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Four-Dimensional Almost Kähler Einstein Manifolds (*).

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Summary. – *We prove that a four-dimensional compact almost Kähler manifold which is Einsteinian and $*$-Einsteinian is a Kähler manifold.*

1. – Introduction.

An almost Hermitian manifold $M = (M, g, J)$ is called an *almost Kähler manifold* if the corresponding Kähler form Ω is closed (i.e. $d\Omega = 0$ or equivalently,

$$\underset{X,Y,Z}{\mathfrak{S}} g((\nabla_X J)Y, Z) = 0$$

for all smooth vector fields X, Y, Z on M; $\mathfrak{S}$ denotes the cyclic sum). M is a Kähler manifold if $\nabla J = 0$ and so it is necessarily an almost Kähler manifold. Moreover, if the almost complex structure of an almost Kähler manifold M is integrable, then M is Kählerian.

An almost Kähler manifold is also necessarily a symplectic manifold and as is well-known, the first compact example of non-Kählerian almost Kähler manifolds appeared in [4], [11]. In this context, the following conjecture of S. GOLDBERG is interesting [1]:

CONJECTURE. – *The almost complex structure of a compact almost Kähler Einstein manifold is integrable.*

As concerns this conjecture some progress has been made under some curvature conditions [1], [6], [7], [8]. For example, in [8] the first author of this paper proved that the above conjecture is true when the scalar curvature is non-negative.

In [10] S. TACHIBANA introduced the Ricci $*$-tensor of an almost Hermitian manifold. On a Kähler manifold this tensor coincides with the Ricci tensor. An almost Hermitian manifold is said to be a $*$-*Einstein manifold* if the Ricci $*$-tensor

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is a constant multiple of the Riemannian metric. The main purpose of this paper is to prove

THEOREM A. – *Let $M = (M, g, J)$ be a four-dimensional compact almost Kähler manifold which is Einsteinian and $*$-Einsteinian. Then M is a Kähler manifold.*

From this we get immediately

COROLLARY B. – *Let $M = (M, g, J)$ be a compact four-dimensional almost Kähler manifold of constant sectional curvature. Then M is a locally flat Kähler manifold.*

This extends a result of Z. OLSZAK who proved a similar result for arbitrary almost Kähler manifolds M with $\dim M \geqslant 8$ [6].

2. – Preliminaries.

Let $M = (M, g, J)$ be a $2n$-dimensional almost Hermitian manifold. We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X, Y) = g(X, JY)$ and $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for $X, Y \in \mathfrak{X}(M)$, the Lie algebra of all smooth vector fields on M. Note that the Nijenhuis tensor field satisfies the condition

$$N(JX, Y) = -JN(X, Y), \qquad X, Y \in \mathfrak{X}(M).$$

Further, we denote by ∇, R, ϱ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. Here the curvature tensor R is defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{2.1}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Furthermore, we denote by ϱ^* and τ^* the *Ricci $*$-tensor* and the *$*$-scalar curvature* defined respectively by

$$\varrho^*(x, y) = g(Q^*x, y) = \operatorname{trace}(z \mapsto R(x, Jz)Jy),$$

$$\tau^* = \operatorname{trace} Q^*,$$

where $x, y, z \in T_pM$, $p \in M$. Hence, by using the first Bianchi identity, we have

$$\varrho^*(x, y) = -\tfrac{1}{2}\sum_{i=1}^{2n} R(x, Jy, e_i, Je_i)$$

for an arbitrary orthonormal basis $\{e_i, i = 1, \dots, 2n\}$. (See [10], [12].)

The following decomposition for the vector bundle $\Lambda^2 M$ of real 2-forms over M is well-known [5]:

$$\Lambda^2 M = \mathbf{R}\Omega \oplus \Lambda_0^{1,1} M \oplus LM\,, \quad \text{(Whitney sum)}\,, \tag{2.2}$$

where $\Lambda_0^{1,1} M$ denotes the bundle of real primitive J-invariant 2-forms and LM the bundle of real primitive J-skew invariant 2-forms over M.

