Sard and Bertini Type Theorems for Complex Spaces (*) (**).

Mirella Manaresi (Bologna)

Summary. – *We prove that if X is a normal [resp. reduced, maximal] complex space and $f: X \to \boldsymbol{C}$ is a holomorphic function, then $f^{-1}(c)$ is normal [resp. reduced, maximal] for all but countably many $c \in \boldsymbol{C}$. This Sard type theorem, together with a Bănică's result on the fibers of a flat map, allows us to prove Bertini type theorems for reduced and normal complex spaces.*

Introduction.

The classical theorem of Sard (see: [M]) says that if $f: X \to Y$ is a differentiable map between differentiable manifolds, then the image of the critical points has Lebesgue measure zero in Y. Moreover, it implies:

(S) if X and Y are regular complex spaces (i.e. complex manifolds) and f is holomorphic, then the image of the critical points is even analytically meagre;

(S') if X is a regular complex space and $f: X \to \boldsymbol{C}$ is a holomorphic function, then $f^{-1}(c)$ is regular for all but countably many $c \in \boldsymbol{C}$ (see (I.1)).

It is also easy to see that from (S') one can deduce a Bertini type theorem such as

(B) let L be a holomorphic line bundle on a regular complex space X and let V be a finite dimensional linear subspace of $\Gamma(X, L)$ which generates L. Then the subspace of zeros of a « general » section of V is regular (see (II.5) for the precise meaning of « general »).

The aim of this note is to prove (S) and (S') when « regular » is replaced by other properties such as « reduced », « normal », « maximal » and to see when (S') implies (B).

In section 0 we recall some preliminaries; in section I we first prove (S') and (S) for « reduced » and « normal » (see: (I.4) and (I.5)). In order to prove (S) we use the Bănică's result (see: (0.3)a) saying that for a flat map $f: X \to Y$ the set N of points $x \in X$ such that the fiber $f^{-1}(f(x))$ is not normal [resp.: not reduced] in x is an

(*) Entrata in Redazione il 13 marzo 1982.
(**) This research was done when the author was a member of the G.N.S.A.G.A. of the C.N.R.

analytic subset of X (but is is not clear a priori that $f(N)$ is analitycally meagre). Later on we prove (S') for « maximal » (see: (I.12)), by using a lemma (see: (I.10)), which says, roughly, that normalization commutes with taking fibers, at least outside a suitable meagre subset.

From (S') for « maximal », we can deduce (S) for such property only when Y is one-dimensional (see: (I.14)), because an analogue of Bănică's result for maximality is not yet available.

Statement (S') and the above mentioned theorem of Bănică allow us, in section II, to prove (B) for « regular », « normal », « reduced » (see: (II.5)). As a consequence we have that the above mentioned properties are preserved by general hyperplane sections for complex spaces embedded in $\boldsymbol{P}^n$ or $\boldsymbol{C}^n$. This is well known in the algebraic case, hence in $\boldsymbol{P}^n$ (see e.g. [Fl], [K], [Se]), but it seems to be new for normal or reduced (not necessarily compact) analytic spaces in $\boldsymbol{C}^n$.

A Bertini theorem for maximality is known only in the algebraic case (see: [CGM]), but it is clear from the above discussion, that it could be deduced from the results of section II if one had the analogue of Bănică's theorem.

Part of this research was done during a stay at the Mathematisches Institut, der Universität Münster (BRD). The author wishes to thank prof. O. FORSTER for his helpful advice.

Standing notations.

All complex spaces are supposed to have a countable base of open subsets.

Holomorphic map always means holomorphic map between complex spaces.

Whenever $f\colon X \to Y$ [resp.: $f\colon X \to \boldsymbol{C}$] is a holomorphic map [resp.: a holomorphic function on the complex space X] for every $y \in Y$ [resp.: $c \in \boldsymbol{C}$] we denote $X_y := f^{-1}(y)$ [resp.: $X_c := f^{-1}(c)$] with its natural complex structure.

0. – Preliminaries.

We recall some well known facts we will use in the following:

(0.1) DEFINITION. – A subset A of a complex space X is said to be *analytically meagre* (« *négligeable* » according to [Fr] (IV.11)) if $A \subset \bigcup_{n \in N} Y_n$ where each Y_n is a locally analytic subset of X of codimension $\geqslant 1$.

