of A such that: $e_1 \leqslant e_0$, $ae_1 = a$ and $e_1 = \bigvee_{e\in\varepsilon} e_1 e$ is a linked join (i.e. $e_1 \prec \varepsilon$ in $\mathcal{F}_\varrho \boldsymbol{A}$).

In such a case the projections of $\varrho c\boldsymbol{A}$ are the equivalence-classes $[\varepsilon]$, where ε is any ϱ-set of parallel projections of $\boldsymbol{A}$. However, the stronger but simpler condition (PCH.4ϱ) is sufficient for our purposes.

5. – Inverse categories and cohesion.

Inverse categories are the obvious generalization of inverse semigroups. They are used here to supply « glueing morphisms » for generalized manifolds; for instance, the usual C^r-manifolds will be constructed in ch. 6, 7 by means of open euclidean spaces and partial C^r-diffeomorphisms between open subsets, forming the inverse category $\operatorname{Inv} \mathcal{C}^r$ associated to $\mathcal{C}^r$.

After a review of elementary properties of inverse categories from [G1, G2], we introduce here their canonical cohesion structure and study their ϱ-cohesive completion. Other references on inverse categories can be found in [G3].

5.1. *Review of inverse categories.* A category $\boldsymbol{K}$ is *inverse* if every morphism $a\colon A \to A'$ has precisely one *generalized inverse* $\tilde{a}\colon A' \to A$:

$$a\tilde{a}a = a\,, \qquad \tilde{a}a\tilde{a} = \tilde{a}\,. \tag{1}$$

Then [G1, thm. 1.25] the mapping $a \mapsto \tilde{a}$ defines an *involution* of $\boldsymbol{K}$ (i.e. a contravariant functor, identical on the objects and involutive), which is selfdual:

$$\tilde{1} = 1\,, \qquad (ba)^\sim = \tilde{a}\tilde{b}\,, \qquad (\tilde{a})^\sim = a\,. \tag{2}$$

A *projection* of the object A is any idempotent endomorphism $e\colon A \to A$; clearly $\tilde{e} = e$. The projections of A are closed with respect to composition ($ef = ef\cdot(ef)^\sim\cdot(ef) = ef\cdot fe\cdot ef = ef\cdot ef$) and commute ($ef = (ef)^\sim = fe$): they form a unitary semilattice $\operatorname{Prj} A$.

Every morphism $a\colon A \to B$ defines two mappings, the covariant and contravariant transfer of projections:

$$a_P\colon \operatorname{Prj} A \to \operatorname{Prj} B\,, \qquad a_P(e) = ae\tilde{a}\,, \tag{3}$$

$$a^P\colon \operatorname{Prj} B \to \operatorname{Prj} A\,, \qquad a^P(f) = \tilde{a}fa = \tilde{a}_P(f)\,, \tag{4}$$

which are easily seen to be homomorphisms of semilattices and to behave functorially $((ba)_P = b_P a_P,\ (ba)^P = a^P b^P)$. Clearly:

(5) a is monic $\Leftrightarrow a^P(1) = \tilde{a}a = 1 \Leftrightarrow a$ has some left inverse,

(6) a is epi $\Leftrightarrow a_P(1) = a\tilde{a} = 1 \Leftrightarrow a$ has some right inverse,

(7) a is monic and epi $\Leftrightarrow (\tilde{a}a = 1,\ \ a\tilde{a} = 1) \Leftrightarrow a$ is an isomorphism.

Last, the category $\boldsymbol{K}$ is provided with a *canonical order* (generalizing the canonical order of inverse semigroups) $a \leqslant b$, characterized by the following equivalent conditions (for $a, b\colon A \to B$):

i) $a = a\tilde{b}a$;

ii) $a = b\cdot\tilde{a}a$;

iii) $a = a\tilde{a}\cdot b$;

iv) $a = a\tilde{a}\cdot b\cdot\tilde{a}a$;

v) $\exists e \in \operatorname{Prj} A\colon\ a = b\cdot e$;

vi) $\exists f \in \operatorname{Prj} B\colon\ a = f\cdot b$;

vii) $\exists e \in \operatorname{Prj} A,\ \exists f \in \operatorname{Prj} B\colon\ a = f\cdot b\cdot e$.

Notice that the endomorphisms $x \leqslant 1$ are precisely the projections and that $x \geqslant 1$ implies $x = 1$. Since for each morphism a:

(8) $$a\tilde{a}a = a\,,\quad \tilde{a}a\tilde{a} = \tilde{a}\,,\quad \tilde{a}a \leqslant 1\,,\quad a\tilde{a} \leqslant 1\,;$$

it follows that a is monic iff it has a right-adjoint b ($ba \geqslant 1$, $ab \leqslant 1$); then $b = \tilde{a}$ is also left-inverse to a.

A functor between inverse categories preserves all the notions considered above.

The paradigmatic inverse category is the category $\mathfrak{J}$ of small sets and partial bijections: any inverse category may be embedded in this ([Ks, G3]). Other examples of interest for our context are given in 5.8 and ch. 8.

5.2. *Inverse categories and regularity.* Let the category $\boldsymbol{A}$ be *regular in the sense of von Neumann* (vN-regular): each morphism $a\colon A \to A'$ has *some* generalized inverse $a'\colon A' \to A$ (verifying: $aa'a = a$, $a'aa' = a'$). Then $\boldsymbol{A}$ is inverse (i.e. the generalized inverses are uniquely determined) iff the idempotents of $\boldsymbol{A}$ commute ([G1], 1.25).

More particularly, let the category $\boldsymbol{A}$ be provided with a *regular* involution $a \mapsto \tilde{a}$, regular meaning that: $a\tilde{a}a = a$, for all a. Call projection of A any symmetrical idempotent, i.e. any endomorphism $e\colon A \to A$ such that $e = ee = \tilde{e}$ (or equivalently: $e = e\tilde{e}$, or also $e = \tilde{e}e$). Then each idempotent a is the product of two projections ($a = a\tilde{a}a = a\tilde{a}\cdot\tilde{a}a$), so that $\boldsymbol{A}$ is inverse iff its idempotents commute, iff its projections commute, iff every idempotent is symmetrical; in this case the involution of $\boldsymbol{A}$ yields the (unique) generalized inverse of every morphism.

5.3. *The canonical cohesion structure.* From now on, $\boldsymbol{K}$ is an inverse category.

It is easy to see that the projections of $\boldsymbol{K}$ satisfy the axioms (PCH.1) and (ECH.1, 2), defining an *e-cohesive* structure with $\boldsymbol{e}(a) = \tilde{a}a = a^P(1)$. Indeed: $a\cdot\boldsymbol{e}(a) = a\cdot\tilde{a}a = a$; if $a = ae$ then $\tilde{a}a = \tilde{a}a\cdot e$ and $\tilde{a}a \leqslant e$; $a\cdot\boldsymbol{e}(ba) = a\cdot\tilde{a}\tilde{b}ba = \tilde{b}b\cdot a\tilde{a}\cdot a = \boldsymbol{e}(b)\cdot a$.

Now, the involution of $\boldsymbol{K}$ determines also an *e*-cohesive* structure (3.9), with cosupports given by: $\boldsymbol{e}^*(a) = \boldsymbol{e}(\tilde{a}) = a\tilde{a} = a_P(1)$.

Thus $\boldsymbol{K}$ is provided with a *first* cohesion structure (determined by supports) and with a *second* one (determined by cosupports):

$$(1)\qquad a \leqslant' b \ \text{ iff } \ a = b\cdot\tilde{a}a\,, \qquad a\,!'\,b \ \text{ iff } \ a\cdot\tilde{b}b = b\cdot\tilde{a}a \ \text{ iff } \ b\tilde{a} \in \operatorname{Prj} B \ (^6)\,,$$

$$(2)\qquad a \leqslant'' b \ \text{ iff } \ a = a\tilde{a}\cdot b\,, \qquad a\,!''\,b \ \text{ iff } \ b\tilde{b}\cdot a = a\tilde{a}\cdot b \ \text{ iff } \ \tilde{b}a \in \operatorname{Prj} A\,.$$

These orders *coincide* with the canonical order $\leqslant$ of $\boldsymbol{K}$ (5.1), while the two linking relations are generally different [7], and related by the involution:

$$(3)\qquad a\,!'\,b \quad \text{iff } \ \tilde{a}\,!''\,\tilde{b}\,.$$

The *canonical* cohesion structure of $\boldsymbol{K}$ will be given by the canonical order $\leqslant$ together with the following linking relation (*preserved by the involution* of $\boldsymbol{K}$):

$$(4)\qquad \begin{aligned} a\,!\,b \quad & \text{iff } \ (a\,!'\,b \text{ and } a\,!''\,b)\,,\\ & \text{iff } \ (a\cdot\tilde{b}b = b\cdot\tilde{a}a \text{ and } b\tilde{b}\cdot a = a\tilde{a}\cdot b)\,,\\ & \text{iff } \ (b\tilde{a} \in \operatorname{Prj} B \text{ and } \tilde{b}a \in \operatorname{Prj} A)\,. \end{aligned}$$

$\boldsymbol{K}$ need not be link-filtered: e.g. consider the inverse subcategory of $\mathfrak{J}$ formed by those partial bijections whose definition-set has no more than five elements.

Every functor between inverse categories preserves the canonical cohesion structure.

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(6) If $a\cdot\tilde{b}b = b\cdot\tilde{a}a$ then: $b\tilde{a}\cdot b\tilde{a} = b\cdot\tilde{b}b\tilde{a}\cdot b\tilde{a} = b\cdot\tilde{a}a\tilde{b}\cdot b\tilde{a} = b\tilde{b}b\cdot\tilde{a}a\tilde{a} = b\tilde{a}$; conversely, if $b\tilde{a}$ is a projection: $b\cdot\tilde{a}a = b\tilde{a}\cdot a = a\tilde{b}b\tilde{a}\cdot a = a\cdot\tilde{b}b\cdot\tilde{a}a = a\tilde{a}a\cdot\tilde{b}b = a\cdot\tilde{b}b$.

(7) For instance, take the inverse category $\mathfrak{J}$ of partial bijections: the projections of $\mathfrak{J}$ coincide with the ones of $\mathfrak{S}$, thus $a\,!'\,b$ iff a and b are compatible functions, while $a\,!''\,b$ iff $\tilde{a}$ and $\tilde{b}$ are compatible functions. Thus, any pair a, b with $\operatorname{Def} a \cap \operatorname{Def} b = \emptyset$ and $\operatorname{Val} a \cap \operatorname{Val} b \neq \emptyset$ yields a counterexample.