We define the 2-forms φ and ψ on M by

$$\varphi(x, y) = \operatorname{trace}\left(z \mapsto J(\nabla_x J)(\nabla_y J) z\right) \tag{2.3}$$

and

$$\psi(x, y) = \operatorname{trace}\left(z \mapsto R(x, y) Jz\right) \tag{2.4}$$

for $x, y, z \in T_p M$, $p \in M$. Then the first Chern form γ of M is given by

$$8\pi\gamma = -\varphi + 2\psi\,. \tag{2.5}$$

It is well-known that the first Chern class $c_1(M)$ of M (in the de Rham cohomology group) is represented by the 2-form γ [2].

In this paper, we shall adopt the following notational conventions:

$$\begin{aligned} & R_{hijk} = g\left(R(e_h, e_i) e_j, e_k\right), \\ & R_{\bar{h}ijk} = g\left(R(Je_h, e_i) e_j, e_k\right), \\ & \dots \\ & R_{\bar{h}\bar{i}\bar{j}\bar{k}} = g\left(R(Je_h, Je_i) Je_j, Je_k\right); \\ & \varrho_{ij} = \varrho(e_i, e_j), \dots, \varrho_{\bar{i}\bar{j}} = \varrho(Je_i, Je_j)\,, \\ & \varrho^*_{ij} = \varrho^*(e_i, e_j), \dots, \varrho^*_{\bar{i}\bar{j}} = \varrho^*(Je_i, Je_j); \\ & J_{ij} = g(Je_i, e_j)\,, \quad \nabla_i J_{jk} = g\left((\nabla_{e_i} J) e_j, e_k\right), \end{aligned} \tag{2.6}$$

and so on, where $\{e_i,\ i = 1, \dots, 2n\}$ is an orthonormal basis of the tangent space $T_p M$ at $p \in M$.

3. – Some results in almost Kähler geometry.

In this section we shall prove some elementary formulas and results for almost Kähler manifolds which are Einsteinian and $*$-Einsteinian.

LEMMA 3.1. – *Let $M = (M, g, J)$ be a $2n$-dimensional $*$-Einstein almost Hermitian manifold. Then we have*

$$\psi = -\frac{\tau^*}{n}\,\Omega\,.$$

PROOF. – By the definition (2.4) of the 2-form ψ we get

$$\psi_{ij} = \sum_k R_{ij\bar{k}k} = \sum_k R_{\bar{i}\bar{j}k\bar{k}} = -2\varrho^*_{i\bar{j}} = -\frac{\tau^*}{n}\,\delta_{i\bar{j}}\,,$$

which proves the required formula.

Further, the following equalities are known for an almost Kähler manifold [13]:

$$\|\nabla J\|^2 = 2(\tau^* - \tau)\,, \tag{3.1}$$

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0\,, \tag{3.2}$$

$$g(N(X, Y), Z) = 2g(J(\nabla_Z J)X, Y)\,, \tag{3.3}$$

for $X, Y, Z \in \mathfrak{X}(M)$. (3.2) means that M is a quasi-Kähler manifold.

From now on we assume that $M = (M, g, J)$ is in addition Einsteinian and $*$-Einsteinian. Then, by Lemma 3.1 and (3.2), we see immediately that *the 2-form ψ is closed and coclosed.* Further we have

LEMMA 3.2. – $\sum_{j,k,a,b} (\nabla_j J_{ik})(\nabla_k J_{ab})(\nabla_j J_{ab}) = 0.$

PROOF. – Using (3.2) we get

$$\sum_{j,k,a,b} (\nabla_j J_{ik})(\nabla_k J_{ab})\nabla_j J_{ab} = \sum_{j,k,a,b} (\nabla_{\bar{j}} J_{i\bar{k}})(\nabla_{\bar{k}} J_{ab})\nabla_{\bar{j}} J_{ab} =$$
$$= -\sum_{j,k,a,b} (\nabla_j J_{ik})(\nabla_k J_{\bar{a}b})\nabla_j J_{\bar{a}b} = -\sum_{j,k,a,b} (\nabla_j J_{ik})(\nabla_k J_{ab})\nabla_j J_{ab}\,,$$

from which the desired result follows.

