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Sobolev, Besov and Nikolskii Fractional Spaces: Imbeddings and Comparisons for Vector Valued Spaces on an Interval (*).

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Summary. – *We consider various fractional properties of regularity for vector valued functions defined on an interval I. In other words we study the functions in the Sobolev spaces $W^{s,p}(I;E)$, in the Nikolskii spaces $N^{s,p}(I;E)$, or in the Besov spaces $B_\lambda^{s,p}(I;E)$. Theses spaces are defined by integration and translation, and E is a Banach space. In particular we study the dependence on the parameters s, p and λ, that is imbeddings for different parameters. Moreover we compare each space to the others, and we give Lipschitz continuity, existence of traces and q-integrability properties. These results rely only on integration techniques.*

Introduction.

Purpose. – Let us consider a p-integrable function f defined on an interval I of $\boldsymbol{R}$, with values in a Banach space E. We are interested in the following regularity properties:

$$\text{(i)}\quad \iint_{I\times I}\left(\frac{\|f(y)-f(x)\|}{|y-x|^s}\right)^p\frac{dy\,dx}{|y-x|}<\infty$$

$$\text{(ii)}\quad \left(\int_{I_h}\|f(x+h)-f(x)\|^p\,dx\right)^{1/p}\leqslant ch^s,\qquad \forall h\geqslant 0$$

where $I_h=\{x\in I\colon x+h\in I\}$, and

$$\text{(iii)}\quad \int_0^\infty\left(\frac{1}{h^s}\left(\int_{I_h}\|f(x+h)-f(x)\|^p\,dx\right)^{1/p}\right)^\lambda\frac{dh}{h}<\infty\,.$$

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These properties are called fractional since the regularity order s is between 0 and 1. They are weaker than the differentiability in L^p, that is each of them is entailed by $\partial f/\partial x \in L^p(I; E)$, which corresponds to the limit case $s = 1$.

These properties characterize respectively, for (i) the Sobolev space $W^{s,p}(I; E)$, for (ii) the Nikoskii space $N^{s,p}(I; E)$, and for (iii) the Besov space $B^{s,p}_\lambda(I; E)$.

Our purpose is to study how these properties depend on s, p and λ, to compare each property with the other ones, and then to deduce either Lipschitz continuity, or traces existence or q-integrability properties.

Contents. – The outline is as follows

Main results. – Sobolev and Nikolskii spaces are particular cases of Besov spaces (proposition 2): $W^{s,p} = B^{s,p}_p$, $N^{s,p} = B^{s,p}_\infty$.

The space $B^{s,p}_\lambda$ increases with λ (theorem 11), thus (corollaries 23, 24, 25): $B^{s,p}_\lambda \subset W^{s,p} \subset B^{s,p}_\mu \subset N^{s,p}$ for $\lambda \leqslant p \leqslant \mu$.

These spaces decrease as s increases (corollary 17, proposition 20, theorem 14): $W^{s,p} \subset W^{r,p}$, $N^{s,p} \subset N^{r,p}$, $B^{s,p}_\lambda \subset B^{r,p}_\lambda$ for $s \geqslant r$.

The variations of s dominate the variations of λ (corollary (15), whence (corollaries 23, 24 and 25): if $s > r$ then $W^{s,p}$, $N^{s,p}$ and $B^{s,p}_\lambda$ are all included in $W^{r,p}$, in $N^{r,p}$ and in $B^{r,p}_\mu$.

Lipschitz continuity is obtained (corollary 26): if $s > 1/p$, $W^{s,p}$, $N^{s,p}$ and $B^{s,p}_\lambda$ are all included in $\mathrm{Lip}^{s-1/p}$.

Traces existence is obtained (corollaries 27, 28 and theorem 29): $W^{s,p}$, $N^{s,p}$ and $B^{s,p}_\lambda$ are all included in $\mathcal{C}$ for $s > 1/p$ (and, when $\lambda = 1$, for the limit coefficient $s = 1/p$).

Integrability properties are obtained (corollaries 31, 32 and 33): $W^{s,p}$, $N^{s,p}$ and

$B^{s,p}_\lambda$ are all included in L^q for $s-1/p>-1/q$, $p\leqslant q$, and in some cases for the limit coefficients $s-1/p=-1/q$.

The Sobolev theorem gives comparisons when both s and p vary (corollaries 18, 21 and theorem 10): $W^{s,p}\subset W^{r,q}$, $N^{s,p}\subset N^{r,q}$ and $B^{s,p}_\lambda\subset B^{r,q}_\lambda$ for $s-1/p=r-1/q$, $s\geqslant r$.

Moreover for all these imbeddings (with the exception of the imbedding in L^q for some limit coefficients) we give estimates for the norms with explicit coefficients. The obtained coefficients are not necessarily the best possible ones, but they give an order of magnitude, and a behaviour with respect to the parameters (s, p, λ and $|I|$).

Motivation. – The regularity properties (i) and (ii) are often used in evolution problems: then I is a time interval an E is a function space built over a domain $\Omega\subset \boldsymbol{R}^n$.

The property (iii) is not so frequently used, however it is very useful for our proofs. Indeed many results for $W^{s,p}=B^{s,p}_p$ are obtained from properties of $B^{s,p}_\lambda$ for $p\neq\lambda$. For example we deduce the Sobolev theorem $W^{s,p}\subset W^{r,q}$ ($s-1/p=$ $=r-1/q$, $p\leqslant q$) from $B^{s,p}_\lambda\subset B^{r,q}_\lambda$ and from $B^{r,q}_\lambda\subset B^{r,q}_\mu$ for $\lambda\leqslant\mu$, by choosing $\lambda=p$, $\mu=q$.

Former results. – The imbeddings given here are already known for real functions on $\boldsymbol{R}$, that is for $E=I=\boldsymbol{R}$. But most of them are not proved for vector valued functions, and there characterizations by translations are not frequently used on an interval.

The main feature of the present paper lies in the methods. We define the fractional spaces by integration properties, and for the proof we use only the integration inequalities of Hölder, of Young and Hardy, and an integral identity given in lemma 7. On the contrary the classic definitions and proofs rely on interpolation and on Fourier transform. For example we prove (corollary 24) that

$$\operatorname*{Sup}_{h>0} h^{-s}\left(\int_{I_h}\|f(x+h)-f(x)\|^p\,dx\right)^{1/p}\leqslant\frac{2}{s}\left(\int_{I\times I}\left(\frac{\|f(y)-f(x)\|}{|y-x|^s}\right)^p\frac{dy\,dx}{|y-x|}\right)^{1/p}.$$

This inequality can be deduced (for $E=\boldsymbol{R}$) from classical results on fractional spaces given in [LP], [LM], [BB] and [T]. But the latter results include many hard properties of interpolation and equivalence of norms. Moreover not all the coefficients are given in these results, and therefore the above coefficient $2/s$ should be replaced by an unknown one.

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1. – Definition of fractional spaces.

Let I be an either bounded or unbounded interval of $\boldsymbol{R}$, and dx be the Lebesgue measure on I. Let E be a Banach space, $\| \ \|$ be the norm on E, and $1 \leqslant p \leqslant \infty$. Then $L^p(I; E)$ is the space of class of measurable functions from I into E, such that $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \Big(\int_I \|f(x)\|^p\,dx\Big)^{1/p} \quad \text{if } p < \infty \qquad \Big(\|f\|_{L^\infty} = \operatorname*{Ess\,sup}_{x\in I} \|f(x)\| \quad \text{if } p = \infty\Big).$$

For any $h \geqslant 0$ we set $I_h = \{x \in I: x + h \in I\}$, and we denote by τ_h the translation operator, that is $(\tau_h f)(x) = f(x+h)$. Given $f \in L^p(I; E)$, then f, $\tau_h f$, and $\tau_h f - f$ are all defined in I_h.

DEFINITION 1. – Let I be an interval of $\boldsymbol{R}$ and let E be a Banach space.

Sobolev spaces are defined for $0 < s < 1$, $1 \leqslant p \leqslant \infty$ by

$$W^{s,p}(I; E) = \{f \in L^p(I; E): \|f\|_{\tilde{W}^{s,p}} \leqslant \infty\},$$

$$\|f\|_{\tilde{W}^{s,p}} = \Big(\int_{I\times I}\Big(\frac{\|f(y)-f(x)\|}{|y-x|^s}\Big)^p \frac{dy\,dx}{|y-x|}\Big)^{1/p} \qquad \text{if } p < \infty,$$

$$\|f\|_{\tilde{W}^{s,p}} = \operatorname*{Ess\,sup}_{x\in I,\, y\in I} \frac{\|f(y)-f(x)\|}{|y-x|^s} \qquad \text{if } p = \infty.$$

Lipschitz spaces are defined for $0 < s < 1$ by

$$\mathrm{Lip}^s(I; E) = \{f \in L^\infty(I; E): \|f\|_{\tilde{\mathrm{Lip}}^s} < \infty\},$$

$$\|f\|_{\tilde{\mathrm{Lip}}^s} = \operatorname*{Ess\,sup}_{x\in I,\, y\in I} \frac{\|f(y)-f(x)\|}{|y-x|^s}.$$

Besov spaces are defined for $0 < s < 1$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \lambda \leqslant \infty$, by

$$B_\lambda^{s,p}(I; E) = \{f \in L^p(I; E): \|f\|_{\tilde{B}_\lambda^{s,p}} < \infty\},$$

$$\|f\|_{\tilde{B}_\lambda^{s,p}} = \Big(\int_0^\infty (h^{-s}\|\tau_h f - f\|_{L^p(I_h; E)})^\lambda \frac{dh}{h}\Big)^{1/\lambda} \qquad \text{if } \lambda < \infty,$$

$$\|f\|_{\tilde{B}_\lambda^{s,p}} = \operatorname{Sup} h^{-s}\|\tau_h f - f\|_{L^p(I_h; E)} \qquad \text{if } \lambda = \infty.$$

Nikolskii spaces are defined for $0 < s < 1$, $1 \leqslant p \leqslant \infty$ by

$$N^{s,p}(I; E) = \{f \in L^p(I; E): \|f\|_{\tilde{N}^{s,p}} < \infty\},$$

$$\|f\|_{\tilde{N}^{s,p}} = \operatorname*{Sup}_{h>0} h^{-s}\|\tau_h f - f\|_{L^p(I_h; E)}. \quad \blacksquare$$

REMARK 1.1. – The interval I_h is as follows.

If $I =]-\infty, \infty[$ or $I =]\alpha, \infty[$, then $I_h = I$.

If $I =]-\infty, \beta[$, then $I_h =]-\infty, \beta - h[$.

If $I =]\alpha, \beta[$, then $I_h =]\alpha, \beta - \alpha[$ for $h < \beta - \alpha$, $I_h = \emptyset$ for $h \geqslant \beta - \alpha$.

REMARK 1.2. – Denote $|I|$ the length of I ($|I| = \infty$ if I is not bounded).
Since $I_h = \emptyset$ for $h \geqslant |I|$, we have

$$\|f\|_{\tilde{B}_\lambda^{s,p}} = \left(\int_0^{|I|} \left(h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)}\right)^\lambda \frac{dh}{h}\right)^{1/\lambda}, \qquad \|f\|_{\tilde{B}_\infty^{s,p}} = \operatorname*{Sup}_{0<h<|I|} h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)}. \quad \blacksquare$$

2. – First properties of fractional spaces.

