Boundary Value Problems with Mixed Lateral Conditions for Parabolic Operators (*) (**).

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Summary. – *We study a parabolic problem in a cylinder with lateral conditions of mixed type. We get an existence, uniqueness and regularity result in dissymmetric function spaces nicely fitting the geometry of the problem.*

Introduction.

In this paper, we study a parabolic problem in a cylinder with lateral conditions of mixed type. Problems of this kind were studied by MAGENES ([15], [16]), who obtained existence and uniqueness results in classes of regular functions. Later, BAIOCCHI considered the same problem in a variational setting ([2], [3], [4]), under very weak hypotheses on the geometrical properties of the regions where different boundary conditions are given. By similar abstract techniques, BERNARDI [5] got some global regularity and non-regularity results. From a completely different point of view, GJUL'MISARJAN [12] studied this problem in a semi-infinite cylinder when the «separating surface» is parallel to the t-axis. In this particular case, Gjul'misarjan proved an existence, uniqueness and regularity theorem in Sobolev spaces taking into account the lack of regularity of the solution near the separating surface in the directions not parallel to the surface; these spaces and the techniques employed to deal with them are analogous to VIŠIK-ESKIN's ones for the elliptic case (see [10], [17]). ESKIN and CHANG ZUY HO [11] stated an existence result for differential systems, parabolic in the sense of PETROVSKIĬ, supposing that the separating surface be nowhere tangent to a hyperplane $t = \text{const}$. In [6] we considered a problem of this kind for the heat equation in a quarter space when the separating surface is a paraboloid of revolution; we obtained existence, uniqueness and regularity results in spaces similar to Gjul'misarjan's ones, but taking into account the essentially different situation due to the presence of the vertex of the

(*) Entrata in Redazione il 27 luglio 1981.
(**) Partially supported by G.N.A.F.A. of C.N.R., Italy.
(***) Partially supported by I.N.F.N., Bologna, Italy.

paraboloid, where the separating surface is tangent to the hyperplane $t = 0$. In [8] we studied a parabolic boundary value problem with mixed lateral conditions in a finite cylinder, when the separating surface is nowhere characteristic; there we solved the problem in function spaces nicely fitting the shape of the separating surface: this was accomplished pulling back, via a suitable change of variables having a global character, function spaces of Gjul'misarjan's type.

The aim of this paper is to study the above mentioned problem in the following situation. We consider a second order parabolic operator with smooth coefficients in a cylinder of finite height whose basis is a bounded regular open subset of $\boldsymbol{R}^n$. The separating surface is an $(n-1)$-submanifold of the lateral boundary of the cylinder and we suppose it is tangent to the hyperplane $t = 0$ at a point, where there is a contact of order exactly 1; furthermore, the separating surface is nowhere else tangent to a characteristic hyperplane. The boundary conditions on the lateral surface may be either two of the the following classical ones: Dirichlet, Neumann and Robin. We prove an existence and uniqueness result and we establish an a priori estimate in function spaces having the following features:

i) if t is bounded away from zero, they are the spaces defined in [8];

ii) near $t = 0$, they are akin to the spaces in [6] but now the weight function on the hyperplane $t = 0$ is of power type; we note that the traces in these spaces on the hyperplane $t = 0$ are forced to be zero.

With respect to [6] there are two kinds of improvements:

a) the problem is more relevant from the geometrical point of view, since we are not confined to a quarter space situation; furthermore, operators with variable coefficients are allowed;

b) a new choice for the weight function on the basis of the cylinder allows to enlarge—in an essential way—the class of data for which the problem is solvable, even for the heat operator.

We wish now to point out differences and connections between our and Baiocchi's results. Indeed, the two points of view are essentially different: the geometrical situation considered in [4] is quite general; to handle such a problem, Baiocchi employs an abstract variational procedure, so that he obtains the solution in a space which does not depend on the geometry involved. This solution cannot be regularized in the usual Sobolev spaces, because of the presence of discontinuities in the boundary conditions. On the other hand, we consider only a particular situation, but the function spaces in which we get the solution are closely connected with the geometrical structure of the problem, so that, if the data are smooth, so is the solution except in the directions normal to the boundary of the cylinder or transverse to the separating surface. Nevertheless, we note that, by a suitable choice of some parameters which are related to the regularity across the separating surface, our spaces can be embedded in BAIOCCHI's [4] ones.

To outline the essential points of the proof, we give a scheme of the paper.

In Chapter 1, we formulate the problem and establish some notations. In Chapter 2, we consider a parabolic problem with constant coefficients in a quarter space with mixed lateral conditions, when the separating surface is a paraboloid of revolution. In this case, analogously to [6], it is possible to use a change of variables introduced by KONDRAT'EV [14], mapping the paraboloid onto a cylinder and the vertex to infinity. In such a way, by a Laplace transformation, we have to study mixed problems in bounded regions and a (non-mixed) problem with unbounded coefficients in an unbounded region; the latter one was solved in [7]. In Chapter 3, we solve the original problem in a thin layer near to $t = 0$. To this end, the results of Chapter 2 and of [9] are used. Finally, in Chapter 4, the original problem is solved; for this, we construct a global diffeomorphism cylindrizing the separating surface far away from the hyperplane $t = 0$ and allowing to define function spaces in the upper part of the cylinder, which fairly fit those we have constructed in the lower part. In such a way, the global existence and regularity theorem follows by the results in Chapter 3 and in [8].

We note explicitly that our problem is meaningless when $n = 1$. Calculations are carried out when $n \geqslant 3$; the case $n = 2$ is obtained by slightly modifying the spaces we consider.

1. – Notations and position of the problem.

In the sequel, we shall always refer to the following notations:

$$\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n;\ x = (x_1, \dots, x_n),\ x_n > 0\};$$

if $x \in \boldsymbol{R}^n$, then $x = (x', x_n) = (x'', x_{n-1}, x_n)$. $N_0 = N \cup \{0\}$. If $x \in \boldsymbol{R}$, $[x]$ is the integral part of x. If $u \in \mathcal{S}(\boldsymbol{R}^n)$, then $\tilde{u}(\xi) = \int_{\boldsymbol{R}^n} \exp(-ix\cdot\xi) u(x)\, dx$, where $x\cdot\xi = \sum_{j=1}^n x_j \xi_j$. If $v \in \mathcal{S}(\boldsymbol{R})$, $\hat{v}(z) = \int_{\boldsymbol{R}} \exp(-zt) v(t)\, dt$. We shall denote by $x\cdot\nabla_x$ the differential operator $\sum_{j=1}^n x_j \partial_j$. If $X \subseteq \boldsymbol{R}^n$, $\alpha, \beta \in [-\infty, +\infty]$, $\alpha < \beta$, we shall put $X_{\alpha,\beta} =]\alpha, \beta] \times X$, if $\beta < +\infty$ and $X_{\alpha,+\infty} =]\alpha, +\infty[\times X$. Finally, $\gamma_k u$ is the k-th order trace of u on $(\partial X)_{\alpha,\beta}$ and $\gamma_\alpha^{(l)}$ is the l-th order trace of u on $\{\alpha\}\times X$.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$ with C^∞ boundary, being locally on one side of $\partial\Omega$. Let $T \in \boldsymbol{R}_+$ and set $I =]0, T]$, $Q = \Omega_{0,T}$, $\Gamma = (\partial\Omega)_{0,T}$. Let Γ_1 be a connected and relatively open subset of $\bar{\Gamma}$ with boundary γ such that:

i) γ is a connected C^∞ $(n-1)$-manifold with boundary $\partial\gamma$;

ii) $\partial\gamma = \gamma \cap (\{T\}\times\partial\Omega)$;

iii) there exists exactly one point $(0, x_0) \in \gamma$, $x_0 \in \partial\Omega$, such that the tangent plane to γ in this point is parallel to the hyperplane $t = 0$;

iv) γ and its tangent plane in $(0, x_0)$ have a contact of order exactly one.

Furthermore, put

$$\Gamma_2 = \bar{\Gamma} \setminus \bar{\Gamma}_1 , \qquad S_{t_0} = \bar{Q} \cap \{(t_0, x);\, x \in \Omega\} ,$$

$$\Gamma_{i,t_0} = \Gamma_i \cap \{(t_0, x);\, x \in \partial\Omega\} , \qquad i = 1, 2 , \qquad \gamma_{t_0} = \gamma \cap \bar{S}_{t_0} , \qquad \text{where } t_0 \in [0, T] .$$

Since the roles of Γ_1 and Γ_2 are interchangeable, for notational convenience we shall suppose Γ_1 is such that $\bar{\Gamma}_1 \cap S_0 = \{(0, x_0)\}$.

Let now $P(t, x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(t, x)\, \partial_x^\alpha$ be an elliptic differential operator with real coefficients such that:

v) $a_\alpha \in C^\infty(\overline{\Omega_{0,T}})$, $\forall \alpha$;

vi) $\sum_{|\alpha|=2} a_\alpha(t, x)\, \xi^\alpha > 0$, $\forall \xi \in \boldsymbol{R}^n \setminus \{0\}$, $\forall (t, x) \in \overline{\Omega_{0,T}}$.

In the sequel, we shall consider the following problem

$$\text{(P)} \qquad \begin{cases} (\partial_t - P(t, x, \partial_x))\, u = f , & \text{in } Q , \\ E_{k_i}(t, x, \partial_x)\, u = g_i , & \text{in } \Gamma_i , \quad i = 1, 2 , \\ \gamma_0^{(0)} u = 0 , & \text{in } \Omega , \end{cases}$$

where $k_i \in \{0, 1, 2\}$, $i = 1, 2$, $E_0 = \gamma_0$, $E_1(t, x, \partial_x) = \gamma_0(\partial/\partial n)$, $E_2 = E_1 + \alpha E_0$, $\alpha \in \boldsymbol{R} \setminus \{0\}$ and $\partial/\partial n$ is the conormal derivative with respect to $P(t, x, \partial_x)$.

2. – Parabolic problems with mixed boundary conditions in a half space.

2.1. *Function spaces.*

In what follows, we shall put $\omega = \{(x', 0) \in \boldsymbol{R}^n;\, |x'| = 1\}$. Let $\{\varphi_1, \dots, \varphi_m\}$ be a C^∞ partition of unity in $\overline{\boldsymbol{R}^n_+}$, such that:

(2.1.*a*) $\operatorname{supp} \varphi_j \cap \omega \neq \emptyset$, if $j = 2, \dots, m' < m$; $\operatorname{supp} \varphi_j \cap \omega = \emptyset$, if $j = 1$ or $j = m' + 1, \dots, m$;

(2.1.*b*) $\varphi_j \in C_0^\infty(\overline{\boldsymbol{R}^n_+})$, $\quad j = 2, \dots, m$.

Furthermore, let W be a fixed neighbourhood of ω in $\overline{\boldsymbol{R}^n_+}$ such that

(2.1.*c*) $\bar{W} \cap \operatorname{supp} \varphi_1 = \emptyset$, and

(2.1.d) if $\operatorname{supp}\varphi_j \cap \complement W \neq \emptyset$ and $\operatorname{supp}\varphi_j \cap (\boldsymbol{R}^{n-1}\times\{0\}) \neq \emptyset$, then $\operatorname{supp}\varphi_j \cap \complement W \cap (\boldsymbol{R}^{n-1}\times\{0\}) \neq \emptyset$, $j = 2, \ldots, m$.

We shall also need a set of functions $\{\psi_j; j = 1, \ldots, m\}$ having the following properties:

put $U_j = \operatorname{supp}\varphi_j$ and let V_j be a neighbourhood of U_j; then

(2.1.e) $\psi_j \in C^\infty(\overline{\boldsymbol{R}^n_+})$ and it is bounded with all its derivatives, $j = 1, \ldots, m$;

(2.1.f) $0 \leqslant \psi_j \leqslant 1$, $j = 1, \ldots, m$;

(2.1.g) $\psi_j(x) = 1$, $\forall x \in U_j$, $j = 1, \ldots, m$;

(2.1.h) $\operatorname{supp}\psi_j$ is a compact set if $j = 2, \ldots, m$ and $\operatorname{supp}\psi_j \subset V_j$, if $j = 1, \ldots, m$;

(2.1.i) if $U_j \cap U_k = \emptyset$, then $\varphi_k\psi_j = 0$, $j, k = 1, \ldots, m$;

(2.1.l) $\operatorname{supp}\psi_1 \subset \complement W$.

We remark that (2.1.g) implies that $\varphi_j\psi_j = \varphi_j$, if $j = 1, \ldots, m$.

DEFINITION 2.1.1. – Let $q \in \boldsymbol{C}\setminus\{0\}$, $s, r, l \in \boldsymbol{R}$. Denote by $H^q_{s,r,l}(\boldsymbol{R}^n)$ the space of functions u such that

$$|||u|||^2_{s,r,l} = \int_{\boldsymbol{R}^n} (|q|^2 + |\xi|^2)^s (|q|^2 + |\xi'|^2)^r (|q|^2 + |\xi''|^2)^l |\tilde{u}(q,\xi)|^2 \, d\xi < +\infty .$$

By $H^q_{s,r,l}(\boldsymbol{R}^n_+)$ we shall denote the natural quotient space. A list of the properties of such spaces is given in [6] and [8]. In a similar way, let us denote by $H^q_{s,l}(\boldsymbol{R}^{n-1})$ the space of functions u such that

$$|||u|||^2_{s,l} = \int_{\boldsymbol{R}^{n-1}} (|q|^2 + |\xi'|^2)^s (|q|^2 + |\xi''|^2)^l |\tilde{u}(q,\xi')|^2 \, d\xi' < +\infty .$$

DEFINITION 2.1.2. – Put $M^q_0(\boldsymbol{R}^n) = L^2(\boldsymbol{R}^n)$, $\forall q \in \boldsymbol{C}$, and, by induction, if

$$2m < s \leqslant 2m + 2 , \quad m \in N_0 , \quad q \in \boldsymbol{C}\setminus\{0\} , \quad \operatorname{Re} q^2 < (n/2) - 2m ,$$

$$M^q_s(\boldsymbol{R}^n) = \mathfrak{D}_{M^q_{2m}(\boldsymbol{R}^n)}\big((1-\Delta)^{(s-2m)/2}\big) \cap \mathfrak{D}_{M^q_{2m}(\boldsymbol{R}^n)}\big((q^2 + x\cdot\nabla)^{(s-2m)/2}\big) ,$$

equipped with the graph norm. Such a definition is well-posed (see [7], Remark 2.5). We shall denote by $(\!(\cdot)\!)_s$ the norm in $M^q_s(\boldsymbol{R}^n)$. Furthermore, we call $M^q_s(\boldsymbol{R}^n_+)$ the natural quotient space. Some properties of these spaces are proved in [7].

In what follows, we shall always suppose that the following hypothesis is satisfied.

HYPOTHESIS 2.1.3. – Whenever the spaces M_s^q are involved, the complex parameter q is such that $\operatorname{Re} q^2 < (n/2) - 2\cdot[s/2] - 2$, $q \neq 0$.

DEFINITION 2.1.4. – If $j \in \{2, ..., m'\}$, call S_j the operator on functions u, induced by a change of coordinates in V_j, mapping $V_j \cap \overline{\boldsymbol{R}_+^n}$ into $\overline{\boldsymbol{R}_+^n}$ and $V_j \cap (\boldsymbol{R}^{n-1}\times\{0\})$ into $\boldsymbol{R}^{n-1}\times\{0\}$, leaving the n-th coordinate unchanged and such that, if $(y_1, ..., y_n)$ are the new coordinates, y_{n-1} is equal to the (signed) distance between the projection onto the x_n-plane of the original point and ω. If $j \in \{m'+1, ..., m\}$, we put $S_j = I$.

Choose $y_{(j)} \in U_j$ in such a way that:

(2.1.4.a) if $U_j \cap \omega \neq \emptyset$, then $y_{(j)} \in \omega$;

(2.1.4.b) if $U_j \cap (\boldsymbol{R}^{n-1}\times\{0\}) \neq \emptyset$ and $U_j \cap \omega = \emptyset$, then $y_{(j)} \in \boldsymbol{R}^{n-1}\times\{0\}$;

(2.1.4.c) if $U_j \not\subseteq W$ and $U_j \cap W \neq \emptyset$, then $y_{(j)} \notin W$.

DEFINITION 2.1.5. – Let $q \in \boldsymbol{C}$ and let $s, r, l\colon \overline{\boldsymbol{R}_+^n} \to \boldsymbol{R}$ be continuous functions such that

i) $s|_{\complement W} \equiv s(y_{(1)})$, $r|_{\complement W} = 0$, $l|_{\complement W} = 0$.

Put $s_j = s(y_{(j)})$, $r_j = r(y_{(j)})$, $l_j = l(y_{(j)})$, $j = 1, ..., m$. We shall say that a function u belongs to $M^q_{(s,r,l)}(\boldsymbol{R}^n_+)$ iff

ii) $\varphi_1 u \in M^q_{s_1}(\boldsymbol{R}^n_+)$;

iii) $\varphi_j u \in H^q_{s_j,r_j,l_j}(\boldsymbol{R}^n)$, if $U_j \cap (\boldsymbol{R}^{n-1}\times\{0\}) = \emptyset$, $j \geqslant 2$;

iv) $S_j\varphi_j u \in H^q_{s_j,r_j,l_j}(\boldsymbol{R}^n_+)$, if $U_j \cap (\boldsymbol{R}^{n-1}\times\{0\}) \neq \emptyset$, $j \geqslant 2$.

If $u \in M^q_{(s,r,l)}(\boldsymbol{R}^n_+)$, we write

(2.1.5.a)
$$(\!(u)\!)^2_{(s,r,l)} = (\!(\varphi_1 u)\!)^2_{s_1} + \sum_{j=2}^{m} |\!|\!| S_j \varphi_j u |\!|\!|^2_{s_j,r_j,l_j},$$

where the norms $|\!|\!|\cdot|\!|\!|_{s_j,r_j,l_j}$ are computed in $\boldsymbol{R}^n_+$ when condition iv) is satisfied.

We note that the so defined spaces are independent of the particular choice of the change of variable operators S_j.

In the sequel, we shall always suppose that the functions s, r, l satisfy the condition

v) $|s_j - s_k| < 1/4$, $|r_j - r_k| < 1/4$, $|l_j - l_k| < 1/4$,

provided $U_j \cap U_k \neq \emptyset$, $j, k = 1, ..., m$.