LEMMA 3.3. – $\sum_{j,k,a,b} J_{ik}(\nabla_k J_{ab})\nabla^2_{jj} J_{ab} = 0.$

PROOF. - We use the first Bianchi identity and the Ricci identity to get

$$\sum_{j,k,a,b} J_{ik}(\nabla_k J_{ab})\nabla^2_{jj}J_{ab} = -\sum_{j,k,a,b} J_{ik}(\nabla_k J_{ab})\nabla^2_{ja}J_{bj} - \sum_{j,k,a,b} J_{ik}(\nabla_k J_{ab})\nabla^2_{jb}J_{ja} =$$

$$= \sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})(R_{jabt}J_{tj} + R_{jajt}J_{bt}) +$$

$$+ \sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})(R_{jbjt}J_{ta} + R_{jbat}J_{jt}) =$$

$$= \tfrac{1}{2} \sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})(R_{ajtb} - R_{atjb})J_{tj} -$$

$$-\frac{\tau}{2n}\sum_{k,a,b} J_{ik}(\nabla_k J_{ab})J_{ba} - \frac{\tau}{2n}\sum_{k,a,b} J_{ik}(\nabla_k J_{ab})J_{ba} +$$

$$+ \tfrac{1}{2} \sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})(R_{bjta} - R_{btja})J_{jt} =$$

$$= -\tfrac{1}{2}\sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})R_{abjt}J_{tj} - \tfrac{1}{2}\sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})R_{bajt}J_{jt} =$$

$$= -\sum_{j,k,a,b,t} J_{ik}(\nabla_k J_{ab})R_{abjt}J_{tj} =$$

$$= 2\sum_{k,a,b} J_{ik}(\nabla_k J_{ab})\varrho^*_{ab} = \frac{\tau^*}{n}\sum_{k,a,b} J_{jk}(\nabla_k J_{ab})J_{ba} = 0\,.$$

LEMMA 3.4. - $\sum_{j,k,a,b} J_{ik}(\nabla^2_{jk}J_{ab})\nabla_j J_{ab} = 0.$

PROOF. - In a similar way as for Lemma 3.3 we have

$$\sum_{j,k,a,b} J_{ik}(\nabla^2_{jk}J_{ab})\nabla_j J_{ab} = \sum_{j,k,a,b} J_{ik}(\nabla^2_{kj}J_{ab})\nabla_j J_{ab} - 2\sum_{j,k,a,b,t} R_{jkat}J_{ik}J_{tb}\nabla_j J_{ab} =$$

$$= 2\sum_{j,k,a,b,t} R_{jkat}J_{ik}J_{tb}\nabla_a J_{bj} + 2\sum_{j,k,a,b,t} J_{ik}R_{jkat}J_{tb}\nabla_b J_{ja} =$$

$$= \sum_{j,k,a,b,t} J_{ik}(R_{kjta} - R_{ktja})J_{tb}\nabla_a J_{bj} + \sum_{j,k,a,b,t} J_{ik}(R_{jkat} - R_{akjt})J_{tb}\nabla_b J_{ja} =$$

$$= -\sum_{j,k,a,b,t} J_{ik}R_{kajt}J_{tb}\nabla_a J_{bj} - \sum_{j,k,a,b,t} J_{ik}R_{ajkt}J_{tb}\nabla_b J_{ja} =$$

$$= -\sum_{j,k,a,b,t} J_{ik}R_{katj}J_{jb}\nabla_a J_{bt} - \sum_{j,k,a,b,t} J_{ik}R_{tjka}J_{ab}\nabla_b J_{jt} =$$

$$= 2\sum_{j,k,a,b,t} J_{ik}R_{kajt}J_{ab}\nabla_b J_{jt} =$$

$$= 4\sum_{a,b}\nabla_b(\varrho^*_{ai}J_{ab}) - 2\sum_{j,k,a,b,t}(\nabla_b J_{ik})R_{kajt}J_{ab}J_{jt} - 2\sum_{j,k,a,b,t} J_{ik}(\nabla_b R_{kajt})J_{ab}J_{jt} =$$

$$= \frac{2\tau^*}{n}\sum_b \nabla_b J_{ib} - 4\sum_{k,a,b}(\nabla_b J_{ik})\varrho^*_{kb} - 2\sum_{j,k,a,b,t} J_{ik}R_{abjt}(\nabla_k J_{ab})J_{jt} =$$

$$= -4\sum_{k,a,b} J_{ik}\varrho^*_{ab}\nabla_k J_{ab} = \frac{2\tau^*}{n}\sum_{k,a,b} J_{ik}J_{ab}\nabla_k J_{ab} = 0\,.$$