5.4. PROPOSITION. – This is indeed a cohesion structure on $\boldsymbol{K}$ (if not a prj-cohesion structure in the sense of ch. 3). If $a!b$:

(1) $$a\wedge b = a\tilde{b}b = b\tilde{a}a = b\tilde{b}a = a\tilde{a}b = a\tilde{b}a = b\tilde{a}b\ ,$$

(2) $$(a\wedge b)_P(e) = a_P(e)\wedge b_P(e)\ , \qquad (a\wedge b)^P(f) = a^P(f)\wedge b^P(f)\ ,$$

(3) $$\boldsymbol{e}(a\wedge b) = \boldsymbol{e}(a)\wedge \boldsymbol{e}(b) = \tilde{b}a = \tilde{a}b\ , \qquad \boldsymbol{e}^*(a\wedge b) = \boldsymbol{e}^*(a)\wedge \boldsymbol{e}^*(b) = b\tilde{a} = a\tilde{b}\ .$$

A set α of parallel morphisms in $\boldsymbol{K}$ is linked iff it has a *double resolution* (e_a), (f_a) of projections, verifying:

(4) $$a = a\cdot e_a = f_a\cdot a\ , \qquad a\cdot e_b = b\cdot e_a\ , \qquad f_b\cdot a = f_a\cdot b\ , \qquad (a, b\in\alpha)\ ,$$

the smallest double resolution being given by: $e_a = \boldsymbol{e}(a)$, $f_a = \boldsymbol{e}^*(z)$.

The cartesian compositive property of meets (1.7.3) holds.

PROOF. – The axioms (CH.1-4) are a straightforward consequence of the definition: the first and second structure are both cohesion structures, with the same order relation. Linked meets may be calculated according to the first structure, e-cohesive ($a\wedge b = a\cdot\tilde{b}b = b\cdot\tilde{a}a$) or to the second one, e^*-cohesive ($a\wedge b = b\tilde{b}\cdot a = a\tilde{a}\cdot b$); the last two expressions in (1) follow at once from $b\tilde{a}\in \mathrm{Prj}\, B$ and $\tilde{b}a\in \mathrm{Prj}\, A$.

The cartesian compositive property of meets follows from 3.3 (applied to the first cohesion structure of $\boldsymbol{K}$). For (2):

$$(a\wedge b)_P(e) = (a\wedge b)\cdot e\cdot (a\wedge b)^\sim = (ae\wedge be)\cdot(\tilde{a}\wedge\tilde{b}) = ae\tilde{a}\wedge be\tilde{b} = a_P(e)\wedge b_P(e)\ .$$

The last assertions are obvious.

5.5. REMARK. – It may be noticed that $a!'b$ (or $a!''b$) is a sufficient condition in order that a and b have compositive meet with respect to the canonical order (use the associated e-cohesive or e^*-cohesive structure). However, in an inverse category, compositive intersections are not « satisfactory »: e.g. they do not satisfy 5.4.2, nor 5.4.3. The good notion seems to be linked meets, in the present sense.

5.6. THEOREM (*the ϱ-completion of an inverse category*). – The ϱ-cohesive completion of the inverse category $\boldsymbol{K}$ (with respect to its canonical cohesion structure) is an inverse category, provided with the canonical cohesion structure. The involution of $\varrho c\boldsymbol{K}$ is given by $\tilde{\alpha} = \{\tilde{a}|a\in\alpha\}$, while its projections are the classes $[\varepsilon]$, where ε is any ϱ-set of parallel projections of $\boldsymbol{K}$.

PROOF. – The mapping $\alpha \mapsto \tilde{\alpha} = \{\tilde{a} | a \in \alpha\{$ defines clearly an involution on $\mathcal{T}_\varrho \boldsymbol{K}$, and further in $\varrho c\boldsymbol{K}$; the latter is regular (5.2), as:

(1) $$\alpha\tilde{\alpha} = \{a\tilde{b} | a, b \in \alpha\} \sim \{a\tilde{a} | a \in \alpha\} ,$$

(2) $$\alpha\tilde{\alpha}\alpha \sim \{a\tilde{a}\cdot b | a, b \in \alpha\} \sim \{a\tilde{a}a | a \in \alpha\} = \alpha .$$

An endomorphism $[\alpha]$ is a projection (with regard to the regular involution, see 5.2) iff $\alpha \sim \tilde{\alpha}\alpha \sim \{\tilde{a}a | a \in \alpha\}$, iff α *is* a ϱ-set of projections of $\boldsymbol{K}$. Therefore the projections of $\varrho c\boldsymbol{K}$ commute and the latter is an inverse category (5.2); we only need to prove that the cohesion structure of $\varrho c\boldsymbol{K}$ coincides with the canonical one, determined by supports and cosupports.

If $[\alpha] \leqslant [\beta]$ in the « completion » structure of $\varrho c\boldsymbol{K}$, the projection $[\varepsilon] = [\tilde{\alpha}\alpha] = [\{\tilde{a}a | a \in \alpha\}]$ yields:

(3) $$[\beta]\cdot[\varepsilon] = [\{b\cdot\tilde{a}a | a \in \alpha, b \in \beta\}] = [\{a \wedge b | a \in \alpha, b \in \beta\}] = [\alpha] \wedge [\beta] = [\alpha] ,$$

hence $[\alpha] \leqslant [\beta]$ in the « inverse » structure. Conversely, if $[\alpha] = [\beta]\cdot e$ for some projection e of $\varrho c\boldsymbol{K}$, the relation $e \leqslant 1$ in $\varrho c\boldsymbol{K}$ implies $[\alpha] \leqslant [\beta]$ in the completion structure.

Last $[\alpha]![\beta]$ in the completion structure iff $\alpha!\beta$ in $\boldsymbol{K}$, iff $a!b$ for all $a \in \alpha$ and $b \in \beta$, iff all the endomorphisms $b\tilde{a}$ and $\tilde{b}a$ are projections, iff $[\beta\tilde{\alpha}]$ and $[\tilde{\beta}\alpha]$ are projections of $\varrho c\boldsymbol{K}$, iff $[\alpha]![\beta]$ in the inverse structure.

5.7. *The inverse subcategory of a prj-category.* Now, let $\boldsymbol{A}$ be a prj-category. Define $\boldsymbol{K} = \operatorname{Inv} \boldsymbol{A}$ as the subcategory of $\boldsymbol{A}$ having the same objects and those morphisms $u\colon A \to B$ having a *Morita inverse* $u'\colon B \to A$ in $\boldsymbol{A}$, verifying:

a) $u = uu'u, \; u' = u'uu', \; u'u \leqslant 1_A, \; uu' \leqslant 1_B.$

We prove now that $\boldsymbol{K}$ is an inverse category whose projections (i.e. idempotent endomorphisms) coincide with the ones of $\boldsymbol{A}$, the generalized inverse in $\boldsymbol{K}$ being given by the Morita inverse in $\boldsymbol{A}$.

First, $\boldsymbol{K}$ is a subcategory of $\boldsymbol{A}$: if $u\colon A \to B$, $v\colon B \to C$ have Morita inverses u' and v', then $u'v'$ is a Morita inverse for vu:

(1) $$vu\cdot u'v'\cdot vu = v\cdot uu'\cdot v'v\cdot u = vv'v\cdot uu'u = vu ,$$

(2) $$vu\cdot u'v' = v(uu')v' \leqslant vv' \leqslant 1 .$$

Thus, $\boldsymbol{K}$ is vN-regular (5.2). Every idempotent endomorphism e of $\boldsymbol{K}$ is a projection of $\boldsymbol{A}$: if v is a Morita inverse of e: $v = vev = ve\cdot ev$ is a projection, hence an idempotent, and $e = eve = ev\cdot ve$ is a projection.

As the converse is trivial, the idempotent endomorphisms of $\boldsymbol{K}$ coincide with the projections of A, hence commute: by 5.2, K is inverse, and the generalized inverse of a morphism u in K is unique: it will be written $\tilde{u}$.

The embedding $\operatorname{Inv} A \to A$ preserves the cohesion structure: generally, it does not reflect it. The Inv-construction is clearly functorial on prj-cohesive functors.

If A *is* ϱ*-cohesive, so is* K with respect to its canonical cohesion structure: if φ is a linked ϱ-set in K, so is $\tilde{\varphi} = \{\tilde{u} | u \in \varphi\}$; both φ and $\tilde{\varphi}$ are also linked in A, with resolutions $e_u = \tilde{u}u$, $e_{\tilde{u}} = u\tilde{\varphi}$ $(u \in \varphi)$, and:

$$(3) \qquad (\bigvee \tilde{\varphi})(\bigvee \varphi) = \bigvee \tilde{u}v = \bigvee e_u e_v = \bigvee e_u = e \in \operatorname{Prj} A\,, \qquad (u, v \in \varphi)\,,$$

$$(4) \qquad (\bigvee \varphi)(\bigvee \tilde{\varphi})(\bigvee \varphi) = (\bigvee \varphi)\cdot e = \bigvee u \cdot e_v = \bigvee \varphi\,.$$

It may also be noticed that an adjunction $u \dashv v$ in A $(vu \geqslant 1,\ uv \leqslant 1)$ forces $vu = 1$, hence is « the same » as a monic u of K (with $v = \tilde{u}$).

5.8. EXAMPLES. – If $A = \mathcal{S}$, the prj-category of small sets and partial mappings (1.1), then $\mathcal{I} = \operatorname{Inv} \mathcal{S}$ is the subcategory of small sets and partial bijections.

Analogously $\operatorname{Inv} \mathcal{T}$ (resp. $\operatorname{Inv} \mathcal{C}^r$) is the category of topological spaces (resp. open euclidean sets) and partial homeomorphisms (resp. partial C^r-diffeomorphisms) between open subsets of the domain and codomain. All these inverse categories are totally cohesive (5.7).

6. – Manifolds and glueing completion for inverse categories.

In this chapter K is always an inverse category and ϱ is an infinite section of cardinals (1.8). Manifolds over K are introduced as symmetrical enriched categories over K. If K is ϱ-cohesive, *bilinked* modules between ϱ-manifolds produce the ϱ-glueing completion $\varrho\mathrm{IMf}\, K$ of K.

6.1. *Manifolds.* A diagram $U = (U_i, u^i_j)_I$, consisting of objects U_i (the *charts*) and morphisms $u^i_j\colon U_i \to U_j$ of K (the *glueing maps*), indexed over a small set I, will be said to be a *manifold* in K if:

$$(1) \qquad u^i_i = 1_{U_i} \qquad \text{(identity law)}\,,$$

$$(2) \qquad u^j_k \cdot u^i_j \leqslant u^i_k \qquad \text{(composition law, or triangle inequality)}\,,$$

$$(3) \qquad u^j_i = (u^i_j)^{\sim} \qquad \text{(symmetry law)}\,,$$

in other words, U is a small symmetrical category enriched over the involutive ordered 2-category K [Be, Wa, BC]: notice that the first condition is equivalent to the usual one, $u^i_i \geqslant 1$ (by 5.1). We say that U is a ϱ*-manifold* if its object-set I is a ϱ-set.

The *glueing* of the manifold U in $\boldsymbol{K}$ (if existing) will be an object $X = \text{gl}\, U$ provided with a family of morphisms $u^i\colon U_i \to X$ $(i \in I)$, such that:

(4) $$u^j \cdot u^i_j \leqslant u^i\,, \qquad \text{for all } i, j \in I\,,$$

(5) $$\tilde{u}^j \cdot u^i \leqslant u^i_j\,, \qquad \text{for all } i, j \in I\,,$$

and universal in the obvious sense. According to the definition 6.3, the family (u^i) is a « bilinked » module from U to the trivial manifold $(X, 1_X)$.