We give some elementary properties which will be used further. At first let us see that Sobolev, Nikolskii and Lipschitz spaces are particular Besov spaces.

PROPOSITION 2. – Assume $0 < s < 1$ and $1 \leqslant p \leqslant \infty$. Then

$$W^{s,p}(I;E) = B_p^{s,p}(I;E),$$
$$N^{s,p}(I;E) = B_\infty^{s,p}(I;E),$$
$$\mathrm{Lip}^s(I;E) = W^{s,\infty}(I;E) = N^{s,\infty}(I;E) = B_\infty^{s,\infty}(I;E).$$

In addition, $\forall f \in L^p(I;E)$,

$$\|f\|_{\tilde{W}^{r,p}} = 2^{1/p}\|f\|_{\tilde{B}_p^{s,p}},$$
$$\|f\|_{\tilde{N}^{s,p}} = \|f\|_{\tilde{B}_\infty^{s,p}},$$
$$\|f\|_{\widetilde{\mathrm{Lip}}^s} = \|f\|_{\tilde{W}^{s,\infty}} = \|f\|_{\tilde{N}^{s,\infty}} = \|f\|_{\tilde{B}_\infty^{s,\infty}}.$$

PROOF. – Use the change of variable $y - x = h$ in the definitions of $\|f\|_{\tilde{W}^{s,p}}$ and $\|f\|_{\widetilde{\mathrm{Lip}}^s}$. ■

Next we see that these spaces have the restriction property.

PROPOSITION 3. – Let J be an interval of $\mathbf{R}$, $J \subset I$, and $0 < s < 1$, $1 \leqslant p \leqslant \infty$, $0 \leqslant \lambda \leqslant \infty$.

The restriction operator from I onto J maps $W^{s,p}(I;E)$, $N^{s,p}(I;E)$ and $B_\lambda^{s,p}(I;E)$ respectively onto $W^{s,p}(J;E)$, $N^{s,p}(J;E)$ and $B_\lambda^{s,p}(J;E)$.

In addition, $\forall f \in L^p(I; E)$,

$$\|f\|_{\tilde{W}^{s,p}(J;E)} \leqslant \|f\|_{\tilde{W}^{s,p}(I;E)},$$

$$\|f\|_{\tilde{N}^{s,p}(J;E)} \leqslant \|f\|_{\tilde{N}^{s,p}(I;E)},$$

$$\|f\|_{\tilde{B}_\lambda^{s,p}(J;E)} \leqslant \|f\|_{\tilde{B}_\lambda^{s,p}(I;E)}. \quad \blacksquare$$

PROOF. – This follows from the restriction property for the integral and for L^p. ■

When there is no possible confusion, we do not use a special notation for the restriction. For example for any function f on I we shall write $f \in W^{s,p}(J; E)$ meaning that «the restriction of f to J belongs to $W^{s,p}(J; E)$».

We see now that theses spaces have the translation property. We set $I + k = \{x + k: x \in I\}$.

PROPOSITION 4. – Let $k > 0$ and $0 < s < 1$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \lambda \leqslant \infty$.

The translation operator τ_k maps $W^{s,p}(I; E)$, $N^{s,p}(I; E)$ and $B_\lambda^{s,p}(I; E)$ respectively onto $W^{s,p}(I - k; E)$, $N^{s,p}(I - k; E)$ and $B_\lambda^{s,p}(I - k; E)$.

In addition, $\forall f \in L^p(I; E)$,

$$\|\tau_k f\|_{\tilde{W}^{s,p}(I-k;E)} = \|f\|_{\tilde{W}^{s,p}(I;E)},$$

$$\|\tau_k f\|_{\tilde{N}^{s,p}(I-k;E)} = \|f\|_{\tilde{N}^{s,p}(I;E)},$$

$$\|\tau_k f\|_{\tilde{B}_\lambda^{s,p}(I-k;E)} = \|f\|_{\tilde{B}_\lambda^{s,p}(I;E)}. \quad \blacksquare$$

PROOF. – This follows from the translation property for the integral and for L^p.

3. – Young and Hardy inequalities.

We will use the following modified Young's inequality.

LEMMA 5. – Let $f \in L^p(I; E)$ and $g \in L^r(0, a)$ where $a > 0$ and $1/p + 1/r \geqslant 1$. Define F in $I_a = \{x \in I: x + a \in I\}$, by $F(x) = \int_0^a f(x + t) g(t)\, dt$.

Then $F \in L^q(I_a; E)$ where $1/q = 1/p + 1/r - 1$, and

$$\|F\|_{L^q(I_a,E)} \leqslant \|f\|_{L^p(I;E)} \|g\|_{L^r(0,a)}. \quad \blacksquare$$

PROOF. – At first assume that f is continuous and has a compact support. Given $x \in I_a$ we have

$$\|F(x)\| \leqslant \int_0^a \|f(x + t)\| |g(t)|\, dt = \int_0^a \left(\|f(x + t)\|^{p/q} |g(t)|^{r/q}\right) \|f(x + t)\|^{1-p/q} |g(t)|^{1-r/q}\, dt.$$

The Hölder inequality with $1/q + (1/p - 1/q) + (1/r - 1/q) = 1$ gives

$$\|F(x)\| \leqslant \left(\int_0^a \|f(x+t)\|^p |g(t)|^r dt\right)^{1/q} \left(\int_0^a \|f(x+t)\|^p dt\right)^{1/p-1/q} \left(\int_0^a |g(t)|^r dt\right)^{1/r-1/q}.$$

When t spans $]0, a[$, $y = x + t$ spans a subset of I. Thus

$$\|F(x)\|^q \leqslant \left(\int_0^a \|f(x+t)\|^p |g(t)|^r dt\right)\left(\int_I \|f(y)\|^p dy\right)^{q/p-1} \left(\int_0^a |g(t)|^r dt\right)^{q/r-1}.$$

By integrating in x on I_a, we obtain

$$\int_{I_a} \|F(x)\|^q dx \leqslant \left(\int_0^a |g(t)|^r \int_{I_a} \|f(x+t)\|^p dx\, dt\right)\left(\int_I \|f(y)\|^p dy\right)^{q/p-1} \left(\int_0^a |g(t)|^r dt\right)^{q/r-1}.$$

When x spans I_a, $y = x + t$ spans a subset of I. Thus $\int_{I_a} \|f(x+t)\|^p dx \leqslant \int_I \|f(y)\|^p dy$, and

$$\left(\int_{I_a} \|F(x)\|^q dx\right)^{1/q} \leqslant \left(\int_I \|f(y)\|^p dy\right)^{1/p} \left(\int_0^a |g(t)|^r dt\right)^{1/r}.$$

This is the desired inequality. The continuous functions with compact support being dense in $L^p(I; E)$ by continuity this inequality is satisfied for all $f \in L^p(I; E)$. ∎

Let us remind Hardy's inequality.

LEMMA 6. – Let g be a real non-negative measurable function on $]0, T[$, $T \leqslant \infty$, $s > 0$ and $1 \leqslant \lambda < \infty$. Then

$$\left(\int_0^T \left(t^{-s} \int_0^T g(h) \frac{dh}{h}\right)^\lambda \frac{dt}{t}\right)^{1/\lambda} \leqslant \frac{1}{s} \left(\int_0^T (t^{-s} g(t))^\lambda \frac{dt}{t}\right)^{1/\lambda}$$

$$\operatorname*{Sup}_{0<t<T} t^{-s} \int_0^t g(h) \frac{dh}{h} \leqslant \frac{1}{s} \operatorname*{Ess\,sup}_{0<t<T} t^{-s} g(t). \quad ∎$$

PROOF. – We assume for the moment that $g = 0$ on a neighborhood of 0, and we denote $G(t) = \int_0^t g(t)\, dh/h$. Then

$$\frac{1}{\lambda} \frac{d}{dt} (t^{-s} G(t))^\lambda = -s t^{-s\lambda-1} G(t)^\lambda + t^{-s\lambda-1} g(t) G(t)^{\lambda-1}.$$

Let us integrate from 0 to T. Since $G = 0$ on a neighborhood of 0, the integral of the left hand side is non-negative. Therefore

$$s\int_0^T t^{-s\lambda-1}G(t)^\lambda\,dt \leqslant \int_0^T t^{-s\lambda-1}g(t)\,G(t)^{\lambda-1}\,dt\,.$$

The right hand side equals

$$=\int_0^T \big(t^{-s\lambda-1}G(t)^\lambda\big)^{(\lambda-1)/\lambda}\big(t^{-s\lambda-1}g(t)^\lambda\big)^{1/\lambda}\,dt\,.$$

Thus, by Hölder inequality with $(\lambda-1)/\lambda + 1/\lambda = 1$ it is bounded by

$$\leqslant\Big(\int_0^T t^{-s\lambda-1}G(t)^\lambda\,dt\Big)^{(\lambda-1)/\lambda}\Big(\int_0^T t^{-s\lambda-1}g(t)^\lambda\,dt\Big)^{1/\lambda}.$$

Therefore

$$s\Big(\int_0^T t^{-s\lambda-1}G(t)^\lambda\,dt\Big)^{1/\lambda}\leqslant\Big(\int_0^T t^{-s\lambda-1}g(t)^\lambda\,dt\Big)^{1/\lambda}.$$

This proves the first desired inequality in the case where g equals 0 in a neighborhood of 0. The general result follows by approaching g by functions which equal 0 in a neighborhood of 0, for example $g_\varepsilon = 1_{\varepsilon>0}\,g$.

Let us prove the second inequality. Denote $c = \operatorname*{Sup}_{0<t<T} t^{-s}g(t)$. Thus $g(h)\leqslant ch^s$ and therefore

$$\operatorname*{Sup}_{0<t<T} t^{-s}\int_0^t g(h)\,\frac{dh}{h}\leqslant \operatorname*{Sup}_{0<t<T} t^{-s}\int_0^t ch^s\,\frac{dh}{h}=\frac{c}{s}\,. \qquad\blacksquare$$

4. – An integral identity.

The proof of the Sobolev imbedding will rely on the following identity.

LEMMA 7. – Let $f\in L^p(I;E)$, $p<\infty$ and $a>0$. Assume that

$$\int_0^a \|f-\tau_h f\|_{L^p(I_h;E)}\,\frac{dh}{h}<\infty\,.$$

Then

$$f = \frac{1}{a}\int_0^a \tau_h f\,dh + \int_a^a dh \int_0^{a-h} (I - \tau_h)\tau_t f \frac{dt}{(t+h)^2} \quad \text{in } L^p(I_a; E)\,. \quad \blacksquare$$

PROOF. – The function $h \to \tau_h f$ is continuous from $[0, a]$ into $L^p(I_a; E)$. Thus the integral

$$Y(a) = \frac{1}{a}\int_0^a \tau_h f\,dh$$

converges in $L^p(I_a; E)$.

