Annali di Matematica pura ed applicata
(IV), Vol. CLVII (1990), pp. 183-197

Some Remarks on the Schemes W_d^r (*).

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Summary. – *Let X be an irreducible smooth projective curve of genus g. Let $\varrho_d^r(g)$ be the Brill-Noether Number. In this paper we prove some results concerning the schemes W_d^r of special divisors.* 1) *Suppose* $\dim(W_{d-1}^r) = \varrho_{d-1}^r(g) \geqslant 0$ *and* $\varrho_d^r(g) < g$. *If W_{d-1}^r is a reduced (resp. irreducible) scheme, then W_d^r is a reduced (resp. irreducible) scheme.* 2) *Under certain conditions, if Z is a generically reduced irreducible component of W_{d-1}^r then $Z \oplus W_1^0$ is a generically reduced irreducible component of W_d^r. For $r = 1$, we obtain some further results in this direction.* 3) *As an application of it we are able to prove some dimension theorems for the schemes W_d^1.*

1. – Introduction.

Let X be a smooth irreducible projective curve of genus $g \geqslant 1$ and let $J(X)$ be the jacobian of X. This is an abelian variety of dimension g which can be identified with $\mathrm{Pic}^0(X)$, the Picard scheme of the invertible sheafs of degree 0 on X. We always make this identification. Let P_0 be a fixed base point on X and let $X^{(d)}$ be the d-th symmetric product. We have a natural morphism

$$I(d)\colon\ X^{(d)} \to J(X)\colon\ D \to [\mathcal{O}_X(D - dP_0)]$$

(if L is an invertible sheaf of degree 0 on X, then $[L]$ is the corresponding point on $J(X)$). Consider

$$W_d^r = \{x \in J(X)\colon \dim([I(d)]^{-1}(x)) \geqslant r\} = \{[L] \in \mathrm{Pic}^0(X)\colon h^0(L(dP_0)) \geqslant r+1\}\,.$$

Those subsets are different from $J(X)$ if $r > d - g$. Those subsets play a central role in the study of special linear systems. The Riemann-Roch Theorem tells us that it is enough to study the case $d \leqslant g-1$. On W_d^r there exists a natural scheme structure. For more details concerning the general theory of the schemes W_d^r we refer to [2], especially Chapter IV.

(*) Entrata in Redazione il 23 settembre 1988.
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Those schemes W_d^r are very well-known if X is a general curve. This is the so-called Brill-Noether Theory. We refer to [2], Chapter V, for a summary of the most important results of that theory. If X is an arbitrary curve, the behaviour of the schemes W_d^r is far from being well-understood. It is the aim of this paper to present some results in this direction.

An important known result is the following.

$$\text{If } \dim(W_d^r) \geqslant r+1 \text{ then } W_{d-1}^r \neq \emptyset \text{ and } \dim(W_{d-1}^r) \geqslant \dim(W_d^r) - (r+1)\,.$$

This is proved in [7] as a consequence of the theory develloped in [6]. A consequence of this statement is the following statement.

Suppose $W_d^r \neq J(X)$ and $\varrho_{d-1}^r(g) \geqslant 0$ (Brill-Noether Number).

$$\text{If } \dim(W_d^r) > \varrho_d^r(g) \text{ then } \dim(W_{d-1}^r) > \varrho_{d-1}^r(g)\,.$$

Hence failure with respect to Brill-Noether behaviour for large d implies failure for $d_0(g, r)$, where

$$d_0(g, r) = \min\{d\colon \varrho_d^r(g) \geqslant 0\}\,.$$

By making a closer analysis of the proof of the mentioned result we can push on this philosophy a little bit as follows.

RESULT 1 (Theorem 4). – Suppose $\dim(W_{d-1}^r) = \varrho_{d-1}^r(g) \geqslant 0$ and $\varrho_d^r(g) < g$.

a) If W_{d-1}^r is a reduced scheme then W_d^r is a reduced scheme.

b) If W_{d-1}^r is an irreducible scheme then W_d^r is an irreducible scheme.

In the proof of this result we use the description of the tangent space to W_d^r at a point $x \in W_d^r \setminus W_d^{r+1}$ by means of the Petri map (see [2], Chapter IV). Using this description we are able to prove the following fact relating W_d^r to W_{d+1}^r.

RESULT 2 (Theorem 5). – Suppose Z is a generically reduced irreducible component of $W_d^r \neq J(X)$. Suppose that, for a general point z on Z one has

$$h^1(L_z^{\otimes 2}(2dP_0)) \neq 0$$

(i.e. $2D$, where D is a divisor of degree d associated to z, is a special divisor). Then $Z \oplus W_1^0$ is a generically reduced irreducible component of W_{d+1}^r.

(If $A, B \subset J(X)$ then $A \oplus B = \{a + b\colon a \in A \text{ and } b \in B\}$. We also use $A \ominus B = \{a - b\colon a \in A \text{ and } b \in B\}$.)