$\boldsymbol{K}$ will be said to be *ϱ-glueing* (as an inverse category) if it is ϱ-cohesive and every ϱ-manifold has a glueing; *totally glueing* (inverse) category, or just *glueing*, will mean Ω-glueing.

From now on, *we assume that* $\boldsymbol{K}$ *is ϱ-cohesive.*

6.2. PROPOSITION. – With the previous notations, a family of morphisms u^i: $U_i \to X$ $(i \in I)$ is the glueing of the manifold U iff, for all $i, j \in I$:

(1) $$u^j \cdot u^i_j \leqslant u^i\,,$$

(2) $$\tilde{u}^j \cdot u^i = u^i_j\,,$$

(3) $$\bigvee_i u^i \cdot \tilde{u}^i = 1_X \; (^8)\,;$$

the condition (2) can be replaced with:

(2′) $$\tilde{u}^j \cdot u^i \leqslant u^i_j\,,$$

(2″) $$\tilde{u}^i \cdot u^i = 1_{u_i}\,.$$

Moreover, if $y^i\colon U_i \to Y$ $(i \in I)$ is any family of morphisms verifying 6.1.4-5, the unique morphism $y\colon X \to Y$ such that $y^i = y \cdot u^i$ is given by:

(4) $$y = \bigvee_i y^i \cdot \tilde{u}^i \quad \text{(linked join)}\,.$$

Every ϱ-cohesive functor between ϱ-cohesive inverse categories preserves the existing glueings of ϱ-manifolds.

PROOF. – First, assume that X is the glueing of U; (1) and (2′) hold by definition. To prove (3), consider the projection $e = \bigvee_i u^i \cdot \tilde{u}^i\colon X \to X$; clearly $eu^i = u^i$, for all i; by the uniqueness in the universal property of the glueing, it follows that $e = 1$. Now, for (2″), fix some $h \in I$ and consider the family of morphisms z^i:

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(8) These conditions mean that $u = (u^i)\colon U \to X$ is an isomorphism, in the category of manifolds over $\boldsymbol{K}$ (6.3, 6.4), between U and the one-index manifold $X = (X, 1_X)$.

$U_i \to U_h$, $z^i = u^i_h$ $(i \in I)$; since it satisfies the conditions 6.1.4-5:

$$z^j \cdot u^i_j = u^j_h u^i_j \leqslant u^i_h = z^i \,, \tag{5}$$

$$\tilde{z}^j \cdot z^i = u^h_j u^i_h \leqslant u^i_j \,, \tag{6}$$

there is exactly one morphism $z\colon X \to U_h$ such that $z^i = z \cdot u^i$ for all i; in particular $z \cdot u^h = z^h = u^h_h = 1$, whence u^h is monic and $\tilde{u}^h \cdot u^h = 1$.

Secondly, (1), (2′), (2″) imply (2): $u^i_j = 1_{U_j} \cdot u^i_j = \tilde{u}^j \cdot u^j \cdot u^i_j \leqslant \tilde{u}^j \cdot u^i$.

Last, if (1)-(3) hold, it is easy to check the universal property for (X, u^i) by means of the formula (4), which concerns the join of a linked ϱ-set, since:

$$u^j \tilde{y}^j \cdot y^i \tilde{u}^i \leqslant u^j u^i_j \tilde{u}^i \leqslant u^i \tilde{u}^i \leqslant 1 \,, \qquad y^j \tilde{u}^j \cdot u^i \tilde{y}^i \leqslant y^j u^i_j \tilde{y}^i \leqslant y^i \tilde{y}^i \leqslant 1 \,, \tag{7}$$

The final assertion on ϱ-cohesive functors is now trivial.

6.3. *Bilinked modules.* We form here the category ϱIMf $\boldsymbol{K}$ of ϱ-manifolds over $\boldsymbol{K}$ and « bilinked modules » between them; we shall show below that this category is the inverse ϱ-glueing completion of $\boldsymbol{K}$.

A *bilinked module* $a = (a^i_h)_{I,H}\colon (U_i, u^i_j)_I \to (V_h, v^h_k)_H$ between the ϱ-manifolds U and V will be a family of $\boldsymbol{K}$-morphisms $a^i_h\colon U_i \to V_h$, verifying (for $i, j \in I$ and $h, k \in H$):

$$v^h_k \cdot a^i_h \leqslant a^i_k \,, \qquad a^j_h \cdot u^i_j \leqslant a^i_h \qquad \text{(module laws)}\,, \tag{1}$$

$$\tilde{a}^j_h \cdot a^i_h \leqslant u^i_j \,, \qquad a^i_k \cdot \tilde{a}^i_h \leqslant v^h_k \qquad \text{(linking laws)}\,, \tag{2}$$

where (1) is the usual condition for a module $a\colon U \to V$ between categories enriched over an ordered category ([Be, Wa]), while (2) expresses the linking property on domain and codomains. Once that the category of bilinked modules is constructed (here below), and provided with its canonical order as an inverse category (6.4), the condition (2) may be thought to mean that the modules $a = (a^i_h)_{I,H}$ and $\tilde{a} = (\tilde{a}^i_h)_{H,I}$ form a Morita context [Bi]: $\tilde{a}a \leqslant 1_U$ and $a\tilde{a} \leqslant 1_V$. Notice, however, that arbitrary modules can *not* be composed, because of the lack of arbitrary joins in $\boldsymbol{K}$.

The (*matrix*) composition with $(b^h_m)_{H,M}\colon (V_h, v^h_k)_H \to (W_m, w^m_n)_M$ is given by:

$$(b^h_m)_{H,M} \cdot (a^i_h)_{I,H} = (c^i_m)_{I,M} \,, \qquad c^i_m = \textstyle\bigvee_h (b^h_m \cdot a^i_h) \,, \tag{3}$$

where the join is legitimate and produces a bilinked module, as:

$$\tilde{a}^i_k \tilde{b}^k_m \cdot b^h_m a^i_h \leqslant \tilde{a}^i_k v^h_k a^i_h \leqslant \tilde{a}^i_k a^i_k \leqslant 1_{U_i} \,, \qquad (h, k \in H) \,, \tag{4}$$

$$c^j_m \cdot u^i_j = \textstyle\bigvee_h (b^h_m a^j_h \cdot u^i_j) \leqslant \bigvee_h (b^h_m \cdot a^i_h) = c^i_m \,, \tag{5}$$

$$\tilde{c}^j_m \cdot c^i_m = \textstyle\bigvee_k (\tilde{a}^j_k \tilde{b}^k_m) \cdot \bigvee_h (b^h_m a^i_h) = \bigvee_{h,k} (\tilde{a}^j_k \cdot \tilde{b}^k_m b^h_m \cdot a^i_h) \leqslant \bigvee_{h,k} (\tilde{a}^j_k v^h_k a^i_h) \leqslant \bigvee_k (\tilde{a}^j_k a^i_k) \leqslant u^i_j \,. \tag{6}$$

It is easy to see that this is indeed a category, with identity of $U = (U_i, u^i_j)$ given by the bilinked endomodule $1_U = (u^i_j)_I$. $\boldsymbol{K}$ has an obvious embedding in $\varrho\mathrm{IMf}\,\boldsymbol{K}$, identifying the object U with the ϱ-manifold $(U, 1_U)$.

6.4. THEOREM (*the inverse structure*). – The category $\boldsymbol{M} = \varrho\mathrm{IMf}\,\boldsymbol{K}$ is inverse. The following conditions are equivalent:

i) $e = (a^i_j)_{I,I}\colon (U_i, u^i_j)_I \to (U_i, u^i_j)_I$ is a projection of $\boldsymbol{M}$,

ii) $e = (a^i_j)_{I,I}$ is an endomorphism and $a^i_j \leqslant u^i_j$, for all i, j,

iii) $e = (a^i_j)_{I,I}$ is an endomorphism, $e_i = a^i_i \in \mathrm{Prj}\, U_i$ and $a^i_j = u^i_j e_i = e_j u^i_j$,

iv) $e = (u^i_j e_i)_{I,I}$ where $e_i \in \mathrm{Prj}\, U_i$ and $u^j_i e_j u^i_j \leqslant e_i$, for all i, j.

If $a = (a^i_h)_{I,H}$ and $b = (v^i_h)_{I,H}$ are maps from $U = (U_i, u^i_j)_I$ to $V = (V_h, v^h_k)_H$, $e = (u^i_j e_i)_{I,I}$ and $f = (v^h_k f_h)_{H,H}$ are projections of U and V respectively:

(1) $$(fae)^i_h = f_h \cdot a^i_h \cdot e_i\,,$$

(2) $$a \leqslant b \Leftrightarrow a^i_h \leqslant b^i_h \text{ in } \boldsymbol{K}\,, \text{ for all } i, h\,;$$

(3) $$a\,!\,b \Leftrightarrow \tilde{b}^j_h a^i_h \leqslant u^i_j \text{ and } b^i_k \tilde{a}^i_h \leqslant v^h_k\,, \text{ for all } i, j \text{ and } h, k\,,$$
$$\Leftrightarrow a^i_h\,!\,b^i_h \text{ for all } i, h \text{ and } (a^i_h \vee b^i_h)_{I,H} \text{ is a linked module}\,.$$

(4) $$a \wedge b = (a^i_h \wedge b^i_h)_{I,H}\,, \quad a \vee b = (a^i_h \vee b^i_h)_{I,H} \quad (\text{for } a\,!\,b)\,.$$

Last, if $e_i \in \mathrm{Prj}\, U_i$ $(i \in I)$ is an arbitrary family of projections of our charts, the least projection $\hat{e} = (a^i_j)_{I,I}$ of the manifold U, with $\hat{e}_i \geqslant e_i$ for all i, is given by:

(5) $$a^i_j = \textstyle\bigvee_h u^h_j e_h u^i_h\,.$$

PROOF. – See 9.3.

6.5. INVERSE GLUEING COMPLETION THEOREM. – The category $\boldsymbol{M} = \varrho\mathrm{IMf}\,\boldsymbol{K}$ is the inverse ϱ-glueing completion of $\boldsymbol{K}$.

PROOF. – See 9.4.

6.6. EXAMPLES. – The inverse category $\mathcal{J} = \mathrm{Inv}\,\mathcal{S}$ of small sets and partial bijections (5.8) is totally glueing: the glueing of the manifold $(U_i, u^i_j)_I$ is the set $X = \mathrm{gl}\, U = (\coprod U_i)/R$, where R is the equivalence relation identifying every $x \in \mathrm{Def}\, u^i_j \subset U_i$ with $u^i_j(x) \in U_j$. The partial bijections $u^i\colon U_i \to X$ are obvious (and everywhere defined).

Analogously for $\mathrm{Inv}\,\mathcal{T}$: take on X the finest topology making continuous all the mappings u^i.

Instead Inv $\mathcal{C}^r$ is totally cohesive and not glueing, even finitely: its (total) glueing completion is (can be interpreted as) the category of C^r-manifolds and partial C^r-diffeomorphisms.