REMARK. – Clearly if $\dim X = 1$, then an analytically meagre subset of X is a countable set of points.

(0.2) If $f\colon X \to Y$ is a proper holomorphic map, then, by Remmert's mapping theorem (see [R] satz 23), the image of any analytic set is again analytic.

This is no longer true if the map is not proper. However we have the following:

LEMMA ([R] satz 20). – Let $f\colon X \to Y$ be a holomorphic map between complex spaces and let Z be an analytic subset of X. Then $f(Z)$ is a countable union of locally analytic subsets of Y.

We can also observe that if $\operatorname{Int} Z = \emptyset$, then $f(Z)$ is analytically meagre.

(0.3) Let $f\colon X \to Y$ be a holomorphic map and let us consider the following subsets of X:

$$\begin{aligned} S_f(X) &:= \{x \in X \mid X_{f(x)} \text{ is not a manifold in } x\}\\ N_f(X) &:= \{x \in X \mid X_{f(x)} \text{ is not normal in } x\}\\ R_f(X) &:= \{x \in X \mid X_{f(x)} \text{ is not reduced in } x\}\end{aligned}$$

and the following subsets of Y:

$$\begin{aligned} \tilde{S}_f(Y) &:= f(S_f(X)) = \{y \in Y \mid X_y \text{ is not a manifold}\}\\ \tilde{N}_f(Y) &:= f(N_f(X)) = \{y \in Y \mid X_y \text{ is normal}\}\\ \tilde{R}_f(Y) &:= f(R_f(X)) = \{y \in Y \mid X_y \text{ is not reduced}\}\,. \end{aligned}$$

It is well known that:

a) PROPOSITION (Bănică [*B*]). – If $f\colon X \to Y$ is flat, then the sets $S_f(X)$, $N_f(X)$, $R_f(X)$ are analytic subsets of X.

It is not clear a priori that $f(S_f(X))$, $f(N_f(X))$, $f(R_f(X))$ are analytically meagre. However, we have

b) COROLLARY (Bănică [*B*]). – If $f\colon X \to Y$ is flat and proper, then the sets $\tilde{S}_f(Y)$, $\tilde{N}\,(Y)$, $\tilde{R}_f(Y)$ are analytic subsets of Y.

(0.4) We recall that a reduced complex space X is said to be *maximal* (according to FISCHER [F], p. 111) or *weakly normal* (according to ANDREOTTI-NORGUET [AN]) if the sheaf $\hat{\mathcal{O}}_X$ of continuous weakly holomorphic functions on X is equal to the structural sheaf $\mathcal{O}_X$ of X. For a summary of the results about maximality we refer to [AN], [AAL], [F] ch. II; however we recall the following fact we shall use later on:

If $\tilde{X} \xrightarrow{\pi} X$ is the normalization of the reduced complex space X, then X is maximal if and only if the following sequence of complex spaces is exact:

$$(\mathrm{I}) \qquad (\tilde{X} \times_X \tilde{X})_{\mathrm{red}} \underset{g_2}{\overset{g_1}{\rightrightarrows}} \tilde{X} \xrightarrow[\pi]{} X$$

where g_1, g_2 are induced by the projections $p_1, p_2\colon \tilde{X} \times_X \tilde{X} \rightrightarrows \tilde{X}$ (see: [F], p. 123-124); or equivalently, if we put $\pi' := g_1 \circ \pi = g_2 \circ \pi\colon (\tilde{X} \times_X \tilde{X})_{\mathrm{red}} \to X$, if and only if the sequence of coherent analytic sheaves over X

$$\text{(II)} \qquad 0 \to \mathcal{O}_X \xrightarrow[\pi^*]{} \pi_* \mathcal{O}_{\tilde{X}} \xrightarrow[(\varrho_1-\varrho_2)^*]{} \pi'_* \mathcal{O}_{(\tilde{X}\times_X \tilde{X})_{\mathrm{red}}} \quad \text{is exact.}$$

I. – Sard type theorems.

In this section we prove Sard type theorems for the properties « normal », « reduced » and « maximal ».