We prove that the condition on the general point z is always satisfied if $Z \not\subset W_{d-1}^r \oplus W_1^0$ and $\dim(Z) > 2d - g - 2r$ (this follows from Remark 6). In particular the condition in Result 2 is always satisfied if $r = 1$ and $\dim(Z) > \varrho_d^1(g)$.

In the case $r = 1$ we make some further remarks on Result 2 (see Corollary 8; Proposition 9 and Problem 10). The conclusion of Result 2 can also be expressed as follows. If g_d^r is the linear system on X associated to a general point on Z and if P is a general point on X, then $g_d^r + P = \{D + P: D \in g_d^r\}$ is not the specialization of a g_{d+1}^r on X without fixed points.

We also discuss the following dimension problem (see [12], p. 280). Let $j \in \mathbf{Z}_{\geqslant 0}$.

$P(j)$ Suppose $g \geqslant 2j + 4$ and $j + 3 \leqslant d \leqslant g - 1 - j$.

Suppose $\dim(W_d^1) = d - 2 - j$.

Is it true that $\dim(W_{j+3}^1) = 1$?

This question is answered affirmatively in the following cases.

$j = 0$: H. Martens' Theorem (see [13]; see also [2], p. 191);

$j = 1$: D. Mumford's Theorem (see [15]; see also [2], p. 193);

$j = 2$: C. Keem, but only for the cases $g \geqslant 11$ (see [11]);

$j = 3$: G. Martens, but only for the cases $g \geqslant 15$;

(G. Martens also has a proof for the cases $4 \leqslant j \leqslant 7$ for g sufficiently large).

As an application of our methods we are able, amongst others, to prove the following contributions to this problem.

RESULT 3 (Propositions 12 and 13 and Theorem 15).

1) $P(2)$ is true.

2) $P(3)$ holds for the cases $g \geqslant 12$.

3) $P(j)$ holds for arbitrary j in the cases $g \geqslant (j + 1)(2j + 1)$.

The reason why we cannot prove $P(3)$ in the case $g = 11$ is very much related to problems mentioned earlier in this paper (Remark 14).

2. – Notations and conventions.

Besides those from [2], we use the following notations and conventions. Let $x \in J(X)$. Then L_x is the corresponding invertible sheaf of degree 0 on X and $g_d(x)$ is the complete linear system associated to $L_x(dP_0)$ (see [9], p. 157). We write ω_X for the canonical sheaf on X and K_X to denote some effective canonical divisor on X. We write k to denote $[I(2g - 2)](K_X)$. For $x \in J(X)$, the Gieseker-Petri homomorphism associated to $L_x(dP_0)$ is the cup-product homomorphism

$$\mu_d(x)\colon H^0(X, L_x(dP_0)) \otimes H^0(X, \omega_X \otimes L_x^{-1}(-dP_0)) \to H^0(X, \omega_X)\,.$$

If Z is some projective set then we write $\dim(Z)$ to denote

$$\sup(\{\dim(A)\colon A \text{ is an irreducible component of } Z\})\,.$$

3. – Results.

The starting point for our investigations is the next proposition which is proved in [7].

Proposition 1. – Let $r \in \mathbf{Z}_{\geqslant 1}$. Let $A \subset W_d^r$ be an irreducible closed subset satisfying $\dim(A) \geqslant r+1$. Then $A \cap W_{d-1}^r$ contains some irreducible component B satisfying $\dim(B) \geqslant \dim(A) - (r+1)$. (This can also be proved from the results in [6] using the arguments of Theorem 11 in [4].)

If, in Proposition 1, we have $B \not\subset W_d^{r+1}$, then $g_d(x)$ is a complete linear system g_d^r on X for x a general point of B. Since $B \subset W_{d-1}^r$ the base point P_0 is a fixed point of $g_d(x)$. We are going to get more information from Proposition 1 by varying the base point.

Theorem 2. – Let A be as in Proposition 1. If $\dim(W_{d-1}^r) \leqslant \dim(A) - (r+1)$, then there exists an irreducible component B of W_{d-1}^r satisfying $\dim(B) = \dim(A) - (r+1)$ such that $B \oplus W_1^0 \subset A$.

Proof. – For $P \in X$ let $W_{d,P}^r \subset J(X)$ be defined in the same way as W_d^r using P as a base point instead of P_0. Then $A \oplus [\mathcal{O}_X(dP_0 - dP)]$ is a closed subset of $W_{d,P}^r$. From Proposition 1 we obtain $B'_P \subset W_{d-1,P}^r$ satisfying

$$\dim(B'_P) \geqslant \dim(A) - (r+1)\,,$$
$$B'_P \subset A \oplus [\mathcal{O}_X(dP_0 - dP)]\,.$$

It follows that

$$B_P := B'_P \oplus [\mathcal{O}_X((d-1)P - (d-1)P_0)] \subset W_{d-1}^r\,,$$
$$B_P \oplus [I(1)](P) \subset A\,,$$
$$\dim(B_P) \geqslant \dim(A) - (r+1)\,.$$

Suppose that $\dim[(W_{d-1}^r \oplus W_1^0) \cap A] \leqslant \dim(A) - (r+1)$ (hence we have equality).