Indeed, the inclusion Inv $\mathcal{C}^r \to$ Inv $\mathcal{T}$ extends, by the universal property of the glueing completion, to a unique glueing functor IMf (Inv $\mathcal{C}^r$) $\to$ Inv $\mathcal{T}$ (the *topological realization* of manifolds), transforming the manifold $U = (U_i, u_j^i)_I$ into the space $X = \mathrm{gl}\, U$, the glueing of U in $\mathcal{T}$. This space X is locally euclidean (with *locally* constant dimension), because of the partial homeomorphisms $u^i\colon U_i \to X$ (everywhere defined), whose images cover X; it is not necessarily paracompact nor Hausdorff. It allows to reconstruct the manifold in the usual setting: a topological space X provided with an open covering (V_i) and a C^r-atlas of charts (onto open euclidean sets) $v^i\colon V_i \to U_i$; take $V_i = u^i(U_i)$ and v^i as the restriction of $(u^i)^\sim$ to its definition-set V_i; the partial C^r-diffeomorphisms u_j^i are thus the coordinate changes.

6.7. *Cauchy-completion and maximal manifolds.* The notion of Cauchy-complete enriched category was introduced by LAWVERE [La] for a monoidal base and extended by BETTI [Be] to enrichment over a bicategory. This notion has a straightforward adaptation to our case: symmetrical categories over a ϱ-cohesive inverse category $\boldsymbol{K}$. However, the interest of such a notion in the present case is small: since the natural morphisms for manifolds are modules, the Cauchy-completion of a manifold would just produce an isomorphic object: the associated maximal glueing atlas; moreover these completions are still *small* manifolds provided that $\boldsymbol{K}$ is small, which in our examples may be true (e.g. for Inv $\mathcal{C}^r$) or not (e.g. Inv $\mathcal{F}$, Inv $\mathcal{B}$ in ch. 8).

Recall that, in the *inverse* category ϱIMf $\boldsymbol{K}$, the datum of an adjoint pair $a \dashv b$ (i.e. a pair of bilinked modules verifying $ba \geqslant 1$ and $ab \leqslant 1$) is just equivalent to giving a monomorphism a (5.1; take $b = \tilde{a}$).

Now, a *linked functor* $f\colon (U_i, u_j^i)_I \to (V_h, v_k^h)_H$ between manifolds over $\boldsymbol{K}$ will be a mapping $f\colon I \to H$ between their index-sets, such that: $U_i = V_{fi}$, $u_j^i = v_{fj}^{fi}$ (9) for $i, j \in I$. It produces a bilinked module $f = (f_h^i)\colon (U_i, u_j^i)_I \to (V_h, v_k^h)_H$, $f_h^i = v_h^{fi}$, which is monic $(f \dashv \tilde{f})$:

$$(1)\qquad (\tilde{f}f)_j^i = \vee_h \tilde{f}_h^j f_h^i = \vee_h v_{fj}^h v_h^{fi} = v_{fj}^{fi} = u_j^i\,.$$

Actually, the only case we are interested in is a (trivially linked) functor $f\colon W \to M = (U_i, u_j^i)_I$ defined on a one-index manifold $W = (W, 1)$: this is just the same as selecting an index $h \in I$ such that $W = U_h$, and produces the monic bilinked module $(f_i)\colon W \to M$, $f_i = u_i^h$.

The manifold $M = (U_i, u_j^i)_I$ over $\boldsymbol{K}$ is said to be *Cauchy-complete* if, for every

(9) For a functor, one would here require $\leqslant$ instead of equality.

W in $\boldsymbol{K}$, every monic bilinked module $(u_i)\colon W \to M$ is produced by such a functor f: i.e. there is some $h \in I$ such that $W = U_h$, $u_i = u_i^h$.

Now, it is easy to see that the datum of a monic bilinked module $u = (u_i)\colon W \to M$ ($\tilde{u}u = 1$) is equivalent to « adding to M a redundant chart »: in other words, giving a larger glueing atlas $M' = (U_i, u_j^i)_{I'}$ with $I' = I \cup \{k\}$ ($k \notin I$) and requiring that the bilinked module $(u_j^i)_{I,I'}\colon M \to M'$ be an isomorphism. The correspondence between these notions is established by the equations: $U_k = W$, $u_i^k = u_i$, $u_k^i = \tilde{u}_i$.

Thus the manifold M is Cauchy-complete iff it is a *maximal* glueing atlas, that is if « every compatible chart is already in M ».

If $\boldsymbol{K}$ is small, every manifold is contained in a maximal isomorphic one, its Cauchy-completion.

7. – Manifolds and glueing completion for prj-categories.

$\boldsymbol{A}$ is always a ϱ-cohesive prj-category and $\boldsymbol{K} = \operatorname{Inv} \boldsymbol{A}$ the associated ϱ-cohesive inverse category (5.7). The simpler, more particular case of a ϱ-cohesive e-category is treated in 7.8.

7.1. *Manifolds and glueing.* A manifold over $\boldsymbol{A}$ will be a diagram $U = (U_i, u_j^i)_I$ in $\boldsymbol{A}$, with $u_j^i\colon U_i \to U_j$ ($i, j \in I$) verifying:

(1) $u_i^i = 1_{U_1}$ (identity law),

(2) $u_k^j \cdot u_j^i \leqslant u_k^i$ (composition law, or triangle inequality),

(3) $u_j^i = u_j^i u_i^j u_j^i$ (symmetry law).

Since $u_i^j u_j^i \leqslant u_i^i = 1_{U_1}$ and because of (3), all the morphisms u_j^i are actually in $\boldsymbol{K} = \operatorname{Inv} \boldsymbol{A}$, and verify: $(u_j^i)^\sim = u_i^j$: in other words the manifolds of $\boldsymbol{A}$ *are precisely those of* $\boldsymbol{K}$.

The *glueing* $X = \operatorname{gl} U$ of the manifold U in $\boldsymbol{A}$ (if existing) will be, by definition, its *lax-colimit*, that is an object X provided with a universal lax cocone $u^i\colon U_i \to X$ ($i \in I$) in $\boldsymbol{A}$:

a) $u^j \cdot u_j^i \leqslant u^i$, for all i, j,

b) for any lax cocone $y^i\colon U^i \to Y$ ($y^j \cdot u_j^i \leqslant y^i$), there exists a unique $y\colon X \to Y$ in $\boldsymbol{A}$ such that $y^i = y \cdot u^i$ ($i \in I$),

c) if $y', y''\colon X \to Y$ and $y' \cdot u^i \leqslant y'' \cdot u^i$ ($i \in I$), then $y' \leqslant y''$.

We show below that this problem is *equivalent* to the glueing of U in $\boldsymbol{K}$ (6.1-2).

A prj-category will be said to be *ϱ-glueing* if it is ϱ-cohesive and every ϱ-manifold has a glueing.

7.2. THEOREM. – *Let* $U = (U_i, u_j^i)_I$ be a manifold over $\boldsymbol{A}$ (and $\boldsymbol{K}$), and u^i: $U_i \to X$ $(i \in I)$ a family of morphisms in $\boldsymbol{A}$.

(X, u^i) is the glueing of U in $\boldsymbol{A}$ iff it is so in $\boldsymbol{K}$. In such a case the morphisms u^i are monomorphisms of $\boldsymbol{K}$ and for every lax cocone y^i: $U^i \to Y$ in $\boldsymbol{A}$, the appropriate morphism y: $X \to Y$ is given by:

$$y = \bigvee_i y^i \cdot \tilde{u}^i \quad \text{(linked join in } \boldsymbol{A}) . \tag{1}$$

$\boldsymbol{A}$ is ϱ-glueing iff Inv $\boldsymbol{A}$ is so. Every ϱ-cohesive functor between ϱ-cohesive prj-categories preserves the existing glueings of ϱ-manifolds.

PROOF. – If (X, u^i) is the glueing of U in $\boldsymbol{K}$, the formula (1) concerns the join of a linked ϱ-set in $\boldsymbol{A}(X, Y)$, with resolution $e_i = u^i \tilde{u}^i \in \text{Prj}\, X$ $(i \in I)$:

$$y^i \tilde{u}^i \cdot e_i = y^i , \qquad y^i \tilde{u}^i \cdot e_j = y^i \cdot \tilde{u}^i u^j \tilde{u}^j = y^i u_j^i \tilde{u}^j \leqslant y^j \tilde{u}^j . \tag{2}$$

It is now easy to check, as in 6.2, the universal properties 7.1 *b*), *c*) in $\boldsymbol{A}$.

Conversely, assume that (X, U^i) is the glueing of U in $\boldsymbol{A}$. Fix an index $h \in I$ and consider, as in the proof of 6.2, the family $z^i = u_h^i$: $U_i \to U_h$ $(i \in I)$ of morphisms of $\boldsymbol{A}$: they form a lax cocone from U (as in 6.2.5), hence there is one morphism z: $X \to U_h$ of $\boldsymbol{A}$ such that $z^i = zu^i$ $(i \in I)$. In particular, $zu^h = 1$; moreover $(u^h z) \cdot u^i = u^h z^i = u^h u_h^i \leqslant u^i$, for all i, so that $u^h z \leqslant 1$ (7.1 *c*)); therefore u^h is in Inv $\boldsymbol{A}$, with generalized inverse $(u^h)^\sim = z$.

It suffices now to verify the conditions 6.2.2-3; the relation:

$$\tilde{u}^h \cdot u^i = zu^i = z^i = u_h^i , \tag{3}$$

gives the first, by the arbitrariness of $h \in I$; the second follows from:

$$(\bigvee_i u^i \tilde{u}^i) \cdot u^j = u^j , \tag{4}$$

by means of the uniqueness property in 7.1 *a*): $(\bigvee_i u^i \tilde{u}^i) = 1$.

The last statement follows now from the last assertion in 6.2.

7.3. *Linked modules.* We form the category ϱMf $\boldsymbol{A}$ of ϱ-manifolds over $\boldsymbol{A}$ and linked modules between them.

A module $(a_h^i)_{I,H}$: $(U_i, u_j^i)_I \to (V_h, v_k^h)_H$ is a family of $\boldsymbol{A}$-morphisms a_h^i: $U_i \to V_h$ verifying, for all $i, j \in I$ and $h, k \in H$:

$$v_k^h \cdot a_h^i \leqslant a_k^i , \qquad a_h^j \cdot u_j^i \leqslant a_h^i \quad \text{(module laws)} ; \tag{1}$$

it will be said to be *linked* (or *compatible*) if it has a *resolution* $e_{ih} \in \text{Prj}\, U_i$ $(i \in I$, $h \in H)$:

$$a_h^i e_{ik} = v_h^k \cdot a_k^i \quad \text{(linking law)} , \tag{2}$$

or equivalently:

$$(2')\qquad a^i_h e_{ih} = a^i_h ,$$

$$(2'')\qquad a^i_h e_{ik} \leqslant v^k_h \cdot a^i_k ,$$

as, from (2′) and (2″): $v^k_h \cdot a^i_k = v^k_h \cdot a^i_k \cdot e_{ik} \leqslant a^i_h \cdot e_{ik}$. Moreover each e_{ih} can be clearly replaced with any e'_{ih} with $e'_{ih} \leqslant e_{ih}$ and $a^i_h e'_{ih} = a^i_h$. Thus, in the e-cohesive case, the linking condition (2) may be more simply expressed by means of supports: $e_{ih} = \boldsymbol{e}(a^i_h)$ (7.8).