By the definition of the 2-form φ (see (2.3)) we get

$$\varphi_{ij} = \sum_{k,a,b} J_{ik}(\nabla_k J_{ab})\nabla_j J_{ab} . \tag{3.4}$$

Further, from (3.4) we obtain

$$\sum_j \nabla_j \varphi_{ij} = \sum_{j,k,a,b} (\nabla_j J_{ik})(\nabla_k J_{ab})\nabla_j J_{ab} + \\ + \sum_{j,k,a,b} J_{ik}(\nabla^2_{jk} J_{ab})\nabla_j J_{ab} + \sum_{j,k,a,b} J_{ik}(\nabla_k J_{ab})\nabla^2_{jj} J_{ab} . \tag{3.5}$$

By using Lemma 3.2, Lemma 3.3 and Lemma 3.4, (3.5) yields

$$\delta\varphi = 0 . \tag{3.6}$$

Moreover, since $d\gamma = 0$, by (3.6) and Lemma 3.1, we have the following

PROPOSITION 3.5. – *Let $M = (M, g, J)$ be an Einstein, $*$-Einstein almost Kähler manifold. Then the 2-forms φ and ψ are both closed and coclosed.*

Note that when M is almost Kählerian, then φ is J-invariant.

4. – Proof of Theorem A.

Let $M = (M, g, J)$ be a four-dimensional Einstein, $*$-Einstein almost Kähler manifold. By (3.2) we have

$$\nabla\Omega = \alpha \otimes \Phi - J\alpha \otimes J\Phi , \tag{4.1}$$

where $\{\Phi, J\Phi\}$ is a local orthonormal frame field of the bundle LM in (2.2), α a local 1-form and $J\alpha$ the local 1-form defined by $(J\alpha)(X) = -\alpha(JX)$, for $X \in \mathfrak{X}(M)$ (see [7], [9]). Then, by (2.3) and (3.1) we get

$$\varphi = 2\alpha \wedge J\alpha \tag{4.2}$$

[7]. By (4.1) we also have

$$\|\nabla J\|^2 = 2\|\nabla\Omega\|^2 = 4\|\alpha\|^2 . \tag{4.3}$$

Now, we assume that M is not Kählerian, i.e. $\tau^* - \tau > 0$. Further, we define a local vector field A' by $g(A', X) = \alpha(X)$, $X \in \mathfrak{X}(M)$. Taking into account (3.1), (4.2) and (4.3), we may define a differentiable 2-dimensional J-invariant distribution

$\mathfrak{D}$ on M by

$$\mathfrak{D}(p) = \{x \in T_pM \mid \alpha(x) = (J\alpha)(x) = 0\} ,$$

$p \in M$. We denote by $\mathfrak{D}^\perp$ the orthogonal complement of $\mathfrak{D}$ in the tangent bundle TM over M. Then we see immediately that the distribution $\mathfrak{D}^\perp$ is also J-invariant. By Proposition 3.5 we have from (4.2)

$$(4.4) \qquad (\operatorname{div} A')g(JA', X) + g([A', JA'], X) - (\operatorname{div} JA')g(A', X) = 0 ,$$

for $X \in \mathfrak{X}(M)$ and

$$(4.5) \qquad g(JA', Z)g(A', [X, Y]) - g(A', Z)g(JA', [X, Y]) = 0 ,$$

for $X, Y \in \mathfrak{D}$, $Z \in \mathfrak{X}(M)$. From (4.4) we get

$$(4.6) \qquad g([A', JA'], X) = 0$$

for $X \in \mathfrak{D}$, and from this we see that $\mathfrak{D}^\perp$ is involutive. (Note that $\{A', JA'\}$ is a basis of $\mathfrak{D}^\perp$.) Similarly, from (4.5) we get

$$(4.7) \qquad g([X, Y], A') = g([X, Y], JA') = 0$$

for $X, Y \in \mathfrak{D}$. Hence $\mathfrak{D}$ is also involutive. So we have

PROPOSITION 4.1. - *Let M be a four-dimensional Einstein, $*$-Einstein almost Kähler manifold. Then the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are both involutive.*