Consider the diagram

$$\begin{array}{ccccc} W^0_1 & \xleftarrow{p_1} & Z \subset W^r_{d-1} \times W^0_1 & & (x, y) \\ & & \downarrow p_2 \qquad \downarrow « + » & & \downarrow \\ & & A \subset \quad J(X) & & x + y \end{array}$$

where $Z = (« + »)^{-1}(A)$ and p_1 is the restriction to Z of the projection morphism $W^r_{d-1} \times W^0_1 \to W^0_1$. We obtain that p_1 is surjective and each fibre of p_1 contains an irreducible component of dimension at least $\dim(A) - (r+1)$. Hence, there exists an irreducible component $\tilde{Z}$ of Z dominating W^0_1 with

$$\dim(\tilde{Z}) \geqslant \dim(A) - r .$$

Our assumption gives us that $\dim\big(p_2(\tilde{Z})\big) < \dim(\tilde{Z})$. On the other hand, p_2 is injective on the fibres of p_1. It follows that $\dim\big(p_2(\tilde{Z})\big) = \dim(\tilde{Z}) - 1$ and for each $x \in p_2(\tilde{Z})$ and for each $P \in X$ there exists $y \in W^r_{d-1}$ such that

$$y + [I(1)](P) = x .$$

If $x \notin W^{r+1}_d$ then each point P on X would be a fixed point of $g_d(x)$. This is of course impossible, hence $p_2(\tilde{Z}) \subset W^{r+1}_d$. But this would imply that

$$\dim(W^{r+1}_d) \geqslant \dim(A) - (r+1) .$$

But $W^r_{d-1} \supset W^{r+1}_d \ominus W^0_1$, hence we would obtain that

$$\dim(W^r_{d-1}) \geqslant \dim(A) - r ,$$

a contradiction to the assumptions. As a corollary, we obtain that

$$\dim[(W^r_{d-1} \oplus W^0_1) \cap A] \geqslant \dim(A) - r .$$

But then, our assumptions give us the existence of an irreducible component B of W^r_{d-1} satisfying $\dim(B) = \dim(A) - (r+1)$ such that $B \oplus W^0_1 \subset A$.

In order to get a more detailed result, we study the following situation. Let $r, d \in \mathbf{Z}_{\geqslant 1}$ such that $\varrho^r_d(g) < g$. Let $x \in W^r_{d-1} \setminus W^{r+1}_{d-1}$ and consider $x \oplus W^0_1 \subset W^r_d$.

LEMMA 3. – Let y be a general point of W^0_1.

(i) If $\dim\big(T_x(W^r_{d-1})\big) = \varrho^r_{d-1}(g)$ then $\dim\big(T_{x+y}(W^r_d)\big) = \varrho^r_d(g)$.

(ii) If $\dim\big(T_x(W^r_{d-1})\big) > \varrho^r_{d-1}(g)$, then

$$\dim\Big(\ker\big(\mu_d(x+y)\big)\Big) \leqslant \dim\Big(\ker\big(\mu_{d-1}(x)\big)\Big) - 1 .$$

PROOF. – We are going to make use of the following well-known fact. If $x \in W_d^r \setminus W_d^{r+1}$ then, for the tangent space $T_x(W_d^r)$ of W_d^r at x we have

$$\dim\left(T_x(W_d^r)\right) = \varrho_d^r(g) + \dim\left(\ker\left(\mu_d(x)\right)\right).$$

Let $y = [I(1)](P)$ with P a general point on X. Since $g_{d-1}(x)$ is special, P is a fixed point of $g_d(x+y)$ and $x + y \in W_d^r \setminus W_d^{r+1}$. Hence we have a natural identification between

$$H^0\big(X, L_x((d-1)P_0)\big) \quad \text{and} \quad H^0\big(X, L_x((d-1)P_0 + P)\big) = H^0\left(X, L_{x+y}(dP_0)\right).$$

Also

$$H^0\left(X, \omega_X \otimes L_{x+y}^{-1}(-dP_0)\right) = H^0\big(X, \omega_X \otimes L_x^{-1}(-(d-1)P_0 - P)\big)$$

can be considered in a natural way as a subspace of $H^0\big(X, \omega_X \otimes L_x^{-1}(-(d-1)P_0)\big)$. Under those identifications, we obtain a commutative triangle

$$\begin{array}{ccc} H^0\left(X, L_{x+y}(dP_0)\right) \otimes H^0\left(X, \omega_X \otimes L_{x+y}^{-1}(-dP_0)\right) & & \\ \big\downarrow & \overset{\mu_d(x+y)}{\searrow} & \\ & & H^0(X, \omega_X) \\ & \underset{\mu_{d-1}(x)}{\nearrow} & \\ H^0\big(X, L_x((d-1)P_0)\big) \otimes H^0\big(X, \omega_X \otimes L_x^{-1}(-(d-1)P_0)\big). & & \end{array}$$

It follows that $\ker\left(\mu_d(x+y)\right) \subset \ker\left(\mu_{d-1}(x)\right)$. From this, (i) follows immediately.