DEFINITION 2.1.6. – We put $W' = W \cap (\boldsymbol{R}^{n-1}\times\{0\})$ and we think of W' as a subset of $\boldsymbol{R}^{n-1}$. Let $\sigma, \varrho\colon \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be continuous functions such that $\sigma|_{\complement W'} = \sigma(y'_{(1)})$,

$\varrho|_{\complement W'} = 0$ and put $\sigma_j = \sigma(y'_{(j)})$, $\varrho_j = \varrho(y'_{(j)})$, where $y_{(j)} = (y'_{(j)}, 0)$, if $U_j \cap (\mathbf{R}^{n-1}\times\{0\}) \neq \emptyset$, $j = 1, \ldots, m$; if $q \in \mathbf{C}$, denote by $M^q_{(\sigma,\varrho)}(\mathbf{R}^{n-1})$ the space of functions v, such that

i) $\varphi_1(\cdot, 0)v \in M^q_{\sigma_1}(\mathbf{R}^{n-1})$;

ii) $S'_j\varphi_j(\cdot, 0)v \in H^q_{\sigma_j,\varrho_j}(\mathbf{R}^{n-1})$, if $U_j \cap (\mathbf{R}^{n-1}\times\{0\}) \neq \emptyset$, $j \geqslant 2$,

where S'_j is the operator induced by S_j on the boundary.

If $v \in M^q_{(\sigma,\varrho)}(\mathbf{R}^{n-1})$, we write

$$((v))^2_{(\sigma,\varrho)} = ((\varphi_1(\cdot, 0)v))^2_{\sigma_1} + \sum_{\substack{j=2 \\ U_j\cap(\mathbf{R}^{n-1}\times\{0\})\neq\emptyset}}^{m} |||S'_j\varphi_j(\cdot, 0)v|||^2_{\sigma_j,\varrho_j}.$$

In the sequel, we shall always suppose that the functions σ, ϱ satisfy the condition:

iii) $|\sigma_j - \sigma_k| < 1/4$, $|\varrho_j - \varrho_k| < 1/4$, provided $U_j \cap U_k \cap (\mathbf{R}^{n-1}\times\{0\}) \neq \emptyset$, $j, k = 1, \ldots, m$.

DEFINITION 2.1.7. – Let $u \in M^q_{(s,r,l)}(\mathbf{R}^n_+)$; then $\varphi_j u$ belongs to the corresponding H-or M-space with fixed indices. Let $k \in N_0$ be such that $s_j > k + 1/2$ if $U_j \cap (\mathbf{R}^{n-1}\times\{0\}) \neq \emptyset$. Then there exists the k-th order trace of $\varphi_j u$, $\gamma_k(\varphi_j u)$, on the hyperplane $x_n = 0$. Put

$$\gamma_k u = \sum_{\substack{j=1 \\ U_j\cap(\mathbf{R}^{n-1}\times\{0\})\neq\emptyset}}^{m} \gamma_k(\varphi_j u).$$

For functions $u \in C^\infty(\overline{\mathbf{R}^n_+}) \cap M^q_{(s,r,l)}(\mathbf{R}^n_+)$, we have $(\gamma_k u)(x') = (\partial^k_n u)(x', 0)$. By Proposition C.1 in [6] and Theorem 2.14 in [7], we have:

PROPOSITION 2.1.8. – Suppose that the hypotheses of Definition 2.1.7 hold. Then γ_k is continuous from $M^q_{(s,r,l)}(\mathbf{R}^n_+)$ into $M^q_{(s+r-k-\frac{1}{2},l)}(\mathbf{R}^{n-1})$ and its norm is uniformly bounded in q.

In what follows, we shall always write $q^2 = \gamma + i\mu$; hence, by Hypothesis 2.1.3, γ will satisfy suitable conditions, depending on the order of the spaces involved.

DEFINITION 2.1.9. – Denote by $M_{((s+r+l)/2;s,r,l);\gamma}(\mathbf{R}\times\mathbf{R}^n_+)$ the space of functions u such that

$$(u)^2_{((s+r+l)/2;s,r,l);\gamma} = \int_{\mathbf{R}} ((\hat{u}(\gamma + i\mu, \cdot)))^2_{(s,r,l)}\, d\mu < +\infty.$$

DEFINITION 2.1.10. – Denote by $M_{((\sigma+\varrho)/2;\sigma,\varrho);\gamma}(\mathbf{R}\times\mathbf{R}^{n-1})$ the space of functions v such that

$$(v)^2_{((\sigma+\varrho)/2;\sigma,\varrho);\gamma} = \int_{\mathbf{R}} ((\hat{v}(\gamma + i\mu, \cdot)))^2_{(\sigma,\varrho)}\, d\mu < +\infty.$$

DEFINITION 2.1.11. – Let $s, r, l, \gamma \in \boldsymbol{R}$. Denote by $H_{(s+r+l)/2;\,s,r,l;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n)$ the space of functions u such that

$$\|u\|^2_{(s+r+l)/2;\,s,r,l;\,\gamma} = \int_{\boldsymbol{R}} |||\hat{u}(\gamma+i\mu,\cdot)|||^2_{s,r,l}\,d\mu < +\infty\,.$$

The space $H_{(s+r+l)/2;\,s,r,l;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$ is defined analogously. Moreover, if J is an interval of the real axis, $H_{(s+r+l)/2;\,s,r,l;\,\gamma}(J\times\boldsymbol{R}^n)$ and $H_{(s+r+l)/2;\,s,r,l;\,\gamma}(J\times\boldsymbol{R}^n_+)$ are defined as the natural quotient spaces.

Here and in what follows, we agree to omit the index γ if $\gamma = 0$.

DEFINITION 2.1.12. – Let $\varrho, \sigma, \gamma \in \boldsymbol{R}$. Denote by $H_{(\sigma+\varrho)/2;\,\sigma,\varrho;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1})$ the space of functions v such that

$$\|v\|^2_{(\sigma+\varrho)/2;\,\sigma,\varrho;\,\gamma} = \int_{\boldsymbol{R}} |||\hat{v}(\gamma+i\mu,\cdot)|||^2_{\sigma,\varrho}\,d\mu < +\infty\,.$$

Moreover, if J is an interval of the real axis, $H_{(\sigma+\varrho)/2;\,\sigma,\varrho;\,\gamma}(J\times\boldsymbol{R}^{n-1})$ is defined as the natural quotient space.

PROPOSITION 2.1.13. – Let $k \in N_0$. If $s > k + 1/2$, γ_k is continuous from $H_{(s+r+l)/2;\,s,r,l;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$ onto $H_{(s+r+l-k-\frac{1}{2})/2;\,s+r-k-\frac{1}{2},l;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1})$.

The proof is given in [19], Theorem 6.1 and [13], Theorem 2.2.8.

DEFINITION 2.1.14. – Let $s \geqslant 0$; denote by $M_{s/2;\,s;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$ the space of functions u such that

$$(u)^2_{s/2;\,s;\,\gamma} = \int_{\boldsymbol{R}} ((\hat{u}(\gamma+i\mu,\cdot)))^2_s\,d\mu < +\infty\,.$$

If J is an interval of the real axis, $M_{s/2;\,s;\,\gamma}(J\times\boldsymbol{R}^n_+)$ is defined as the natural quotient space.

By Proposition 2.10 in [7], we get

PROPOSITION 2.1.15. – The spaces $M_{s/2;\,s;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$ are interpolation spaces.

DEFINITION 2.1.16. – Let $\sigma \geqslant 0$; denote by $M_{\sigma/2;\,\sigma;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1})$ the space of functions v such that

$$(v)^2_{\sigma/2;\,\sigma;\,\gamma} = \int_{\boldsymbol{R}} ((\hat{v}(\gamma+i\mu,\cdot)))^2_\sigma\,d\mu < +\infty\,.$$

If J is an interval of the real axis, $M_{\sigma/2;\,\sigma;\,\gamma}(J\times\boldsymbol{R}^{n-1})$ is defined as the natural quotient space.

PROPOSITION 2.1.17. – Let $k \in N_0$, $s > k + 1/2$; then γ_k is continuous from

$$M_{s/2;\,s;\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+) \text{ onto } M_{(s-k-\frac{1}{2})/2;\,s-k-\frac{1}{2};\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1}).$$

The proof is straightforward once the corresponding theorems for « elliptic » M-spaces are known ([7], Theorem 2.14).

REMARK 2.1.18. – Let $u \in M_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$; then

$$(u)^2_{((s+r+l)/2;\,s,r,l);\,\gamma} = (\varphi_1 u)^2_{s_1/2;\,s_1;\,\gamma} + \sum_{j=2}^{m} \|S_j\,\varphi_j u\|^2_{(s_j+r_j+l_j)/2;\,s_j,r_j,l_j;\,\gamma}\,.$$

An analogous statement holds for the spaces on the boundary.

DEFINITION 2.1.19. – Let $u \in M_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$; then $\varphi_j u$ belongs to the corresponding M- or H-space with fixed indices. Let $k \in N_0$, $s_j > k + 1/2$, provided $U_j \cap (\boldsymbol{R}^{n-1}\times\{0\}) \neq \emptyset$. So, there exists the k-th order trace of $\varphi_j u$ on the hyperplane $x_n = 0$. Put

$$\gamma_k u = \sum_{\substack{j=1 \\ U_j \cap (\boldsymbol{R}^{n-1}\times\{0\})\neq\emptyset}}^{m} \gamma_k(\varphi_j u)\,.$$

We note that, if $u \in C_0^\infty(\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+})$, $(\gamma_k u)(t, x') = (\partial_n^k u)(t, x', 0)$.

By Propositions 2.1.13 and 2.1.17, γ_k is bounded from

$$M_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$$

to

$$M_{((s+r+l-k-\frac{1}{2})/2;\,s+r-k-\frac{1}{2},l);\,\gamma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1})\,.$$

In $\boldsymbol{R}_+\times\boldsymbol{R}^n_+$ we define the following change of variables ([14]):

$$T\colon (t, x) \mapsto (\tau, \omega)\,, \qquad \begin{cases} \tau = -\frac{1}{2}\log t \\ \omega = t^{-1/2}x\,. \end{cases}$$

It is obvious that T is a C^∞-diffeomorphism onto $\boldsymbol{R}\times\boldsymbol{R}^n_+$. Let $u \in C^\infty(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$ and put $v = u\circ T^{-1}$; then we have:

$$\partial_x^\beta \partial_t^p u = \exp\left[(2p + |\beta|)\tau\right] \sum_{j=0}^{p} \sum_{|\alpha|\leqslant j} \sum_{\gamma\leqslant\beta} C_{\alpha\gamma j}\,\omega^{\alpha-\gamma}\partial_\tau^{j-|\alpha|}\partial_\omega^{\alpha+\beta-\gamma} v\,,$$

$$\partial_\omega^\beta \partial_\tau^p v = \sum_{j=0}^{p} \sum_{|\alpha|\leqslant j} \sum_{\gamma\leqslant\beta} d_{\alpha\gamma j}\, t^{j-|\alpha|+|\beta|/2} x^{\alpha-\gamma}\partial_t^{j-|\alpha|}\partial_x^{\alpha+\beta-\gamma} u\,,$$

where $C_{\alpha\gamma j}$, $d_{\alpha\gamma j}$ are suitable constants, equal to zero if $\gamma \not\leqslant \alpha$.

DEFINITION 2.1.20. – Let $s, r, l, \gamma \in \boldsymbol{R}$. We denote by $\mathcal{H}_{(s+r+l)/2;s,r,l;\gamma}(\boldsymbol{R}_+ \times \boldsymbol{R}^n)$ the space of functions u such that $u \circ T^{-1} = v \in H_{(s+r+l)/2;s,r,l;\gamma}(\boldsymbol{R} \times \boldsymbol{R}^n)$ equipped with the norm

$$\|u\|^*_{(s+r+l)/2;s,r,l;\gamma} = \|v\|_{(s+r+l)/2;s,r,l;\gamma}.$$

In an analogous way, «italic» spaces in $\boldsymbol{R}_+ \times \boldsymbol{R}^n_+$ and in $\boldsymbol{R}_+ \times \boldsymbol{R}^{n-1}$ are defined.

DEFINITION 2.1.21. – Let $s \geqslant 0$; denote by $\mathcal{M}_{s/2;s;\gamma}(\boldsymbol{R}_+ \times \boldsymbol{R}^n)$ the space of functions u such that $u \circ T^{-1} = v \in M_{s/2;s;\gamma}(\boldsymbol{R} \times \boldsymbol{R}^n)$ equipped with the norm

$$(u)^*_{s/2;s;\gamma} = (v)_{s/2;s;\gamma}.$$

In the same way spaces in $\boldsymbol{R}^n_+$ are defined. If J is an interval of the real positive axis, $\mathcal{M}_{s/2;s;\gamma}(J \times \boldsymbol{R}^n)$ and $\mathcal{M}_{s/2;s;\gamma}(J \times \boldsymbol{R}^n_+)$ are defined as natural quotient spaces.

PROPOSITION 2.1.22. – Let $k \in N_0$, $s > k + 1/2$. Then γ_k is continuous from $\mathcal{M}_{s/2;s;\gamma}(\boldsymbol{R}_+ \times \boldsymbol{R}^n_+)$ onto $\mathcal{M}_{(s-k-\frac{1}{2})/2;s-k-\frac{1}{2};\gamma+k}(\boldsymbol{R}_+ \times \boldsymbol{R}^{n-1})$.

The proof is a straightforward consequence of Proposition 2.1.17.

DEFINITION 2.1.23. – Let $s \in N_0$, $T \in]0, +\infty]$, X an open subset of $\boldsymbol{R}^n$; denote by $K^s_\gamma(X_{0,T})$ the space of functions u such that

$$[u; T]^2_{s;\gamma} = \int\limits_{X_{0,T}} \left(\frac{t}{1+t}\right)^\gamma |u(t,x)|^2\,dt\,dx + \int\limits_{X_{0,T}} \left(\frac{t}{1+t}\right)^{\gamma+2s} |\partial^s_t u(t,x)|^2\,dt\,dx + \sum_{i=1}^n \int\limits_{X_{0,T}} \left(\frac{t}{1+t}\right)^{\gamma+2s} |\partial^{2s}_i u(t,x)|^2\,dt\,dx < +\infty\,.$$

The spaces K^s_γ, where s is not an integer, are defined by interpolation.

Further results about these spaces can be found in [9].

PROPOSITION 2.1.24. – Let $s \geqslant 0$, $t_0 > 0$, $\gamma' = \gamma - n/2 - 1$. Then

$$K^{s/2}_{\gamma'}((\boldsymbol{R}^n)_{0,t_0}) = \mathcal{M}_{s/2;s;\gamma}((\boldsymbol{R}^n)_{0,t_0}),$$

algebraically and topologically.

An analogous assertion holds true for the half space.

PROOF. – We note that the spaces $\mathcal{M}_{s/2;s;\gamma}((\boldsymbol{R}^n)_{0,t_0})$ are interpolation spaces. This follows by standard techniques from Proposition 2.1.15. Hence we need only to prove the assertion in the case $s = 2m$, $m \in N_0$. At first, we prove that

$$(w)_{m;2m;\gamma} = \left(\sum_{k=1}^m \sum_{h+l=k} \int\limits_{\boldsymbol{R}^n} \left(\int\limits_{\boldsymbol{R}_+} t^{\gamma+k-n/2-1} |\partial^h_t (1-\Delta)^l w|^2\,dt\right) dx + \int\limits_{\boldsymbol{R}^n} \left(\int\limits_{\boldsymbol{R}_+} t^{\gamma-n/2-1} |w|^2\,dt\right) dx\right)^{\frac{1}{2}}$$

is a norm equivalent to $(\cdot)^*_{m;2m;\gamma}$ in $\mathcal{M}_{m;2m;\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n)$. By Lemma 2.3 in [7], putting $w\circ T^{-1}=g$, $C_1=q^2+x\cdot\nabla_x$, $C_2=1-\Delta_x$, $\check{C}_1=\partial_\tau+\omega\cdot\nabla_\omega$, $\check{C}_2=1-\Delta_\omega$, we have

$$(w)^{*2}_{m;2m;\gamma}=(w\circ T^{-1})^2_{m;2m;\gamma}=\int_{\boldsymbol{R}}(\!(\hat{g}(\gamma+i\sigma,\cdot))\!)^2_{2m}\,d\sigma=$$

$$=\sum_{k=1}^m c_{k,m}\sum_{i_1,\dots,i_k=1}^2\int_{\boldsymbol{R}}\|C_{i_1}\dots C_{i_k}\hat{g}(\gamma+i\sigma,\cdot)\|_0^2\,d\sigma+\int_{\boldsymbol{R}}\|\hat{g}(\gamma+i\sigma,\cdot)\|_0^2\,d\sigma=$$

$$=2\pi\left(\sum_{k=1}^m c_{k,m}\sum_{i_1,\dots,i_k=1}^2\int_{\boldsymbol{R}^n}\left(\int_{\boldsymbol{R}}e^{-2\gamma\tau}|\check{C}_{i_1}\dots\check{C}_{i_k}g|^2\,d\tau\right)d\omega+\right.$$

$$\left.+\int_{\boldsymbol{R}^n}\left(\int_{\boldsymbol{R}}e^{-2\gamma\tau}|g|^2\,d\tau\right)d\omega\right)\leqslant\eta_1(w)^2_{m;2m;\gamma}.$$

The converse inequality can be proved in a similar way.

Let now p be a bounded continuation operator from $K^m_{\gamma'}((\boldsymbol{R}^n)_{0,t_0})$ into $K^m_{\gamma'}(\boldsymbol{R}_+\times\times\boldsymbol{R}^n)$, such that $\operatorname{supp}(pu)\subset\overline{(\boldsymbol{R}^n)_{0,2t_0}}$, $\forall u\in K^m_{\gamma'}((\boldsymbol{R}^n)_{0,t_0})$. Then, if $u\in K^m_{\gamma'}((\boldsymbol{R}^n)_{0,t_0})$, we have

$$(pu)^*_{m;2m;\gamma}\leqslant\eta_2(pu)_{m;2m;\gamma}\leqslant\eta_3[pu;+\infty]_{m;\gamma'}\leqslant\eta_4[u;t_0]_{m;\gamma'};$$

hence, $u\in\mathcal{M}_{m;2m;\gamma}((\boldsymbol{R}^n)_{0,t_0})$.