We show next the convergence of the following integral

$$X(a) = \int_0^a dh \int_0^{a-h} (I - \tau_h)\tau_t f \frac{dt}{(t+h)^2}\,.$$

The integrated function is continuous, and therefore measurable. In addition if $0 \leqslant t \leqslant a - h$, then $x + t$ spans a subset of I_h as x spans I_a. Thus

$$\int_0^a dh \int_0^{a-h} \|(I - \tau_h)\tau_t f\|_{L^p(I_a; E)} \frac{dt}{(t+h)^2} \leqslant \int_0^a dh \int_0^{a-h} \|f - \tau_h f\|_{L^p(I_h; E)} \frac{dt}{(t+h)^2} =$$

$$= \int_0^a \|f - \tau_h f\|_{L^p(I_h; E)} \left(\frac{1}{h} - \frac{1}{a}\right) dh\,.$$

This last term is finite by hypothesis, thus the integral $X(a)$ converges in $L^p(I_a; E)$.

The integrals $X(a)$ and $Y(a)$ being convergent, it remains to prove that their sum equals f. By the change of variable $u = t$, $k = t + h$ we obtain

$$X(a) = \int_0^a \frac{dk}{k^2} \int_0^k (\tau_u - \tau_k) f\,du \qquad \text{in } L^p(I_a, E)\,.$$

Let $\alpha > 0$ be given, and consider $0 < a \leqslant \alpha$. Then $I_\alpha \subset I_a$. Therefore by restriction the above equality also holds in $L^p(I_\alpha; E)$. Thus $a \to X(a)$ is differentiable from $[0, \alpha]$ into $L^p(I_\alpha, E)$ and

$$\frac{\partial X}{\partial a}(a) = \frac{1}{a^2}\int_0^a (\tau_u - \tau_a) f\,du = -\frac{\partial}{\partial a}\left(\frac{1}{a}\int_0^a \tau_h f\,dh\right)(a) = -\frac{\partial Y}{\partial a}(a) \quad \text{in } L^p(I_\alpha; E)\,.$$

Thus the restriction to I_α of $X(a)+Y(a)$ is independent of a. Moreover, if $a\to 0$ then, in $L^p(I_\alpha; E)$, $X(a)\to 0$ (by the last expression for $X(a)$), and $Y(a)\to f$. Therefore

$$X(a)+Y(a)=f \quad \text{in } L^p(I_\alpha; E) \text{ for all } a\leqslant\alpha\,.$$

In particular this is satisfied for $\alpha=a$, which proves the lemma. ∎

REMARK 4.1. – If f is just in $L^p(I; E)$, the identity given in lemma 7 holds in $W^{-1,p}(I_a; E)$.

Indeed the double integral $X(a)$ converges in $W^{-1,p}(I_a; E)$ since

$$\|(I-\tau_h)\tau_t f\|_{W^{-1,p}(I_a;E)}\leqslant h\|\tau_t f\|_{L^p(I_{a-h};E)}\leqslant h\|f\|_{L^p(I;E)}\,. \quad ∎$$

REMARK 4.2. – If $I=\boldsymbol{R}$, letting $a\to\infty$, it is possible to prove that:

$$f=\int_0^\infty dh\int_0^\infty (I-\tau_h)\tau_t f\,\frac{dt}{(t+h)^2} \quad \text{in } L^p(\boldsymbol{R}; E)\,. \quad ∎$$

REMARK 4.3. – In the case $I=\boldsymbol{R}$ and $\partial f/\partial t\in L^p$, the identity of lemma 7 is due to IL'IN [I]. It was used in [G], p. 260, to prove trace results.

5. – A preliminary imbedding from $B_1^{s,p}$ into L^q.

We give an inequality which will be used to prove the Sobolev theorem for Besov spaces.

LEMMA 8. – Suppose that $s-1/p=-1/q$ $(0<s<1, 1\leqslant p<q\leqslant\infty)$. Then

$$B_1^{s,p}(I; E)\subset L^q(I; E)\,.$$

Moreover for all $a\leqslant|I|/2$ ($|I|$ being the length of I) and $f\in B_1^{s,p}$,

$$\|f\|_{L^q(I;E)}\leqslant 2^{1/q}\left(\int_0^a h^{-s}\|\tau_h f-f\|_{L^p(I_h;E)}\frac{dh}{h}+a^{-s}\|f\|_{L^p(I;E)}\right);$$

for unbounded I, $2^{1/q}$ may be replaced by 1, and the inequality holds for all a.

For $a=|I|/2$ this inequality implies

$$\|f\|_{L^q}\leqslant 2^{1/q}\left(\|f\|_{\tilde{B}_1^{s,p}}+\left(\frac{2}{|I|}\right)^s\|f\|_{L^p}\right). \quad ∎$$

PROOF – Let $f \in B_1^{s,p}(I;E)$ and $a > 0$ The assumption of lemma 7 is satisfied since

$$\int_0^a \|f - \tau_h f\|_{L^p(I_h;E)} \frac{dh}{h} \leqslant a^s \int_0^a h^{-s} \|f - \tau_h f\|_{L^p(I_h;E)} \frac{dh}{h} \leqslant a^s \|f\|_{\tilde{B}_1^{s,p}}.$$

Therefore the identity of lemma 7 yields:

$$f = \frac{1}{a}\int_0^a \tau_h f\, dh + \int_0^a dh \int_0^{a-h} (I - \tau_h)\tau_t f \frac{dt}{(t+h)^2} \quad \text{in } L^p(I_a;E).$$

Let us estimate the first integral by Young's inequality, that is by lemma 5, with $g(t) = 1$ and $1/r = 1 - s$, thus $1/q = 1/p + 1/r - 1$. This gives $1/a \int_0^a \tau_h f\, dh \in \in L^q(I_a;E)$, and

$$\left\| \frac{1}{a}\int_0^a \tau_h f\, dh \right\|_{L^q(I_a;E)} \leqslant \frac{1}{a} \|f\|_{L^p(I;E)} \|1\|_{L^r(0,a)} = a^{-s} \|f\|_{L^p(I;E)}.$$

We now estimate the double integral. Using lemma 5 for $\tau_h f - f \in L^p(I_h;E)$ and $g(t) = (t+h)^{-2}$, we obtain

$$\left\| \int_0^{a-h} (I - \tau_h)\tau_t f \frac{dt}{(t+h)^2} \right\|_{L^q(I_a;E)} \leqslant \|f - \tau_h f\|_{L^p(I_h;E)} \|g\|_{L^r(0,a-h)}.$$

Here

$$\|g\|_{L^r(0,a-h)} = \left(\frac{h^{1-2r} - a^{1-2r}}{2r-1} \right)^{1/r} \leqslant c h^{1/r-2} = c h^{-s-1}$$

where $c = (2r-1)^{-1/r} \leqslant 1$. Thus

$$\int_0^a dh \left\| \int_0^{a-h} (I - \tau_h)\tau_t f \frac{dt}{(t+h)^2} \right\|_{L^q(I_a;E)} \leqslant \int_0^a h^{-s} \|f - \tau_h f\|_{L^p(I_h;E)} \frac{dh}{h}.$$

This last term is finite since $f \in B_1^{s,p}$. Thus the integral

$$\int_0^a dh \int_0^{a-h} (I - \tau_h)\tau_t f \frac{dt}{(t+h)^2}$$

converges in $L^q(I_a, E)$. Therefore the identity of lemma 7 yields $f \in L^q(I_a; E)$ and

$$\|f\|_{L^q(I_a;E)} \leqslant \int_0^a h^{-s}\|f-\tau_h f\|_{L^p(I_h;E)} \frac{dh}{h} + a^{-s}\|f\|_{L^p(I;E)} .$$

If the interval I is unbounded on the right side, then $I_a = I$ and the first desired inequality of lemma 8 is proved, with $2^{1/q}$ replaced by 1.

If the interval I is bounded on the right side, we invert the time direction. More precisely we use the above inequality for $\tilde f(t) = f(-t)$ which is defined on $\tilde I = \{-t : t \in I\}$. Since $\|\tilde f - \tau_h \tilde f\|_{L^p(\tilde I_h;E)} = \|f-\tau_h f\|_{L^p(I;E)}$, this gives

$$\|\tilde f\|_{L^q(\tilde I_a;E)} \leqslant \int_0^a h^{-s}\|f-\tau_h f\|_{L^p(I_h;E)} \frac{dh}{h} + a^{-s}\|f\|_{L^p(I;E)} .$$

If the interval I is unbounded on the left side, then $\tilde I_a = \tilde I$ and $\|\tilde f\|_{L^q(\tilde I_a;E)} = \|f\|_{L^q(I;E)}$. Thus the first desired inequality is proved again.

There remains the case where I is bounded on both sides, that is $I =]\alpha; \beta[$. Then, if $a \leqslant (\beta-\alpha)/2$,

$$\|f\|_{L^q(I;E)} \leqslant \left(\int_\alpha^{\beta-a} \|f(x)\|^q dx + \int_{\alpha+a}^{\beta} \|f(x)\|^q dx \right)^{1/q} = \left((\|f\|_{L^q(I_a;E)})^q + (\|\tilde f\|_{L^q(\tilde I_a;E)})^q \right)^{1/q} .$$

The previous two inequalities then yield

$$\|f\|_{L^q(I;E)} \leqslant 2^{1/q} \left(\int_0^a h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)} \frac{dh}{h} + a^{-s}\|f\|_{L^p(I;E)} \right).$$

Thus the first desired inequality, and therefore the lemma, is proved for any interval I. ∎

6. – A preliminary imbedding from $B^{s,p}_\lambda$ into $B^{r,p}_1$.

We now give an imbedding which will be used to prove the Sobolev theorem for Besov spaces.

LEMMA 9. – Assume $s > r$ $(0 < r < s < 1, 1 \leqslant p \leqslant \infty, 1 \leqslant \lambda \leqslant \infty)$. Then

$$B^{s,p}_\lambda(I; E) \subset B^{r,p}_1(I; E) .$$

Moreover, $|I|$ being the length of I, $\forall f \in B_\lambda^{s,p}$:

$$\|f\|_{\tilde{B}_1^{r,p}} \leqslant \begin{cases} \dfrac{|I|^{s-r}}{s-r}\, \|f\|_{\tilde{B}_\lambda^{s,p}} & \text{for bounded } I \\ \dfrac{1}{s-r}\, \|f\|_{\tilde{B}_\lambda^{s,p}} + \dfrac{2}{r}\, \|f\|_{L^p} & \text{for all } I . \end{cases} \qquad \blacksquare$$

PROOF. – Let $f \in B_\lambda^{s,p}$ and $a > 0$. By Hölder's inequality for $(\lambda - 1)/\lambda + 1/\lambda = 1$ we have

$$\int_0^a h^{s-r+1/\lambda-1}\left(h^{-s-1/\lambda}\|\tau_h f - f\|_{L^p(I_h;E)}\right) dh \leqslant c\left(\int_0^a \left(h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)}\right)^\lambda \frac{dh}{h}\right)^{1/\lambda}$$

where

$$c = \left(\int_0^a h^{(s-r+1/\lambda-1)\lambda/(\lambda-1)}\, dh\right)^{(\lambda-1)/\lambda} = \left(\frac{1}{s-r}\,\frac{\lambda-1}{\lambda}\right)^{(\lambda-1)/\lambda} a^{s-r} \leqslant \frac{a^{s-r}}{s-r}.$$

Thus, for $\lambda < \infty$,

$$\int_0^a h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)}\,\frac{dh}{h} \leqslant \frac{a^{s-r}}{s-r}\left(\int_0^a \left(h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)}\right)^\lambda \frac{dh}{h}\right)^{1/\lambda}.$$

For $\lambda = \infty$ we obtain

$$\int_0^a h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)}\,\frac{dh}{h} \leqslant \frac{a^{s-r}}{s-r}\ \operatorname*{Sup}_{0<h<a} h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)} .$$

When I is bounded, we obtain the first desired inequality by choosing $a = |I|$. Indeed (see remark 1.2) the integrals and the supremum for $0 \leqslant h \leqslant |I|$ are equal to the integrals and to the supremum for $0 \leqslant h \leqslant \infty$.