Next, let $\{A, JA\}$, respectively $\{B, JB\}$, be a local orthonormal frame field of $\mathfrak{D}^\perp$, respectively $\mathfrak{D}$. Let L be any leaf of $\mathfrak{D}$ and $L^\perp$ any leaf of $\mathfrak{D}^\perp$. We denote by ∇^1, respectively ∇^2, the induced Riemannian connection on L, respectively $L^\perp$, and by σ_1, σ_2 the corresponding second fundamental forms of L and $L^\perp$ in M. Since $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are J-invariant, L and $L^\perp$ are holomorphic submanifolds of M. Further, since M is a quasi-Kähler manifold, L and $L^\perp$ are necessarily minimal submanifolds of M.

Now, we denote by D^1, respectively D^2, the Hermitian connection on the holomorphic vector bundle $\mathfrak{D}$, respectively $\mathfrak{D}^\perp$, defined by

$$(4.8) \qquad D^1_X\xi = g(\nabla_X\xi, B)B + g(\nabla_X\xi, JB)JB ,$$

$$(4.8') \qquad D^2_X\eta = g(\nabla_X\eta, A)A + g(\nabla_X\eta, JA)JA$$

for any $\xi \in \mathfrak{D}$, $\eta \in \mathfrak{D}^\perp$ and $X \in \mathfrak{X}(M)$. Finally, we denote by S^1_η, respectively S^2_η, the shape operator of L, respectively $L^\perp$, corresponding to η, respectively ξ.

We now prepare some formulas which we will need later on. Using (3.3) we obtain easily

$$g(\sigma_2(A, JA), B) - g(J\sigma_2(A, A), B) = \tfrac{1}{2} g(N(A, JB), A) , \tag{4.9}$$

$$g(\sigma_2(A, A), B) - g(\sigma_2(A, JA), JB) = \tfrac{1}{2} g(N(A, B), A) . \tag{4.10}$$

On the other hand, (3.3) and (4.1) yield

$$g(\sigma_1(X, JY), Z) = g(J\sigma_1(X, Y), Z) \tag{4.11}$$

for $X, Y \in \mathfrak{D}$, $Z \in \mathfrak{D}^{\perp}$. Hence (4.11) implies that each $L = (L, J = J|_L)$ is a 2-dimensional σ-submanifold.

Moreover, let K_1, respectively K_2, denote the curvature tensor of D^1, respectively D^2. Then we obtain

LEMMA 4.2. – *We have*

$$g(K_1(B, JB)B, JB) = g(R(B, JB)B, JB) + 2\|\sigma_1(B, B)\|^2 , \tag{4.12}$$

$$\begin{aligned} g(K_2(A, JA)A, JA) &= g(R(A, JA)A, JA) + 2\|\sigma_2(A, A)\|^2 - \\ &\quad - g\big(N(A, \sigma_2(A, A)), A\big) + \tfrac{1}{4}\|N(A, B)\|^2 , \end{aligned} \tag{4.13}$$

$$\begin{aligned} g(K_1(A, JA)B, JB) &= g(R(A, JA)B, JB) - 2\|\sigma_2(A, A)\|^2 + \\ &\quad + g(N(A, \sigma_2(A, A)), A) , \end{aligned} \tag{4.14}$$

$$g(K_2(B, JB)A, JA) = g(R(B, JB)A, JA) - 2\|\sigma_1(B, B)\|^2 . \tag{4.15}$$

PROOF. – First, (4.8) and (4.8′) yield

$$\begin{aligned} g(K_1(X, Y)B, JB) &= g(R(X, Y)B, JB) + g(\nabla_Y B, (\nabla_X J)B) - \\ &\quad - g(\nabla_X B, (\nabla_Y J)B) + g(\nabla_Y B, J\nabla_X B) - g(\nabla_X B, J\nabla_Y B) , \end{aligned} \tag{4.16}$$

$$\begin{aligned} g(K_2(X, Y)A, JA) &= g(R(X, Y)A, JA) + g(\nabla_Y A, (\nabla_X J)A) - \\ &\quad - g(\nabla_X A, (\nabla_Y J)A) + g(\nabla_Y A, J\nabla_X A) - g(\nabla_X A, J\nabla_Y A) . \end{aligned} \tag{4.16′}$$