The converse inequality, as well as the assertion about the half space, can be proved in a similar way.

DEFINITION 2.1.25. – We denote by $\mathcal{M}_{((s+r+l)/2;s,r,l);\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$ the space of functions u such that $u\circ T^{-1}=v\in M_{((s+r+l)/2;s,r,l);\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n_+)$ and put

$$(u)^*_{((s+r+l)/2;s,r,l);\gamma}=(v)_{((s+r+l)/2;s,r,l);\gamma}.$$

If J is an interval of the positive real axis, $\mathcal{M}_{((s+r+l)/2;s,r,l);\gamma}(J\times\boldsymbol{R}^n_+)$ is defined as the natural quotient space.

It is straightforward to prove the following result.

PROPOSITION 2.1.26. – Let $\beta>0$. The functions in $\mathcal{M}_{((s+r+l)/2;s,r,l);\gamma}((\boldsymbol{R}^n_+)_{0,\beta})$ can be continued to functions in $\mathcal{M}_{((s+r+l)/2;s,r,l);\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$ so that the continuation operator has its norm uniformly bounded with respect to β.

DEFINITION 2.1.27. – We denote by $\mathcal{M}_{((\sigma+\varrho)/2;\sigma,\varrho);\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^{n-1})$ the space of functions u such that $u\circ T^{-1}=v\in M_{((\sigma+\varrho)/2;\sigma,\varrho);\gamma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1})$ and put

$$(u)^*_{((\sigma+\varrho)/2;\sigma,\varrho);\gamma}=(v)_{((\sigma+\varrho)/2;\sigma,\varrho);\gamma}.$$

If J is an interval of the positive real axis, $\mathcal{M}_{((\sigma+\varrho)/2;\,\sigma,\varrho);\,\gamma}(J\times\boldsymbol{R}^{n-1})$ is defined as the natural quotient space.

PROPOSITION 2.1.28. – i) Let $s_1\geqslant 1$; then the map $u\mapsto\partial_{x_j}u$ is continuous from

$$\mathcal{M}_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)\ \text{to}\ \mathcal{M}_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+),\ j=1,\dots,n.$$

ii) Let $s_1\geqslant 2$; then the map $u\mapsto\partial_t u$ is continuous from

$$\mathcal{M}_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)\ \text{to}\ \mathcal{M}_{((s+r+l-2)/2;\,s-2,r,l);\,\gamma+2}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)\,.$$

PROOF. – Let us prove i). At first, we note that the continuity of spatial derivatives in spaces $M^q_{(s,r,l)}(\boldsymbol{R}^n_+)$ is proved quite analogously to Proposition 2.1.17 in [8]. Now

$$\begin{aligned}(\partial_{x_j}u)^{*2}_{((s+r+l)/2;\,s,r,l);\,\gamma+1}&=\big((\partial_{x_j}u)\circ T^{-1}\big)^2_{((s+r+l)/2;\,s,r,l);\,\gamma+1}=\\&=\big(\exp(\tau)\,\partial_{\omega_j}(u\circ T^{-1})\big)^2_{((s+r+l)/2;\,s,r,l);\,\gamma+1}=\\&=\int_{\boldsymbol{R}}\big(\!\big(\partial_{\omega_j}\widehat{(\exp(\tau)(u\circ T^{-1}))}(\gamma+1+i\sigma,\cdot)\big)\!\big)^2_{(s,r,l)}\,d\sigma\leqslant\\&\leqslant C_1\int_{\boldsymbol{R}}\big(\!\big(\partial_{\omega_j}\widehat{(u\circ T^{-1})}(\gamma+i\sigma,\cdot)\big)\!\big)^2_{(s,r,l)}\,d\sigma\leqslant\\&\leqslant C_2\int_{\boldsymbol{R}}\big(\!\big(\widehat{(u\circ T^{-1})}(\gamma+i\sigma,\cdot)\big)\!\big)^2_{(s+1,r,l)}\,d\sigma=C_2(u)^{*2}_{((s+r+l+1)/2;\,s+1,r,l);\,\gamma}\,.\end{aligned}$$

So, i) is completely proved.

Assertion ii) is proved in a similar way. Indeed,

$$\begin{aligned}(\partial_t u)^{*2}_{((s+r+l)/2;\,s,r,l);\,\gamma+2}&=\tfrac14\big(\exp(2\tau)(\partial_\tau+\omega\cdot\nabla_\omega)(u\circ T^{-1})\big)^2_{((s+r+l)/2;\,s,r,l);\,\gamma+2}\leqslant\\&\leqslant C_3\int_{\boldsymbol{R}}\big(\!\big((\gamma+i\sigma+\omega\cdot\nabla_\omega)\widehat{(u\circ T^{-1})}(\gamma+i\sigma,\cdot)\big)\!\big)^2_{(s,r,l)}\,d\sigma;\end{aligned}$$

by the continuity of $q^2+\omega\cdot\nabla_\omega$ in spaces with fixed indices (see [6], Appendix C and [7], Proposition 2.12), the assertion follows.

PROPOSITION 2.1.29. – Let $\alpha\in\mathcal{S}(\boldsymbol{R}\times\boldsymbol{R}^n)$; then there exist $C_0>0$, $C_\alpha>0$, such that, $\forall\delta>0$, $\forall u\in\mathcal{M}_{(s/2;\,s,-l,l);\,\gamma}((\boldsymbol{R}^n_+)_{0,\delta})$,

$$(\alpha u)^*_{(s/2;\,s,-l,l);\,\gamma}\leqslant C_0\Big(\sup_{(\boldsymbol{R}^n_+)_{0,\delta}}|\alpha|+C_\alpha\delta^{\frac12}\Big)(u)^*_{(s/2;\,s,-l,l);\,\gamma}\,.$$

PROOF. – By Proposition 2.1.24, we have:

$$(\alpha u)^*_{(s/2;s,-l,l);\gamma} \leq C_1\Big([\alpha(\varphi_1\circ T)u]_{s_1/2;\gamma-n/2-1} + \\ + \sum_{j=2}^{m} \|S_j(\varphi_j(\alpha\circ T^{-1})(u\circ T^{-1}))\|_{s_j/2;s_j,-l_j,l_j;\gamma}\Big) = C_1\sum_{j=1}^{m} I_j .$$

In order to estimate I_1, we first prove the following assertion: there exist C'_0, $C'_\alpha > 0$ such that

$$(2.1.29.a) \qquad \forall v \in K^{\sigma}_{\gamma-n/2-1}((\boldsymbol{R}^n_+)_{0,\delta}) \\ [\alpha v]_{\sigma/2;\gamma-n/2-1} \leq C'_0\Big(\sup_{(\boldsymbol{R}^n_+)_{0,\delta}} |\alpha| + C'_\alpha \delta^{\frac12}\Big)[v]_{\sigma/2;\gamma-n/2-1} .$$

By interpolation, we may suppose $\sigma = 2m$, $m \in N_0$. Then, we have, by Remark 2.10 and Proposition 2.12 in [9],

$$[\alpha v]^2_{m;\gamma-n/2-1} \leq C_2\Big(\int_{(\boldsymbol{R}^n_+)_{0,\delta}} \Big(t^{\gamma-n/2-1+2m}|\partial_t^m(\alpha v)(t,x)|^2 + \\ + t^{\gamma-n/2-1+2m}\sum_{i=1}^{n}|\partial_i^{2m}(\alpha v)(t,x)|^2 + t^{\gamma-n/2-1}|(\alpha v)(t,x)|^2\Big)\,dt\,dx\Big) \leq \\ \leq C_3\Big(\int_{(\boldsymbol{R}^n_+)_{0,\delta}} \Big(t^{\gamma-n/2-1+2m}\sum_{l=0}^{m}|(\partial_t^l\alpha)(\partial_t^{m-l}v)(t,x)|^2 + \\ + t^{\gamma-n/2-1+2m}\sum_{i=1}^{n}\sum_{\lambda=0}^{2m}|(\partial_i^\lambda\alpha)(\partial_i^{2m-\lambda}v)(t,x)|^2 + t^{\gamma-n/2-1}|(\alpha v)(t,x)|^2\Big)\,dt\,dx\Big) \leq \\ \leq C'\Big(\sup_{(\boldsymbol{R}^n_+)_{0,\delta}} |\alpha|^2[v]^2_{m;\gamma-n/2-1} + C'^2_\alpha[v]^2_{m;\gamma-n/2}\Big) \leq C'^2_0\Big(\sup_{(\boldsymbol{R}^n_+)_{0,\delta}} |\alpha| + C'_\alpha\delta^{\frac12}\Big)^2[v]^2_{m;\gamma-n/2-1} .$$

So (2.1.29.*a*) is proved.

Then, by Proposition 2.1.24,

$$I_1 \leq C'_1\Big(\sup_{(\boldsymbol{R}^n_+)_{0,\delta}} |\alpha| + C'_\alpha\delta^{\frac12}\Big)(u)^*_{(s/2;s,-l,l);\gamma} .$$

In order to estimate I_j, $j > 1$, we need only to prove the following assertion:

(2.1.29.*b*) there exist $C''_0 > 0$, $C''_\alpha > 0$ such that,

$$\forall v \in H_{\varrho/2;\varrho,-\lambda,\lambda;\gamma}\,(]-\tfrac12\log\delta, +\infty[\times \boldsymbol{R}^n) , \\ \|S_j(\varphi_j(\alpha\circ T^{-1})v)\|_{\varrho/2;\varrho,-\lambda,\lambda;\gamma} \leq C''_0\Big(\sup_{(\boldsymbol{R}^n)_{0,\delta}} |\alpha| + C''_\alpha\delta^{\frac12}\Big)\|\varphi_j v\|_{\varrho/2;\varrho,-\lambda,\lambda;\gamma} .$$

First, we note that, if $v' \in H_{\varrho/2;\varrho,-\lambda,\lambda;\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n)$, $\|v'; H_{\varrho/2;\varrho,-\lambda,\lambda;\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n)\|^2$ is equivalent to

$$\int_{\boldsymbol{R}^{n+1}} (|\gamma| + |\sigma| + |\xi|^2)^{\varrho}(|\gamma| + |\sigma| + |\xi'|^2)^{-\lambda}(|\gamma| + |\sigma| + |\xi''|^2)^{\lambda}|\widetilde{w}'(\sigma, \xi)|^2\, d\sigma\, d\xi\,,$$

where $w'(t, x) = \exp(-\gamma t)v'(t, x)$.

Put (see also the proof of Proposition 2.2.8 in [8])

$$\hat{Q}_1^+ = \sigma + i|\gamma| + i|\xi|^2\,,$$

$$\hat{Q}_2^+ = \sigma + i|\gamma| + i|\xi'|^2\,,$$

$$\hat{Q}_3^+ = \sigma + i|\gamma| + i|\xi''|^2\,.$$

Call E^+ the pseudo-differential operator whose symbol is

$$\hat{E}^+ = (\hat{Q}_1^+)^{\varrho}(\hat{Q}_2^+)^{-\lambda}(\hat{Q}_3^+)^{\lambda}\,.$$

We remark that $\hat{E}^+$ is analytic in σ, if $\operatorname{Im}\sigma > 0$.

As in the proof of Lemma 4.3 in [18], we have:

$$\|v'; H_{\varrho/2;\varrho,-\lambda,\lambda;\gamma}(]-\tfrac{1}{2}\log\delta, +\infty[\times\boldsymbol{R}^n)\|^2 = \int_{-\frac{1}{2}\log\delta}^{+\infty}\Big(\int_{\boldsymbol{R}^n}|E^+(lw')|^2\,dx\Big)\,dt\,,$$

where $lw' = e^{-\gamma t}l_0 v'$ and $l_0 v'$ is an arbitrary continuation of v' in $H_{\varrho/2;\varrho,-\lambda,\lambda;\gamma}(\boldsymbol{R}\times\boldsymbol{R}^n)$.

As in the proof of Proposition 2.2.8 in [8], if $\psi(t, x) = S_j\big(\psi_j(t, x)(\alpha(t, x) - \alpha(0, 0))\circ T^{-1}\big)$, we have:

$$\begin{aligned}\hat{E}^+(\tau, \xi)(\tilde{\psi} * \widetilde{lw})(\tau, \xi) &= (\tilde{\psi} * (\hat{E}^+\widetilde{lw}))(\tau, \xi) + \\ &\quad + \int_{\boldsymbol{R}^{n+1}}\Big(\int_0^1 \nabla\hat{E}^+(\theta + \mu(\tau - \theta), \eta + \mu(\xi - \eta))\cdot \\ &\quad \cdot(\tau - \theta, \xi - \eta)\tilde{\psi}(\tau - \theta, \xi - \eta)\,\widetilde{lw}(\tau, \xi)\,d\mu\Big)\,d\theta\,d\eta = \\ &= (\tilde{\psi} * (\hat{E}^+\widetilde{lw}))(\tau, \xi) + (B\,\widetilde{lw})(\tau, \xi)\,.\end{aligned}$$

Then, (2.1.29.*b*) follows by arguments analogous to those in Lemma 1.2 in [10] and Proposition 2.2.8 in [8]. Now, by (2.1.29.*b*), I_j $(j > 1)$ can be estimated analogously to I_1. So, the assertion is completely proved.

In an analogous way, we can prove the following

PROPOSITION 2.1.30. – Suppose $s+r+l\geqslant 0$; then, $\forall\varepsilon>0$, there exists $\bar\delta>0$, such that

$$(u)^*_{((s+r+l)/2;\,s,r,l);\,\gamma+1}\leqslant\varepsilon(u)^*_{((s+r+l)/2;\,s,r,l);\,\gamma}\,,$$

for every $\delta\leqslant\bar\delta$; here the norms are computed in $(\boldsymbol{R}^n_+)_{0,\delta}$.

DEFINITION 2.1.31. – Let $u\in\mathcal{M}_{((s+r+l)/2;\,s,r,\,l);\,\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$; then, if $v=u\circ T^{-1}$, $(\varphi_j v)\circ T$ belongs to a corresponding $\mathcal{K}$- or $\mathcal{M}$-space with fixed indices. Let $k\in N_0$, $s_j>k+\frac12$ if $U_j\cap(\boldsymbol{R}^{n-1}\times\{0\})\neq\emptyset$; then there exists the k-th order trace of $(\varphi_j v)\circ T$ on the hyperplane $x_n=0$. Set

$$\gamma_k u=\sum_{\substack{j=1\\ U_j\cap(\boldsymbol{R}^{n-1}\times\{0\})\neq\emptyset}}^{m}\gamma_k\big((\varphi_j v)\circ T\big)\,.$$

We note that, if u is a smooth function, $(\gamma_k u)(t,x')=(\partial^k_n u)(t,x',0)$.

PROPOSITION 2.1.32. – Let $k\in N_0$, $s_j>k+\frac12$, if $U_j\cap(\boldsymbol{R}^{n-1}\times\{0\})\neq\emptyset$; then γ_k is a bounded operator from $\mathcal{M}_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$ into

$$\mathcal{M}_{((s+r+l-k-\frac12)/2;\,s+r-k-\frac12,l);\,\gamma+k}(\boldsymbol{R}_+\times\boldsymbol{R}^{n-1})\,.$$

PROOF. – We have

$$\gamma_k\big((\varphi_j v)\circ T\big)=\gamma_0\Big(\partial^k_{x_n}\big((\varphi_j v)\circ T\big)\Big)=t^{-k/2}\gamma_0\Big(\big(\partial^k_{\omega_n}(\varphi_j v)\big)\circ T\Big)$$

and so, by Propositions 2.1.13 and 2.1.22,

$$(2.1.32.a)\qquad(\gamma_k u)^{*2}_{((s+r+l-k-\frac12)/2;\,s+r-k-\frac12,l);\,\gamma+k}\leqslant$$
$$\leqslant C_1\sum_{j=1}^{m}\Bigg(\sum_{i=2}^{m}\big\|S'_i\big((\gamma_0\varphi_i)\,e^{k\tau}\gamma_k(\varphi_j v)\big)\big\|^2_{(s_i+r_i+l_i-k-\frac12)/2;\,s_i+r_i-k-\frac12,l_i;\,\gamma+k}+$$
$$+\big((\gamma_0\varphi_1)\,e^{k\tau}\gamma_k(\varphi_j v)\big)^2_{(s_1-k-\frac12)/2;s_1-k-\frac12;\,\gamma+k}\Bigg).$$

Now, we have:

$$\big\|S'_i\big((\gamma_0\varphi_i)\,e^{k\tau}\gamma_k(\varphi_j v)\big)\big\|^2_{(s_i+r_i+l_i-k-\frac12)/2;\,s_i+r_i-k-\frac12,l_i;\,\gamma+k}=$$
$$=\int_{\boldsymbol{R}}\Big|\!\Big|\!\Big|\widehat{\Big(S'_i\big((\gamma_0\varphi_j)\,e^{k\tau}\gamma_k(\varphi_j v)\big)\Big)}(\gamma+k+i\sigma,\,\cdot)\Big|\!\Big|\!\Big|^2_{s_i+r_i-k-\frac12,l_i}d\sigma\leqslant$$

$$\leqslant C_2\int_{\boldsymbol{R}} \|\!|\widehat{\left(S_i'((\gamma_0\varphi_i)\gamma_k(\varphi_j v))\right)}(\gamma+i\sigma,\cdot)\|\!|^2_{s_i+r_i-k-\frac12,l_i}\,d\sigma\leqslant$$

$$\leqslant C_3\int_{\boldsymbol{R}} \|\!|S_i(\varphi_i\partial_n^k\widehat{(\varphi_j v)}(\gamma+i\sigma,\cdot))\|\!|^2_{s_i-k,r_i,l_i}\,d\sigma\leqslant$$

$$\leqslant C_3\big(\partial_n^k(\varphi_j v)\big)^2_{((s-k+r+l)/2;\,s-k,r,l);\,\gamma}\leqslant C_4(u)^{*2}_{((s+r+l)/2;\,s,r,l);\,\gamma}\,.$$

The last inequality is obtained by the continuity of the spatial derivatives in the space $M_{((s+r+l)/2;\,s,r,l)\,;\,\gamma}$, which is almost obvious.