Clearly $\varrho I\mathrm{Mf}\,\boldsymbol{K} \subset \varrho\mathrm{Mf}\,\boldsymbol{A}$. But note that a linked module over $\boldsymbol{A}$ whose components a^i_h are in $\boldsymbol{K}$ need not belong to $\varrho I\mathrm{Mf}\,\boldsymbol{K}$: this happens iff also the « reverse » module $(\tilde{a}^i_h)$ is linked. It is easy to give counterexamples in the categories $\mathcal{S}$ and $\mathcal{T}$, where the linking condition (2) forces the module (a^i_h) (more precisely, its glueing) to be « single-valued » but not « injective », even if all the components are so. We shall prove in 7.6 that $\varrho I\mathrm{Mf}\,\boldsymbol{K}$ coincides with $\mathrm{Inv}\,(\varrho\mathrm{Mf}\,\boldsymbol{A})$.

Again, the composition is matrix-like: if $(b^h_m)_{H,M}\colon (V_h, v^k_h)_H \to (W_m, w^n_m)_M$ is a linked module:

$$(3)\qquad (b^h_m)_{H,M} \cdot (a^i_h)_{I,H} = (c^i_m)_{I,M} , \qquad c^i_m = \bigvee_h (b^h_m \cdot a^i_h) .$$

We prove that ba is well-defined. Let (f_{hm}) be a resolution of $b = (b^h_m)$ and choose projections $e_{ihm} \in \mathrm{Prj}\, U_i$ such that:

$$(4)\qquad f_{hm} a^i_h = a^i_h e_{ihm} , \qquad e_{ihm} \leqslant e_{ih} , \qquad (i \in I,\ h \in H,\ m \in M) .$$

Then each family $(b^h_m a^i_h)_{h\in H}$ is linked, with resolution $(e_{ihm})_{h\in H}$:

$$(5)\qquad (b^h_m a^i_h) \cdot e_{ihm} = b^h_m f_{hm} a^i_h = b^h_m a^i_h ,$$

$$(6)\qquad (b^h_m a^i_h) \cdot e_{ikm} = b^h_m a^i_h \cdot e_{ik} e_{ikm} \leqslant b^h_m \cdot v^k_h a^i_k \leqslant b^k_m a^i_k .$$

More generally, for $n \in M$:

$$(7)\qquad (b^h_m a^i_h) \cdot e_{ikn} = b^h_m a^i_h e_{ik} e_{ikn} = b^h_m v^k_h a^i_k e_{ikn} \leqslant b^k_m a^i_k e_{ikn} = b^k_m f_{kn} a^i_k = w^n_m b^k_n a^i_k ,$$

and (c^i_m) is a linked module, with resolution $\hat{e}_{im} = \bigvee_h e_{ihm}$:

$$(8)\qquad c^j_m \cdot u^i_j = \bigvee_h (b^h_m a^j_h \cdot u^i_j) \leqslant \bigvee_h (b^h_m \cdot a^i_h) = c^i_m ,$$

$$(9)\qquad c^i_m \cdot \hat{e}_{im} = \bigvee_{h,k} ((b^h_m a^i_h e_{ikm}) = \bigvee_h b^h_m a^i_h = c^i_m \qquad \text{(by (5))} ,$$

$$(10)\qquad c^i_m \cdot \hat{e}_{in} = \bigvee_{h,k} (b^h_m a^i_h e_{ikn}) \leqslant \bigvee_k w^n_m \cdot b^k_n a^i_k = w^n_m \cdot \bigvee_k b^k_n a^i_k = w^{n'}_m \cdot c^i_n \qquad \text{(by (7))} .$$

This is indeed a category and $\boldsymbol{A}$ embeds in $\varrho\mathrm{Mf}\,\boldsymbol{A}$ as in the inverse case (6.3): $U \mapsto (U, 1_U)$.

7.4. *The prj-structure.* Define the *projections* of ϱMf $\boldsymbol{A}$ to be those of ϱIMf $\boldsymbol{K}$, described in 6.4. Note that, as in 6.4.1 and with the same proof, if $a\colon U \to V$ is a morphism in ϱ Mf $\boldsymbol{A}$, $e \in \mathrm{Prj}\ U$ and $f \in \mathrm{Prj}\ V$:

$$(1) \qquad (fae)^i_h = f_h a^i_h e_i .$$

The axiom (PCH.1) holds, because ϱIMf $\boldsymbol{K}$ is inverse. As to (PCH.2), given the linked module $a\colon U \to V$ in ϱ Mf $\boldsymbol{A}$, with resolution (e_{ih}), and $f \in \mathrm{Prj}\ V$, choose projections $e'_{ih} \in \mathrm{Prj}\ U_i$ such that:

$$(2) \qquad f_h a^i_h = a^i_h e'_{ih} , \qquad e'_{ih} \leqslant e_{ih} \qquad (i \in I ,\ h \in H) .$$

Further, let:

$$(3) \qquad e'_i = \bigvee_h e'_{ih} , \qquad \hat{e}_i = \bigvee_j (u^j_i e'_j u^i_j) ,$$

so that, by 6.4.5, $\hat{e} = (u^i_j \hat{e}_i)$ is the projection of the manifold U spanned by the family (e'_i). We prove that $fa = a\hat{e}$:

$$(4) \qquad a^i_h \cdot e'_i = a^i_h \cdot \bigvee_k e'_{ik} = \bigvee_k (a^i_h \cdot e_{ik} e'_{ik}) = \bigvee_k (v^k_h a^i_k \cdot e'_{ik}) = \\ = \bigvee_k (v^k_h f_k a^i_k) = \bigvee_k (f_h v^k_h a^i_k) = f_h a^i_h ,$$

$$(5) \qquad (fa)^i_h = f_h a^i_h = a^i_h e'_{ih} \leqslant a^i_h e'_i \leqslant a^i_h \hat{e}_i = (a\hat{e})^i_h ,$$

$$(6) \qquad (a\hat{e})^i_h = a^i_h \hat{e}_i = \bigvee_j (a^i_h u^j_i e'_j u^i_j) \leqslant \bigvee_j (a^j_h e'_j u^i_j) = \qquad \text{(by (4))} \\ = \bigvee_j (f_h a^j_h u^i_j) = f_h a^i_h = (fa)^i_h .$$

7.5. LEMMA. – If $a = (a^i_h)$ and $b = (b^i_h)$ are parallel morphisms in ϱMf $\boldsymbol{A}$:

(1) $a \leqslant b \Leftrightarrow$ the modules a, b have resolutions (e_{ih}), (f_{ih}) such that:

$$e_{ih} \leqslant f_{ih} \text{ and } a^i_h = b^i_h e_{ih} \qquad (\text{for all } i, h) ,$$

$\Leftrightarrow a^i_h \leqslant b^i_h$ (for all i, h) ;

(2) $a\,!\,b \Leftrightarrow$ the modules a, b have resolutions (e_{ih}), (f_{ih}) such that:

$$a^i_h f_{ik} \leqslant b^i_h \text{ and } b^i_h e_{ik} \leqslant a^i_h \qquad (\text{for all } i, h, k) ,$$

$\Leftrightarrow a^i_h\,!\,b^i_h$ (for all i, h) and $(a^i_h \vee b^i_h)$ is a linked module ;

(3) $a \wedge b = (a^i_h \wedge b^i_h)_{I,H}$, $\quad a \vee b = (a^i_h \vee b^i_h)_{I,H}$ $\quad$ (for $a\,!\,b$) .

PROOF. – If $a \leqslant b$ then $a = b \cdot e$ and $a^i_h = b^i_h \cdot e_i \leqslant b^i_h$. Assume now that $a^i_h \leqslant b^i_h$, for all i and h; let (e'_{ih}), (f_{ih}) be resolutions of a and b, respectively, and choose projections e''_{ih} such that $a^i_h = b^i_h \cdot e''_{ih}$; then the family $e_{ih} = e'_{ih} \cdot e''_{ih} \cdot f_{ih}$ is a resolution of a (by 7.3) satisfying, with (f_{ih}), our conditions. Last, if the resolutions (e_{ih})

and (f_{ih}) verify these conditions, write $e_i = \bigvee_h e_{ih}$ and $\hat{e}$ the projection of U spanned by the family (e_i), as in 7.4.3, so that $a = b\hat{e}$:

$$(4) \qquad a^i_h = b^i_h \cdot e_{ih} \leqslant b^i_h \cdot \hat{e}_i = (b\hat{e})^i_h ,$$

$$(5) \qquad (b\hat{e})^i_h = b^i_h \cdot \hat{e}_i = b^i_h \cdot \bigvee_j (u^j_i e_j u^i_j) = \bigvee_j (b^i_h \cdot u^j_i e_j u^i_j) \leqslant \bigvee_{j,k} b^j_h \cdot e_{jk} u^i_j =$$

$$\leqslant \bigvee_{j,k} b^j_h \cdot f_{jk} e_{jk} u^i_j = \bigvee_{j,k} v^k_h b^j_k e_{jk} u^i_j = \bigvee_{j,k} v^k_h a^j_k u^i_j = a^i_h .$$

The proof of (2) and (3) is similar (see also 6.4).

7.6. We prove now that, for the ϱ-cohesive prj-category $\boldsymbol{A}$, the inverse ϱ-glueing completion of $\boldsymbol{K} = \operatorname{Inv} \boldsymbol{A}$ coincides with the inverse subcategory of the ϱ-glueing completion of $\boldsymbol{A}$:

$$(1) \qquad \varrho \mathrm{IMf}\, \boldsymbol{K} = \operatorname{Inv}(\varrho\, \mathrm{Mf}\, \boldsymbol{A}) .$$

Trivially, a bilinked module $a = (a^i_h)_{I,H}$ over $\boldsymbol{K} = \operatorname{Inv} \boldsymbol{A}$ is a linked module over $\boldsymbol{A}$, provided with a Morita inverse $(\tilde{a}^i_h)_{H,I}$ (5.7) in $\varrho \mathrm{Mf}\, \boldsymbol{A}$.

Conversely, let $a = (a^i_h)_{I,H}\colon (U_i, u^i_j)_I \to (V_h, v^h_k)_H$ be a linked module, with resolution (e_{ih}) and having a Morita inverse $b = (b^h_i)_{H,I}$. Then ba and ab are projections of $\varrho \mathrm{Mf}\, \boldsymbol{A}$, hence so are all the compositions $b^h_i a^i_h$ and $a^i_h b^h_i$:

$$(2) \qquad b^h_i a^i_h \leqslant (ba)^i_i \leqslant 1 ;$$

moreover $(ba)^i_i \cdot e_{ih} = b^h_i a^i_h$, as:

$$(3) \qquad (ba)^i_i \cdot e_{ih} = \bigvee_k (b^k_i a^i_k \cdot e_{ih}) \leqslant \bigvee_k (b^k_i v^h_k a^i_h) \leqslant b^h_i a^i_h \leqslant (ba)^i_i \cdot e_{ih} ,$$

and finally $a^i_h = a^i_h b^h_i a^i_h$, because:

$$(4) \qquad a^i_h = (aba)^i_h \cdot e_{ih} = \big(\bigvee_j a^j_h (ba)^i_j\big) \cdot e_{ih} = \bigvee_j (a^j_h \cdot u^i_j (ba)^i_i \cdot e_{ih}) \leqslant$$

$$\leqslant a^i_h \cdot (ba)^i_i e_{ih} = a^i_h b^h_i a^i_h \leqslant (aba)^i_h \leqslant a^i_h .$$

7.7. GLUEING COMPLETION THEOREM. – The prj-category $\varrho \mathrm{Mf}\, \boldsymbol{A}$ is the ϱ-glueing completion of $\boldsymbol{A}$.