Suppose $\dim\left(\ker\left(\mu_{d-1}(x)\right)\right) > 0$. Assume that

$$(*) \qquad \ker\left(\mu_d(x+y)\right) = \ker\left(\mu_{d-1}(x)\right).$$

Suppose $\{s_0, \dots, s_r\}$ is a $\mathbf{C}$-basis for $H^0\big(X, L_x((d-1)P_0)\big)$ and fix $\sum (a_i s_i \otimes t_i : 0 \leqslant i \leqslant r)$, a nonzero element of $\ker\left(\mu_{d-1}(x)\right)$ for some $t_i \in H^0\big(X, \omega_X \otimes L_x^{-1}(-(d-1)P_0)\big)$. It follows from $(*)$ that $t_i(P) = 0$ for $0 \leqslant i \leqslant r$ while P is a general point on X. This is of course impossible. This proves (ii).

THEOREM 2(bis). – In the situation of Theorem 2, if $\dim(A) > \varrho_d^r(g)$ then B is a multiple irreducible component of W_{d-1}^r.

PROOF. – Let x be a general point on B and let y be a general element of W_1^0. From [2], p. 182, Lemma 3.5, it follows that $x \notin W_{d-1}^{r+1}$ and since it concerns special divisors, also $x + y \notin W_d^{r+1}$. Since $B \oplus W_1^0 \subset A$, we have

$$\dim\left(T_{x+y}(W_d^r)\right) = \varrho_d^r(g) + \dim\left(\ker\left(\mu_d(x+y)\right)\right) \geqslant \dim(A).$$

Hence $\dim\big(\ker(\mu_d(x+y))\big) \geqslant \dim(A) - \varrho^r_d(g)$. Since $\dim(A) > \varrho^r_d(g)$ also $\dim(B) > \varrho^r_{d-1}(g)$. Applying Lemma 3 (ii) we obtain that

$$\dim\big(\ker(\mu_{d-1}(x))\big) \geqslant \dim\big(\ker(\mu_d(x+y))\big) + 1 \geqslant \dim(A) + 1 - \varrho^r_d(g) =$$

$$= \dim(B) + (r+1) + 1 - \varrho^r_{d-1}(g) - (r+1) = \dim(B) - \varrho^r_{d-1}(g) + 1 .$$

It follows that

$$\dim\big(T_x(W^r_{d-1})\big) = \varrho^r_{d-1}(g) + \dim\big(\ker(\mu_{d-1}(x))\big) \geqslant \dim(B) + 1 .$$

Assume that $\varrho^r_d(g) \geqslant 0$ and $W^r_{d+1} \neq J(X)$. The next theorem indicates that good behaviour with respect to Brill-Noether Theory for W^r_d implies the same for W^r_{d+1}.

THEOREM 4. – (i) If $\dim(W^r_d) = \varrho^r_d(g)$ then $\dim(W^r_{d+1}) = \varrho^r_{d+1}(g)$. Suppose that $\dim(W^r_d) = \varrho^r_d(g)$.

(ii) If W^r_d is a reduced scheme then W^r_{d+1} is a reduced scheme.

(iii) If W^r_d is an irreducible scheme then W^r_{d+1} is an irreducible scheme.

PROOF. – (i) follows immediately from Proposition 1. In order to prove (ii) and (iii) we start by making the observation that, since $\dim(W^r_d) = \varrho^r_d(g)$, we already know from (i) that $\dim(W^r_{d+1}) = \varrho^r_{d+1}(g)$. It follows that W^r_{d+1} is a Cohen-Macauly scheme (see [6], Remark 2.8). The Unmixedness Theorem (see e.g. [14], 16.D) gives us that W^r_{d+1} has a multiple component A if W^r_{d+1} would not be a reduced scheme. Suppose that A is a multiple irreducible component of W^r_{d+1} and assume that W^r_d would be reduced. We can apply Theorem 2 which proves the existence of an irreducible component B of W^r_d satisfying $B \oplus W^0_1 \subset A$. Since W^r_d is reduced, a general point x on B satisfies $\dim\big(T_x(W^r_d)\big) = \varrho^r_d(g)$. From Lemma 3 (i) it follows that a general point y on W^0_1 satisfies $\dim\big(T_{x+y}(W^r_d)\big) = \varrho^r_{d+1}(g)$. This is a contradiction to the fact that $x+y$ belongs to a multiple irreducible component of W^r_{d+1}. This proves (ii). Next, suppose that W^r_d is irreducible while W^r_{d+1} is not. Let A and B be two different components of W^r_{d+1}. Using Theorem 2 again, we obtain that $W^r_d \oplus W^0_1 \subset A \cap B \subset \mathrm{Sing}(W^r_{d+1})$. But, as before, we find that, if x is a general point on W^r_d and if y is a general point on W^0_1 then $\dim\big(T_{x+y}(W^r_{d+1})\big) = \varrho^r_{d+1}(g)$, a contradiction. This proves (iii).