The other terms in (2.1.32.a) can be handled analogously.

REMARK 2.1.33. – Let $k\in N_0$, $\gamma-n/2-1+2s_1\leqslant 0$, $s_1>2k+1$, and let $u\in C^\infty(\overline{\boldsymbol{R}}_+\times\overline{\boldsymbol{R}}^n_+)\cap\mathcal{M}_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$. Then, if we put $(\gamma_0^{(k)}u)(x)=(\partial_t^k u)(0,x)$, we have $\gamma_0^{(k)}u=0$. Indeed, $u=\sum_{j=1}^m(\varphi_j\circ T)u$; now, if $x\in\boldsymbol{R}^n\setminus\{0\}$, then $|\omega|=t^{-\frac12}|x|\underset{t\to0+}{\longrightarrow}+\infty$, so that $(\varphi_j\circ T)(t,x)=0$ for $j=2,\dots,m$, provided t is small enough. On the other hand, $(\varphi_1\circ T)\,u\in K^{s_1}_{\gamma-n/2-1}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$, so that, by Proposition 2.16 in [9], $\partial_t^k((\varphi_1\circ T)u)(0,x)=0$. This justifies to set

$$\gamma_0^{(k)}u=0\,,$$

$\forall u\in\mathcal{M}_{((s+r+l)/2;\,s,r,l);\,\gamma}(\boldsymbol{R}_+\times\boldsymbol{R}^n_+)$, provided $\gamma-n/2-1+2s_1\leqslant0$, $s_1>2k+1$.

2.2. *Elliptic estimates depending on a parameter.*

In this section we shall recall some known results.

In what follows, $L(\partial)=\sum_{i,j=1}^n c_{ij}\partial_i\partial_j$ will denote a differential operator with constant real coefficients such that the matrix $\mathrm{C}=(c_{ij})_{i,j=1,\dots,n}$ is positive definite. Furthermore, we set

$$E_1=\gamma_0\Big(\sum_{j=1}^n c_{jn}\partial_j\Big)\,,$$

$$E_0=\gamma_0\,.$$

THEOREM 2.2.1 ([7], Theorems 4.1, 4.2). – Let $s\geqslant2$; if $\operatorname{Re}q^2<q_0$ (q_0 suitable) the operator $(L(\partial)+x\cdot\nabla+q^2)\oplus E_k$ is an isomorphism from $M^q_s(\boldsymbol{R}^n_+)$ onto $M^q_{s-2}(\boldsymbol{R}^n_+)\times M^q_{s-k-\frac12}(\boldsymbol{R}^{n-1})$, $k=0,1$; furthermore, it has, together with its inverse, a norm uniformly bounded in q.

THEOREM 2.2.2 ([12]; see also [8]). Let $s\geqslant2$, $l\geqslant0$; if $\operatorname{Re}q^2<0$, $|q|\geqslant q_0$ (q_0 suitable), $(k_1+k_2)/2<s-l<(k_1+k_2)/2+1$,

$$f\in H^q_{s-2,-l,l}(\boldsymbol{R}^n_+)\,,\qquad g_1\in H^q_{s-l-k_1-\frac12,l}(\boldsymbol{R}^{n-1}_+)\,,\qquad g_2\in H^q_{s-l-k_2-\frac12,l}(\boldsymbol{R}^{n-1}_-)\,,$$

then there exists a unique solution $u \in H^q_{s,-l,l}(\boldsymbol{R}^n_+)$ of the problem

$$(2.2.2.a)_{(k_1,k_2)} \qquad \begin{cases} (L(\partial) + q^2)u = f\,, & \text{in } \boldsymbol{R}^n_+\,,\\ E_{k_1}u = g_1\,, & \text{in } \boldsymbol{R}^{n-1}_+\,,\\ E_{k_2}u = g_2\,, & \text{in } \boldsymbol{R}^{n-1}_-\,,\end{cases}$$

where $k_i \in \{0, 1\}$, $i = 1, 2$.

Furthermore, the following a priori estimate holds

$$|||u|||_{s,-l,l} \leqslant C(|||f|||_{s-2,-l,l} + |||g_1|||_{s-l-k_1-\frac{1}{2},l} + |||g_2|||_{s-l-k_2-\frac{1}{2},l})\,,$$

where C is independent of q.

THEOREM 2.2.3 (cfr. [6], Theorems 3.3.2, 3.3.3 and 3.3.4). - Let $s \geqslant 2$, $l \geqslant 0$; if $\operatorname{Re} q^2 < 0$, $|q| \geqslant q_0$ (q_0 suitable), then the operator $(L(\partial) + q^2) \oplus E_k$ is an isomorphism from $H^q_{s,-l,l}(\boldsymbol{R}^n_+)$ onto $H^q_{s-2,-l,l}(\boldsymbol{R}^n_+) \times H^q_{s-l-k-\frac{1}{2},l}(\boldsymbol{R}^{n-1})$, $k = 0, 1$; furthermore, it has, together with its inverse, a norm uniformly bounded in q.

Analogously, $L(\partial) + q^2$ is an isomorphism from $H^q_{s,-l,l}(\boldsymbol{R}^n)$ onto $H^q_{s-2,-l,l}(\boldsymbol{R}^n)$; furthermore it has, together with its inverse, a norm uniformly bounded in q.

2.3. *Existence theorems and a priori estimates for elliptic problems depending on a parameter.*

THEOREM 2.3.1. - Let $q \in \boldsymbol{C}$, $\operatorname{Re} q^2 < 0$, $s, l\colon \overline{\boldsymbol{R}^n_+} \to \boldsymbol{R}$ such that

i) the triple $(s, -l, l)$ satisfies all the conditions of Definition 2.1.5;

ii) $s_j \geqslant 2$, $l_j \geqslant 0$, $j = 1, \ldots, m$;

iii) $(k_1 + k_2)/2 < s_j - l_j < (k_1 + k_2)/2 + 1$, if $j = 2, \ldots, m'$.

Furthermore, let $f \in M^q_{(s-2,-l,l)}(\boldsymbol{R}^n_+)$,

$$g_1 \in M^q_{(s-l-k_1-\frac{1}{2},l)}(\{x' \in \boldsymbol{R}^{n-1}; |x'| < 1\})\,,$$
$$g_2 \in M^q_{(s-l-k_2-\frac{1}{2},l)}(\{x' \in \boldsymbol{R}^{n-1}; |x'| > 1\})\,.$$

Let u be a solution of the problem

$$(2.3.1.a)_{(k_1,k_2)} \qquad \begin{cases} (L(\partial) + x\cdot\nabla + q^2)u = f\,, & \text{in } \boldsymbol{R}^n_+\,,\\ E_{k_1}u = g_1\,, & \text{in } \{x';\ |x'| < 1\}\,,\\ E_{k_2}u = g_2\,, & \text{in } \{x';\ |x'| > 1\}\,,\end{cases}$$

where $k_i \in \{0, 1\}$, $i = 1, 2$. Then, if $\operatorname{Re} q^2 \leq q_0 < 0$, the following a priori estimate holds

$$((u))_{(s,-l,l)} \leq C\Big(((f))_{(s-2,-l,l)} + \sum_{i=1}^{2} ((g_i))_{(s-l-k_i-\frac{1}{2},l)}\Big), \tag{2.3.1.b}$$

where C is independent of q.

PROOF. – We agree that all the constants appearing throughout this proof will be independent of q.

1) Multiply the first equation in $(2.3.1.a)_{(k_1,k_2)}$ by φ_1 and the third one by $\gamma_0\varphi_1$; we obtain

$$\begin{cases} (L(\partial) + x\cdot\nabla + q^2)(\varphi_1 u) = \varphi_1 f - T_1\psi_1 u = \Phi_1, \\ E_{k_2}(\varphi_1 u) = (\gamma_0\varphi_1) g_2 - F^{(1)}_{k_2}(\psi_1 u), \end{cases}$$

where

$$T_1 v = (L(\partial)\varphi_1) v + 2(\mathrm{C}\,\nabla\varphi_1)\cdot\nabla v + (x\cdot\nabla\varphi_1) v,$$

$$F^{(1)}_0 v = 0, \qquad F^{(1)}_1 v = \gamma_0\Big(\Big(\sum_{j=1}^{n} c_{jn}\partial_j\varphi_1\Big) v\Big).$$

By Theorem 2.2.1, we have

$$((\varphi_1 u))_{s_1} \leq C_1\Big(((\varphi_1 f))_{s_1-2} + ((T_1\psi_1 u))_{s_1-2} + (((\gamma_0\varphi_1)g_2))_{s_1-k_2-\frac{1}{2}} + ((F^{(1)}_{k_2}(\psi_1 u)))_{s_1-k_2-\frac{1}{2}}\Big),$$

provided $\operatorname{Re} q^2 < \gamma_0 < 0$ (γ_0 suitable).

Now, as $\varphi_1 = $ const. outside of a compact set, by Lemma 2.13 and Proposition 2.12 in [7], we get

$$((T_1\psi_1 u))_{s_1-2} \leq C_2((\psi_1 u))_{s_1-1} \leq C_2 \sum_{k=1}^{m} ((\psi_1\varphi_k u))_{s_1-1} = C_2\Big(((\varphi_1 u))_{s_1-1} + \sum_{k=2}^{m} ((\psi_1\varphi_k u))_{s_1-1}\Big).$$

By Proposition 2.17 in [7],

$$((\varphi_1 u))_{s_1-1} \leq C_3(1 + |\operatorname{Re} q^2|)^{-\frac{1}{2}}((\varphi_1 u))_{s_1};$$

furthermore, we remark that, if $\psi_1\varphi_k \not\equiv 0$, by (2.1.$i$) and (2.1.$l$), $U_k = \operatorname{supp}\varphi_k \not\subset W$, and this implies that $y_{(k)} \in \complement W$, by (2.1.4.$c$); so $s_k = s_1$ and $l_k = 0$. Hence, by (2.1.b) and Propositions C.2, C.3 in [6],

$$\sum_{k=2}^{m} ((\psi_1\varphi_k u))_{s_1-1} \leq C_4 \sum_{k=2}^{m} |||S_k\psi_1\varphi_k u|||_{s_k-1,-l_k,l_k} \leq C_5|q|^{-1} \sum_{k=2}^{m} |||S_k\varphi_k u|||_{s_k,-l_k,l_k}.$$

Finally, we get

$$((T_1\psi_1 u))_{s_1-2} \leqslant C_6(1+|\mathrm{Re}\, q^2|)^{-\frac{1}{2}}((u))_{(s,-l,l)}\,. \tag{2.3.1.c}$$

In an analogous way, if $k_2 = 1$, by Theorem 2.14 and Lemma 2.13 in [7],

$$((F_1^{(1)}(\psi_1 u)))_{s_1-\frac{3}{2}} \leqslant C_7((\psi_1 u))_{s_1-1} \leqslant C_7\Big(((\varphi_1\psi_1 u))_{s_1-1} + \sum_{k=2}^{m}((\varphi_k\psi_1 u))_{s_1-1}\Big)\,,$$

which can be estimated as above. At last, we get

$$((\varphi_1 u))_{s_1} \leqslant C_8\Big(((\varphi_1 f))_{s_1-2} + (((\gamma_0\varphi_1)g_2))_{s_1-k_2-\frac{1}{2}} + \big(1+|\mathrm{Re}\, q^2|\big)^{-\frac{1}{2}}((u))_{s,-l,l}\Big)\,. \tag{2.3.1.d}$$

2) As in [12] and in [6], Theorem 4.1, we can prove that, for every $\varepsilon > 0$, if the partition of unity $\{\varphi_j;\, j = 1, \dots, m\}$ is sufficiently refined,

$$\begin{aligned} |||S_j\varphi_j u|||_{s_j,-l_j,l_j} \leqslant C_9\Big(& |||S_j\varphi_j f|||_{s_j-2,-l_j,l_j} + \\ &+ \sum_{i=1}^{2} |||S_j'(\gamma_0\varphi_j)g_i|||_{s_j-l_j-k_i-\frac{1}{2},l_j} + (1+|\mathrm{Re}\, q^2|)^{-\frac{1}{2}}((u))_{(s,-l,l)} + \varepsilon|||S_j\varphi_j u|||_{s_j,-l_j,l_j}\Big)\,, \end{aligned} \tag{2.3.1.e}$$

$j = 2, \dots, m'$. We note that, by Proposition 2.1.16 in [8], we may refine the partition of unity as we want, without changing the spaces; hence (2.3.1.e) is proved in our spaces.

3) As in [6], Theorem 4.1, we can prove that, if $j = m' + 1, \dots, m$,

$$\begin{aligned} |||\varphi_j u|||_{s_j,-l_j,l_j} \leqslant C_{10}\Big(& |||\varphi_j f|||_{s_j-2,-l_j,l_j} + \\ &+ \sum_{i=1}^{2} |||(\gamma_0\varphi_j)g_i|||_{s_j-l_j-k_i-\frac{1}{2},l_j} + (1+|\mathrm{Re}\, q^2|)^{-\frac{1}{2}}((u))_{(s,-l,l)}\Big)\,. \end{aligned} \tag{2.3.1.f}$$

4) Summing up (2.3.1.d)-(2.3.1.f), we obtain

$$((u))_{(s,-l,l)} \leqslant C_{11}\Big(((f))_{(s-2,-l,l)} + \sum_{i=1}^{2}((g_i))_{(s-l-k_i-\frac{1}{2},l)} + \big(\varepsilon + m(1+|\mathrm{Re}\, q^2|)^{-\frac{1}{2}}\big)((u))_{(s,-l,l)}\Big)\,.$$

As C_{11} does not depend on ε, m, q, we can choose ε, q in such a way that $C_{11}\big(\varepsilon + m(1+|\mathrm{Re}\, q^2|)^{-\frac{1}{2}}\big) < 1$.
Thus, (2.3.1.b) is completely proved.

THEOREM 2.3.2. – Let the hypotheses of Theorem 2.3.1 be satisfied; then there exists a unique solution $u \in M^q_{(s,-l,l)}(\boldsymbol{R}^n_+)$ of the problem $(2.3.1.a)_{(k_1,k_2)}$.

PROOF. – The proof is the same as that of Theorem 4.2 in [6].

2.4. *The parabolic problem.*

THEOREM 2.4.1. – Let $s, l\colon \overline{\mathbf{R}^n_+} \to [0, +\infty[$ satisfy the hypotheses of Theorem 2.3.1 and let $\delta \in]0, +\infty]$. Suppose that

i) $f \in \mathcal{M}_{((s-2)/2; s-2, -l, l); \gamma+2}((\mathbf{R}^n_+)_{0,\delta})$;

ii) $g_i \in \mathcal{M}_{((s-k_i-\frac12)/2; s-l-k_i-\frac12, l); \gamma+k_i}((\mathbf{R}^{n-1})_{0,\delta})$, $i = 1, 2$.

Then there exists $\gamma_0 \in \mathbf{R}$ such that, if $\gamma \leqslant \gamma_0$, there is a unique function u satisfying the following conditions:

iii) $u \in \mathcal{M}_{(s/2; s, -l, l); \gamma}((\mathbf{R}^n_+)_{0,\delta})$;

iv) $(\partial_t - L(\partial_x))u = f$, in $(\mathbf{R}^n_+)_{0,\delta}$;

v) $E_{k_1} u = g_1$, in $\{(t, x') \in (\mathbf{R}^{n-1})_{0,\delta};\ |x'|^2 < t\}$;

vi) $E_{k_2} u = g_2$, in $\{(t, x') \in (\mathbf{R}^{n-1})_{0,\delta};\ |x'|^2 > t\}$;

vii) $\gamma_0^{(0)} u = 0$.

Furthermore, the following a priori estimate holds:

$$(u)^*_{(s/2; s, -l, l); \gamma} \leqslant C\Big((f)^*_{((s-2)/2; s-2, -l, l); \gamma+2} + \sum_{i=1}^{2} (g_i)^*_{((s-k_i-\frac12)/2; s-l-k_i-\frac12, l); \gamma+k_i}\Big), \tag{2.4.1.a}$$

where C is independent of δ.

PROOF. – Suppose, first, $\delta = +\infty$. By Remark 2.1.33, if

$$u \in \mathcal{M}_{(s/2; s, -l, l); \gamma}(\mathbf{R}_+ \times \mathbf{R}^n_+),$$

then $\gamma_0^{(0)} u = 0$. We use the notations of section 2.1 and put

$$U = u \circ T^{-1}, \quad F = -\exp(-2\tau)(f \circ T^{-1}), \quad G_j = \exp(-k_j \tau)(g_j \circ T^{-1}),$$

$j = 1, 2$; then u satisfies iii)-vi) iff

iii′) $U \in M_{(s/2; s, -l, l); \gamma}(\mathbf{R} \times \mathbf{R}^n_+)$;

iv′) $\frac12 \partial_\tau U + \frac12 \omega \cdot \nabla_\omega U + L(\partial_\omega) U = F$, in $\mathbf{R} \times \mathbf{R}^n_+$;

v′) $E_{k_1} U = G_1$, in $\{(\tau, \omega') \in \mathbf{R} \times \mathbf{R}^{n-1};\ |\omega'| < 1\}$;

vi′) $E_{k_2} U = G_2$, in $\{(\tau, \omega') \in \mathbf{R} \times \mathbf{R}^{n-1};\ |\omega'| > 1\}$.

By means of a bilateral Laplace transform, iii′)-vi′) are equivalent to

iii″) $\hat{U} \in M^{\sqrt{p}}_{(s, -l, l)}(\mathbf{R}^n_+)$;

iv$''$) $\frac{1}{2}p\hat{U}+\frac{1}{2}\omega\cdot\nabla_\omega\hat{U}+L(\partial_\omega)\hat{U}=\hat{F}$, in $\boldsymbol{R}^n_+$;

v$''$) $E_{k_1}\hat{U}=\hat{G}_1$, if $|\omega'|<1$;

vi$''$) $E_{k_2}\hat{U}=\hat{G}_2$, if $|\omega'|>1$;

where $\operatorname{Re} p=\gamma$ and by $\sqrt{p}$ we mean the square root of p with positive real part.