Now, we prove (4.12). Using (4.11) we get

$$\begin{aligned} g(\nabla_{JB} B, J\nabla_B B) &= g\big(\nabla^1_{JB} B + \sigma_1(JB, B), J(\nabla^1_B B + \sigma_1(B, B))\big) = \\ &= g\big(\sigma_1(JB, B), J\sigma_1(B, B)\big) = \|\sigma_1(B, B)\|^2 . \end{aligned} \tag{4.17}$$

Similarly, we have

$$g(\nabla_B B, J\nabla_{JB} B) = -\|\sigma_1(B, B)\|^2 . \tag{4.18}$$

Hence, (4.12) follows from (4.16), (4.17) and (4.18). (4.13) may be derived in a similar way. Next, we prove (4.14). By a straightforward calculation we get

$$\begin{aligned} g(\nabla_{JA} B, (\nabla_A J)B) &= g(-S_B^2 JA + D_{JA}^2 B, \nabla_A(JB) - J\nabla_A B) = \\ &= g(-S_B^2 JA + D_{JA}^2 B, -S_{JB}^2 A + D_A^2 JB - J(-S_B^2 A + D_A^2 B)) = \\ &= g(S_B^2 JA, S_{JB}^2 A - JS_B^2 A) = \\ &= g(S_B^2 JA, A) g(S_{JB}^2 A - JS_B^2 A, A) + g(S_B^2 JA, JA) g(S_{JB}^2 A - JS_B^2 A, JA) = \\ &= g(\sigma_2(A, JA), B)^2 + g(\sigma_2(A, A), B)^2 + g(\sigma_2(A, JA), B) g(\sigma_2(A, A), JB) - \\ &- g(\sigma_2(A, A), B) g(\sigma_2(A, JA), JB) . \end{aligned} \tag{4.19}$$

Similarly, we have

$$\begin{aligned} g(\nabla_A B, (\nabla_{JA} J)B) = &- g(\sigma_2(A, A), B)^2 - g(\sigma_2(A, JA), B)^2 + \\ &+ g(\sigma_2(A, A), B) g(\sigma_2(A, JA), JB) - g(\sigma_2(A, A), JB) g(\sigma_2(A, JA), B) , \end{aligned} \tag{4.20}$$

$$\begin{aligned} g(\nabla_{JA} B, J\nabla_A B) = &- g(\sigma_2(A, JA), B)^2 - g(\sigma_2(A, A), B)^2 - \\ &- g(\sigma_2(A, JA), B) g(\sigma_2(A, JA), B) + g(\sigma_2(JA, JA), B) g(\sigma_2(A, A), B) , \end{aligned} \tag{4.21}$$

$$g(\nabla_A B, J\nabla_{JA} B) = g(\sigma_2(A, A), B)^2 + g(\sigma_2(A, JA), B)^2 . \tag{4.22}$$

Further, by (4.10), we obtain

$$g(\sigma_2(A, JA), JB) = g(\sigma_2(A, A), B) - \tfrac{1}{2} g(N(A, B), A)$$

and hence

$$\begin{aligned} g(\sigma_2(A, A), B) g(\sigma_2(A, JA), JB) = \\ = g(\sigma_2(A, A), B)^2 - \tfrac{1}{2} g(N(A, B), A) g(\sigma_2(A, A), B) . \end{aligned} \tag{4.23}$$

Similarly, (4.9) yields

$$\begin{aligned} g(\sigma_2(A, A), JB) g(\sigma_2(A, JA), B) = \\ = g(\sigma_2(A, A), JB)\{- g(\sigma_2(A, A), JB) + \tfrac{1}{2} g(N(A, JB), A)\} = \\ = - g(\sigma_2(A, A), JB)^2 + \tfrac{1}{2} g(N(A, JB), A) g(\sigma_2(A, A), JB) . \end{aligned} \tag{4.24}$$

Therefore, by (4.16) and (4.19)-(4.24), we obtain

$$g(K_1(A, JA)B, JB) = g(R(A, JA)B, JB) + \\ + 2g(\sigma_2(A, JA), B)g(\sigma_2(A, A), JB) - 2g(\sigma_2(A, A), B)g(\sigma_2(A, JA), JB) = \\ = g(R(A, JA)B, JB) + 2\Big\{\|\sigma_2(A, A)\|^2 - \tfrac{1}{2}\, g\big(N(A, \sigma_2(A, A)), A\big)\Big\} = \\ = g(R(A, JA)B, JB) + 2\|\sigma_2(A, A)\|^2 - g\big(N(A, \sigma_2(A, A)), A\big) .$$

(4.15) is obtained in a similar way.