For all I we have the following estimate

$$\int_0^\infty h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)}\,\frac{dh}{h} \leqslant \int_0^1 h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)}\,\frac{dh}{h} + 2\|f\|_{L^p(I;E)}\int_1^\infty h^{-r}\,\frac{dh}{h} .$$

By using the previous inequality for $a = 1$ we obtain

$$\int_0^\infty h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)}\, dh \leqslant \frac{1}{s-r}\left(\int_0^1 \left(h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)}\right)^\lambda \frac{dh}{h}\right)^{1/\lambda} + \frac{2}{r}\,\|f\|_{L^p(I;E)} .$$

For $\lambda = \infty$ the integral in the right-hand side is replaced by a Sup. This implies the second desired inequality. Then the lemma holds.

7. – Sobolev theorem for Besov spaces: dependence of $B_\lambda^{s,p}$ on s and p.

Let us see that $B_\lambda^{s,p}$ decreases when s increases and p decreases, for any fixed $s - 1/p$.

THEOREM 10. – Suppose $s \geqslant r$ and $s - 1/p = r - 1/q$ $(0 < r \leqslant s < 1, 1 \leqslant p \leqslant q \leqslant \infty, 1 \leqslant \lambda \leqslant \infty)$. Then

$$B_\lambda^{s,p}(I; E) \subset B_\lambda^{r,q}(I; E),$$

$$\|f\|_{\tilde{B}_\lambda^{r,q}} \leqslant \frac{3\theta}{r} \|f\|_{\tilde{B}_\lambda^{s,p}} \qquad \forall f \in B_\lambda^{s,p}.$$

This holds for $\theta = 2^{1/q} 3^{1-r}$, thus $\theta \leqslant 6$ and, if I is unbounded, for $\theta = 1$. ∎

PROOF. – If $s = r$ the spaces to be compared coincide. Now let $s > r$, $s - 1/p = r - 1/q$ and $f \in B_\lambda^{s,p}(I; E)$. By lemmata 9 and 8, $B_\lambda^{s,p} \subset B_1^{s-r,p} \subset L^q$, therefore $f \in L^q(I; E)$.

Let $t > 0$. By restriction (proposition 3), $f \in B_\lambda^{s,p}(I_t; E)$. By translation and restriction (propositions 4 and 3), $\tau_t f \in B_\lambda^{s,p}(I_t; E)$. Thus $\tau_t f - f \in B_\lambda^{s,p}(I_t; E)$ and, by lemma 9, $\tau_t f - f \in B_1^{s-r,p}(I_t; E)$.

Now we can use the lemma 8 for $\tau_t f - f$, $(s - r) - 1/p = -1/q$ and $a = t$.

Unbounded I. – If I is not bounded the lemma 8 yields, for all t

$$\|\tau_t f - f\|_{L^q(I_t; E)} \leqslant \int_0^t h^{r-s} \|(\tau_h(\tau_t f - f) - (\tau_t f - f))\|_{L^p(I_{t+h}; E)} \frac{dh}{h} + t^{r-s} \|\tau_t f - f\|_{L^p(I_t; E)}.$$

Since $\tau_h(\tau_t f - f) - (\tau_t f - f) = \tau_t(\tau_h f - f) - (\tau_h f - f)$, it's norm in $L^p(I_{t+h}; E)$ is bounded by 2 times the norm of $\tau_h f - f$ in $L^p(I_h; E)$.

Case $\lambda < \infty$. Thus the above inequality yields

$$\left(\int_0^\infty (t^{-r} \|\tau_t f - f\|_{L^q(I_t; E)})^\lambda \frac{dt}{t} \right)^{1/\lambda} \leqslant$$

$$\leqslant 2 \left(\int_0^\infty \left(t^{-r} \int_0^t h^{r-s} \|\tau_h f - f\|_{L^p(I_h; E)} \frac{dh}{h} \right)^\lambda \frac{dt}{t} \right)^{1/\lambda} + \left(\int_0^\infty (t^{-s} \|\tau_t f - f\|_{L^p(I_t; E)})^\lambda \frac{dt}{t} \right)^{1/\lambda}.$$

Bounding the first integral of the right hand side by Hardy's inequality, that is by lemma 6, we find

$$\left(\int_0^\infty (t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda} \leqslant \left(\frac{2}{r}+1\right)\left(\int_0^\infty (t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda}.$$

This yields the desired inequality since $2/r + 1 \leqslant 3/r$.

Case $\lambda = \infty$. Replacing the integration in t by a supremum, and using the second inequality of lemma 6, we obtain

$$\operatorname*{Sup}_{t>0} t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)} \leqslant 2 \operatorname*{Sup}_{t>0}\left(t^{-r}\int_0^t h^{r-s}\|\tau_h f - f\|_{L^p(I_h;E)}\frac{dh}{h}\right) + \operatorname*{Sup}_{t>0} t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)} \leqslant$$

$$\leqslant \left(\frac{2}{r}+1\right)\operatorname*{Sup}_{t>0} t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)}.$$

This yields the desired inequality for $\lambda = \infty$.

Bounded I. - If $I =)\alpha, \beta($ is bounded, then the condition on a in lemma 8 yields $t \leqslant |I_t|/2$, that is $t \leqslant (\beta - t - \alpha)/2$ thus $t \leqslant (\beta - \alpha)/3$. Under this condition

$$\|\tau_t f - f\|_{L^q(I_t;E)} \leqslant 2^{1/q}\left(\int_0^t h^{r-s}\|\tau_h(\tau_t f - f) - (\tau_t f - f)\|_{L^p(I_{t+h};E)}\frac{dh}{h} + t^{r-s}\|\tau_t f - f\|_{L^q(I_t;E)}\right).$$

As in the unbounded case $\|\tau_h(\tau_t f - f) - (\tau_t f - f)\|_{L^p} \leqslant 2\|\tau_h f - f\|_{L^p}$.

Case $\lambda < \infty$. Thus

$$\left(\int_0^{(\beta-\alpha)/3} (t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda} \leqslant$$

$$\leqslant 2^{1/q}\left(2\left(\int_0^\infty\left(t^{-r}\int_0^t h^{r-s}\|\tau_h f - f\|_{L^p(I_h;E)}\frac{dh}{h}\right)^\lambda \frac{dt}{t}\right)^{1/\lambda} + \left(\int_0^\infty (t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda}\right).$$

Therefore, by Hardy's inequality, that is by lemma 6,

$$\left(\int_0^{(\beta-\alpha)/3} (t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda} \leqslant 2^{1/q}\left(\frac{2}{r}+1\right)\left(\int_0^\infty (t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda}.$$

In the left hand side we use $t = h/3$. Then $\tau_h f - f = \tau_{2t}(\tau_t f - f) + \tau_t(\tau_t f - f) + (\tau_t f - f)$, and therefore $\|\tau_h f - f\|_{L^q(I_h;E)} \leq 3\|\tau_t f - f\|_{L^q(I_t;E)}$. Thus this change of variable gives

$$\left(\int_0^{\beta-\alpha} (h^{-r}\|\tau_h f - f\|_{L^q(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} \leq 3^{1-r}\left(\int_0^{(\beta-\alpha)/3} (t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda}.$$

If $h \geq \beta - \alpha$, then $I_h = \emptyset$ and $\|\tau_h f - f\|_{L^q(I_h;E)} = 0$. Thus $\beta - \alpha$ may be replaced by ∞ in the left hand side. Then the last two inequalities yield,

$$\left(\int_0^\infty (h^{-r}\|\tau_h f - f\|_{L^q(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} \leq 3^{1-r}2^{1/q}\left(\frac{2}{r}+1\right)\left(\int_0^\infty (t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)})^\lambda \frac{dt}{t}\right)^{1/\lambda}.$$

This gives the desired inequality.

Case $\lambda = \infty$. Replacing the integration in t by a supremum, and using the second inequality of lemma 6 we obtain

$$\operatorname*{Sup}_{t<(\beta-\alpha)/3} t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)} \leq 2^{1/q} \operatorname*{Sup}_{t<\infty} 2t^{-r}\int_0^t h^{r-s}\|\tau_h f - f\|_{L^p(I_h;E)}\frac{dh}{h} + t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)} \leq$$

$$\leq 2^{1/q}\left(\frac{2}{r}+1\right)\operatorname*{Sup}_{t<\infty} t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)}.$$

Now, using again $t = h/3$ and $\|\tau_h f - f\|_{L^q} \leq 3\|\tau_t f - f\|_{L^q}$, we obtain

$$\operatorname*{Sup}_{h<\infty} h^{-r}\|\tau_h f - f\|_{L^q(I_h;E)} = \operatorname*{Sup}_{h<\beta-\alpha} h^{-r}\|\tau_h f - f\|_{L^q(I_h;E)} \leq 3^{1-r}\operatorname*{Sup}_{t<(\beta-\alpha)/3} t^{-r}\|\tau_t f - f\|_{L^q(I_t;E)}.$$

Finally

$$\operatorname*{Sup}_{h<\infty} h^{-r}\|\tau_h f - f\|_{L^q(I_h;E)} \leq 3^{1-r}2^{1/q}\left(\frac{2}{r}+1\right)\operatorname*{Sup}_{t<\infty} t^{-s}\|\tau_t f - f\|_{L^p(I_t;E)}.$$

This is the desired inequality, which is proved for all I and λ. ∎

8. – Dependence of $B_\lambda^{s,p}$ on λ.

Let us see that $B_\lambda^{s,p}$ increases with λ.

THEOREM 11. – Suppose $\lambda \leqslant \mu$ $(0 < s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \mu \leqslant \infty)$. Then

$$B^{s,p}_\lambda(I;E) \subset B^{s,p}_\mu(I;E)\,,$$
$$\|f\|_{\tilde{B}^{s,p}_\mu} \leqslant \frac{2}{s}\,\|f\|_{\tilde{B}^{s,p}_\lambda} \qquad \forall f \in B^{s,p}_\lambda\,. \quad \blacksquare$$

The theorem will be proved in three steps, by using the following semi-norm on $B^{s,p}_\lambda$. Given f in $L^p(I;E)$, we set

$$\omega_p(h) = \sup_{0 \leqslant t \leqslant h} \|\tau_t f - f\|_{L^p(I_t;E)}$$

$$\|f\|^{*}_{\tilde{B}^{s,p}_\lambda} = \left(\int_0^\infty (h^{-s}\omega_p(h))^\lambda \frac{dh}{h}\right)^{1/\lambda} \qquad \left(\|f\|^{*}_{\tilde{B}^{s,\infty}_\lambda} = \sup_{h>0} h^{-s}\omega_p(h) \text{ if } \lambda = \infty\right).$$

In the first step let us prove that this defines an equivalent semi-norm on $B^{s,p}_\lambda$.