Due to our hypotheses, $\hat{F}\in M^{\sqrt{p}}_{(s-2,-l,l)}(\boldsymbol{R}^n_+)$, $\hat{G}_j\in M^{\sqrt{p}}_{(s-k_j-l-\frac{1}{2},l)}(\boldsymbol{R}^{n-1})$ ($j=1,2$), so that the problem iii$''$)-vi$''$) has a unique solution satisfying (2.4.1.a), by Theorems 2.3.1 and 2.3.2.

By an argument quite analogous to that of [1], § 11, it follows that the problem considered has a unique solution in a cylinder of finite height, too, and that the constant appearing in the corresponding a priori estimate can be chosen independent of the height of the cylinder.

3. – The problem in the lower part of the cylinder.

3.1. *Some preliminaries.*

Denote by $\overline{\delta^{(0)}}$, $\overline{\eta^{(0)}}$ two, arbitrarily chosen, positive numbers and by $\delta^{(0)}$, $\eta^{(0)}$ two positive numbers such that $\delta^{(0)}<\overline{\delta^{(0)}}$, $\eta^{(0)}<\overline{\eta^{(0)}}$ and satisfying conditions α_0)-α_3) specified below; moreover, set

$$Q^{(0)}=\Omega_{0,\delta^{(0)}}\,,\qquad \Gamma^{(0)}=(\partial\Omega)_{0,\delta^{(0)}}\,,\qquad \Gamma^{(0)}_i=\Gamma_i\cap\Gamma^{(0)}\,,\qquad i=1,2\,.$$

We shall suppose that

α_0) $\eta^{(0)}$, $\delta^{(0)}$ are small enough, so that $\gamma\cap([0,\delta^{(0)}]\times S(x_0,2\eta^{(0)}))$ is the graph of a C^∞ real function defined on a subset of $\partial\Omega$.

Now, let $\{\varphi^{(0)}_1,\varphi^{(0)}_2\}$ be a partition of unity in $\bar{\Omega}$ subordinated to the covering $\{S(x_0,2\eta^{(0)}),\ \complement S(x_0,\eta^{(0)})\}$. We shall denote by the same symbol, $\varphi^{(0)}_i$, the function $1_t\otimes\varphi^{(0)}_i$, $i=1,2$. Obviously, $\{\varphi^{(0)}_1,\varphi^{(0)}_2\}$ is a partition of unity in $\overline{Q^{(0)}}$.

Multiply by $\varphi^{(0)}_i$ problem (P) in $Q^{(0)}$ (i.e. multiply by $\varphi^{(0)}_i$ the differential equation, by $\gamma_0\varphi^{(0)}_i$ the boundary conditions and by $\gamma^{(0)}_0\varphi^{(0)}_i$ the initial condition). Thus, we obtain the problem

$$(\mathrm{P}^{(0)}_i)\quad\begin{cases}(\partial_t-P(t,x,\partial_x))(\varphi^{(0)}_i u)=\varphi^{(0)}_i f+T^{(0)}_i(t,x,\partial_x)u\,, & \text{in } Q^{(0)}\,,\\ E_{k_j}(t,x,\partial_x)(\varphi^{(0)}_i u)=(\gamma_0\varphi^{(0)}_i)g_j+S^{(j)(0)}_i(t,x)(\gamma_0 u)\,, & \text{in } \Gamma_j,\ j=1,2\,,\\ \gamma^{(0)}_0(\varphi^{(0)}_i u)=0\,,\end{cases}$$

where $T^{(0)}_i$ is a first order differential operator and $S^{(j)(0)}_i$ is the multiplication operator by a suitable smooth function, if $k_j=1,2$, and is zero if $k_j=0$, $i=1,2$.

We suppose that

α_1) $\eta^{(0)}$ is sufficiently small so that there exists a C^∞-diffeomorphism h_{x_0} from $\bar{\Omega} \cap S(x_0, 2\eta^{(0)})$ onto $U^{(0)} \subseteq \{y \in \boldsymbol{R}^n;\ |y| \leqslant 1,\ y_n \geqslant 0\}$, such that $h_{x_0}(x_0) = 0$ and y_n is the distance from $\partial\Omega$; hence $h_{x_0}(\partial\Omega \cap \operatorname{supp} \varphi_1^{(0)}) = \{y \in U^{(0)},\ y_n = 0\}$.

Problem $(\mathrm{P}_1^{(0)})$ is transformed, by h_{x_0}, into the following one:

$$(\mathrm{P}_1^{(0)\prime}) \qquad \begin{cases} (\partial_t - P_0'(t, y, \partial_y))(\varphi_1^{(0)\prime} u') = F', & \text{in } (U^{(0)})_{0,\delta^{(0)}}, \\ E_{kj}'(t, y, \partial_y)(\varphi_1^{(0)\prime} u') = G_j', & \text{in } \Gamma_j^{(0)\prime}, \quad j = 1, 2, \\ \gamma_0^{(0)}(\varphi_1^{(0)\prime} u') = 0, \end{cases}$$

where P_0', E_{kj}' are the operators P, E_{kj} written in the new variables and $\varphi_1^{(0)\prime}$, u' are the functions $\varphi_1^{(0)}$, u written in the y-variables. The functions F', G_j' are obtained via h_{x_0}, from the right-hand-side terms of $(\mathrm{P}_1^{(0)})$ and $\Gamma_j^{(0)\prime}$ is the image, through h_{x_0}, of $\Gamma_j \cap \overline{(S(x_0, 2\eta^{(0)}))_{0,\delta^{(0)}}}$, $j = 1, 2$.

Set $\gamma' = h_{x_0}\big(\gamma \cap \overline{(S(x_0, 2\eta^{(0)}))_{0,\delta^{(0)}}}\big)$; by hypothesis α_0),

$$\gamma' = \{(t, y);\ t \in [0, \delta^{(0)}],\ y \in U^{(0)},\ y_n = 0,\ t = \Phi(y')\},$$

where Φ is a real C^∞-function defined in a neighbourhood of the origin in $\boldsymbol{R}^{n-1}$. We remark that, by hypotheses iii), iv) of section 1, $\Phi(0) = 0$, $(\nabla\Phi)(0) = 0$ and $\mathcal{H}_\Phi(0)$ is positive definite.

We now make the following assumption:

α_2) $\eta^{(0)}$ is so small that in $h_{x_0}(\partial\Omega \cap S(x_0, 2\eta^{(0)}))$ there exists a C^∞ change of variables $y' = \chi(z')$, such that $(\Phi \circ \chi)(z') = |z'|^2$ (Morse Theorem).

Hence, we may now make the following change of variables:

$$(y', y_n) \mapsto (z', z_n), \qquad \text{where} \qquad z' = (\chi \otimes 1_{z_n})^{-1}(y'), \qquad z_n = y_n.$$

Problem $(\mathrm{P}_1^{(0)\prime})$ is transformed into the following one

$$(\mathrm{P}_1^{(0)\prime\prime}) \qquad \begin{cases} (\partial_t - P''(t, z, \partial_z))(\varphi_1^{(0)\prime\prime} u'') = F'', & \text{in } ((\chi \otimes 1_{z_n})^{-1}(U^{(0)}))_{0,\delta^{(0)}}, \\ E_{kj}''(t, z, \partial_z)(\varphi_1^{(0)\prime\prime} u'') = G_j'', & \text{in } \Gamma_j'', \quad j = 1, 2, \\ \gamma_0^{(0)}(\varphi_1^{(0)\prime\prime} u'') = 0, \end{cases}$$

where the new symbols have an obvious meaning. Here

$$\Gamma_1'' = \{(t, z) \in \boldsymbol{R}^{n+1};\ t \in]0, \delta^{(0)}],\ z = (z', 0),\ z \in (\chi \otimes 1_{z_n})^{-1}(U^{(0)}),\ t > |z'|^2\}.$$

Furthermore, we note that Γ_2'' is the complement in $]0, \delta^{(0)}]\times((\chi\otimes 1_{z_n})^{-1}(U^{(0)})\cap$ $\cap\{z\in \boldsymbol{R}^n;\, z_n=0\})$ of $\overline{\Gamma_1''}$.

At last, we suppose that

α_3) we choose $\delta^{(0)}$ such that

$$((1_t\otimes h_{x_0})^{-1}\circ(1_t\otimes\chi\otimes 1_{z_n})\circ T^{-1})\Big([-\tfrac{1}{2}\log 2\delta^{(0)},\, +\infty[\times\Big(\bigcup_{j=2}^{m} U_j\Big)\Big)\subset (S(x_0,\eta^{(0)}))_{0,2\delta^{(0)}}\,.$$

This hypothesis implies, in particular, that the boundary condition corresponding to $j=1$ in problem $(\mathrm{P}_2^{(0)})$ is empty.

3.2. *Function spaces.*

DEFINITION 3.2.1. – We denote by $\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$ the space of all functions u such that

$$(\varphi_1^{(0)}u)\circ(1_t\otimes h_{x_0})^{-1}\circ(1_t\otimes\chi\otimes 1_{z_n})\in\mathcal{M}_{((s+r+l)/2;\,s,r,l);\,\gamma}((\boldsymbol{R}^n_+)_{0,\delta^{(0)}})$$

and

$$\varphi_2^{(0)}u\in K^{s_1/2}_{\gamma-n/2-1}(Q^{(0)})\,,$$

equipped with the norm

$$[u]_{(s,r,l);\,\gamma}=\Big(\Big(\big((\varphi_1^{(0)}u)\big(t,h_{x_0}^{-1}(\chi(z'),z_n)\big)\big)\Big)^{*2}_{((s+r+l)/2;\,s,r,l);\,\gamma}+[\varphi_2^{(0)}u]^2_{s_1/2,;\,\gamma-n/2-1}\Big)^{\frac{1}{2}}\,.$$

Analogously, if $0<\alpha<\beta\leqslant\delta^{(0)}$, we can define the space

$$\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(\Omega_{\alpha,\beta})\,,$$

whose norm will be denoted by $[\,\cdot\,;\alpha,\beta]_{(s,r,l);\,\gamma}$.

Since these spaces are isomorphic when γ varies, if $\alpha>0$, we shall omit γ in the symbol of the space.

Furthermore, let $\psi_1^{(0)},\psi_2^{(0)}\in C^\infty(\bar\Omega)$ be two functions such that

$$\psi_i^{(0)}\varphi_i^{(0)}=\varphi_i^{(0)}\,,\qquad 0\leqslant\psi_i^{(0)}\leqslant 1\,,\qquad i=1,2\,,$$

$$\operatorname{supp}\psi_1^{(0)}\subset\bar\Omega\cap S(x_0,2\eta^{(0)})\,,\qquad \operatorname{supp}\psi_2^{(0)}\subset\bar\Omega\setminus\overline{S(x_0,\eta^{(0)})}\,.$$

We shall still call $\psi_i^{(0)}$ the function $1_t\otimes\psi_i^{(0)}$, $i=1,2$.

We note that, if the boundary conditions in problem (P) are not both Dirichlet conditions, we may choose functions s, l such that $s-l\geqslant 1$; in this case, the spaces $\mathcal{K}$ are contained in BAIOCCHI's [4] ones when $s\geqslant 2$.

DEFINITION 3.2.2. – Let $s+r+l\geqslant 1$, $\gamma-n/2-1+s_1<0$. Then, by Proposition 2.16 in [9] and Remark 2.1.33, the functions $\varphi_1^{(0)}u$ and $\varphi_2^{(0)}u$ have null traces on the hyperplane $t=0$. So, if $u\in\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$, we can put

$$\gamma_0^{(0)}u=0\,.$$

DEFINITION 3.2.3. – We denote by $\mathcal{K}_\gamma^{((\sigma+\varrho)/2;\,\sigma,\varrho)}(\Gamma^{(0)})$ the set of all functions u such that

$$\big((\gamma_0\varphi_1^{(0)})u\big)\circ(1_t\otimes h_{x_0})^{-1}\circ(1_t\otimes\chi)\in\mathcal{M}_{((\sigma+\varrho)/2;\,\sigma,\varrho);\,\gamma}\big((\boldsymbol{R}^{n-1})_{0,\delta^{(0)}}\big)$$

and

$$(\gamma_0\varphi_2^{(0)})u\in K^{\sigma_1/2}_{\gamma-n/2-\frac12}(\Gamma^{(0)})\,,$$

equipped with the norm

$$[u]_{(\sigma,\varrho);\,\gamma}=\Big(\Big(\big((\gamma_0\varphi_1^{(0)})u\big)\circ(1_t\otimes h_{x_0})^{-1}\circ(1_t\otimes\chi)\Big)^{*2}_{((\sigma+\varrho)/2;\,\sigma,\varrho);\,\gamma}+[(\gamma_0\varphi_2^{(0)})u]^2_{\sigma_1/2;\,\gamma-n/2-\frac12}\Big)^{\frac12}.$$

In what follows we shall put

$$\begin{aligned}\Lambda&=(1_t\otimes\chi\otimes 1_{2n})^{-1}\circ(1_t\otimes h_{x_0})=1_t\otimes\psi\,,\\ \Lambda'&=(1_t\otimes\chi)^{-1}\circ\big((1_t\otimes h_{x_0})|_{\Gamma^{(0)}}\big)\,.\end{aligned}$$

By Proposition 2.6 in [9] and Proposition 2.1.29, we have:

PROPOSITION 3.2.4. – Let $\alpha\in C^\infty(\overline{Q^{(0)}})$; then, the map $u\mapsto\alpha u$ is continuous in $\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$.

PROPOSITION 3.2.5. – i) Let $s\geqslant 1$; then the map $u\mapsto\partial_j u$ is continuous from $\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$ into $\mathcal{K}_{\gamma+1}^{((s+r+l-1)/2;\,s-1,r,l)}(Q^{(0)})$, $j=1,\dots,n$.

ii) Let $s+r+l\geqslant 2$; then the map $u\mapsto\partial_t u$ is continuous from

$$\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})\quad\text{into}\quad\mathcal{K}_{\gamma+2}^{((s+r+l-2)/2;\,s-2,r,l)}(Q^{(0)})\,.$$

Moreover, the norm of each one of these maps is uniformly bounded with respect to $\delta^{(0)}$.

PROOF. – ii) is a straightforward consequence of Proposition 2.1.28 and Proposition 2.13 in [9].

Let us now prove i).

$$[\partial_j u]^2_{(s-1,r,l);\,\gamma+1}=\big((\varphi_1^{(0)}\partial_j u)\circ\Lambda^{-1}\big)^{*2}_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1}+\\ +[\varphi_2^{(0)}\partial_j u]^2_{(s_1-1)/2;\,\gamma-n/2}=I_1^2+I_2^2\,.$$

Now, keeping in mind Proposition 2.1.28, we have:

$$I_1 \leqslant \big((\partial_j(\varphi_1^{(0)}u))\circ\Lambda^{-1}\big)^*_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1} + \big(((\partial_j\varphi_1^{(0)})u)\circ\Lambda^{-1}\big)^*_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1} =$$
$$= \big(\nabla((\varphi_1^{(0)}u)\circ\Lambda^{-1})\cdot(\partial_j\Lambda)\circ\Lambda^{-1}\big)^*_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1} +$$
$$+ \big(((\partial_j\varphi_1^{(0)})\circ\Lambda^{-1})((\varphi_1^{(0)}+\varphi_2^{(0)})\psi_1^{(0)}u)\circ\Lambda^{-1}\big)^*_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1} \leqslant$$
$$\leqslant C_1\Big[\big((\varphi_1^{(0)}u)\circ\Lambda^{-1}\big)^*_{((s+r+l)/2;\,s,r,l);\,\gamma} + \big(((\partial_j\varphi_1^{(0)})\varphi_1^{(0)}u)\circ\Lambda^{-1}\big)^*_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1} +$$
$$+ \big((\varphi_2^{(0)}\psi_1^{(0)}u)\circ\Lambda^{-1}\big)^*_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1}\Big] = C_1[J_1+J_2+J_3]\,.$$

Let us now deal with J_3. Denote by $p_{\delta^{(0)}}$ a continuation operator from $K_\gamma^{s_1/2}((\boldsymbol{R}^n_+)_{0,\delta^{(0)}})$ into $K_\gamma^{s_1/2}((\boldsymbol{R}^n_+)_{0,+\infty})$, with norm uniformly bounded in $\delta^{(0)}$. So, by Hypothesis α_3), Proposition 2.1.24, Propositions 2.11, 2.12 in [9] and 2.1.5.i), we have

$$J_3 \leqslant \Big(\big(p_{\delta^{(0)}}((\varphi_2^{(0)}\psi_1^{(0)}u)\circ\Lambda^{-1})\big)\circ T^{-1}\Big)_{((s+r+l-1)/2;\,s-1,r,l);\,\gamma+1} \leqslant$$
$$\leqslant C_2[p_{\delta^{(0)}}((\varphi_2^{(0)}\psi_1^{(0)}u)\circ\Lambda^{-1})]_{(s_1-1)/2;\,\gamma-n/2} \leqslant C_3[\varphi_2^{(0)}u]_{(s_1-1)/2;\,\gamma-n/2} \leqslant C_4[\varphi_2^{(0)}u]_{s_1/2;\gamma-n/2-1}\,.$$

J_2 can be handled in an analogous way. So, the estimate for I_1 is obtained. The estimate for I_2 follows analogously.

DEFINITION 3.2.6. – Let $u \in \mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$, $k \in N_0$, $s_j > k + \frac{1}{2}$, provided $U_j \cap (\boldsymbol{R}^{n-1}\times\{0\}) \neq \emptyset$. We put

$$\gamma_k u = \big(\gamma_k((\varphi_1^{(0)}u)\circ(1_t\otimes h_{x_0})^{-1}\circ(1_t\otimes\chi\otimes 1_{z_n}))\big)\circ\big((1_t\otimes\chi^{-1})\circ(1_t\otimes h_{x_0})\big)|_{\Gamma^{(0)}} + \gamma_k(\varphi_2^{(0)}u)\,.$$

By Definition 2.1.31 and Theorem 2.15 in [9], this definition is well posed.

REMARK 3.2.7. – It is easy to show that, if $u \in \mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$, then $\gamma_k u = \gamma_0(\partial_\nu^k u)$, where ∂_ν denotes the inward normal derivative to the hypersurface $\Gamma^{(0)}$. Furthermore, if $u \in C^\infty(\overline{Q^{(0)}})$, we have $\gamma_k u = \partial_\nu^k u|_{\Gamma^{(0)}}$.