Next, we define the 2-forms $\gamma_1 = \gamma(\mathfrak{D})$ and $\gamma_2 = \gamma(\mathfrak{D}^\perp)$ by

$$(4.25) \quad \gamma_1 = \gamma(\mathfrak{D}) = \\ = -\frac{1}{2\pi}\{g(K_1(B, JB)B, JB)\, i(B)\wedge i(JB) + g(K_1(B, A)B, JB)\, i(B)\wedge i(A) + \\ + g(K_1(B, JA)B, JB)\, i(B)\wedge i(JA) + g(K_1(JB, A)B, JB)\, i(JB)\wedge i(A) + \\ + g(K_1(JB, JA)B, JB)\, i(JB)\wedge i(JA) + g(K_1(A, JA)B, JB)\, i(A)\wedge i(JA)\} ,$$

$$(4.26) \quad \gamma_2 = \gamma(\mathfrak{D}^\perp) = \\ = -\frac{1}{2\pi}\{g(K_2(B, JB)A, JA)\, i(B)\wedge i(JB) + g(K_2(B, A)A, JA)\, i(B)\wedge i(A) + \\ + g(K_2(B, JA)A, JA)\, i(B)\wedge i(JA) + g(K_2(JB, A)A, JA)\, i(JB)\wedge i(A) + \\ + g(K_2(JB, JA)A, JA)\, i(JB)\wedge i(JA) + g(K_2(A, JA)A, JA)\, i(A)\wedge i(JA)\} .$$

Here i denotes the duality $TM \to \Lambda^1 M$ defined by means of the metric. Then the first Chern class $c_1(\mathfrak{D})$, respectively $c_1(\mathfrak{D}^\perp)$, of the Hermitian vector bundle $\mathfrak{D}$, respectively $\mathfrak{D}^\perp$, is represented by γ_1, respectively γ_2 (in the de Rham cohomology group).

In what follows we suppose that M is compact. First, we recall an integral formula for a four-dimensional, compact, Einsteinian almost Kähler manifold, established in [7]:

$$(4.27) \quad \int_M \Big\{\|R_{\Lambda^2_+ M}\|^2 - \Big\|R\Big(\frac{1}{\sqrt{2}}\,\Omega\Big)\Big\|^2\Big\}\, dM = -\frac{1}{8}\int_M \Big(\frac{\tau}{4} + \frac{1}{8}\,\|\nabla J\|^2\Big)\, \|\nabla J\|^2\, dM \geqslant 0 ,$$

where $dM = \frac{1}{2}\Omega^2$ is the volume form of M. We note that $\Lambda^2_+ M = \mathbf{R}\Omega \oplus LM$ and $\Lambda^2_- M = \Lambda^{1,1}_0 M$.

Now, (3.1) yields

$$(4.28) \quad \frac{\tau}{4} + \frac{1}{8}\,\|\nabla J\|^2 = \frac{\tau^*}{4} .$$

Thus, by (4.27) and (4.28), we have

PROPOSITION 4.3. – *Let $M = (M, g, J)$ be a four-dimensional, compact, Einsteinian almost Kähler manifold such that $\tau^* \geqslant 0$. Then M is a Kähler manifold.*

This result is a slight generalization of the result in [7] and it was also claimed by B. WATSON in his unpublished work.

Further, the 2-form φ may be written as

$$\varphi = 2\alpha \wedge J\alpha = 2\|\alpha\|^2 i(A) \wedge i(JA) \tag{4.29}$$

and moreover, we have for the Kähler form Ω:

$$\Omega = -i(B) \wedge i(JB) - i(A) \wedge i(JA) . \tag{4.30}$$

Since the tangent bundle TM over M is represented by the Whitney sum of the Hermitian vector subbundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$, the total Chern class $c(M) = c(TM)$ of M is given by