PROOF. – It is an easy consequence of the inverse glueing completion theorem (6.5) and of the previous arguments. A direct proof, in the simpler e-cohesive case, can be found in [G4].

By 7.6 and the inverse glueing completion theorem, $\operatorname{Inv}(\varrho \mathrm{Mf}\, \boldsymbol{A}) = \varrho \mathrm{IMf}\, \boldsymbol{K}$ is ϱ-glueing (as an inverse category); hence the prj-category $\varrho \mathrm{Mf}\, \boldsymbol{A}$ is ϱ-glueing (7.2). Now, if $F\colon \boldsymbol{A} \to \boldsymbol{B}$ is a totally cohesive prj-functor with values in a glueing prj-category, F transforms manifolds and linked modules over $\boldsymbol{A}$ into manifolds and linked modules over $\boldsymbol{B}$, which can be glued in $\boldsymbol{B}$.

7.8. *The e-cohesive case.* Let $\boldsymbol{A}$ be a ϱ-cohesive *e-category*: the previous results take a simpler form. Notice that, for every u in $\boldsymbol{K} = \operatorname{Inv} \boldsymbol{A}$, the support of u in $\boldsymbol{A}$ is: $\boldsymbol{e}(u) = \tilde{u}u$.

A module $a = (a_h^i)_{I,H}\colon (U_i, u_j^i)_I \to (V_h, v_k^h)_H$ between ϱ-manifolds over $\boldsymbol{A}$ (satisfying the module laws 7.3.1) is linked iff it verifies the equivalent conditions:

$$a_h^i \boldsymbol{e}(a_k^i) = v_h^k \cdot a_k^i \qquad \text{(linking law)}, \tag{1}$$

$$a_h^i \boldsymbol{e}(a_k^i) \leqslant v_h^k \cdot a_k^i , \tag{1'}$$

iff the family $(\boldsymbol{e}(a_h^i))$ is a resolution of a (the least one).

The prj-category $\varrho\mathrm{Mf}\,\boldsymbol{A}$ of ϱ-manifolds and linked modules over $\boldsymbol{A}$ is now e-cohesive, with:

$$(\boldsymbol{e}(a))_i = \bigvee\nolimits_h \boldsymbol{e}(a_h^i) , \qquad (\boldsymbol{e}(a))_j^i = u_j^i \cdot (\boldsymbol{e}(a))_i = (\boldsymbol{e}(a))_j \cdot u_j^i . \tag{2}$$

Indeed, $\boldsymbol{e}(a)$ is a projection of the manifold $(U_i, u_j^i)_I$, according to 6.4 iv):

$$a_h^i \cdot (u_i^j \cdot \boldsymbol{e}(a_h^j) \cdot u_j^i) \leqslant a_h^j \cdot \boldsymbol{e}(a_h^j) \cdot u_j^i = a_h^j \cdot u_j^i \leqslant a_h^i , \tag{3}$$

$$(u_i^j \cdot \boldsymbol{e}(a_h^j) \cdot u_j^i) \leqslant \boldsymbol{e}(a_h^i) , \tag{4}$$

$$\big(u_i^j \cdot (\boldsymbol{e}(a))_j \cdot u_j^i\big) = (u_i^j \cdot \bigvee\nolimits_h \boldsymbol{e}(a_h^j) \cdot u_j^i) \leqslant \bigvee\nolimits_h \boldsymbol{e}(a_h^i) = (\boldsymbol{e}(a))_i . \tag{5}$$

We verify now the axioms of the e-structure. For (ECH.1): $(a \cdot \boldsymbol{e}(a))_h^i = a_h^i$, as it follows from the argument below (for $b = 1$). On the other hand, $a = ae$ in $\varrho\mathrm{Mf}\,\boldsymbol{A}$ implies $a_h^i = a_h^i e_i$ (by (4)), hence $e_i \geqslant \boldsymbol{e}(a_h^i)$ for every h, and $e \geqslant \boldsymbol{e}(a)$. Last, for (ECH.2), given a second module $b = (b_m^h)\colon (V_h, v_k^h) \to (W_m, w_n^m)$:

$$\begin{aligned}(a \cdot \boldsymbol{e}(ba))_h^i &= a_h^i \cdot \boldsymbol{e}(ba)_i = a_h^i \cdot \bigvee\nolimits_m (\boldsymbol{e}(ba)_m^i) = a_h^i \cdot \bigvee\nolimits_{k,m} \boldsymbol{e}(b_m^k a_k^i) = \\ &= a_h^i \cdot \bigvee\nolimits_{k,m} (\boldsymbol{e}(a_k^i) \cdot \boldsymbol{e}(b_m^k a_k^i)) = \bigvee\nolimits_{k,m} (a_h^i \cdot \boldsymbol{e}(a_k^i) \cdot \boldsymbol{e}(b_m^k a_k^i)) = \\ &= \bigvee\nolimits_{k,m} (v_h^k a_k^i \cdot \boldsymbol{e}(b_m^k a_k^i)) = \bigvee\nolimits_k \big(v_h^k (\bigvee\nolimits_m \boldsymbol{e}(b_m^k)) \cdot a_k^i\big) = \\ &= \bigvee\nolimits_k \big(v_h^k (\boldsymbol{e}(b))_k \cdot a_k^i\big) = \bigvee\nolimits_k \big((\boldsymbol{e}(b))_h \cdot v_h^k a_k^i\big) = (\boldsymbol{e}(b))_h \cdot a_h^i = (\boldsymbol{e}(b) \cdot a)_h^i .\end{aligned} \tag{6}$$

Finally, from 7.5, for parallel linked modules a, b:

$$a \leqslant b \Leftrightarrow a_h^i \leqslant b_h^i \cdot \boldsymbol{e}(a_h^i) \qquad \text{(for all } i, h) , \tag{7}$$

$$a\,!\,b \Leftrightarrow a_h^i \boldsymbol{e}(b_k^i) \leqslant b_h^i \quad \text{and} \quad b_h^i \boldsymbol{e}(a_k^i) \leqslant a_h^i \qquad \text{(for all } i, h, k) . \tag{8}$$

7.9. *Differentiable manifolds.* The e-categories $\mathcal{S}$ and $\mathcal{T}$ are glueing. The e-category $\mathcal{C}^r$ is totally cohesive and not glueing (even finitely): its glueing completion is the category of C^r-manifolds (as in 6.6) with partial C^r-mappings (defined on open subsets). Also here, the inclusion $\mathcal{C}^r \to \mathcal{T}$ extends to a glueing functor $\mathrm{Mf}\,\mathcal{C}^r \to \mathcal{T}$, the topological realization of C^r-manifolds.

Manifolds with boundary can be obtained in a similar way, by glueing the open subspaces of the spaces $H^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_n \geqslant 0\}$.

The category $\mathrm{Mf}\,\mathcal{C}^r$ (more precisely, an equivalent one) can also be obtained by glueing completion of the full subcategory of $\mathcal{C}^r$ whose objects are the euclidean spaces $\mathbb{R}^n$, since each open euclidean space is a union (and a glueing in $\mathcal{C}^r$) of open balls. It can be noticed that our totally cohesive e-subcategory of $\mathcal{C}^r$ yields back $\mathcal{C}^r$ by a projection-completion procedure (analogous to the well-known idempotent completion).

8. – Fibre bundles, vector bundles and foliations.

We sketch here a definition of fibre bundles, vector bundles and foliations as « manifolds » over the e-cohesive categories of the corresponding trivial structures. For fibre and vector bundles, the topological realization takes place in a (glueing) category $\mathcal{F}$ of « fibrations » $p: X \to B$, playing the role of $\mathcal{T}$ for differentiable manifolds.

8.1. *A glueing category.* A *fibration* will be *just* a continuous, surjective (everywhere defined) mapping $p: X \to B$ between topological spaces.

Form the category $\mathcal{F}$ of fibrations and *partial maps* $(f, \bar{f}): p \to p'$, provided by commutative diagrams in $\mathcal{T}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{f}} & B' . \end{array} \tag{1}$$

Thus f and $\bar{f}$ are partial continuous mappings, defined on open subsets of X and B respectively, with:

$$\mathrm{Def}\, f = p^{-1}(\mathrm{Def}\, \bar{f}) , \quad \mathrm{Def}\, \bar{f} = p(\mathrm{Def}\, f) , \tag{2}$$

and $\bar{f}$ is determined by f.

A projection $(e, \bar{e}): p \to p$ of $\mathcal{F}$ will be any pair of partial identities on a *distinguished* pair $(p^{-1}(W), W)$ of the fibration p, determined by any open subset W of the base B. $\mathcal{F}$ becomes thus an e-category.

The inverse category $\mathrm{Inv}\,\mathcal{F}$ has the same objects and for morphisms the pairs

$(u, \bar{u})\colon p \to p'$ composed by partial homeomorphisms between distinguished pairs of p and p', making (1) commutative.

$\mathcal{F}$ is totally cohesive and glueing: if $M = (p_i\colon X_i \to B_i, (u^i_j, \bar{u}^i_j))_I$ is a manifold over $\mathcal{F}$, its glueing $p\colon X \to B$ in $\mathcal{F}$ can be obtained by glueing in $\mathcal{C}$ the spaces X_i, the bases B_i and the module determined by the fibrations p_i:

$$(3) \qquad X = \mathrm{gl}\,(X_i, u^i_j)_I\,, \qquad B = \mathrm{gl}\,(B_i, \bar{u}^i_j)_I\,, \qquad p = \mathrm{gl}\,(\bar{u}^i_j \cdot p_i\colon X_i \to B_j)\,.$$

The full subcategory $\mathcal{F}_0$ determined by *Serre fibrations* has similar properties; it can be substituted to $\mathcal{F}$ in the following, yielding straightforwardly the homotopy-lifting property for fibre bundles.

8.2. *Fibre bundles.* The « elementary spaces » we want to patch together are the *trivial fibre bundles*, i.e. the cartesian projections [10] $p\colon B \times F \to B$, where B and F are topological spaces and $B \times F$ has the product topology.

Let $\mathcal{B}$ be the full subcategory of $\mathcal{F}$ determined by such objects, with the induced e-cohesive structure: this is totally cohesive but not glueing. For a morphism $(f, \bar{f})\colon p \to p'$ in $\boldsymbol{B}$, we have:

$$(1) \qquad \mathrm{Def}\, f = p^{-1}(\mathrm{Def}\, \bar{f}) = (\mathrm{Def}\, \bar{f}) \times F\,,$$

$$(2) \qquad f(b, y) = \big(\bar{f}(b), f_2(b, y)\big)\,,$$

so that a morphism can also be given by two morphisms in $\mathcal{C}$, $\bar{f}\colon B \to B'$ and $f_2\colon B \times F \to F'$, with $\mathrm{Def}\, f_2 = (\mathrm{Def}\, \bar{f}) \times F$.