PROPOSITION 3.2.8. – Under the hypotheses of Definition 3.2.6, γ_k is a continuous operator from $\mathcal{K}_\gamma^{((s+r+l)/2;\,s,r,l)}(Q^{(0)})$ into $\mathcal{K}_{\gamma+k}^{((s+r+l-k-\frac{1}{2})/2;\,s+r-k-\frac{1}{2},l)}(\Gamma^{(0)})$.

PROOF. – By Proposition 3.2.5, we need only to prove the assertion in the case $k=0$. We have

$$[\gamma_0 u]^2_{(s+r+l-\frac{1}{2},l);\,\gamma} \leqslant 2\Big(((\gamma_0\varphi_1^{(0)})\circ\Lambda'^{-1})\big(\gamma_0((\varphi_1^{(0)}u)\circ\Lambda^{-1})\big)\Big)^{*2}_{((s+r+l-\frac{1}{2})/2:\,s+r-\frac{1}{2},l);\,\gamma} +$$
$$+ 2\Big(((\gamma_0\varphi_1^{(0)})\circ\Lambda'^{-1})\big(\gamma_0(\varphi_2^{(0)}u)\big)\circ\Lambda'^{-1}\Big)^{*2}_{((s+r+l-\frac{1}{2})/2;\,s+r-\frac{1}{2},l);\,\gamma} +$$
$$+ 2\Big[(\gamma_0\varphi_2^{(0)})\big(\gamma_0((\varphi_1^{(0)}u)\circ\Lambda^{-1})\big)\circ\Lambda'\Big]^2_{s_1/2-\frac{1}{4};\,\gamma-n/2-\frac{1}{2}} +$$
$$+ 2[(\gamma_0\varphi_2^{(0)})(\gamma_0\varphi_2^{(0)}u)]^2_{s_1/2-\frac{1}{4};\,\gamma-n/2-\frac{1}{2}} = 2\sum_{j=1}^4 I_j\,.$$

Now, by Propositions 2.1.32 and 3.2.4,

$$I_1 = \big(\gamma_0((\varphi_1^{(0)2}u)\circ\Lambda^{-1})\big)^{*2}_{((s+r+l-\frac{1}{2})/2;\, s+r-\frac{1}{2},\, l);\,\gamma} \leqslant$$
$$\leqslant C_1\big((\varphi_1^{(0)2}u)\circ\Lambda^{-1}\big)^{*2}_{((s+r+l)/2;\, s,r,l);\,\gamma} \leqslant C_2[u]^2_{(s,r,l);\,\gamma};$$

furthermore, by Hypothesis α_3), (2.1.5.i) and by Proposition 2.6 in [9], we have

$$I_2 = \big((\gamma_0(\varphi_1^{(0)}\varphi_2^{(0)}u))\circ\Lambda'^{-1}\big)^{*2}_{((s+r+l-\frac{1}{2})/2;\, s+r-\frac{1}{2},\, l);\,\gamma} \leqslant$$
$$\leqslant C_3\Big[(\gamma_0(\varphi_1\circ T))\big((\gamma_0(\varphi_1^{(0)}\varphi_2^{(0)}u))\circ\Lambda'^{-1}\big)\Big]^2_{(s_1-\frac{1}{2})/2;\,\gamma-n/2-\frac{1}{2}} \leqslant$$
$$\leqslant C_4[\gamma_0(\varphi_1^{(0)}\varphi_2^{(0)}u)]^2_{(s_1-\frac{1}{2})/2;\,\gamma-n/2-\frac{1}{2}} \leqslant C_5[\varphi_1^{(0)}\varphi_2^{(0)}u]^2_{s_1/2;\,\gamma-n/2-1} \leqslant$$
$$\leqslant C_6[\varphi_2^{(0)}u]^2_{s_1/2;\,\gamma-n/2-1} \leqslant C_6[u]^2_{(s,r,l);\,\gamma}\,.$$

I_3 and I_4 can be estimated in an analogous way. So, the assertion is proved.

3.3. *Solution in the lower part of the cylinder.*

THEOREM 3.3.1. – Suppose that the functions $s, l: \overline{\boldsymbol{R}^n_+} \to \boldsymbol{R}$ satisfy the hypotheses of Theorem 2.3.1 and that $\eta^{(0)}$, $\delta^{(0)}$ satisfy Hypotheses α_0)-α_3).

Then, there exists $\bar{\gamma}$, such that, if $\gamma \leqslant \bar{\gamma}$, the operator $\mathcal{A}$ corresponding to the problem

$$(3.3.1.a)_{(k_1,k_2)} \qquad \begin{cases} (\partial_t - P(t, x, \partial_x))u = f\,, & \text{in } Q^{(0)}\,, \\ E_{k_i}(t, x, \partial_x)u = g_i\,, & \text{in } \Gamma_i^{(0)},\ i = 1, 2\,, \\ \gamma_0^{(0)}u = 0\,, & \text{in } \Omega\,, \end{cases}$$

is an isomorphism from $\mathcal{K}_\gamma^{(s/2;\, s,\, -l,l)}(Q^{(0)})$ onto

$$\mathcal{K}_{\gamma+2}^{((s-2)/2;\, s-2,\, -l,l)}(Q^{(0)}) \times \Big(\mathop{\times}_{i=1}^{2} \mathcal{K}_{\gamma+k_i}^{((s-k_i-\frac{1}{2})/2;\, s-l-k_i-\frac{1}{2},\, l)}(\Gamma_i^{(0)})\Big)\,,$$

provided $\eta^{(0)}$, $\delta^{(0)}$ are small enough.

PROOF. – In the sequel, by $\varphi\mathcal{A}$ we mean the «product» of the C^∞-function φ by the operator $\mathcal{A}$ (see Section 3.1). We write the operator $\psi_1^{(0)}\mathcal{A}\varphi_1^{(0)}$ in the local coordinates defined by Λ^{-1}; let us now call it $\psi_1^{(0)'}\mathcal{A}_1'\varphi_1^{(0)'}$, where $\psi_1^{(0)'} = \psi_1^{(0)}\circ\Lambda^{-1}$, $\varphi_1^{(0)'} = \varphi_1^{(0)}\circ\Lambda^{-1}$ and $\mathcal{A}_1'$ is a parabolic operator with coefficients suitably continued by constants outside $\overline{K_{0,\delta^{(0)}}}$, where K is a compact subset of $\overline{\boldsymbol{R}^n_+}$.

Let $\{\varrho_j';\ j = 0, \dots, N\}$ be a C^∞ partition of unity in $\overline{\boldsymbol{R}^n_+}$, such that $\varrho_j' \in C_0^\infty(\overline{\boldsymbol{R}^n_+})$, $j = 1, \dots, N$. Let $\{\sigma_j';\ j = 0, \dots, N\}$, $\{\chi_j';\ j = 0, \dots, N\}$ be two sets of C^∞-functions such that $\chi_j', \sigma_j' \in C_0^\infty(\overline{\boldsymbol{R}_+^n})$, $j = 1, \dots, N$, and $0 \leqslant \sigma_j' \leqslant 1$, $0 \leqslant \chi_j' \leqslant 1$, $\sigma_j' \varphi_j' = \varphi_j'$, $\chi_j' \sigma_j' = \sigma_j'$, $j = 0, \dots, N$. Let $z_j \in \operatorname{supp} \varrho_j'$, $j = 0, \dots, N$.

Denote by $\mathcal{A}_{1(0)}'$ the parabolic principal part of $\mathcal{A}_1'$. We have

$$\sigma_j' \mathcal{A}_1' = \sigma_j' (\mathcal{A}_{1(0)}'(0, z_j) + K_{1j}' + T_{1j}') \,, \qquad j = 0, \dots, N \,, \tag{3.3.1.b}$$

where $\mathcal{A}_{1(0)}'(0, z_j)$ is the operator $\mathcal{A}_{1(0)}'$ with coefficients frozen at the point $(0, z_j)$,

$$\begin{aligned} \sigma_j' K_{1j}' &= \sigma_j' (\mathcal{A}_{1(0)}' - \mathcal{A}_{1(0)}'(0, z_j)) \,, \\ \sigma_j' T_{1j}' &= \sigma_j' (\mathcal{A}_1' - \mathcal{A}_{1(0)}') \,, \qquad j = 0, \dots, N \,. \end{aligned}$$

We can think of K_{1j}' and T_{1j}' as defined everywhere.

Since $\mathcal{A}_{1(0)}'(0, z_j)$ has constant coefficients, then, if γ is suitable, by Theorem 2.4.1 it has an inverse operator $R_{1(0)j}'$ which is a bounded operator between the spaces where it naturally operates. For instance, if, for a given j, $((\operatorname{supp} \varrho_j')_{0, \delta^{(0)}}) \cap$ $\cap \Lambda(\gamma) \neq \emptyset$, we have:

$$\begin{aligned} R_{1(0)j}' \colon \mathcal{M}_{((s-2)/2; s-2, -l, l); \gamma+2}((\boldsymbol{R}_+^n)_{0, \delta^{(0)}}) \times \\ \times \Big(\mathop{\times}_{i=1}^{2} \mathcal{M}_{((s-k_i-\frac{1}{2})/2; s-k_i-l-\frac{1}{2}, l); \gamma+k_i}((\boldsymbol{R}^{n-1})_{0, \delta^{(0)}}) \Big) \to \mathcal{M}_{(s/2; s, -l, l); \gamma}((\boldsymbol{R}_+^n)_{0, \delta^{(0)}}) \,. \end{aligned}$$

Let us now consider the « differential equation component » P_{1j}' of K_{1j}'; an analogous argument holds, if necessary, for the other components. We get (the norms are computed in $(\boldsymbol{R}_+^n)_{0, \delta^{(0)}}$):

$$(\chi_j' P_{1j}'(t, z, \partial_z) v)^{*2}_{((s-2)/2; s-2, -l, l); \gamma+2} \leqslant C_1 \sum_{|\alpha|=2} (\chi_j' a_{\alpha j}'(t, z) \, \partial_z^\alpha v)^{*2}_{((s-2)/2; s-2, -l, l); \gamma+2} \,,$$

where $a_{\alpha j}'$ are the coefficients of P_{1j}'.

So, by Proposition 2.1.29 and 2.1.28, if we shrink $\operatorname{supp} \chi_j'$ (and hence $\operatorname{supp} \sigma_j'$ and $\operatorname{supp} \varrho_j'$), we have, provided $\delta^{(0)}$ is small enough,

$$(\chi_j' P_{1j}'(t, z, \partial_z) v)^{*2}_{((s-2)/2; s-2, -l, l); \gamma+2} \leqslant \varepsilon (v)^{*2}_{(s/2; s, -l, l); \gamma} \,.$$

By Propositions 2.1.28, 2.1.29 and 2.1.30, an analogous estimate holds for T_{1j}', provided $\delta^{(0)}$ is small enough.

Thus, $\mathcal{A}_{1(0)}'(0, z_j) + \chi_j' K_{1j}' + \chi_j' T_{1j}'$ has a continuous inverse R_{1j}', which operates between the same spaces as $R_{1(0)j}'$. Let $F = (f, g_1, g_2)$ belong to the « space of data », and put

$$\overline{R_1'} F = \sum_{j=1}^{N} \sigma_j' R_{1j}' \varrho_j' F \,;$$

we have

$$\mathcal{A}_1'\bar{R}_1'F = \sum_{j=1}^{N} \sigma_j'\mathcal{A}_1'R_{1j}'\varrho_j'F + \sum_{j=1}^{N} [\sigma_j', \mathcal{A}_1']R_{1j}'\varrho_j'F = F + \mathcal{B}F\,.$$

Let us now estimate the «differential equation component» BF of $\mathcal{B}F$; the other components can be estimated analogously. We get, by Proposition 2.1.30,

$$\begin{aligned}(BF)^*_{((s-2)/2;\, s-2,\, -l, l);\, \gamma+2} &\leqslant C_2 \sum_{|\alpha|\leqslant 1} (\partial_z^\alpha R_{1j}'\varrho_j'F)^*_{((s-2)/2;\, s-2,\, -l, l);\, \gamma+2} \leqslant \\ &\leqslant C_3 (R_{1j}'\varrho_j'F)^*_{((s-1)/2;\, s-1,\, -l, l);\, \gamma+1} \leqslant C_4\,\varepsilon (R_{1j}'\varrho_j'F)^*_{(s/2;\, s,\, -l, l);\, \gamma} \leqslant \\ &\leqslant C_5\,\varepsilon\Big[(f)^*_{((s-2)/2;\, s-2,\, -l, l);\, \gamma+2} + \sum_{i=1}^{2} (g_i)^*_{((s-k_i-\frac12)/2;\, s-k_i-l-\frac12,\, l);\, \gamma+k_i}\Big]\,,\end{aligned}$$

provided $\delta^{(0)}$ is small enough.

So, $\mathcal{A}_1'\bar{R}_1'$ has an inverse; analogously, $\bar{R}_1'\mathcal{A}_1'$ has an inverse, too. Furthermore,

$$\bar{R}_1'(\mathcal{A}_1'\bar{R}_1')^{-1} \quad \text{and} \quad (\bar{R}_1'\mathcal{A}_1')^{-1}\bar{R}_1'$$

are a right and a left inverse of $\mathcal{A}_1'$, respectively. Denote by R_1' the inverse of $\mathcal{A}_1'$ and by R_1 the operator in the (t, x)-variables obtained by R_1' changing back coordinates via the map Λ^{-1}; in other words, R_1 is such that $\psi_1^{(0)}R_1\varphi_1^{(0)}$ is the operator $\psi_1^{(0)'}R_1'\varphi_1^{(0)'}$ written in the original variables.

Let $\mathcal{A}_2 = (\partial_t - P(t, x, \partial_x)) \oplus E_{k_2}(t, x, \partial_x)$; by α_3), we have $\psi_2^{(0)}\mathcal{A}\varphi_2^{(0)} = \psi_2^{(0)}\mathcal{A}_2\varphi_2^{(0)}$. If $\gamma \leqslant \gamma_1$, γ_1 suitable, by Theorem 3.1 in [9], $\mathcal{A}_2$ is an isomorphism from $K_\gamma^{s_1/2}(Q^{(0)})$ onto $K_{\gamma+2}^{(s_1-2)/2}(Q^{(0)}) \times K_{\gamma+i+\frac12}^{(s_1-i-\frac12)/2}(\Gamma^{(0)})$, where i is either 0 or 1, depending on the order of the boundary operator E_{k_2} in problem $(3.3.1.a)_{(k_1, k_2)}$. If $R_2 = \mathcal{A}_2^{-1}$, set

$$R = \psi_1^{(0)}R_1\varphi_1^{(0)} + \psi_2^{(0)}R_2\varphi_2^{(0)}\,.$$

We shall prove that $\mathcal{A}R$ and $R\mathcal{A}$ have an inverse, provided $\delta^{(0)}$ is small enough.

Let $\chi_j^{(0)}$ $(j = 1, 2)$ a function enjoying the same properties as $\psi_j^{(0)}$ and such that $\chi_j^{(0)}\psi_j^{(0)} = \psi_j^{(0)}$, $j = 1, 2$. We have

$$\mathcal{A}R = \sum_{j=1}^{2} \chi_j^{(0)}\mathcal{A}\psi_j^{(0)}R_j\varphi_j^{(0)}\,.$$

Let us write the first term in the local coordinate system; we get, with an obvious meaning of the notations,

$$\chi_1^{(0)'}\mathcal{A}_1'\psi_1^{(0)'}R_1'\varphi_1^{(0)'} = \psi_1^{(0)'}\mathcal{A}_1'R_1'\varphi_1^{(0)'} + \chi_1^{(0)'}[\mathcal{A}_1', \psi_1^{(0)'}]R_1'\varphi_1^{(0)'}\,.$$

Let $\chi_1^{(0)}N_1\varphi_1^{(0)}$ be the operator $\chi_1^{(0)'}[\mathcal{A}_1, \psi_1^{(0)'}]R_1'\varphi_1^{(0)'}$ written in the original coordinates; arguing as above, the norm of $\chi_1^{(0)}N_1\varphi_1^{(0)}$ in the space of data turns out to be small, if $\delta^{(0)}$ is small enough.

Furthermore,

$$\chi_2^{(0)} \mathcal{A} \psi_2^{(0)} R_2 \varphi_2^{(0)} = \varphi_2^{(0)} I + \chi_2^{(0)} [\mathcal{A}, \psi_2^{(0)}] R_2 \varphi_2^{(0)} ,$$

where the norm in the space of data of the second term in the right hand side is small if $\delta^{(0)}$ is small enough. Thus, by a suitable choice of $\delta^{(0)}$, $\mathcal{A}R$ has an inverse. Analogously, $R\mathcal{A}$ has an inverse, too. Furthermore, $R(\mathcal{A}R)^{-1}$ and $(R\mathcal{A})^{-1}R$ are a right and a left inverse of A, respectively.

This completes the proof of the assertion.

4. – The parabolic problem in the whole cylinder.

4.1. *Geometric preliminaries.*

Aim of this section is to construct a diffeomorphism which «cylindrizes» the separating surface γ far away from $t = 0$. Actually, the exhibited diffeomorphism has further good properties; indeed, it maps the upper part of the cylinder (i.e. the cylinder we obtain by cutting off a layer near $t = 0$) onto itself, it maps the boundary onto itself and it leaves time unchanged. Furthermore, it allows to define function spaces in the upper part of the cylinder which fairly fit those we have constructed in the lower part.

Let δ_1 be a real positive number verifying Hypotheses α_0)-α_3) in Section 3.1. If $t_0 \in]0, \delta_1[$, denote by L_{t_0} the map

$$(\tau, \omega) \mapsto \left(\exp(-2\tau), \sqrt{t_0/2}\,\omega\right) .$$

THEOREM 4.1.1. – There exists a C^∞ diffeomorphism r_{t_0} from $\overline{\Omega_{t_0/2,T}}$ onto itself, such that

i) $r_{t_0}\left(\overline{(\partial\Omega)_{t_0/2,T}}\right) = \overline{(\partial\Omega)_{t_0/2,T}}$;

ii) $r_{t_0}\left(\gamma \cap \overline{\Omega_{t_0/2,T}}\right) = [t_0/2, T] \times \{x \in \boldsymbol{R}^n;\, (t_0, x) \in \gamma_{t_0}\}$;

iii) $r_{t_0}(\bar{S}_t) = \bar{S}_t\,, \quad \forall t \in [t_0/2, T]$;

iv) $r_{t_0}|_{(\operatorname{supp} \varphi_1^{(0)})_{t_0/2,\delta_1}} = (\Lambda^{-1} \circ L_{t_0} \circ T \circ \Lambda)|_{(\operatorname{supp} \varphi_1^{(0)})_{t_0/2,\delta_1}}$.