LEMMA 12. – For all $f \in B^{s,p}_\lambda(I;E)$ $(0 < s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$,

$$\|f\|_{\tilde{B}^{s,p}_\lambda} \leqslant \|f\|^{*}_{\tilde{B}^{s,p}_\lambda} \leqslant \frac{1}{2^s - 1}\,\|f\|_{\tilde{B}^{s,p}_\lambda}\,. \quad \blacksquare$$

PROOF. – The first inequality is obvious since $\|\tau_h f - f\|_{L^p(I_t;E)} \leqslant \omega_p(h)$.

Let us prove the second one. For $h \leqslant t \leqslant 2h$ we have $\tau_t f - f = \tau_h f - f + \tau_h(\tau_{t-h} f - f)$, thus

$$\|\tau_t f - f\|_{L^p(I_t;E)} \leqslant \|\tau_h f - f\|_{L^p(I_h;E)} + \omega_p(h)$$

For $t \leqslant h$ this inequality still holds. Thus, calculating the supremum for $0 \leqslant t \leqslant 2h$ we obtain

$$\omega_p(2h) \leqslant \|\tau_h f - f\|_{L^p(I_h;E)} + \omega_p(h)$$

The case $\lambda < \infty$. Thus

$$\left(\int_0^\infty (h^{-s}\omega_p(2h))^\lambda \frac{dh}{h}\right)^{1/\lambda} \leqslant \left(\int_0^\infty (h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} + \left(\int_0^\infty (h^{-s}\omega_p(h))^\lambda \frac{dh}{h}\right)^{1/\lambda}.$$

By the change of variable $2h \to h$, in the left hand side we obtain

$$(2^s - 1)\left(\int_0^\infty h^{-s}\omega_p(h))^\lambda \frac{dh}{h}\right)^{1/\lambda} \leqslant \left(\int_0^\infty (h^{-s}\|\tau_h - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda}$$

which is the second desired inequality.

The case $\lambda = \infty$. Replacing now the integration in t by a supremum, we obtain

$$2^s \operatorname*{Sup}_{h>0} h^{-s}\omega_p(h) = \operatorname*{Sup}_{h>0} h^{-s}\omega_p(2h) \leqslant \operatorname*{Sup}_{h>0} h^{-s}\|\tau_h f - f\|_{L^p(I_h;\,E)} + h^{-s}\omega_p(h)\ .$$

This gives the second desired inequality for $\lambda = \infty$. ■

In the second step let us prove the result of theorem 11 for $\mu = \infty$.

LEMMA 13. – For all $f \in B_\lambda^{s,p}(I;\,E)$ $(0 < s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$,

$$\|f\|^*_{\tilde{B}^{s,p}_\infty} \leqslant (s\lambda)^{1/\lambda}\|f\|^*_{\tilde{B}^{s,p}_\lambda} \quad ■$$

PROOF. – Let $t > 0$. Since $\omega_p(h)$ increases with h we have

$$\left(\int_0^\infty (h^{-s}\omega_p(h))^\lambda \frac{dh}{h}\right)^{1/\lambda} \geqslant \left(\int_t^\infty (h^{-s}\omega_p(h))^\lambda \frac{dh}{h}\right)^{1/\lambda} \geqslant \omega_p(t)\left(\int_t^\infty h^{-s\lambda}\frac{dh}{h}\right)^{1/\lambda} = (s\lambda)^{-1/\lambda}t^{-s}\omega_p(t)\ .$$

Thus

$$\operatorname*{Sup}_{t>0} t^{-s}\omega_p(t) \leqslant (s\lambda)^{1/\lambda}\left(\int_0^\infty (h^{-s}\omega_p(h))^\lambda \frac{dh}{h}\right)^{1/\lambda}. \quad ■$$

In the last step let us prove the imbedding for all $\mu \geqslant \lambda$.

PROOF OF THEOREM 11. – Let $f \in B_\lambda^{s,p}$ and $\mu \geqslant \lambda$. We have

$$\left(\int_0^\infty (h^{-s}\|\tau_h f - f\|_{L^p(I_h;\,E)})^\mu \frac{dh}{h}\right)^{1/\mu} \leqslant$$

$$\leqslant \left(\operatorname*{Sup}_{h>0} h^{-s}\|\tau_h f - f\|_{L^p(I_h;\,E)}\right)^{(\mu-\lambda)/\mu}\left(\int_0^\infty (h^{-s}\|\tau_h f - f\|_{L^p(I_h;\,E)})^\lambda \frac{dh}{h}\right)^{1/\mu},$$

This is

$$\|f\|_{\tilde{B}^{s,p}_\mu} \leqslant \left(\|f\|_{\tilde{B}^{s,p}_\infty}\right)^{(\mu-\lambda)/\mu}\left(\|f\|_{\tilde{B}^{s,p}_\lambda}\right)^{\lambda/\mu}.$$

In addition, by lemmata 12 and 13,

$$\|f\|_{\tilde{B}^{s,p}_\infty} \leqslant \|f\|^*_{\tilde{B}^{s,p}_\infty} \leqslant (s\lambda)^{1/\lambda}\|f\|^*_{\tilde{B}^{s,p}_\lambda} \leqslant \frac{(s\lambda)^{1/\lambda}}{2^s - 1}\|f\|_{\tilde{B}^{s,p}_\lambda}.$$

The maximum of $(s\lambda)^{1/\lambda}$ with respect to λ is attained for $s\lambda = e$. Thus $(s\lambda)^{1/\lambda}/(2^s-1) \leqslant e^{s/e}/(2^s-1) \leqslant 2/s$. Therefore

$$\|f\|_{\tilde{B}_\mu^{s,p}} \leqslant \left(\frac{2}{s}\right)^{1-\lambda/\mu} \|f\|_{\tilde{B}_\lambda^{s,p}}.$$

This proves the inequality in theorem 11, and therefore the theorem. ∎

9. – Dependence of $B_\lambda^{s,p}$ on s.

First we show that $B_\lambda^{s,p}$ decreases when s increases, for fixed p and λ.

THEOREM 14. – Suppose $s \geqslant r$ $(0 < r \leqslant s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$. Then

$$B_\lambda^{s,p}(I; E) \subset B_\lambda^{r,p}(I, E).$$

And, $|I|$ being the length of I, $\forall f \in B_\lambda^{s,p}$

$$\|f\|_{\tilde{B}_\lambda^{r,p}} \leqslant \begin{cases} |I|^{s-r}\|f\|_{\tilde{B}_\lambda^{s,p}} & \text{for bounded } I, \\ \|f\|_{\tilde{B}_\lambda^{s,p}} + \dfrac{2}{r}\|f\|_{L^p} & \text{for all } I. \end{cases}$$ ∎

PROOF. – Let $f \in B_\lambda^{s,p}$ and $a > 0$.

Bounded I. – For $\lambda < \infty$,

$$\left(\int_0^a (h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} \leqslant a^{s-r}\left(\int_0^a (h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda}.$$

For $\lambda = \infty$,

$$\operatorname*{Sup}_{0<h<a} h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)} \leqslant a^{s-r} \operatorname*{Sup}_{0<h<a} h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)}.$$

These inequalities give the first desired inequality by choosing $a = |I|$. Indeed (see remark 1.2) the integrals and supremum for $0 \leqslant h \leqslant |I|$ are equal to the integrals and to the supremum for $0 \leqslant h \leqslant \infty$.

Bounded or unbounded I. – We bound

$$\left(\int_0^\infty (h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} \leqslant$$
$$\leqslant \left(\int_0^1 (h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} + 2\|f\|_{L^p(I;E)}\left(\int_1^\infty h^{-r\lambda}\frac{dh}{h}\right)^{1/\lambda}.$$

Thus

$$\left(\int_0^\infty (h^{-r}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} \leqslant \left(\int_0^1 (h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} + 2\|f\|_{L^p(I;E)} \left(\frac{1}{r}\right)^{1/\lambda} \leqslant$$

$$\leqslant \left(\int_0^\infty (h^{-s}\|\tau_h f - f\|_{L^p(I_h;E)})^\lambda \frac{dh}{h}\right)^{1/\lambda} + \frac{2}{r}\|f\|_{L^p(I;E)}.$$

For $\lambda = \infty$ the integrals are replaced by Sup. This proves the second inequality, and therefore the theorem. ■

Let us see that $B^{s,p}_\lambda$ decreases when s increases, even if λ varies.

COROLLARY 15. – Suppose $s > r$ $(0 < r < s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty,\ 1 \leqslant \mu \leqslant \infty)$. Then

$$B^{s,p}_\lambda(I; E) \subset B^{r,p}_\mu(I; E)$$

And, $\forall f \in B^{s,p}_\lambda$,

$$\|f\|_{\tilde{B}^{r,p}_\mu} \leqslant \begin{cases} \dfrac{2|I|^{s-r}}{r(s-r)}\|f\|_{\tilde{B}^{s,p}_\lambda} & \text{for bounded } I\,, \\[2mm] \dfrac{2}{r(s-r)}\|f\|_{\tilde{B}^{s,p}_\lambda} + \dfrac{4}{r^2}\|f\|_{L^p} & \text{for all } I\,. \end{cases}$$

Moreover, if $\lambda \leqslant \mu$,

$$\|f\|_{\tilde{B}^{r,p}_\mu} \leqslant \begin{cases} \dfrac{2}{s}|I|^{s-r}\|f\|_{\tilde{B}^{s,p}_\lambda} & \text{for bounded } I\,, \\[2mm] \dfrac{2}{s}\|f\|_{\tilde{B}^{s,p}_\lambda} + \dfrac{2}{r}\|f\|_{L^p} & \text{for all } I\,. \end{cases}$$

PROOF. – If $\lambda \leqslant \mu$ theorems 11 and 14 yield $B^{s,p}_\lambda \subset B^{s,p}_\mu \subset B^{r,p}_\mu$ and the desired inequality.

For all λ and μ, lemma 9 and theorem 11 yield $B^{s,p}_\lambda \subset B^{r,p}_1 \subset B^{r,p}_\mu$. ■

10. – Dependence of $B^{s,p}_\lambda$ on s, p and e.

Let us see that $B^{s,p}_\lambda$ decreases as s and $s - 1/p$ increase and as p decreases.