In the case $r = 1$, the proof of Theorem 2(bis) also gives us:

If A is a generically reduced irreducible component of W^1_d with $\dim(A) > \varrho^1_d(g)$, then $A \oplus W^0_1$ is a generically reduced irreducible component of W^1_{d+1}.

For $r > 1$ we have the following generalization.

THEOREM 5. – Let A be a generically reduced irreducible component of W_d^r of dimension $\varrho_d^r(g) + s$ with $s > 0$. Suppose that a general point x on A satisfies

$$h^0(\omega_X \otimes L_x^{-2}(-2dP_0)) \neq 0 .$$

Then $A \oplus W_1^0$ is a generically reduced irreducible component of W_{d+1}^r.

PROOF. – The assumptions give us $\dim\big(\ker(\mu_d(x))\big) = s$. Let

$$0 \neq h \in H^0(X, \omega_X \otimes L_x^{-2}(-2dP_0)) .$$

Let $y = [I(1)](P)$ be a general point on W_1^0 (hence P a general point on X). Let $s_0, \ldots, s_r$ be a base of $H^0(X, L_x(dP_0))$ with

$$s_0(P) \neq 0 ; \quad s_i(P) = 0 \quad \text{for } 1 \leqslant i \leqslant r .$$

Consider

$$G = \langle\{s_i \otimes s_j h - s_j \otimes s_i h \colon 0 \leqslant i < j \leqslant r\}\rangle$$

(here $\langle\ \rangle$ means the linear span). This is a subvectorspace of $\ker(\mu_d(x))$ of dimension $(r+1)r/2$. As already mentioned in the proof of Lemma 3, we can consider $\ker(\mu_{d+1}(x+y))$ as a linear subspace of $\ker(\mu_d(x))$. One has

$$G \cap \ker(\mu_{d+1}(x+y)) = \langle\{s_i \otimes s_j h - s_j \otimes s_i h \colon 0 < i < j \leqslant r\}\rangle$$

hence $G \cap \ker(\mu_{d+1}(x+y))$ has codimension r in G. It follows that $\ker(\mu_{d+1}(x+y))$ has codimension at least r in $\ker(\mu_d(x))$. Therefore $\dim(T_{x+y}(W_{d+1}^r)) \leqslant \dim(A) + 1$. This proves the theorem.

REMARK 6. – Suppose A is an irreducible component of W_d^r of dimension $\varrho_d^r(g) + s$ with $s > 0$. Let x be a general point on A and suppose that F is the fixed divisor of $g_d(x)$. Let $F + D$ be a general element of $g_d(x)$. Then D imposes at least

$$h^0(\omega_X \otimes L_x^{-1}(-dP_0)) - h^0(\omega_X \otimes L_x^{-1}(-dP_0 - D))$$

conditions on $\operatorname{im}(\mu_d(x))$, hence

$$\dim\big(\operatorname{im}(\mu_d(x))\big) \geqslant 2h^0(\omega_X \otimes L_x^{-1}(-dP_0)) - h^0(\omega_X \otimes L_x^{-1}(-dP_0 - D)) .$$

In particular

$$\dim\big(\operatorname{im}(\mu_d(x))\big) \geqslant 2(r - d + g) - h^0(\omega_X \otimes L_x^{-2}(-2dP_0)) - \deg(F)$$

and

$$h^0(\omega_X \otimes L_x^{-2}(-2dP_0)) \geqslant 2r - 2d + g - \deg(F) + \dim(A)\,.$$

This gives us information about the assumption made in Theorem 5.

If $g_d(x)$ is birationally ample, we can use the so-called Accola-Griffiths-Harris Theorem (see [8], p. 73) which gives us

$$h^0(\omega_X \otimes L_x^{-2}(-2dP_0)) \geqslant \dim(A) - 2d + g + 3r - 1\,.$$

In the case $r = 1$, we have equality in Remark 6, namely

LEMMA 7. – Let $x \in W_d^1 \setminus W_d^2$ and let F be the fixed divisor of the associated linear system $g_d(x)$. One has

$$\dim(T_x(W_d^1)) = h^0(L_x^2(2dP_0 - F)) - 3 + \deg(F)\,.$$

PROOF. – Let s_1, s_2 be a base for $H^0(X, L_x(dP_0))$. For the associated divisors one has

$$(s_1) = E_1 + F\,; \qquad (s_2) = E_2 + F$$

and $\mathrm{Supp}(E_1) \cap \mathrm{Supp}(E_2) = \emptyset$. Suppose

$$s_1 \otimes t_1 + s_2 \otimes t_2 \in \ker(\mu_d(x))\,.$$

It is easy to see that this is equivalent to the existence of $s \in H^0(X, \omega_X \otimes \otimes L_x^{-2}(F - 2dP_0))$ such that

$$t_1 = s_2 s \quad \text{and} \quad t_2 = -s_1 s\,.$$

It follows that

$$\dim\big(\ker(\mu_d(x))\big) = h^0(\omega_X \otimes L_x^{-2}(F - 2dP_0))$$

hence

$$\dim(T_x(W_d^1)) = h^0(L_x^2(2dP_0 - F)) - 3 + \deg(F)\,.$$

COROLLARY 8. – Let A be a generically reduced irreducible component of W_d^1 of dimension $\varrho_d^1(g) + s$ with $s > 0$. Then, for $0 \leqslant s' \leqslant s$, $A \oplus W_{s'}^0$ is a generically reduced irreducible component of $W_{d+s'}^1$.