The trivial fibre bundle $p\colon B \times F \to B$ will also be written $B \times F$; the morphism $(f, \bar{f})$ will then be denoted by its component f (determining $\bar{f}$).

The inverse category $\mathrm{Inv}\, \mathcal{B}$ has the same objects and for morphisms the pairs $(u, \bar{u})\colon p \to p'$ composed by partial homeomorphisms between distinguished pairs of p and p', making 8.1.1 to commute. As in (2), this is equivalent to giving two mappings of $\mathrm{Inv}\, \mathcal{C}$, $\bar{u}\colon B \to B'$ and $u_2\colon B \times F \to F'$ (partial homeomorphism between open subsets), such that $\mathrm{Def}\, u_2 = (\mathrm{Def}\, \bar{u}) \times F$ and for every $b \in \mathrm{Def}\, \bar{u}$, $u_2(b, -)\colon F \to F'$ is a homeomorphism. Thus, provided that the morphism u is not empty, the fibres F and F' are homeomorphic.

The glueing completion $\mathrm{Mf}\, \mathcal{B}$ has for objects the « manifolds » $M = (B_i \times F_i, u^i_j)_I$ over $\mathcal{B}$, for morphisms their bilinked modules: it is the category of fibre bundles and partial maps. The inclusion $\mathcal{B} \to \mathcal{F}$ (or, more tightly, $\mathcal{B} \to \mathcal{F}_0$) extends to the *topological realization* functor $\mathrm{Mf}\, \mathcal{B} \to \mathcal{F}$ ($\mathrm{Mf}\, \mathcal{B} \to \mathcal{F}_0$), taking the above object M into its glueing (8.1.3).

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(10) Not to be confused, of course, with the selected endomaps which we call projections.

By the above characterization of the morphisms of Inv $\mathcal{B}$, the topological type of the fibre $F_b = p^{-1}(\{b\})$ at the point b of the base $B = \mathrm{gl}\,(B_i, \bar{u}_j^i)_I$ is locally constant, hence constant on every connected component of B.

8.3. *Vector bundles.* A *trivial vector bundle* is a trivial fibre bundle $p\colon B\times F \to B$, where B is a topological space and F is a finite-dimensional, real vector space (provided with the linear topology).

Let $\mathcal{V}$ be the subcategory of $\mathcal{B}$ (and $\mathcal{F}$) having such objects, with « fiberwise linear » morphisms $f\colon B\times F \to B'\times F'$: this means that, for every $b \in \mathrm{Def}\, f_0$, the (everywhere defined) mapping $f_2(b, -)\colon F \to F'$ (8.2.2) is $\mathbb{R}$-linear. A morphism $u\colon B\times F \to B'\times F'$ of Inv $\mathcal{V}$ is in Inv $\mathcal{B}$ (8.2); moreover, for every b in Def $\bar{u}$, $u_2(b, -)\colon F \to F'$ is a linear isomorphism.

The glueing completion Mf $\mathcal{V}$ yields bundles and their usual morphisms (partially defined, on distinguished pairs). Also here we have the topological realization into $\mathcal{F}$, or into $\mathcal{F}_0$.

8.4. *Differentiable manifolds and tangent bundles.* Consider again the category $\mathcal{C}^r$ (of trivial C^r-manifolds), with $r\geqslant 1$. The (trivial) tangent bundle functor, with the abuse of notations described in 8.2, is:

(1) $$T\colon \mathcal{C}^r \to \mathcal{V}\,, \qquad U \mapsto U\times\mathbb{R}^{\dim U}\,, \qquad f\mapsto Tf\,,$$

(2) $$Tf(x, h) = (fx, D_h f(x))\,, \qquad \text{for } x\in \mathrm{Def}\, f \text{ and } h\in\mathbb{R}^{\dim U}\,,$$

where $D_h f(x)$ is the derivative of f at x, along the vector h.

Since T is totally cohesive, it extends to a glucing functor, the tangent bundle functor Mf $\mathcal{C}^r \to$ Mf $\mathcal{V}$ for C^r-manifolds.

8.5. *Foliations.* A *trivial foliations* is a cartesian product $U\times V$, where U and V are open euclidean spaces; the subsets $V_x = \{x\}\times V$ are its *leaves* (for $x\in U$). A *partial C^r-map* $f\colon U\times V \to U'\times V'$ (of trivial foliations) is a partial C^r-mapping, defined on an open subset of $U\times V$, which preserves leaves: if (x, y_1) and (x, y_2) are in V_x, their f-images are in the same leaf of $U'\times V'$ ([11]).

All this forms the category $\mathcal{C}^r\mathcal{F}$ of trivial C^r-foliations and partial C^r-maps, ordered by restriction. It is a totally cohesive e-category, whose glueing completion Mf $\mathcal{C}^r\mathcal{F}$ yields C^r-foliations, with partial C^r-maps.

([11]) In other words: there exists a partial map $\bar{f}\colon U \to U'$ (also of class C^r) defined on $p(\mathrm{Def}\, f)$, such that $p' f \leqslant \bar{f} p$, where $p\colon U\times V \to V$ and analogously p'. Compare this with the stronger condition of commutativity in 8.1: a partial map of foliations need not be defined on a *union* of leaves.

9. – Proof of some completion theorems.

We prove here the ϱ-cohesive completion theorem (2.7) and the two theorems on the ϱ-glueing conpletion of an inverse category (6.4, 6.5).

9.1. *The category of linked ϱ-sets.* Let $\boldsymbol{A}$ be a category provided with a proximity relation ! (and no order): we embed $\boldsymbol{A}$ in a category $\mathcal{I}_\varrho\boldsymbol{A}$ with order and proximity satisfying (CH.1-3) and that part of (CH.5ϱ) which concerns joins.

The objects are the same. A morphism $\alpha \in \mathcal{I}_\varrho\boldsymbol{A}(A, B)$ is given by any ϱ-set $\alpha \subset \boldsymbol{A}(A, B)$, linked in $\boldsymbol{A}$ (including the empty subset O^A_B, if $O \in \varrho$). The composition of $\alpha: A \to B$ with $\beta: B \to C$ is obviously:

$$\beta\alpha = \{ba | a \in \alpha, b \in \beta\}, \tag{1}$$

which is again a linked ϱ-set of $\boldsymbol{A}$-morphisms from A to C.

$\mathcal{I}_\varrho\boldsymbol{A}$ is obviously a category, with identity of A given by the subset $\{1_A\}$; provide $\mathcal{I}_\varrho\boldsymbol{A}$ with the inclusion relation $\alpha \subset \alpha'$ (for parallel maps) and the linking relation:

$$\alpha ! \alpha \quad \text{if } a ! a' \text{ in } \boldsymbol{A}, \quad \text{for all } a \in \alpha, \ a' \in \alpha'. \tag{2}$$

Now (CH.1-3) are trivially satisfied. Let $\Sigma \subset \mathcal{I}_\varrho\boldsymbol{A}(A, B)$ be a linked ϱ-set of $\mathcal{I}_\varrho\boldsymbol{A}$ and let $\beta = \bigcup \Sigma \subset \boldsymbol{A}(A, B)$: this is again a ϱ-set (1.8) of parallel morphisms of $\boldsymbol{A}$, clearly linked; β is the join of the set Σ with respect to the order of $\mathcal{I}_\varrho\boldsymbol{A}$; the join is compositive: if $\gamma: A' \to A$ and $\delta: B \to B'$ are in $\mathcal{I}_\varrho\boldsymbol{A}$:

$$\delta\beta\gamma = \{dbc | c \in \gamma, b \in \bigcup \Sigma, d \in \delta\} = \bigcup_{\alpha \in \Sigma} \{dbc | c \in \gamma, b \in \alpha, d \in \delta\} = \bigvee \delta\alpha\gamma. \tag{3}$$

It may be noticed that $\mathcal{I}_\varrho\boldsymbol{A}$ has arbitrary non-empty meets; however these are not compositive, even in the binary case, and will play no role in the following steps.

9.2. PROOF OF THE ϱ-COHESIVE COMPLETION THEOREM (2.7). – Now $\boldsymbol{A}$ is a cohesive category and $\mathcal{I}_\varrho\boldsymbol{A}$ is the category of its linked ϱ-sets, constructed on $(\boldsymbol{A}, !)$.

Consider the following binary relation on parallel morphisms of $\mathcal{I}_\varrho\boldsymbol{A}$:

$$\alpha \prec \beta \quad \text{iff } \alpha ! \beta \text{ and } \forall a \in \alpha, \quad a = \bigvee_{b \in \beta} (a \wedge b) \quad \text{(linked join)}. \tag{1}$$

It is a preorder of categories: if $\alpha \prec \beta \prec \gamma$: $a = \bigvee a \wedge b$ $(b \in \beta)$ and $b = \bigvee b \wedge c$ $(c \in \gamma)$; let $b \in \beta$, $c \in \gamma$: from $b ! c$ it follows that $(a \wedge b) ! c$ (for $a \in \alpha$) and $a = \bigvee (a \wedge b) ! c$;

thus $a!c$ for all $c \in \gamma$; moreover, by the property 1.7 c), we have a compositive join: $a = \bigvee_b \big(a \wedge (\bigvee_c (b \wedge c))\big) = \bigvee_{b,c} (a \wedge b \wedge c) = \bigvee (a \wedge c)$ (for $b \in \beta$, $c \in \gamma$), which is easily seen to be distributive (in the sense of 2.3.1). This preorder is consistent with composition because linked joins and meets are so.

Let $\sim$ be the congruence associated to $\prec$ and consider the quotient category:

(2) $$\varrho c\boldsymbol{A} = \mathcal{P}_\varrho \boldsymbol{A}/\!\sim\,,$$

provided with the order $\leqslant$ induced by the preorder $\prec$: $[\alpha] \leqslant [\beta]$ iff $\alpha \prec \beta$ (independently from the choice of representatives). The linking relation is defined by: $[\alpha]![\beta]$ iff $\alpha!\beta$ as linked sets of $\boldsymbol{A}$ (again, independently from choice).

For (CH.4, 5ϱ), linked meets and linked ϱ-joins are calculated in $\varrho c\boldsymbol{A}$ by the following formulas:

(3) $$[\alpha] \wedge [\beta] = \{a \wedge b \,|\, a \in \alpha,\, b \in \beta\}\,, \qquad \text{for } [\alpha]![\beta]\,;$$

(4) $$\bigvee \Sigma' = [\bigcup \Sigma]\,,$$

where Σ is any linked ϱ-set of ϱ-sets of $\boldsymbol{A}$ ($\alpha!\alpha'$, for all $\alpha, \alpha' \in \Sigma$) and $\Sigma' = \{[\alpha] \,|\, \alpha \in \Sigma\}$.