THEOREM 16. – Suppose $s \geqslant r$, $p \leqslant q$ and either $s - 1/p > r - 1/q$ or $s - 1/p = r - 1/q$, $\lambda \leqslant \mu$. Then

$$B^{s,p}_\lambda(I; E) \subset B^{r,q}_\mu(I; E)\,.$$

Moreover, $\forall f \in B_\lambda^{s,p}$, if $\lambda \leqslant \mu$

$$\|f\|_{\tilde{B}_\mu^{r,q}} \leqslant \begin{cases} \dfrac{36}{rs} |I|^{s-r-1/p+1/q} \|f\|_{\tilde{B}_\lambda^{s,p}} & \text{for bounded } I\,, \\ \dfrac{6}{rs} \|f\|_{\tilde{B}_\lambda^{s,p}} + \dfrac{6}{r^2} \|f\|_{L^p} & \text{for unbounded } I \end{cases}$$

and, if $s - 1/p > r - 1/q$,

$$\|f\|_{\tilde{B}_\mu^{r,q}} \leqslant \begin{cases} \dfrac{36|I|^{s-r-1/p+1/q}}{r^2(s-r-1/p+1/q)} \|f\|_{\tilde{B}_\lambda^{s,p}} & \text{for bounded } I\,, \\ \dfrac{6}{r^2(s-r-1/p+1/q)} \|f\|_{\tilde{B}_\lambda^{s,p}} + \dfrac{12}{r^3} \|f\|_{L^p} & \text{for unbounded } I\,. \quad \blacksquare \end{cases}$$

PROOF. – Denote $S = r - 1/q + 1/p$. Then $r \leqslant S \leqslant s$ and $S - 1/p = r - 1/q$.

If $\lambda \leqslant \mu$ theorems 11, 14 and 10 sucessively yield $B_\lambda^{s,p} \subset B_\mu^{s,p} \subset B_\mu^{S,p} \subset B_\mu^{r,q}$, and the desired inequalities.

If $s - 1/p > r - 1/q$, lemma 9 and theorems 10 and 11 successively yield $B_\lambda^{s,p} \subset B_1^{S,p} \subset B_1^{r,q} \subset B_\mu^{r,q}$, and the desired inequalities. $\blacksquare$

REMARK 10.1. – The embedding of theorem 16 contains all the embeddings for Besov spaces given in sections 7, 8 and 9. But, for each particular case, the coefficients of the inequalities are better there than here. $\blacksquare$

11. – Sobolev theorem, and other dependence of $W^{s,p}$ on s and p.

Imbeddings for Sobolev spaces are particular cases of imbeddings for Besov spaces since $W^{s,p} = B_p^{s,p}$. First let us see that $W^{s,p}$ decreases as s increases.

COROLLARY 17. – Suppose $s \geqslant r$ $(0 < r \leqslant s < 1,\ 1 \leqslant p \leqslant \infty)$. Then

$$W^{s,p}(I; E) \subset W^{r,p}(I; E)$$

and, $\forall f \in W^{s,p}$,

$$\|f\|_{\tilde{W}^{r,p}} \leqslant \begin{cases} |I|^{s-r} \|f\|_{\tilde{W}^{s,p}} & \text{for bounded } I\,, \\ \|f\|_{\tilde{W}^{s,p}} + \dfrac{4}{r} \|f\|_{L^p} & \text{for all } I\,. \quad \blacksquare \end{cases}$$

PROOF. – Proposition 2 and theorem 14 yield $W^{s,p} = B_p^{s,p} \subset B_p^{r,p} = W^{r,p}$ and the desired inequalities. $\blacksquare$

Now we shall see that $W^{s,p}$ decreases when s increases and p decreases, $s - 1/p$ being fixed. That is the Sobolev theorem.

COROLLARY 18. – Suppose $s - 1/p = r - 1/q$ and $s \geq r$ $(0 < r \leq s < 1,\ 1 \leq p \leq \leq q \leq \infty)$. Then

$$W^{s,p}(I;E) \subset W^{r,q}(I;E)$$

$$\|f\|_{\tilde{W}^{r,q}} \leq \frac{36}{sr}\|f\|_{\tilde{W}^{s,p}} \quad \forall f \in W^{s,p}. \quad \blacksquare$$

For unbounded I, 36 may be replaced by 6. ∎

PROOF. – Theorems 11 and 10 give $B^{s,p}_p \subset B^{s,p}_q \subset B^{r,q}_q$. Then the proposition 2 give $W^{s,p} = B^{s,p}_p \subset B^{r,q}_q = W^{r,q}$. And these results yield the inequality. ∎

Finally we shall see that $W^{s,p}$ decreases when s and $s - 1/p$ increase and p decreases.

COROLLARY 19. – Suppose $s \geq r$, $p \leq q$ and $s - 1/p \geq r - 1/q$ $(0 < r \leq s < 1,\ 1 \leq p \leq \leq q \leq \infty)$. Then

$$W^{s,p}(I;E) \subset W^{r,q}(I;E)$$

and, $\forall f \in W^{s,p}$,

$$\|f\|_{\tilde{W}^{r,q}} \leq \begin{cases} \dfrac{36}{rs}|I|^{s-r-1/p+1/q}\|f\|_{\tilde{W}^{s,p}} & \text{for bounded } I, \\ \dfrac{6}{rs}\|f\|_{\tilde{W}^{s,p}} + \dfrac{12}{r^2}\|f\|_{L^p} & \text{for unbounded } I. \quad \blacksquare \end{cases}$$

PROOF. – Proposition 2 and theorem 16 give $W^{s,p} = B^{s,p}_p \subset B^{r,q}_q = W^{r,q}$, and the desired inequalities. ∎

12. – Dependence of $N^{s,p}$ on s and p.

Imbeddings for Nikoslkii spaces are particular case of imbeddings for Besov spaces, since $N^{s,p} = B^{s,p}_\infty$.

First let us see that $N^{s,p}$ decreases when s increases, for fixed p.

PROPOSITION 20. – Suppose $s \geq r$ $(0 < r \leq s < 1,\ 1 \leq p \leq \infty)$, Then

$$N^{s,p}(I;E) \subset N^{r,p}(I;E)$$

and, $\forall f \in N^{s,p}$,

$$\|f\|_{\tilde{N}^{r,p}} \leq \begin{cases} |I|^{s-r}\|f\|_{\tilde{N}^{s,p}} & \text{for bounded } I, \\ \operatorname{Sup}\{\|f\|_{\tilde{N}^{s,p}};\ 2\|f\|_{L^p}\} & \text{for unbounded } I. \quad \blacksquare \end{cases}$$

PROOF. – Let $f \in N^{s,p}$. When I is bounded we have, for $h \leqslant |I|$,

$$h^{-r}\|\tau_h f - f\|_{L^p(I_h;\,E)} \leqslant |I|^{s-r} h^{-s}\|\tau_h f - f\|_{L^p(I_h;\,E)} \,.$$

For $h \geqslant |I|$ this inequality is still satisfied since the two sides equal 0. The first inequality is then obtained by taking the supremum for $h \geqslant 0$.

When I is unbounded we have

$$h^{-r}\|\tau_h f - f\|_{L^p(I_h;\,E)} \leqslant \begin{cases} h^{-s}\|\tau_h f - f\|_{L^p(I_h;\,E)} & \text{for } h \leqslant 1\,, \\ 2\|f\|_{L^p(I;\,E)} & \text{for } h \geqslant 1\,. \end{cases}$$

The second inequality, and therefore the proposition, is then obtained by taking the supremum for $h \geqslant 0$. ■

Now let us see that $N^{s,p}$ decreases when s increases and p decreases, $s - 1/p$ being fixed. That is the Sobolev theorem for Nikolskii spaces.

COROLLARY 21. – Suppose $s - 1/p = r - 1/q$ and $s \geqslant r$ $(0 < r \leqslant s < 1,\ 1 \leqslant p \leqslant q \leqslant \infty)$. Then

$$N^{s,p}(I;\,E) \subset N^{r,q}(I;\,E)\,,$$

$$\|f\|_{\bar{N}^{r,q}} \leqslant \frac{18}{r}\,\|f\|_{\bar{N}^{s,p}} \qquad \forall f \in N^{s,p}\,.$$

If I is unbounded, 18 may be replaced by 3. ■

PROOF. – Proposition 2 and theorem 10 give $N^{s,p} = B^{s,p}_\infty \subset B^{r,q}_\infty = N^{r,q}$, and the inequality. ■

REMARK 12.1. – The corollary 21 may be written as follows:

Let $f \in L^p(I;\,E)$ satisfy: $\|\tau_h f - f\|_{L^p(I_h,\,E)} \leqslant ch^s$ $\forall h > 0$.

Let q be such that: $p \leqslant q \leqslant \infty$ if $sp > 1$, $p \leqslant q < p^* = p/(1 - sp)$ if $sp \leqslant 1$.

Then $f \in L^q(I;\,E)$ and $\|\tau_h f - f\|_{L^q(I_h;\,E)} \leqslant c' h^{s-1/p+1/q}$ $\forall h > 0$. ■

Finally we shall show that $N^{s,p}$ decreases when s and $s - 1/p$ increase and p decreases.

COROLLARY 22. – Assume $s \geqslant r$, $p \leqslant q$ and $s - 1/p \geqslant r - 1/q$ $(0 < r \leqslant s < 1,$ $1 \leqslant p \leqslant q \leqslant \infty)$. Then

$$N^{s,p}(I;\,E) \subset N^{r,q}(I;\,E)$$

and, $\forall f \in N^{s,p}$,

$$\|f\|_{\tilde{N}^{r,q}} \leqslant \begin{cases} \dfrac{18}{r}\,|I|^{s-r-1/p+1/q}\|f\|_{\tilde{N}^{s,p}} & \text{for bounded } I\,, \\ \dfrac{3}{r}\,\mathrm{Sup}\,\{\|f\|_{\tilde{N}^{s,p}};\, 2\|f\|_{L^p}\} & \text{for unbounded } I\,. \end{cases} \quad \blacksquare$$

PROOF. – Set $S = r - 1/q + 1/p$. Thus $r \leqslant S \leqslant s$ and $S - 1/p = r - 1/q$. Proposition 20 and corollary 21 yield $N^{s,p} \subset N^{S,p} \subset N^{r,q}$, and the desired inequality. ■

13. – Comparison of $W^{s,p}$, $N^{s,p}$ and $B_\lambda^{s,p}$.

Let us first compare Sobolev and Besov spaces.

COROLLARY 23. – Assume $s > r$ $(0 < r < s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$.

If $\lambda = p$ then

$$W^{s,p}(I; E) = B_p^{s,p}(I; E)$$

$$\|f\|_{\tilde{W}^{s,p}} = 2^{1/p}\|f\|_{\tilde{B}_p^{s,p}} \qquad \forall f \in W^{s,p}\,.$$

If $\lambda \geqslant p$ then

$$W^{s,p}(I; E) \subset B_\lambda^{s,p}(I; E) \subset W^{r,p}(I; E)$$

and, $\forall f \in W^{s,p}$,

$$\|f\|_{\tilde{B}_\lambda^{s,p}} \leqslant \frac{2}{s}\,\|f\|_{\tilde{W}^{s,p}}$$

and, $\forall f \in B_\lambda^{s,p}$,

$$\|f\|_{\tilde{W}^{r,p}} \leqslant \begin{cases} \dfrac{4}{r(s-r)}\,|I|^{s-r}\|f\|_{\tilde{B}_\lambda^{s,p}} & \text{for bounded } I\,, \\ \dfrac{4}{r(s-r)}\,\|f\|_{\tilde{B}_\lambda^{s,p}} + \dfrac{8}{r^2}\,\|f\|_{L^p} & \text{for all } I\,. \end{cases}$$

If $\lambda \leqslant p$ then

$$W^{s,p}(I; E) \subset B_\lambda^{r,p}(I; E) \subset W^{r,p}(I; E)$$

and, $\forall f \in W^{s,p}$,

$$\|f\|_{\tilde{B}_\lambda^{r,p}} \leqslant \begin{cases} \dfrac{2}{r(s-r)}\,|I|^{s-r}\|f\|_{\tilde{W}^{s,p}} & \text{for bounded } I\,, \\ \dfrac{2}{r(s-r)}\,\|f\|_{\tilde{W}^{s,p}} + \dfrac{4}{r^2}\,\|f\|_{L^p} & \text{for all } I \end{cases}$$

and, $\forall f \in B_\lambda^{r,p}$,

$$\|f\|_{\tilde{W}^{r,p}} \leqslant \frac{4}{r} \|f\|_{\tilde{B}_\lambda^{r,p}} . \quad \blacksquare$$

PROOF. – When $\lambda = p$ proposition 2 yields $W^{s,p} = B_p^{s,p}$ and the equality for norms.