PROOF. – This can be proved immediately from Lemma 7 (see [5], Corollary 2.11).

Using Lemma 7, we can also prove

PROPOSITION 9. – Let A be an irreducible component of W_d^1 of dimension $\varrho_d^1(g) + s$. Suppose that, for a general point x on A, one has $\dim(T_x(W_d^1)) =$

$= \varrho_d^1(g) + s + 1$. Then, for $0 \leqslant s' \leqslant s$, one has that $A \oplus W_{s'}^0$ is a multiple irreducible component of $W_{d+s'}^1$.

PROOF. – Suppose $s \geqslant s' > 0$ and let B be an irreducible component of $W_{d+s'}^1$ containing $A \oplus W_{s'}^0$. Let x be a general point on A and let y be a general point on $W_{s'}^0$ (i.e. $y = [I(s')](D)$ with D a general point on $W^{(s')}$). From [5], Corollary 2.8 (proved as an application of Lemma 7) we obtain that

$$\dim\big(T_{x+y}(W_{d+s'}^1)\big) = \dim(A) + s' + 1\,.$$

If $B \neq A \oplus W_{s'}^0$ then $\dim(B) = \dim(A) + s' + 1$. Suppose s' is the smallest value for which such a component B exists. Then, for a general point z on B, the linear system $g_{d+s'}(z)$ is a linear system $g_{d+s'}^1$ without fixed points. From Lemma 7 we obtain that

$$h^0\big(\omega_X \otimes L_z^{-2}(-2(d+s')P_0)\big) \geqslant s - s' + 1\,.$$

Using the Semicontinuity Theorem (see [9], p. 288) we obtain that

$$h^0(\omega_X \otimes L_x^{-2}(-2dP_0 - 2D)) \geqslant s - s' + 1$$

and, since $D \in X^{(s')}$ is general, we have

$$h^0(\omega_X \otimes L_x^{-2}(-2dP_0)) \geqslant s + s' + 1\,.$$

Since $s' \geqslant 1$ this gives us a contradiction to the assumption that

$$\dim\big(T_x(W_d^1)\big) = \varrho_d^1(g) + s + 1\,.$$

PROBLEM 10. – It would be very interesting to know whether or not the following situation occurs for some smooth curve X.

> A is an irreducible component of $W_d^1 \neq J(X)$ such that, for a general point x on A, one has
>
> $$\dim\big(T_x(W_d^1)\big) \geqslant \dim(A) + 2$$
>
> and $A \oplus W_1^0$ is not an irreducible component of W_{d+1}^1.

This problem is strongly related to dimension problems on the schemes W_d^1, as we shall see.

EXAMPLES 11. – Multiple irreducible components A for the schemes W_d^1 satisfying $\dim\big(T_x(W_d^1)\big) = \dim(A) + 1$ for a general point x on A occur. In [3], it is proved that, for each $2d - 2 \leqslant g \leqslant (d-1)^2$, there exists a smooth curve X of genus g having a point x on W_d^1 such that $\{x\}$ is an irreducible component of W_d^1 and $\dim\big(T_x(W_d^1)\big) = 1$.

Using Proposition 9, one finds higher dimensional such multiple irreducible components.

Another very explicite example is found in [5], Theorem 5.9: if X is a smooth plane curve of degree $d \geq 9$, then W^1_{3d-8} has an irreducible component A such that $\dim (T_x(W^1_{3d-8})) = \dim (A) + 1$ for x a general point on A.

We are going to study problem $P(j)$ mentioned in the Introduction.

As already mentioned Statements $P(0)$ and $P(1)$ are true.

PROPOSITION 12. – Statement $P(2)$ is true.

PROOF. – As already mentioned in the Introduction Statement $P(2)$ is proved by C. KEEM in [11] (see also the batch of exercises in [2], pp. 200, 201, 202) for the cases $g \geq 11$. So we only have to prove $P(2)$ for the cases $g = 10$ and $g = 9$.

Suppose $g = 10$. In [11] it is also proved that $\dim (W^1_6) = 2$ implies $\dim (W^1_5) = 1$. So, we only have to prove that $\dim (W^1_7) = 3$ implies $\dim (W^1_6) = 2$.