PROOF. – Let $\varepsilon > 0$ sufficiently small. In $\left(\bar{\Omega} \cap S(x_0, 2\eta^{(0)})\right)_{0,\delta_1+\varepsilon}$ we define a C^∞-vector field as follows: put

$$\Phi = L_{t_0} \circ T \circ \Lambda .$$

By definition, $\Lambda(t, x) = (t, \psi(x))$, where ψ is a C^∞ diffeomorphism from $\bar{\Omega} \cap \cap S(x_0, 2\eta^{(0)})$ into $\overline{\boldsymbol{R}^n_+}$. So, we have

$$\Phi(t, x) = \left(t, \sqrt{t_0/2}\, t^{-\frac{1}{2}} \psi(x)\right) .$$

Then, we define the vector field X_0 in the following way:

$$X_0(t, x) = d\Phi^{-1}(\Phi(t, x))(1, 0) = \big(1, d\psi^{-1}(\psi(x))(\psi(x)/2t)\big) .$$

Obviously, the restriction of X_0 to the lateral boundary of the cylinder is a tangent vector field and the restriction of X_0 to γ is tangent to γ itself.

Let now $(\bar{t}, \bar{x}) \in \gamma$, $\delta_1 + \varepsilon \leqslant \bar{t} < T$; by 2. iii), there exists a neighbourhood $U_{(\bar{t},\bar{x})}$ of $(\bar{t}, \bar{x})$ in $\overline{\Omega_{\delta_1+\varepsilon/2, T}}$ and a C^∞ diffeomorphism $\varrho_{(\bar{t},\bar{x})}\colon U_{(\bar{t},\bar{x})} \to \boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$, such that

$$\varrho_{(\bar{t},\bar{x})}\big((\partial\Omega)_{\delta_1+\varepsilon/2, T} \cap U_{(\bar{t},\bar{x})}\big) \subseteq \boldsymbol{R}\times\boldsymbol{R}^{n-1}\times\{0\}; \tag{4.1.1.a}$$

$$\varrho_{(\bar{t},\bar{x})}(\gamma \cap U_{(\bar{t},\bar{x})}) \subseteq \boldsymbol{R}\times\boldsymbol{R}^{n-2}\times\{(0, 0)\} . \tag{4.1.1.b}$$

By a slight modification of the above formulas, we can recover the case $\bar{t} = T$. Without loss of generality, we may suppose that

$$\varrho_{(\bar{t},\bar{x})}(t, x) = \big(t, \psi_{(\bar{t},\bar{x})}(t, x)\big) , \tag{4.1.1.c}$$

$$(\varrho_{(\bar{t},\bar{x})})^{-1}(\tau, y) = \Big(\tau, \chi(\tau, y') + y_n \nu\big(\chi(\tau, y')\big)\Big) , \tag{4.1.1.d}$$

where χ is a coordinate system of Γ at $(\bar{t}, \bar{x})$ and ν denotes the unit inward normal field to Γ. We also suppose that:

(4.1.1.e) the $(n-1)$-th component of $\varrho_{(\bar{t},\bar{x})}(t, x)$ is equal to the geodetic distance in Γ between the orthogonal projection of (t, x) and $\gamma \cap \bar{S}_t$.

Now, $\{U_{(\bar{t},\bar{x})}; (\bar{t}, \bar{x}) \in \gamma \cap \overline{\Omega_{\delta_1+\varepsilon/2, T}}\}$ is an open covering of $\gamma \cap \overline{\Omega_{\delta_1+\varepsilon/2, T}}$; call $\{U_1, \ldots, U_p\}$ a finite subcovering and $\varrho_1, \ldots, \varrho_p$ the corresponding diffeomorphisms.

Let us now define the vector fields

$$X_j(t, x) = d\varrho_j^{-1}\big(\varrho_j(t, x)\big)(1, 0) , \qquad (t, x) \in U_j,\ j = 1, \ldots, p .$$

It turns out that X_j is a C^∞-vector field tangent to $\bar{\Gamma}$ on $U_j \cap \bar{\Gamma}$ and to γ on $U_j \cap \gamma$; furthermore, by (4.1.1.c), the t-component of X_j is always equal to 1 and, by (4.1.1.d), X_j is parallel to Γ. Let $\{\omega_j; j = 1, \ldots, p\}$ be a C^∞-partition of unity subordinated to the covering $\{U_j; j = 1, \ldots, p\}$; we agree to think of the ω_j's as continued in a C^∞ way to all of $\overline{\Omega_{0,T}}$, so that $0 \leqslant \omega_j \leqslant 1$, $\omega_j \equiv 0$ in $\overline{\Omega_{0,\delta_1}}$, $j = 1, \ldots, p$. If we put $\omega_0 = 1 - \sum_{j=1}^{p} \omega_j$ in $(\bar{\Omega} \cap S(x_0, 2\eta^{(0)}))_{0,\delta_1+\varepsilon}$, then it turns out that $\{\omega_0, \ldots, \omega_p\}$ is a partition of unity in

$$B = (\bar{\Omega} \cap S(x_0, 2\eta^{(0)}))_{\delta_1+\varepsilon} \cup \bigcup_{j=1}^{p} U_j .$$

Set

$$\tilde{X}(t, x) = \sum_{j=0}^{p} \omega_j(t, x) X_j(t, x) , \qquad (t, x) \in B .$$

Obviously, the vector field $\tilde{X}$ is tangent to Γ (or to γ) at every point of Γ (or of γ) and its t-component is always positive. Let now V be a neighbourhood of $\gamma \cap \bar{\Omega}_{0,T}$ in $\bar{\Omega}_{0,T}$, such that

$$(\operatorname{supp} \varphi_1^{(0)})_{0,\delta_1+\varepsilon} \subset \bar{V} \cap \bar{\Omega}_{0,T} \subset B ,$$

and let $w \in C^\infty(\overline{\Omega_{0,T}})$, $0 \leqslant w \leqslant 1$, $w \equiv 1$ in V, $\operatorname{supp} w \subset B$; then define, in $\overline{\Omega_{0,T}}$,

$$X(t, x) = w(t, x)\tilde{X}(t, x) + \big(1 - w(t, x)\big)(1, 0) .$$

The vector field X has the same properties as $\tilde{X}$.

If X_t is the t-component of X, put $Y = X/X_t = (1, \tilde{Y})$. Let $x \in \bar{\Omega}$ and consider the Cauchy problem

$$\text{(4.1.1.}f\text{)} \qquad \begin{cases} \dot{\eta} = Y(\eta) \\ \eta(t_0/2) = (t_0/2, x) . \end{cases}$$

The solution of (4.1.1.f) has the form $t \mapsto (t, \tilde{\eta}(t; x))$, where $\tilde{\eta}(\cdot\,; x)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{\tilde{\eta}}(t) = \tilde{Y}(t, \tilde{\eta}(t)) , \\ \tilde{\eta}(t_0/2) = x . \end{cases}$$

Since $\bar{\Omega}$ is a compact manifold and $\tilde{Y}$ is tangent to $\partial\Omega$ in $\partial\Omega$, $\tilde{\eta}(\cdot\,; x)$ can be continued to all of $[t_0/2, T]$.

Denote now by r_{t_0} the map $(t, x) \mapsto (t, y)$, where y is such that $\tilde{\eta}(t; y) = x$. Obviously, r_{t_0} is a C^∞ diffeomorphism from $\overline{\Omega_{t_0/2,T}}$ onto itself and verifies conditions i), ii), iii) of the assertion.

To prove that condition iv) holds, it is enough to show that

$$r_{t_0}(t, x) = \Big(t, \psi^{-1}\big(\sqrt{t_0/2}\, t^{-\frac{1}{2}} \psi(x)\big)\Big) ,$$

$\forall (t, x) \in E = (\operatorname{supp} \varphi_1^{(0)})_{t_0/2,\delta_1}$. Since $Y = X_0$ in E, this is easily accomplished by remarking that, in E,

$$r_{t_0}(t, x) = (\mathcal{R}_t \circ \Phi^{-1} \circ \pi_{t_0/2} \circ \Phi)(t, x) ,$$

where

$$\pi_{t_0/2}(s, y) = (t_0/2, y) ,$$

$$\mathcal{R}_t(t_0/2, z) = (t, z) .$$

So, the assertion is completely proved.

REMARK 4.1.2. – Since the vector field Y is independent of the choice of t_0, it easily follows by assertion iv) in the preceding theorem that, if $\delta' > \delta > 0$, $\forall (t, x) \in \in (L_{\delta'}^{-1} \circ \Lambda)((\operatorname{supp} \varphi_1^{(0)})_{\delta'/2, T})$,

$$(L_{\delta}^{-1} \circ \Lambda \circ r_{\delta} \circ r_{\delta'}^{-1} \circ \Lambda^{-1} \circ L_{\delta'})(t, x) = (t, x) .$$

4.2. *Function spaces.*

DEFINITION 4.2.1 (see [8], § 2.1). We use the notations of Section 2.1. Set

$$\zeta_j(t, x) = \varphi_j(\sqrt{t_0/2}\,\psi(t, x)) , \quad j = 2, \ldots, m ,$$

and continue them by zero to all of $\bar{\Omega}$,

$$\zeta_1 = 1 - \sum_{j=2}^{m} \zeta_j .$$

Furthermore, put

$$k_j(t, x) = \big(t, h_j(\sqrt{t_0/2}\,\psi(t, x))\big) , \quad j = 2, \ldots, m .$$

If $t_0/2 \leqslant a < b \leqslant T$, we shall denote by $H_{(s/2;\, s,\, -l, l)}(\Omega_{a,b})$ the space of functions u such that

$$\|u;\, a, b\|^2_{(s/2;\, s,\, -l, l)} = \|\zeta_1 u;\, a, b\|^2_{s_1/2;\, s_1} + \sum_{j=2}^{m} \|(\zeta_j u) \circ k_j^{-1};\, a, b\|^2_{s_j/2;\, s_j,\, -l_j, l_j} < +\infty ,$$

where the norms in the right hand side for $j > 1$ are computed either in $(\boldsymbol{R}^n_+)_{a,b}$ or in $(\boldsymbol{R}^n)_{a,b}$, according to supp ζ_j meets or not Γ. We note explicitly that s_j, l_j are the same as in Definition 2.1.5.

Whenever there is no way to misunderstanding, we shall omit the end-points of the time interval in the notation of the norm.

In an analogous way, spaces on the lateral boundary of the cylinder are defined.

Properties of these spaces (in particular trace theorems and continuity of the derivatives) are established in [8], § 2.1.

DEFINITION 4.2.2. – Let $t_0/2 \leqslant \alpha < \beta \leqslant T$; denote by $\mathcal{R}_{(s/2;s,-l,l)}(\Omega_{\alpha,\beta})$ the space of functions u such that

$$\{u;\alpha,\beta\}_{(s/2;s,-l,l)} = \|u \circ r_{t_0}^{-1};\alpha,\beta\|_{(s/2;s,-l,l)} < +\infty .$$

Spaces on the lateral boundary of the cylinder are defined analogously.

Whenever there is no way to misunderstanding, we shall omit the end-points of the time interval in the notation of the norm.

PROPOSITION 4.2.3 ([8], Proposition 3.2.2). – i) Let $s \geqslant 1$; then the operator ∂_i is bounded from

$$\mathcal{R}_{(s/2;s,-l,l)}(\Omega_{\alpha,\beta}) \quad \text{into} \quad \mathcal{R}_{((s-1)/2;s-1,-l,l)}(\Omega_{\alpha,\beta}), \qquad i = 1, \dots, n;$$

ii) Let $s \geqslant 2$; then the operator ∂_t is bounded from

$$\mathcal{R}_{(s/2;s,-l,l)}(\Omega_{\alpha,\beta}) \quad \text{into} \quad \mathcal{R}_{((s-2)/2;s-2,-l,l)}(\Omega_{\alpha,\beta}) .$$

LEMMA 4.2.4. – Let $t_0/2 \leqslant \alpha < \beta \leqslant t_0$; then

$$\mathcal{R}_{(s/2;s,-l,l)}(\Omega_{\alpha,\beta}) = \mathcal{K}^{(s/2;s,-l,l)}(\Omega_{\alpha,\beta}),$$

algebraically and topologically.

PROOF. – Let $u \in \mathcal{K}^{(s/2;s,-l,l)}(\Omega_{\alpha,\beta})$ and set $v = u \circ r_{t_0}^{-1}$. We have

$$\{u;\alpha,\beta\}^2_{(s/2;s,-l,l)} = \|v;\alpha,\beta\|^2_{(s/2;s,-l,l)} =$$
$$= \|\zeta_1 v;\alpha,\beta\|^2_{s_1/2;s_1} + \sum_{j=2}^{m} \|(\zeta_j v)\circ k_j^{-1};\alpha,\beta\|^2_{s_j/2;s_j,-l_j,l_j} = \sum_{j=1}^{m} I_j^2 .$$

Let us consider the case $j > 1$. We have

$$I_j \leqslant \sum_{k=1}^{2} \|(\zeta_j(\varphi_k^{(0)}u)\circ r_{t_0}^{-1})\circ k_j^{-1};\alpha,\beta\|_{s_j/2;s_j,-l_j,l_j} \leqslant$$
$$\leqslant C_1 \sum_{k=1}^{2} \left\|\left(\varphi_j\left((\varphi_k^{(0)}u)\circ r_{t_0}^{-1}\right)\circ \Lambda^{-1}\circ L_{t_0}\right)\circ h_j^{-1}; -\tfrac{1}{2}\log\beta, -\tfrac{1}{2}\log\alpha\right\|_{s_j/2;s_j,-l_j,l_j},$$

by Proposition 2.1.4 in [8]. As, by Hypothesis α_3) and condition iv) of Theorem 4.1.1,

$$[-\tfrac{1}{2}\log\beta, -\tfrac{1}{2}\log\alpha]\times U_j \subset L_{t_0}^{-1}\circ\Lambda\circ r_{t_0}([\alpha,\beta]\times S(x_0,\eta^{(0)})),$$

the term corresponding to $k=2$ vanishes. We have now only to deal with the term corresponding to $k=1$; so, by condition iv) of Theorem 4.1.1, we get

$$I_j \leqslant C_1 \| \big(\varphi_j((\varphi_1^{(0)}u)\circ\Lambda^{-1}\circ T^{-1})\big)\circ h_j^{-1};\, -\tfrac{1}{2}\log\beta, -\tfrac{1}{2}\log\alpha \|_{s_j/2;\, s_j,\, -l_j,\, l_j} \leqslant C_2[u;\, \alpha, \beta]_{(s,\, -l,\, l);\, \gamma}\,.$$

Let us now consider the case $j=1$; we have

$$I_1 \leqslant \sum_{k=1}^{2} \|\zeta_1(\varphi_k^{(0)}u)\circ r_{t_0}^{-1};\, \alpha, \beta\|_{s_1/2;\, s_1} = J_1 + J_2\,.$$

J_2 is easily estimated by $\|\varphi_2^{(0)}u;\, \alpha, \beta\|_{s_1/2;\, s_1} \leqslant C_3[u;\, \alpha, \beta]_{(s,\, -l,\, l);\, \gamma}$.

In order to estimate J_1, we note that $\zeta_1 = \varphi_1\circ L_{t_0}^{-1}\circ\Lambda$, on $r_{t_0}(\overline{(\operatorname{supp}\varphi_1^{(0)})_{\alpha,\beta}})$, so that, using condition iv) of Theorem 4.1.1, we get

$$J_1 \leqslant C_4 \|\varphi_1(\varphi_1^{(0)}u)\circ r_{t_0}^{-1}\circ\Lambda^{-1}\circ L_{t_0};\, -\tfrac{1}{2}\log\beta,\, -\tfrac{1}{2}\log\alpha\|_{s_1/2;\, s_1} =$$

$$= C_4 \|\varphi_1(\varphi_1^{(0)}u)\circ\Lambda^{-1}\circ T^{-1};\, -\tfrac{1}{2}\log\beta,\, -\tfrac{1}{2}\log\alpha\|_{s_1/2;\, s_1} \leqslant C_5[u;\, \alpha, \beta]_{(s,\, -l,\, l);\, \gamma}\,.$$

This proves the inclusion $\mathcal{K}^{(s/2;\, s,\, -l,\, l)}(\Omega_{\alpha,\beta}) \subset \mathfrak{K}_{(s/2;\, s,\, -l,\, l)}(\Omega_{\alpha,\beta})$; the converse inclusion is proved analogously.

DEFINITION 4.2.5. – We denote by $\mathcal{Q}_\gamma^{(s/2;\, s,\, -l,\, l)}(Q)$ the space of all functions u such that

$$|[u]|^2_{(s/2;\, s,\, -l,\, l);\, \gamma} = [u;\, 0, t_0]^2_{(s,\, -l,\, l);\, \gamma} + \{u;\, t_0/2, T\}^2_{(s/2;\, s,\, -l,\, l)} < +\infty\,.$$

Analogously we define the lateral boundary spaces $\mathcal{Q}_\gamma^{((\sigma+\varrho)/2;\, \sigma,\, \varrho)}(\Gamma)$.

We remark that, by Lemma 4.2.4, the squared and curl bracket norms give the same «regularity» on the layer $\Omega_{t_0/2,\, t_0}$. Furthermore, $\mathcal{Q}$-spaces are embedded in BAIOCCHI's [4] ones, provided $\mathcal{K}$-spaces are (see, e.g., Definition 3.2.1).

The following proposition asserts that $\mathcal{Q}$-spaces do not depend on the choice of t_0; in fact

PROPOSITION 4.2.6. – Let $t_1 \in\,]0, \delta_1[$; then the norms

$$([u;\, 0, t_1]^2_{(s,\, -l,\, l);\, \gamma} + \{u;\, t_1/2, T\}^2_{(s/2;\, s,\, -l,\, l)})^{\frac{1}{2}} \tag{4.2.6.a}$$

and

$$([u;\, 0, t_0]^2_{(s,\, -l,\, l);\, \gamma} + \{u;\, t_0/2, T\}^2_{(s/2;\, s,\, -l,\, l)})^{\frac{1}{2}} \tag{4.2.6.b}$$

are equivalent.