When $\lambda \geqslant p$ proposition 2 and theorem 11 yield $W^{s,p} = B_p^{s,p} \subset B_\lambda^{s,p}$ and the first inequality. Corollary 15 and proposition 2 yield $B_\lambda^{s,p} \subset B_p^{r,p} = W^{r,p}$ and the other inequalities.

When $\lambda \leqslant p$ proposition 2 and corollary 15 yield $W^{s,p} = B_p^{s,p} \subset B_\lambda^{r,p}$ and the first two inequalities. Theorem 11 and proposition 2 give $B_\lambda^{r,p} \subset B_p^{r,p} = W^{r,p}$ and the last inequality. $\blacksquare$

Now let us compare Sobolev and Nikolskii spaces

COROLLARY 24. – Suppose $s > r$ $(0 < r < s < 1,\ 1 \leqslant p \leqslant \infty)$. Then

$$W^{s,p}(I; E) \subset N^{s,p}(I; E) \subset W^{r,p}(I; E)$$

and, $\forall f \in W^{s,p}$,

$$\|f\|_{\tilde{N}^{s,p}} \leqslant \frac{2}{s} \|f\|_{\tilde{W}^{s,p}}$$

and, $\forall f \in N^{s,p}$,

$$\|f\|_{\tilde{W}^{r,p}} \leqslant \begin{cases} \dfrac{4}{r(s-r)} |I|^{s-r} \|f\|_{\tilde{N}^{s,p}} & \text{for bounded } I, \\ \dfrac{4}{r(s-r)} \|f\|_{\tilde{N}^{s,p}} + \dfrac{8}{r^2} \|f\|_{L^p} & \text{for all } I . \quad \blacksquare \end{cases}$$

PROOF. – This is given by corollary 23 with $\lambda = \infty$. Indeed by proposition 2, $N^{s,p} = B_\infty^{s,p}$ and $\|f\|_{\tilde{N}^{s,p}} = \|f\|_{\tilde{B}_\infty^{s,p}}$.

At last let us compare Nikolskii and Besov spaces

COROLLARY 25. – Suppose $s > r$ $(0 < r < s < 1,\ 1 \leqslant p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$. Then

$$N^{s,p}(I; E) \subset B_\lambda^{r,p}(I; E) \subset N^{r,p}(I; E)$$

and, $\forall f \in N^{s,p}$,

$$\|f\|_{B_\lambda^{r,p}} \leqslant \begin{cases} \dfrac{2|I|^{s-r}}{r(s-r)} |I|^{s-r} \|f\|_{\tilde{N}^{s,p}} & \text{for bounded } I, \\ \dfrac{2}{r(s-r)} \|f\|_{\tilde{N}^{s,p}} + \dfrac{4}{r^2} \|f\|_{L^p} & \text{for all } I \end{cases}$$

and, $\forall f \in B_\lambda^{r,p}$,

$$\|f\|_{\tilde{N}^{r,p}} \leqslant \frac{2}{r} \|f\|_{\tilde{B}_\lambda^{r,p}}.$$

If $\lambda = \infty$ then

$$N^{s,p}(I; E) = B_\infty^{s,p}(I; E)$$

$$\|f\|_{\tilde{N}^{s,p}} = \|f\|_{\tilde{B}_\infty^{s,p}} \qquad \forall f \in N^{s,p}. \quad ■$$

PROOF. – Proposition 2 and corollary 15 give $N^{s,p} = B_\infty^{s,p} \subset B_\lambda^{r,p}$ and the first two inequalities. Theorem 11 and proposition 2 give $B_\lambda^{r,p} \subset B_\infty^{r,p} = N^{r,p}$ and the third inequality.

Proposition 2 give $N^{s,p} = B_\infty^{s,p}$ and the equality for norms. ■

14. – Imbeddings from $W^{s,p}$, $N^{s,p}$ and $B_\lambda^{s,p}$ into $\mathrm{Lip}^{s-1/p}$.

Let us see that for s large enough, any function of any one of these spaces has Lipschitz properties.

COROLLARY 26. – Assume $s > 1/p$ $(0 < s < 1,\ 1 < p \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$. Then

$$W^{s,p}(I; E) \subset \mathrm{Lip}^{s-1/p}(I; E)$$

$$N^{s,p}(I; E) \subset \mathrm{Lip}^{s-1/p}(I; E)$$

$$B_\lambda^{s,p}(I; E) \subset \mathrm{Lip}^{s-1/p}(I; E)$$

and

$$\|f\|_{\widetilde{\mathrm{Lip}}^{s-1/p}} \leqslant \frac{36}{s(s-1/p)} \|f\|_{\tilde{W}^{s,p}} \qquad \forall f \in W^{s,p},$$

$$\|f\|_{\widetilde{\mathrm{Lip}}^{s-1/p}} \leqslant \frac{18}{s-1/p} \|f\|_{\tilde{N}^{s,p}} \qquad \forall f \in N^{s,p},$$

$$\|f\|_{\widetilde{\mathrm{Lip}}^{s-1/p}} \leqslant \frac{36}{s(s-1/p)} \|f\|_{\tilde{B}_\lambda^{s,p}} \qquad \forall f \in B_\lambda^{s,p}. \quad ■$$

If I is unbounded 36 and 18 may be replaced by 6 and 3. ■

PROOF. – Proposition 2 and theorems 11 and 10 yield $W^{s,p} = B_p^{s,p} \subset B_\infty^{s,p} \subset B_\infty^{s-1/p,\infty} = \mathrm{Lip}^{s-1/p}$, and the first inequality. Corollary 21 and proposition 2 give $N^{s,p} \subset N^{s-1/p,\infty} = \mathrm{Lip}^{s-1/p}$ and the second inequality. Theorems 11 and 10 and proposition 2 yield $B_\lambda^{s,p} \subset B_\infty^{s,p} \subset B_\infty^{s-1/p,\infty} = \mathrm{Lip}^{s-1/p}$ and the third inequality. ■

15. – Traces of $W^{s,p}$, $N^{s,p}$ and $B_\lambda^{s,p}$: imbeddings into $\mathcal{C}_u$.

We denote by $\mathcal{C}_u(I; E)$ the space of uniformly continuous functions from I into E, that is

$$\mathcal{C}_u(I; E) = \Big\{f \in \mathcal{C}(I; E)\colon \operatorname*{Sup}_{x\in I,\, y\in I,\, |y-x|\leqslant \varepsilon} \|f(y) - f(x)\| \to 0 \ \text{as} \ \varepsilon \to 0\Big\}.$$

First let us give trace results for Sobolev spaces.

COROLLARY 27. – Suppose $s > 1/p$ $(0 < s < 1,\ 1 < p \leqslant \infty)$. Then

$$W^{s,p}(I; E) \subset \mathcal{C}_u(I; E)\,.$$

For any $f \in W^{s,p}$ and $t \in \bar{I}$ (closure of I), $f(t)$ is uniquely defined and

$$\|f(t)\|_E \leqslant \frac{1}{s - 1/p}\, \|f\|_{\widetilde{W}^{s,p}} + \left(2p + \left(\frac{2}{|I|}\right)^{1/p}\right) \|f\|_{L^p}\,. \quad ■$$

PROOF. – By corollary 26 $W^{s,p}$ is included in $\mathrm{Lip}^{s-1/p}$ and therefore in $\mathcal{C}_u$. The inequality is given by lemma 8 with $q = \infty$ and $s = 1/p$, by lemma 9 with $\lambda = p$ and $r = 1/p$ and by proposition 2. ■

Now let us give trace results for Nikolskii spaces

COROLLARY 28. – Suppose $s > 1/p$ $(0 < s < 1,\ 1 < p \leqslant \infty)$. Then

$$N^{s,p}(I; E) \subset \mathcal{C}_u(I; E)\,.$$

For any $f \in N^{s,p}$ and $t \in \bar{I}$, $f(t)$ is uniquely defined and

$$\|f(t)\|_E \leqslant \frac{1}{s - 1/p}\, \|f\|_{\widetilde{N}^{s,p}} + \left(2p + \left(\frac{2}{|I|}\right)^{1/p}\right) \|f\|_{L^p}\,. \quad ■$$

PROOF. – It is same as for corollary 27.

Finally we give trace results for Besov spaces

THEOREM 29. – Suppose either $s = 1/p$, $\lambda = 1$ or $s > 1/p$ $(0 < s < 1,\ 1 < p \leqslant \infty,$ $1 \leqslant \lambda \leqslant \infty)$. Then

$$B_\lambda^{s,p}(I; E) \subset \mathcal{C}_u(I; E)\,.$$

For any $f \in B_\lambda^{s,p}$ and $t \in \bar{I}$, $f(t)$ is uniquely defined, and

$$\|f(t)\|_E \leq \begin{cases} \|f\|_{\tilde{B}_1^{1/p,p}} + \left(\dfrac{2}{|I|}\right)^{1/p} \|f\|_{L^p} & \text{for } s = 1/p\,, \\ \dfrac{1}{s-1/p} \|f\|_{\tilde{B}_\lambda^{s,p}} + \left(2p + \left(\dfrac{2}{|I|}\right)^{1/p}\right) \|f\|_{L^p} & \text{for } s > 1/p\,, \end{cases}$$

where $|I|$ is the length of I, thus $2/|I| = 0$ for unbounded I. ∎

PROOF. – Lemma 8 with $q = \infty$ yields $B_1^{1/p,p} \subset L^\infty$ and the first inequality for almost at t in I. Lemma 9 yields $B_\lambda^{s,p} \subset B_1^{1/p,p}$ for $s > 1/p$, and the second inequality. There remains to prove that $B_1^{1/p,p} \subset \mathcal{C}_u$.