Suppose A is an irreducible component of W^1_7 of dimension 3 while $\dim (W^1_6) \leq 1$. From Theorem 2 it follows that there exists an irreducible component B of A satisfying $\dim (B) = 1$ and $B \oplus W^0_1 \subset A$. Since $A \not\subset W^1_6 \oplus W^0_1$, it follows from Lemma 7 that, for a general point a on A, one has

$$h^0(L_a^2(14P_0)) \geq 6 .$$

Using the Semicontinuity Theorem it follows that, for a general point b on B and a general point P on X, one has

$$h^0(L_b^2(12P_0 + 2P)) \geq 6 .$$

It follows that $L_b^2(12P_0 + 2P)$ is special. In particular also $L_b^2(12P_0)$ is special. Since P is a general point on X it is not an inflection point of the complete linear system associated to $\omega_X \otimes L_b^{-2}(-12P_0)$, in particular

$$h^0(\omega_X \otimes L_b^{-2}(-12P_0 - 2P)) = h^0(\omega_X \otimes L_b^{-2}(-12P_0)) - 2 .$$

From the Theorem of Riemann-Roch, it follows that

$$h^0(L_b^2(12P_0)) \geq 6$$

i.e. $2b \in W^5_{12}$. In particular we obtain that $\dim (W^5_{12}) \geq 1$ hence $\dim (W^1_8) \geq 5$. Now we can apply $P(1)$ which gives us that $\dim (W^1_4) \geq 1$. It would follow that $\dim (W^1_6) \geq 3$, a contradiction to the assumptions.

Suppose $g = 9$. We only have to prove that $\dim(W_6^1) = 2$ implies $\dim(W_5^1) = 1$. The proof is exactly the same as before; we leave it to the reader.

PROPOSITION 13. – Statement $P(3)$ holds for $g \geqslant 12$.

PROOF. – G. MARTENS proved $P(3)$ for $g \geqslant 15$ in [12].

a) Case $g = 14$.

Step 1: $\dim(W_{10}^1) = 5$ implies $\dim(W_9^1) = 4$. Suppose A is an irreducible component of W_{10}^1 of dimension 5 and suppose that $\dim(W_9^1) \leqslant 3$. From Theorem 2 it follows that there exists an irreducible component B of W_9^1 satisfying

$$\dim(B) = 3 \quad \text{and} \quad B \oplus W_1^0 \subset A .$$

Let a be a general point on A. Since $A \not\subset W_9^1 \oplus W_1^0$, $g_{10}(a)$ is a complete g_{10}^1 without fixed points. If b is a general point on B then $h^0(L_b^2(18P_0)) \geqslant 8$ (using the same arguments as in the proof of Proposition 12). It follows that $\dim(W_{18}^7) \geqslant 3$, hence $\dim(W_{12}^1) \geqslant 9$. From $P(1)$ we obtain that $\dim(W_4^1) = 1$, in particular $\dim(W_9^1) \geqslant 6$, hence a contradiction.

Step 2: $\dim(W_9^1) = 4$ implies $\dim(W_8^1) = 3$. This can be proved in the same way as Step 1. We leave it to the reader.

Step 3: $\dim(W_8^1) = 3$ implies $\dim(W_7^1) = 2$. Suppose A is an irreducible component of W_8^1 of dimension 3 and suppose that $\dim(W_7^1) \leqslant 1$. From Theorem 2 we obtain that there exists an irreducible component B of W_7^1 satisfying

$$\dim(B) = 1 \quad \text{and} \quad B \oplus W_1^0 \subset A .$$

Let a be a general point on A and let b be a general point on B. As before one can prove that $h^0(L_{2b}(14P_0)) \geqslant 6$. Consider the cup-product homomorphism

$$\mu\colon H^0(X, L_a(8P_0)) \otimes H^0(X, L_{2b}(14P_0)) \to H^0(X, L_{a+2b}(22P_0)) .$$

If $h^0(L_{a+2b}(22P_0)) \leqslant 10$, then $\dim(\ker(\mu)) \geqslant 2$. Because of the Base point free pencil trick (see [2], p. 126) we obtain that $h^0(L_{2b-a}(6P_0)) \geqslant 2$. As a corollary, we would obtain that $\dim(W_6^1) \geqslant 3$. In particular $\dim(W_7^1) \geqslant 4$, which gives us a contradiction. Suppose that, for a general $a \in A$ and $b \in B$, we have $h^0(L_{a+2b}(22P_0)) > 10$. Then $\dim(W_{22}^{10}) \geqslant 3$ hence $\dim(W_{13}^1) \geqslant 12$. This is impossible.

Step 4: $\dim(W_7^1) = 2$ implies $\dim(W_6^1) = 1$. This can be proved in the same way as Step 3; we leave it to the reader.

b) Case $g = 13$.

Similar arguments as those used in Case *a*) can be used. We leave it to the reader.

c) Case $g = 12$.

Step 1: $\dim(W_8^1) = 3$ implies $\dim(W_7^1) = 2$. This can be proved in the same way as Step 1 of Case *a*). We leave it to the reader.