Last, define the functor $\eta\colon \boldsymbol{A} \to \varrho c\boldsymbol{A}$ taking the object A into itself and the morphism a into the equivalence class of $\{a\}$. Clearly, it reflects the order and linking relations, it is cohesive and preserves the existing linked ϱ-joins of $\boldsymbol{A}$. To verify the universal property, set $G([\alpha]) = \bigvee Fa$ ($a \in \alpha$) and check that G is a ϱ-cohesive functor; its uniqueness is trivial.

9.3. PROOF OF THEOREM 6.4. –

a) $\boldsymbol{M}$ has a natural regular *involution*:

(1) $$\big((a^i_h)_{I,H}\big)^{\sim} = (\tilde{a}^i_h)_{H,I}\,,$$

(2) $$(a^i_h)\cdot(\tilde{a}^i_h)\cdot(a^i_h) = \big(\bigvee_{k,j} a^j_h \tilde{a}^j_k a^i_k\big) = (a^i_h)_{I,H}\,,$$

where the last equality follows from $a^j_h \tilde{a}^j_k a^i_k \leqslant a^j_h u^i_j \leqslant a^i_h$ for all k and j, with equality for $j = i$ and $k = h$.

b) We prove now the equivalence of i)-iv), where a projection is any idempotent endomap, symmetrical with respect to the above involution.

i) $\Rightarrow$ ii) $(a^i_j) = e = \tilde{e}e = \big(\bigvee_h \tilde{a}^j_h a^i_h\big)$ and $a^i_j = \bigvee_h \tilde{a}^j_h a^i_h \leqslant u^i_j$.

ii) $\Rightarrow$ iv) $e_i = a^i_i \leqslant u^i_i = 1$ is a projection of U_i and:

(3) $$a^i_j = a^i_j \tilde{a}^i_j a^i_j \leqslant a^i_j e_i \leqslant u^i_j e_i = u^i_j a^i_i \leqslant a^i_j\,,$$

so that $a^i_j = u^i_j e_i$ and: $u^j_i e_j u^i_j = a^j_i u^i_j \leqslant a^i_i = e_i$, for all i, j.

iv) $\Rightarrow$ iii) It is easy to show that $u^i_j e_i = e_j u^i_j$. The family $e = (e^i_j) = (u^i_j e_i)_{I,I}$ is an endomorphism of U, as (for $i, j, h \in I$):

$$u^j_h e^i_j = u^j_h \cdot (u^i_j e_i) \leqslant u^i_h e_i = e^i_h , \tag{4}$$

$$\tilde{e}^h_j e^i_j = (e_h u^j_h)(u^i_j e_i) \leqslant u^i_h , \tag{5}$$

iii) $\Rightarrow$ ii)

$$(\tilde{e}e)^i_j = \bigvee_h \tilde{a}^j_h a^i_h = \bigvee_h (e_j \tilde{u}^j_h u^i_h e_i) \leqslant e_j u^i_j e_i = a^i_j ; \tag{6}$$

$$(\tilde{e}e)^i_j = \bigvee_h \tilde{a}^j_h a^i_h \geqslant a^j_j a^i_j = e_j u^i_j e_i = a^i_j . \tag{7}$$

c) $\boldsymbol{M}$ is *inverse*. We just need to show that the product of two parallel projections $e = (e^i_j)$, $f = (f^i_j)$ is a projection:

$$(ef)^i_j = \bigvee_h e^h_j f^i_h = \bigvee_h (e_j u^h_j u^i_h f_i) = e_j u^i_j f_i , \tag{8}$$

$$(ef)^i_i = e_i f_i \in \mathrm{Prj}\, A_i , \qquad (ef)^i_j = e_j u^i_j f_i = u^i_j \cdot (ef)^i_i = (ef)^j_j \cdot u^i_j . \tag{9}$$

Moreover, the property 6.4.1 is an easy consequence of the following inequality: $f^k_h a^j_k e^i_j = f_h v^k_h \cdot a^i_k \cdot u^i_j e_i \leqslant f_h a^i_h e_i$ (with equality for $j = i$ and $k = h$).

d) We check now the characterization 6.4.2 of the order of $\boldsymbol{M}$. If $a^i_h \leqslant b^i_h$, for all $i \in I$, $h \in H$:

$$(a\tilde{b}a)^i_h = \bigvee_{k,j} a^j_h \tilde{b}^j_k a^i_k \geqslant \bigvee_{k,j} a^j_h \tilde{a}^j_k a^i_k = (a\tilde{a}a)^i_h = a^i_h , \tag{10}$$

$$(a\tilde{b}a)^i_h = \bigvee_{k,j} a^j_h \tilde{b}^j_k a^i_k \leqslant \bigvee_{k,j} a^j_h \tilde{b}^j_k b^i_k \leqslant \bigvee_{k,j} a^j_h u^i_j \leqslant a^i_h , \tag{11}$$

hence $a\tilde{b}a = a$ and $a \leqslant b$ in $\boldsymbol{M}$. Conversely, if the last property holds, $a = be$ for some projection e of $\boldsymbol{M}$ and:

$$a^i_h = (be)^i_h = \bigvee_j b^j_h e^i_j = \bigvee_j b^j_h u^i_j e_i \leqslant \bigvee_j b^i_h = b^i_h . \tag{12}$$

e) Finally we prove the characterization 6.4.3 of the linking relation in $\boldsymbol{M}$.

First, assume that $a!b$ in the inverse category $\boldsymbol{M}$; then $\tilde{b}a$ and $b\tilde{a}$ are projections, and $\tilde{b}^j_h a^i_h \leqslant (\tilde{b}a)^i_j \leqslant u^i_j$ for all $i, j \in I$ and $h \in H$ (property ii)). Analogously for $b\tilde{a}$.

Now, if the previous conditions hold, $\tilde{b}^j_h a^i_h \leqslant u^i_j = 1$ and $b^i_h \tilde{a}^i_h \leqslant v^h_h = 1$, i.e. $a^i_h ! b^i_h$ in $\boldsymbol{K}$ (for all i, h); moreover: $(a^i_h \vee b^i_h)_{I,H}$ is a linked module:

$$v^h_k(a^i_h \vee b^i_h) = (v^h_k a^i_h) \vee (v^h_k b^i_h) \leqslant a^i_k \vee b^i_k , \tag{13}$$

$$(a^j_h \vee b^j_h)^{\sim} \cdot (a^i_h \vee b^i_h) = (\tilde{a}^j_h a^i_h) \vee (\tilde{a}^j_h b^i_h) \wedge (\tilde{b}^j_h a^i_h) \vee (\tilde{b}^j_h b^i_h) \leqslant u^i_j . \tag{14}$$

Last, if $x = (a^i_h \vee b^i_h)_{I,H}$ is a linked module: $a, b \leqslant x$ (by (2)), and $a!b$.

As to 6.4.4, if $a!b$ one shows as before that $y = (a^i_h \wedge b^i_h)_{I,H}$ is a linked module; by 6.4.2, $x = a \vee b$ and $y = a \wedge b$. The last remark follows now easily from our previous characterization of projections.

9.4. PROOF OF THE INVERSE GLUEING COMPLETION THEOREM (6.5).

a) $\boldsymbol{M}$ is ϱ-cohesive. Assume that α is a linked ϱ-set of parallel maps $(a^i_h)_{I,H}$: $(U_i, u^i_j)_I \to (V_h, v^h_k)_H$ and write $\alpha^i_h\colon U_i \to V_h$ the ϱ-set of its i, h-components, which is linked by the characterization 6.4.3; set $b^i_h\colon U_i \to V_h$ the join of the former set in $\boldsymbol{K}$. It is now easy to check that $b = (b^i_h)$ is the linked join of α.

b) $\boldsymbol{M}$ is ϱ-glueing: we have to show that each ϱ-manifold $U = (U^r, Z^{rs})_R$ of $\boldsymbol{M}$ has a glueing in $\boldsymbol{M}$. The manifold U is given by objects:

$$U^r = (U^r_i, u^r_{ij})_I \ (^{12}), \tag{1}$$

with glueing morphisms:

$$Z^{rs}\colon U^r = (U^r_i, u^r_{ij})_I \to U^s = (U^s_i, u^s_{ij}), \tag{2}$$

$$Z^{rs} = (z^{rs}_{ij}\colon U^r_i \to U^s_j)_{i,j\in I} \qquad (r, s \in R), \tag{3}$$

verifying the following conditions (for $r, s, t \in R$ and $i, j, h \in I$):

$$Z^{rr} = 1, \qquad \text{i.e. } z^{rr}_{ij} = u^r_{ij}, \tag{4}$$

$$Z^{st}\cdot Z^{rs} \leqslant Z^{rt}, \qquad \text{i.e. } z^{st}_{jh}\cdot z^{rs}_{ij} \leqslant z^{rt}_{ih}, \tag{5}$$

$$(Z^{rs})^{\sim} = Z^{sr}, \qquad \text{i.e. } (z^{rs}_{ij})^{\sim} = z^{sr}_{ji}. \tag{6}$$

Now the ϱ-diagram over $\boldsymbol{K}$: $X = (U^r_i, z^{rs}_{ij})_{R\times I}$, is in $\boldsymbol{M}$ by (4)-(6). It is provided with natural maps (for $r \in R$):

$$Z^r\colon U^r = (U^r_i, u^r_{ij})_I \to X = (U^r_i, z^{rs}_{ij})_{R\times I}, \tag{7}$$

$$Z^r = (z^{rs}_{ij}\colon U^r_i \to U^s_j)_{i\in I, (s,j)\in R\times I}, \tag{8}$$

verifying the characterization 6.2.1-3 for the glueing:

$$(Z^s\cdot Z^{rs})_{i,t,h} = \textstyle\bigvee_j (z^{st}_{jh}\cdot z^{rs}_{ij}) \leqslant z^{rt}_{ih} = (Z^r)_{i,t,h}, \tag{9}$$

$$(\tilde{Z}^s\cdot Z^r)_{i,j} = \textstyle\bigvee_{t,h} (\tilde{z}^{st}_{jh}\cdot z^{rt}_{ih}) = \bigvee_{t,h} (z^{ts}_{hj}\cdot z^{rt}_{ih}) = z^{rs}_{ij} = (Z^{rs})_{i,j}, \tag{10}$$

$$(\textstyle\bigvee_r Z^r\cdot \tilde{Z}^r)_{r,j} = \bigvee_{r,i} (z^{rs}_{ij}\tilde{z}^{rs}_{ij}) = \bigvee_{r,i} (z^{rs}_{ji} z^{sr}_{ji}) = z^{ss}_{jj} = (1_X)_{s,j}. \tag{11}$$

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(12) Clearly it is possible to index all the manifolds U^r on the *same* ϱ-set I.

c) Finally the embedding $\boldsymbol{K} \to \boldsymbol{M}$ satisfies this universal property: if F: $\boldsymbol{K} \to \boldsymbol{A}$ is a ϱ-cohesive functor with values into a ϱ-glueing inverse category, there is exactly one ϱ-cohesive functor G: $\boldsymbol{M} \to \boldsymbol{A}$ extending F. Obviously one takes $G(U_i, u^i_j)_I$ to be the glueing of the manifold $(FU_i, Fu^i_j)_I$ in $\boldsymbol{A}$.

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