Let $f \in B_1^{1/p,p}(I; E)$ and $t > 0$. By propositions 3 and 4 $f - \tau_t f \in B_1^{1/p,p}(I_t; E)$. Then, by lemma 8, for $a \leq |I_t|/2$,

$$\|f - \tau_t f\|_{L^\infty(I_t;E)} \leq \int_0^a h^{-s} \|\tau_h(f - \tau_t f) - (f - \tau_t f)\|_{L^p(I_{t+h};E)} \frac{dh}{h} + a^{-s}\|f - \tau_t f\|_{L^p(I_t;E)}\,.$$

Thus

$$\|f - \tau_t f\|_{L^\infty(I_t;E)} \leq 2\int_0^a h^{-s} \|f - \tau_h f\|_{L^p(I_h;E)} \frac{dh}{h} + a^{-s}\|f - \tau_t f\|_{L^p(I_t;E)}\,.$$

Let $t \to 0$. Then $\|f - \tau_t f\|_{L^p(I_t;E)} \to 0$ (see for example [S1], remark 3.2 p. 73). Therefore we can choose $a = a(t)$ such that $a(t) \to 0$ and $a(t)^{-s}\|f - \tau_t f\|_{L^p(I_t;E)} \to 0$. Then the right hand side vanishes as $t \to 0$, and

$$\|f - \tau_t f\|_{L^\infty(I_t;E)} \to 0\,.$$

This yields $f \in \mathcal{C}(I; E)$ (see for example [S1] theorem 1 p. 71, with $F = \{f\}$), and then

$$\operatorname*{Sup}_{x \in I:\, x+t \in I} \|f(x) - f(x+t)\| \to 0 \quad \text{as } t \to 0\,.$$

This is $f \in \mathcal{C}_u(I; E)$. ∎

16. – A limit case of imbedding from $B_\lambda^{s,p}$ into L^q.

In section 5 we proved this imbedding for $s - 1/p = -1/q$, in the case $\lambda = 1$. Now we extend this result for $\lambda \leq q$.

THEOREM 30. – Suppose $s - 1/p = -1/q$, $s < 1/p$ and $\lambda \leqslant q < \infty$ $(0 < s < 1, 1 \leqslant p < q < \infty, 1 \leqslant \lambda \leqslant q)$. Then

$$B_\lambda^{s,p}(I; E) \subset L^q(I; E)$$

and there exists $c > 0$ such that, $\forall f \in B_\lambda^{s,p}$,

$$\|f\|_{L^q} \leqslant c(\|f\|_{\tilde{B}_\lambda^{s,p}} + \|f\|_{L^p}) . \quad ∎$$

REMARK 16.1. – The value of c is unknown. Indeed the proof relies on many interpolation results for which the coefficients are not all known.

A more direct proof with a known c can be given by means of the repartition function, but it is quite long. It is not carried on since the result is of little use in the present paper. ∎

PROOF. – *The case* $I = \boldsymbol{R}$. We use the standard notation $[A, B]_{\theta,\mu}$ for interpolation spaces. For an exact definition the reader is referred to [BB] p. 165. By the characterization of fractional domains of semi-groups (theorem 3.4.2 p. 194 of [BB] with $A = d/dt$ and $X = L^p(\boldsymbol{R}; E)$) there holds

$$B_\lambda^{s,p}(\boldsymbol{R}; E) = [W^{1,p}(\boldsymbol{R}; E), L^p(\boldsymbol{R}; E)]_{1-s,\lambda}, \qquad 0 < s < 1, \quad 1 \leqslant \lambda \leqslant \infty .$$

Thus by the reiteration theorem (theorem 3.2.20 p. 178 and definition 3.2.15 p. 175 of [BB]),

$$B_q^{s,p}(\boldsymbol{R}; E) = [B_1^{s_1,p}(\boldsymbol{R}; E), B_1^{s_2,p}(\boldsymbol{R}; E)]_{\frac{1}{2},q} \quad \text{if } s = (s_1 + s_2)/2, , \quad s_1 \neq s_2 .$$

On other hand the Riesz theorem (theorems 3.3.8 p. 186 and 3.3.10 p. 190 of [BB]) yields

$$L^q(\boldsymbol{R}; E) = [L^{q_1}(\boldsymbol{R}; E), L^{q_2}(\boldsymbol{R}; E)]_{\frac{1}{2},q} \quad \text{if } 1/q = (1/q_1 + 1/q_2)/2 , \quad q_1 \neq q_2 .$$

Suppose now that $s < 1/p$. We choose s_1 and s_2 such that $0 < s_1 < s < s_2 < 1/p$ and $s = (s_1 + s_2)/2$. Let q_1 and q_2 be defined by $s_1 - 1/p = -1/q_1$ and $s_2 - 1/p = -1/q_2$. Then $1/q = (1/q_1 + 1/q_2)/2$. By lemma 8 we have $B_1^{s_1,p}(\boldsymbol{R}; E) \subset L^{q_1}(\boldsymbol{R}; E)$ and $B_1^{s_2,p}(\boldsymbol{R}; E) \subset L^{q_2}(\boldsymbol{R}; E)$. Thus the above interpolation properties yield

$$B_q^{s,p}(\boldsymbol{R}; E) \subset L^q(\boldsymbol{R}; E) .$$

For all $\lambda \leqslant q$ we have $B_\lambda^{s,p} \subset B_q^{s,p}$ by theorem 11, and therefore

$$B_\lambda^{s,p}(\boldsymbol{R}; E) \subset L^q(\boldsymbol{R}; E) .$$

All these imbeddings are continuous and therefore c exists.

The case $I \subset \boldsymbol{R}$. For all $f \in B^{s,p}_\lambda(I; E)$ there exists $\hat{f} \in B^{s,p}_\lambda(\boldsymbol{R}; E)$ whose restriction to I is f; and the map $f \to \hat{f}$ can be choosen linear and continuous.

Then $\hat{f} \in L^q(\boldsymbol{R}, E)$ and therefore $f \in L^q(I; E)$, which proves the theorem. The extension result, that is the existence of $\hat{f}$, is proved for real functions in [T] (definition 4.2.1 p. 310 and theorem 4.4.1 p. 321 successively for $\Omega = \mathrm{I}$ and $\Omega = \boldsymbol{R}$). For vector valued functions a similar proof holds. It is not given here since it is quite long and since the main change is to replace the norm in $\boldsymbol{R}$ by the norm in E.

17. – Imbeddings from $W^{s,p}$, $N^{s,p}$ and $B^{s,p}_\lambda$ into L^q.

At first we give imbeddings for Besov spaces.

COROLLARY 31. – Let s, p, q and λ satisfy

if $s > 1/p$ then $p \leqslant q \leqslant \infty$,

if $s = 1/p$ then either $p \leqslant q < \infty$ or $q = \infty$, $\lambda = 1$,

if $s < 1/p$ then either $p \leqslant q < p_*$ or $q = p_*$, $\lambda \leqslant p_*$, where $s - 1/p = -1/p_*$,

$(0 < s < 1,\ 1 \leqslant p \leqslant q \leqslant \infty,\ 1 \leqslant \lambda \leqslant \infty)$. Then

$$B^{s,p}_\lambda(I; E) \subset L^q(I; E) .$$

If either $s > 1/p$, $p < q$ or $s = 1/p$, $p < q < \infty$ or $s < 1/p$, $p < q < p^*$, then $\forall f \in B^{s,p}_\lambda$,

$$\|f\|_{L^q} \leqslant 2^{1/q}\left(\frac{1}{s - 1/p + 1/q}\, \|f\|_{\tilde{B}^{s,p}_\lambda} + \left(\frac{2}{1/p - 1/q} + \left(\frac{2}{|I|}\right)^{1/p - 1/q}\right) \|f\|_{L^p}\right).$$

If $s = 1/p$, $q = \infty$, $\lambda = 1$, then $\forall f \in B^{1/p,p}_1$,

$$\|f\|_{L^\infty} \leqslant \|f\|_{\tilde{B}^{1/p,p}_1} + \left(\frac{2}{|I|}\right)^{1/p} \|f\|_{L^p} .$$

If I is unbounded, then $2^{1/q}$ can be replaced by 1 in these inequalities, and $2/|I| = 0$.

If $s < 1/p$, $q = p_*$ and $\lambda \leqslant p_*$, there exists $c > 0$ such that, $\forall f \in B^{s,p}_\lambda$,

$$\|f\|_{L^q} \leqslant c\left(\|f\|_{\tilde{B}^{s,p}_\lambda} + \|f\|_{L^p}\right) . \quad \blacksquare$$

PROOF. – If $q = p$ by definition $B^{s,p}_\lambda \subset L^q$.

Now let $q > p$ and set $r = 1/p - 1/q$. Then $r > 0$. If either $s > 1/p$ or $s = 1/p$,

$p<q<\infty$ or $s<1/p$, $p<q<p_*$, then $s>r$. Thus lemmata 9 and 8 yield $B^{s,p}_\lambda \subset B^{r,p}_1 \subset L^q$ and the desired inequality.

If $s=1/p$, $q=\infty$, $\lambda=1$, the result is given by lemma 8.

If $s<1/p$, $q=p_*$, the result is given by theorem 30. ■

Now we give imbeddings for Sobolev spaces.

COROLLARY 32. – Let s, p and q satisfy

if $s>1/p$ then $p\leqslant q\leqslant\infty$,

if $s=1/p$ then $p\leqslant q<\infty$,

if $s<1/p$ then $p\leqslant q\leqslant p_*$, where $s-1/p=-1/p_*$ that is $p_*=p/(1-sp)$,

$(0<s<1,\ 1\leqslant p\leqslant q\leqslant\infty)$. Then

$$W^{s,p}(I;E)\subset L^q(I;E)$$

and, if either $s\geqslant 1/p$ or $s<1/p$, $q<p_*$, $\forall f\in W^{s,p}$,

$$\|f\|_{L^q}\leqslant 2^{1/q}\left(\frac{1}{s-1/p+1/q}\,\|f\|_{\tilde W^{s,p}}+\left(\frac{2}{1/p-1/q}+\left(\frac{2}{|I|}\right)^{1/p-1/q}\right)\|f\|_{L^p}\right).$$

If I is unbounded, then $2^{1/q}$ can be replaced by 1, and $2/|I|=0$. ■

PROOF. – This follows from corollary 31 with $\lambda=p$. Indeed by proposition 2, $W^{s,p}=B^{s,p}_p$ and $\|f\|_{\tilde B^{s,p}_p}=2^{-1/p}\|f\|_{\tilde W^{s,p}}$. ■

Finally we give imbeddings for Nikolskii spaces.

COROLLARY 33. – Let s, p and q satisfy

if $s>1/p$ then $p\leqslant q\leqslant\infty$,

if $s\leqslant 1/p$ then $p\leqslant q<p_*$, where $s-1/p=-1/p_*$ that is $p_*=p/(1-sp)$,

$(0<s<1,\ 1\leqslant p\leqslant q\leqslant\infty)$. Then

$$N^{s,p}(I;E)\subset L^q(I;E)\,,$$

$$\|f\|_{L^q}\leqslant 2^{1/q}\left(\frac{1}{s-1/p+1/q}\,\|f\|_{\tilde N^{s,p}}+\left(\frac{2}{1/p-1/q}+\left(\frac{2}{|I|}\right)^{1/p-1/q}\right)\|f\|_{L^p}\right);\qquad \forall f\in N^{s,p}\,.$$

If I is unbounded, then $2^{1/q}$ can be replaced by 1, and $2/|I|=0$. ■

PROOF. – It follows from corollary 31 with $\lambda=\infty$. Indeed by proposition 2, $N^{s,p}=B^{s,p}_\infty$ and $\|f\|_{\tilde B^{s,p}_\infty}=\|f\|_{\tilde N^{s,p}}$. ■

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