PROOF. – We suppose, for instance, that $t_0 < t_1$. Let u be such that (4.2.6.*a*) is finite. Then, trivially,

$$[u;\, 0,\, t_0]_{(s,-l,l);\,\gamma} \leqslant [u;\, 0,\, t_1]_{(s,-l,l);\,\gamma}\,.$$

Now,

$$\{u;\, t_0/2,\, T\}_{(s/2;\,s,\,-l,l)} = \|u\circ r_{t_0}^{-1};\, t_0/2,\, T\|_{(s/2;\,s,\,-l,l)} \leqslant$$
$$\leqslant C_1\big(\|u\circ r_{t_0}^{-1};\, t_0/2,\, t_1\|_{(s/2;\,s,\,-l,l)} + \|u\circ r_{t_0}^{-1};\, t_1/2,\, T\|_{(s/2;\,s,\,-l,l)}\big)\,.$$

The first term in the r.h.s. of the above inequality is easily estimated, by Lemma 4.2.4, with the first term in (4.2.6.*a*). Let us estimate the second one; we have:

$$\|u\circ r_{t_0}^{-1};\, t_1/2,\, T\|_{(s/2;\,s,\,-l,l)} \leqslant \|\zeta_1(u\circ r_{t_0}^{-1});\, t_1/2,\, T\|_{s_1/2;\,s_1} +$$
$$+ \sum_{j=2}^{m} \|(\zeta_j(u\circ r_{t_0}^{-1}))\circ k_j^{-1};\, t_1/2,\, T\|_{s_j/2;\,s_j,\,-l_j,\,l_j} = \sum_{j=1}^{m} I_j\,.$$

Now,

$$I_1 \leqslant C_2\|(\zeta_1\circ r_{t_0}\circ r_{t_1}^{-1})(u\circ r_{t_1}^{-1});\, t_1/2,\, T\|_{s_1/2;\,s_1} \leqslant$$
$$\leqslant C_2 \sum_{k=1}^{m} \|\bar\zeta_k(\zeta_1\circ r_{t_0}\circ r_{t_1}^{-1})(u\circ r_{t_1}^{-1});\, t_1/2,\, T\|_{s_1/2;\,s_1} = C_2 \sum_{k=1}^{m} J_k\,,$$

where $\{\bar\zeta_k;\ k = 1, \ldots, m\}$ is a partition of unity in $\bar\Omega$, defined quite analogously to $\{\zeta_k;\ k = 1, \ldots, m\}$ in Definition 4.2.1, replacing t_0 by t_1. Now,

$$J_1 \leqslant C_3\|\bar\zeta_1(u\circ r_{t_1}^{-1});\, t_1/2,\, T\|_{s_1/2;\,s_1} \leqslant C_3\|u\circ r_{t_1}^{-1};\, t_1/2,\, T\|_{(s/2;\,s,\,-l,l)}\,.$$

On the other hand, since $\operatorname{supp}\bar\zeta_k \subset [t_1/2, T]\times S(x_0, \eta^{(0)})$, if $k > 1$,

$$J_k = \|\bar\zeta_k(\varphi_1\circ L_{t_0}^{-1}\circ\Lambda\circ r_{t_0}\circ r_{t_1}^{-1})(u\circ r_{t_1}^{-1});\, t_1/2,\, T\|_{s_1/2;\,s_1} =$$
$$= \|\big(((\varphi_1\,\varphi_k)\circ L_{t_1}^{-1}\circ\Lambda)(u\circ r_{t_1}^{-1})\big)\circ h_k^{-1}\circ h_k;\, t_1/2,\, T\|_{s_1/2;\,s_1}\,,$$

by Remark 4.1.2. So, by (2.1.4.*c*) and (2.1.5.*i*),

$$J_k \leqslant C_4\|(\bar\zeta_k(u\circ r_{t_1}^{-1}))\circ\bar k_j;\, t_1/2,\, T\|_{s_k/2;\,s_k,\,-l_k,\,l_k} \leqslant C_4\|u\circ r_{t_1}^{-1};\, t_1/2,\, T\|_{(s/2;\,s,\,-l,l)}\,,$$

where $\bar k_j$, $j = 2, \ldots, m$, is defined quite analogously to k_j in Definition 4.2.1, replacing t_0 by t_1. Let us now estimate the term I_j, $j > 1$. We get, by Remark 4.1.2,

$$I_j = \|((\varphi_j\circ L_{t_0}^{-1}\circ\Lambda)(u\circ r_{t_1}^{-1}\circ r_{t_1}\circ r_{t_0}^{-1}))\circ\Lambda^{-1}\circ L_{t_0}\circ h_j^{-1};\, t_1/2,\, T\|_{s_j/2;\,s_j,\,-l_j,\,l_j} =$$
$$= \|\big(\varphi_j((u\circ r_{t_1}^{-1})\circ\Lambda^{-1}\circ L_{t_1})\big)\circ h_j^{-1};\, t_1/2,\, T\|_{s_j/2;\,s_j,\,-l_j,\,l_j} =$$
$$= \|\big(\varphi_j((u\circ r_{t_1}^{-1})\circ\Lambda^{-1}\circ L_{t_1})\big)\circ L_{t_1}^{-1}\circ\Lambda\circ\bar k_j^{-1};\, t_1/2,\, T\|_{s_j/2;\,s_j,\,-l_j,\,l_j} =$$
$$= \|(\bar\zeta_j(u\circ r_{t_1}^{-1}))\circ\bar k_j^{-1};\, t_1/2,\, T\|_{s_j/2;\,s_j,\,-l_j,\,l_j} \leqslant \|u\circ r_{t_1}^{-1};\, t_1/2,\, T\|_{(s/2;\,s,\,-l,l)}\,.$$

Thus, (4.2.5.*b*) is finite. Analogously we prove the converse inequality.

Let us now recall some trace results about $\mathcal{K}$-spaces.

PROPOSITION 4.2.7 (See Proposition 3.2.5 in [8]). – Let $t_0/2 \leqslant \alpha < \beta \leqslant T$, $k \in N_0$, $s > k + \frac{1}{2}$; we define the k-th order trace on $(\partial\Omega)_{\alpha,\beta}$ as follows:

$$\gamma_k u = \big(\gamma_0((\partial_\nu^k u)\circ r_{t_0}^{-1})\big)\circ r_{t_0}|_{(\partial\Omega)_{\alpha,\beta}}, \tag{4.2.7.a}$$

where ∂_ν^k denotes the k-th order inward normal derivative (see Definition 3.2.3 in [8]). Then γ_k is a bounded operator from

$$\mathcal{K}_{(s/2;s,-l,l)}(\Omega_{\alpha,\beta}) \quad \text{into} \quad \mathcal{K}_{((s-k-\frac{1}{2})/2;s-k-l-\frac{1}{2},l)}((\partial\Omega)_{\alpha,\beta}).$$

LEMMA 4.2.8. – Let $k \in N_0$, $s > k + \frac{1}{2}$ and $u \in \mathcal{Q}_\gamma^{(s/2;s,-l,l)}(Q)$. For sake of simplicity and only in this lemma, we shall denote by $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ the i-th order trace operator in $\mathcal{H}$- and $\mathcal{K}$-spaces respectively. Then, we have

$$\gamma_i^{(1)}(u|_{[t_0/2,t_0]\times\Omega}) = \gamma_i^{(2)}(u|_{[t_0/2,t_0]\times\Omega}), \qquad i = 0, \dots, k. \tag{4.2.8.a}$$

PROOF. – We need only to prove the assertion in the case $i = 0$, as otherwise we can replace the function u by $\partial_\nu^i u$. Put

$$w = u|_{[t_0/2,t_0]\times\Omega};$$

as $w \in \mathcal{K}_{(s/2;s,-l,l)}(\Omega_{t_0/2,t_0})$, we have

$$\gamma_0^{(2)} w = \sum_{k=1}^{2}\big(\gamma_0((\varphi_k^{(0)}w)\circ r_{t_0}^{-1})\big)\circ r_{t_0}|_{(\partial\Omega)_{t_0/2,t_0}} = I_1 + I_2.$$

Now, $I_2 = \gamma_0(\varphi_2^{(0)}w)$, where γ_0 is the trace operator in the usual Sobolev spaces. On the other hand, by condition iv) in Theorem 4.1.1,

$$I_1 = \big(\gamma_0((\varphi_1^{(0)}w)\circ\Lambda^{-1})\big)\circ\Lambda|_{(\partial\Omega)_{t_0/2,t_0}},$$

since Λ, T, L_{t_0} commute with the ordinary trace operator.
Thus, the assertion is proved.

DEFINITION 4.2.9. – Let $k \in N_0$, $s > k + \frac{1}{2}$ and $u \in \mathcal{Q}_\gamma^{(s/2;s,-l,l)}(Q)$; then, by Propositions 3.2.8 and 4.2.7, both $u|_{\Omega_{0,t_0}}$ and $u|_{\Omega_{t_0/2,T}}$ have k-th order traces on the lateral boundary of the cylinder. Furthermore, by Lemma 4.2.8, the restrictions of these traces to the common layer $(\partial\Omega)_{t_0/2,t_0}$ coincide. This justifies the following definition of k-th order trace:

$$\gamma_k u = \begin{cases} \gamma_k(u|_{\Omega_{0,t_0}}), & \text{in } \Omega_{0,t_0}, \\ \gamma_k(u|_{\Omega_{t_0/2,T}}), & \text{in } \Omega_{t_0/2,T}; \end{cases}$$

where the first trace is understood to be a $\mathcal{K}$-space one, while the second trace an $\mathfrak{K}$-space one.

Moreover, the above mentioned Propositions guarantee that γ_k is bounded from $\mathfrak{Q}_{\gamma}^{(s/2;\,s,\,-l,l)}(Q)$ into $\mathfrak{Q}_{\gamma+k}^{((s-k-\frac{1}{2})/2;\,s-l-k-\frac{1}{2},l)}(\Gamma)$.

REMARK 4.2.10. – Let $s>1$, $\gamma-n/2-1+s_1<0$; then, by Definition 3.2.2, if $u\in\mathfrak{Q}_{\gamma}^{(s/2;\,s,\,-l,l)}(Q)$, it is quite natural the following position

$$\gamma_0^{(0)}u=0\,.$$

4.3. *Solution of the global problem.*

Let us consider the problem (P) described in Chapter 2. The results obtained in Section 3.3 and in [8], Theorem 3.3.1, allow to prove an existence, uniqueness and « regularity » theorem; namely, we have

THEOREM 4.3.1. – Let $s\geqslant 2$, $\gamma\in\boldsymbol{R}_-$; then the operator associated to problem (P) is an isomorphism from

$$\mathfrak{Q}_{\gamma}^{(s/2;\,s,\,-l,l)}(Q)\quad\text{onto}\quad \mathfrak{Q}_{\gamma+2}^{((s-2)/2;\,s-2,\,-l,l)}(Q)\times\Big(\mathop{\times}_{i=1}^{2}\mathfrak{Q}_{\gamma+k_i}^{((s-k_i-\frac{1}{2})/2;\,s-l-k_i-\frac{1}{2},l)}(\Gamma_i)\Big)\,,$$

provided $|\gamma|$ is large enough.

Furthermore, we have the a priori estimate

$$(4.3.1.a)\qquad |[u]|_{(s/2;\,s,\,-l,l);\,\gamma}\leqslant C\Big(|[(\partial_t-P(t,x,\partial_x))\,u]|_{((s-2)/2;\,s-2,\,-l,l);\,\gamma+2}+ \\ +\sum_{i=1}^{2}|[E_{k_i}(t,x,\partial_x)u]|_{((s-k_i-\frac{1}{2})/2;\,s-k_i-l-\frac{1}{2},l);\,\gamma+k_i}\Big)\,.$$

PROOF. – By Theorem 3.3.1, there exists a function $u^{(0)}\in\mathcal{K}_{\gamma}^{(s/2;\,s,\,-l,l)}(Q^{(0)})$, which is a solution of problem (P) in $Q^{(0)}$, the lower part of the cylinder, such that

$$(4.3.1.b)\qquad [u^{(0)}]_{(s,\,-l,l);\,\gamma}\leqslant C_1\Big([(\partial_t-P(t,x,\partial_x))\,u^{(0)}]_{(s-2,\,-l,l);\,\gamma+2}+ \\ +\sum_{i=1}^{2}[E_{k_i}(t,x,\partial_x)u^{(0)}]_{(s-l-k_i-\frac{1}{2},l);\,\gamma+k_i}\Big)\,,$$

provided $|\gamma|$ is large.

Let l be a continuation operator to all of Q, which is continuous from $\mathcal{K}_{\gamma}^{(s/2;\,s,\,-l,l)}(Q^{(0)})$ into $\mathfrak{Q}_{\gamma}^{(s/2;\,s,\,-l,l)}(Q)$. Now, put

$$f^*=f-(\partial_t-P(t,x,\partial_x))lu^{(0)}\,,$$
$$g_i^*=g_i-E_{k_i}(t,x,\partial_x)lu^{(0)}\,,\qquad i=1,2\,;$$

then,

$$f^*|_{\Omega_{\delta^{(0)}/2,T}} \in \mathcal{R}_{((s-2)/2;s-2,-l,l)}(\Omega_{\delta^{(0)}/2,T}) ,$$
$$g_i^*|_{\Gamma_{\delta^{(0)}/2,T}} \in \mathcal{R}_{((s-k_i-\frac{1}{2})/2;s-k_i-l-\frac{1}{2},l)}(\Gamma_{\delta^{(0)}/2,T}) , \qquad i = 1, 2 ,$$

and f^* and g_i^* are zero when $t \in]\delta^{(0)}/2, \delta^{(0)}[$; hence, in particular, they are consistent with zero at $t = \delta^{(0)}/2$ (see Definition 3.2.6 in [8]). So, by Theorem 3.3.1 in [8], there exists a unique solution $u_1 \in \mathcal{R}_{(s/2;s,-l,l)}(\Omega_{\delta^{(0)}/2,T})$ of the problem

$$\begin{cases} (\partial_t - P(t, x, \partial_x))u_1 = f^* , & \text{in } \Omega_{\delta^{(0)}/2,T} , \\ E_{k_i}(t, x, \partial_x)u_1 = g_i^* , & \text{in } \Gamma_i \cap (\partial\Omega)_{\delta^{(0)}/2,T} , \quad i = 1, 2 , \end{cases}$$

which is consistent with zero at $t = \delta^{(0)}/2$. Furthermore, the following estimate holds:

$$\text{(4.3.1.}c) \qquad \{u_1; \delta^{(0)}/2, T\}_{(s/2;s,-l,l)} \leqslant C_2\Big(\{f^*; \delta^{(0)}/2, T\}_{((s-2)/2;s-2,-l,l)} + \\ + \sum_{i=1}^{2} \{g_i^*; \delta^{(0)}/2, T\}_{((s-k_i-\frac{1}{2})/2;s-k_i-l-\frac{1}{2},l)}\Big) .$$

By the corrisponding property of data, u_1 is zero in $\Omega_{\delta^{(0)}/2,\delta^{(0)}}$, and we shall think of it as continued by zero in $\Omega_{0,\delta^{(0)}/2}$. Set

$$u = lu^{(0)} + u_1 .$$

It is obvious, by definition, that u is a solution of the problem (P). Moreover, we have, by (4.3.1.b) and (4.3.1.c)

$$\begin{aligned} |[u]|_{(s/2;s,-l,l);\gamma} &\leqslant |[lu^{(0)}]|_{(s/2;s,-l,l);\gamma} + |[u_1]|_{(s/2;s,-l,l);\gamma} \leqslant \\ &\leqslant C_3([u^{(0)}]_{(s,-l,l);\gamma} + \{u_1; \delta^{(0)}/2, T\}_{(s/2;s,-l,l);\gamma}) \leqslant \\ &\leqslant C_4\Big([f; 0, \delta^{(0)}]_{(s-2,-l,l);\gamma+2} + \{f^*; \delta^{(0)}/2, T\}_{((s-2)/2;s-2,-l,l)} + \\ &+ \sum_{i=1}^{2} ([g_i; 0, \delta^{(0)}]_{(s-k_i-l-\frac{1}{2},l);\gamma+k_i} + \{g_i^*; \delta^{(0)}/2, T\}_{((s-k_i-\frac{1}{2})/2;s-k_i-l-\frac{1}{2},l)})\Big) . \end{aligned}$$

Now,

$$\begin{aligned} \{f^*; \delta^{(0)}/2, T\}_{((s-2)/2;s-2,-l,l)} &\leqslant \{f; \delta^{(0)}/2, T\}_{((s-2)/2;s-2,-l,l)} + C_5\{lu^{(0)}\}_{(s/2;s,-l,l)} \leqslant \\ &\leqslant \{f; \delta^{(0)}/2, T\}_{((s-2)/2;s-2,-l,l)} + C_6[u^{(0)}; 0, \delta^{(0)}]_{(s,-l,l);\gamma} \leqslant \\ &\leqslant \{f; \delta^{(0)}/2, T\}_{((s-2)/2;s-2,-l,l)} + C_7\Big([f; 0, \delta^{(0)}]_{(s-2,-l,l);\gamma+2} + \\ &+ \sum_{i=1}^{2} [g_i; 0, \delta^{(0)}]_{(s-k_i-l-\frac{1}{2},l);\gamma+k_i}\Big) \leqslant \\ &\leqslant C_8\Big(|[f]|_{((s-2)/2;s-2,-l,l);\gamma+2} + \sum_{i=1}^{2} |[g_i]|_{((s-k_i-\frac{1}{2})/2;s-k_i-l-\frac{1}{2},l);\gamma+k_i}\Big) . \end{aligned}$$

In an analogous way, $\{g_i^*; \delta^{(0)}/2, T\}_{((s-k_i-\frac{1}{2})/2;s-k_i-l-\frac{1}{2},l)}$, $i = 1, 2$, can be estimated.

So, (4.3.1.a) is proved.

The uniqueness of the solution is an immediate consequence of Theorem 3.3.1 and Theorem 3.3.1 in [8].

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