Step 2: $\dim(W_7^1) = 2$ implies $\dim(W_6^1) = 1$. Suppose A is an irreducible component of W_7^1 of dimension 2. Because of Theorem 2 there exists $b \in W_6^1$ such that $b \oplus W_1^0 \subset A$. Using the base point free pencil trick as we did in the proof of Case *a*) Step 3, we obtain a contradiction unless, for a general point a on A, one has $h^0(L_{a+2b}(19P_0)) \geq 9$. Upper-Semicontinuity gives that, for a general point P on X, $h^0(L_{3b}(18P_0 + P)) \geq 9$. From the Theorem of Riemann-Roch it follows that $L_{3b}(18P_0 + P)$ is special hence, since P is general on X, $h^0(L_{3b}(18P_0)) \geq 9$. Using the Riemann-Roch Theorem, we obtain that $h^0(\omega_X \otimes L_{3b}^{-1}(-18P_0)) \geq 2$ hence $W_4^1 \neq \emptyset$. It follows that $\dim(W_6^1) \geq 2$, which gives us a contradiction.

REMARK 14. – In order to give a complete proof for $P(3)$ we have to prove that, for $g = 11$, $\dim(W_7^1) = 2$ implies $\dim(W_6^1) = 1$. It is enough to prove that the following situation does not occur.

> X is a smooth curve of genus 11; x is an isolated point of W_6^1 satisfying $\dim(T_x(W_6^1)) \geq 2$ and $x \oplus W_1^0$ is not an irreducible component of W_7^1.

This is Problem 10 in a special case.

THEOREM 15. – Suppose $j \geq 4$. $P(j)$ holds for $g > (j+1)(2j+1)$.

PROOF. – It is well-known that, if $g \geq 4j + 3$ and $2j + 2 \leq d \leq g - j$, then $\dim(W_d^1) = d - 2 - j$ implies $\dim(W_{2j+2}^1) = j$ (see [10], Theorem 1). Hence we can assume that for some $j + 3 < d \leq 2j + 2$, one has an irreducible component A of W_d^1 of dimension $d - 2 - j$ while $\dim(W_{d-1}^1) \leq d - 4 - j$. From Theorem 2 we obtain the existence of a component B of W_{d-1}^1 satisfying

$$\dim(B) = d - 4 - j \quad \text{and} \quad B \oplus W_1^0 \subset A .$$

Since $g > (j+1)(2j+1)$ clearly $g > d(d-1)/2$. Also because of the assumptions, it follows that $A \not\subset W_{d-1}^1 \oplus W_1^0$. Hence if a_1 and a_2 are general points on A then $g_d(a_1)$ and $g_d(a_2)$ are linear systems g_d^1 on X without fixed points. From computations in [1] (see also [4], Proposition 3) it follows that $g_d(a_1)$ and $g_d(a_2)$ are compounded of the same involution. If they would be compounded of some rational involution then X would have a linear system g_a^1 for some $a|d$. Since $d \leqslant 2j+2$ it would follow that $a \leqslant j+1$ hence $W_{j+1}^1 \neq \emptyset$. This would imply that $\dim(W_{d-1}^1) \geqslant d-j-2$, a contradiction to the assumption. Hence $g_d(a_1)$ and $g_d(a_2)$ are compounded of the same non-rational involution. But X possesses only a finite number of non-rational involutions of degree $a \leqslant j+1$ (see [16]). As a corollary, there exists a smooth curve X' of genus $g' \geqslant 1$ and a morphism $f\colon X \to X'$ of some degree $a|d$ such that for the associated morphism $f^*\colon J(X') \to J(X)$ one has (taking $f(P_0)$ as base point for X') that there exists an irreducible component A' of $W_{d/a}^1$ of dimension $d-2-j$ satisfying $f^*(A') = A$. In particular $B \oplus W_1^0 \subset f^*(A')$. Since B is an irreducible component of W_{d-1}^1 one has that for b a general point on B, the linear system $g_{d-1}(b)$ is 1-dimensional (see [2], p. 182). In particular, if $x = [I(1)](P)$ is a general point on W_1^0, then $g_d(b+x)$ is a linear system g_d^1 and P is a fixed point of it. Since $b + x \in f^*(A')$, there exists a linear system $g_{d/a}^1$ on X' such that

$$g_d^1 = \{f^{-1}(D)\colon D \in g_{d/a}^1\}\ .$$

But P is a fixed point of g_d^1, hence $f(P)$ is a fixed point of $g_{d/a}^1$. Since P is general, $f^{-1}(f(P))$ contains a point P' different from P and P' is also a fixed point of g_d^1, hence P' is a fixed point of $g_{d-1}(b)$. Varying P on X we obtain infinitely many fixed points for $g_{d-1}(b)$ which is absurd.

REMARK 16. – Using a more detailed analysis using arguments as those used in e.g. [12], we are able to obtain a better lower bound for g. Since this lower bound is still of order $O(j^2)$ as $j \to \infty$ it seems not very useful to reproduce such a long proof here.

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