$$c(M) = c(\mathfrak{D}) \cdot c(\mathfrak{D}^\perp) \tag{4.31}$$

(in $H^*(M, \boldsymbol{R})$). From (4.31) we have in particular

$$c_1(M) = c_1(\mathfrak{D}) + c_1(\mathfrak{D}^\perp) . \tag{4.32}$$

Hence $\gamma - (\gamma_1 + \gamma_2)$ is an exact 1-form on M. Thus, using Proposition 3.5, we obtain

$$\int_M \{\gamma - (\gamma_1 + \gamma_2)\} \wedge \varphi = 0 . \tag{4.33}$$

Now, by Lemma 3.1, (2.5), (3.1), (4.2) and (4.3), we have

$$\begin{aligned} \gamma &= \frac{1}{8\pi}\left\{-\frac{\tau^*}{2}\,\Omega - 2\|\alpha\|^2 i(A) \wedge i(JA)\right\} = \\ &= \frac{1}{16\pi}\,\{\tau^* i(B) \wedge i(JB) + (2\tau - \tau^*)\, i(A) \wedge i(JA)\} , \end{aligned} \tag{4.34}$$

and by Lemma 4.2, (3.1), (4.2), (4.3), (4.25), (4.26), (4.29) and (4.30), we get

$$\begin{aligned} (\gamma_1 + \gamma_2) \wedge \varphi &= -\frac{1}{2\pi}(\tau^* - \tau)\{g(K_1(B, JB)B, JB) + g(K_2(B, JB)A, JA)\} \cdot \\ &\quad \cdot i(B) \wedge i(JB) \wedge i(A) \wedge i(JA) = \frac{\tau^* - \tau}{8\pi}\,\tau^* dM . \end{aligned} \tag{4.35}$$

Further, (4.3), (4.29) and (4.34) yield

$$\gamma \wedge \varphi = \frac{\tau^* - \tau}{16\pi} \tau^* dM . \tag{4.36}$$

Hence, from (4.35) and (4.36) we get

$$\{\gamma - (\gamma_1 + \gamma_2)\} \wedge \varphi = - \frac{\tau^* - \tau}{16\pi} \tau^* dM . \tag{4.37}$$

Therefore (4.33) and (4.37) imply $\tau^* = 0$ and hence, by Proposition 4.3, M is a Kähler manifold. This is a contradition and it completes the proof of Theorem A.

REFERENCES

[1] S. I. GOLDBERG, *Integrability of almost Kähler manifolds*, Proc. Amer. Math. Soc., **21** (1969), pp. 96-100.

[2] A. GRAY - M. BARROS - A. M. NAVEIRA - L. VANHECKE, *The Chern numbers of holomorphic vector bundles and formally holomorphic connections of complex vector bundles over almost complex manifolds*, J. Reine Angew. Math., **314** (1980), pp. 84-98.

[3] S. KOBAYASHI - K. NOMIZU, *Foundations of differential geometry, II*, Interscience Publ., New York (1969).

[4] K. KODAIRA, *On the structure of complex analytic surfaces, I*, Amer. J. Math., **86** (1964), pp. 751-798.

[5] P. LIBERMANN, *Classification and conformal properties of almost Hermitian structures*, Differential Geometry, János Bolyai Soc., **31**, Budapest (1979), pp. 371-391.

[6] Z. OLSZAK, *A note on almost Kähler manifolds*, Bull. Acad. Pol. Sci. sér. Sci. Math. Astronom. Phys., **26** (1978), pp. 139-141.

[7] K. SEKIGAWA, *On some 4-dimensional compact Einstein almost Kähler manifolds*, Math. Ann., **271** (1985), pp. 333-337.

[8] K. SEKIGAWA, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan, **39** (1987), pp. 677-684.

[9] K. SEKIGAWA, *On some 4-dimensional compact almost Hermitian manifolds*, J. Ramanujan Math. Soc., **2** (1987), pp. 101-116.

[10] S. TACHIBANA, *On almost-analytic vectors in almost-Kählerian manifolds*, Tôhoku Math. J., **11** (1959), pp. 247-265.

[11] W. P. THURSTON, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc., **55** (1976), pp. 467-468.

[12] F. TRICERRI - L. VANHECKE, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc., **267** (1981), pp. 365-398.

[13] K. YANO, *Differential geometry on complex and almost complex spaces*, Pergamon Press, New York (1965).