(I.1) THEOREM (Sard). – *Let X be a complex manifold, $f\colon X \to \boldsymbol{C}$ an holomorphic function. Then there exists a countable subset $A \subset \boldsymbol{C}$ such that for each $c \in \boldsymbol{C} - A$ the fiber X_c is a manifold.*

PROOF. – Let S be the set of critical points of f. It is well known that S is analytic in X, moreover by the classical theorem of Sard $f(S)$ has Lebesgue measure zero in Y. By (0.2) the conclusion follows.

(I.2) COROLLARY. – *Let X be a complex space, $f\colon X \to \boldsymbol{C}$ a holomorphic function. Then there exists a countable subset $A \subset \boldsymbol{C}$ such that for each $c \in \boldsymbol{C} - A$* $\operatorname{Sing}(X_c) \subset X_c \cap \operatorname{Sing}(X)$.

(I.3) LEMMA. *Let X be a complex space, let Z be an analytic subset of X and let $f\colon X \to \boldsymbol{C}$ be a holomorphic function.*

Then there exists a countable subset $A \subset \boldsymbol{C}$ such that for each $x \in Z$ with $f(x) \in \boldsymbol{C} - A$ one has

$$\dim_x (X_{f(x)} \cap Z) < \dim_x Z\,.$$

PROOF. – Let $A := \{c \in \boldsymbol{C} \mid X_c$ contains an irreducible component of $Z\}$ and $f' := f|_Z\colon Z \to \boldsymbol{C}$. Clearly for each point $x \in Z$ such that $f(x) \in \boldsymbol{C} - A$ we have

$$\dim_x (X_{f(x)} \cap Z) = \dim_x Z_{f'(x)} = \dim_x Z - 1$$

and the conclusion follows.

(I.4) THEOREM. – *Let X be a normal [resp.: reduced] complex space, $f\colon X \to \boldsymbol{C}$ be a holomorphic function.*

Then there exists a countable subset $A \subset \boldsymbol{C}$ such that, for each $c \in \boldsymbol{C} - A$, X_c is normal [resp. reduced].

PROOF. – We prove the theorem for X normal.

Without restriction we may assume that f is not constant on any irreducible component of X. So, if for each $k \in \boldsymbol{N}$ we consider the analytic subsets of X (see [F] lemma p. 160, or [ST] (1.11))

$$S_k(\mathcal{O}_X) := \{P \in X | \operatorname{prof} \mathcal{O}_{X,P} \leqslant k\}$$
$$S_k(\mathcal{O}_X; f) := \{P \in X | \operatorname{prof} (\mathcal{O}_{X,P}/m_{\boldsymbol{C},f(P)}\mathcal{O}_{X,P}) \leqslant k\}$$

we have (see: [F] cor. p. 154)

$$S_{k+1}(\mathcal{O}_X) = S_k(\mathcal{O}_X; f) \qquad \forall k \in \boldsymbol{N}. \tag{1}$$

Moreover we can observe that

$$S_k(\mathcal{O}_X; f) = \bigcup_{c \in \boldsymbol{C}} S_k(\mathcal{O}_{X_c}) \qquad \forall k \in \boldsymbol{N}. \tag{2}$$

Since X is normal, we have (see: [F] (2.27)) that

$$\forall P \in X \text{ and } \forall k \geqslant 1 \qquad \dim_P (\operatorname{Sing}(X) \cap S_k(\mathcal{O}_X)) \leqslant k-2 \,;$$

so, by (1)

$$\dim_P (\operatorname{Sing}(X) \cap S_{k-1}(\mathcal{O}_X; f)) \leqslant k-2 \,.$$

By lemma (I.3) applied to the analytic subset $Z_{k-1} := \operatorname{Sing}(X) \cap S_{k-1}(\mathcal{O}_X, f)$ for each $k \geqslant 1$ there exists a countable subset $A_{k-1} \subset \boldsymbol{C}$ such that for each $P \in X$ with $f(P) \notin A_{k-1}$

$$\dim_P (X_{f(P)} \cap \operatorname{Sing}(X) \cap S_{k-1}(\mathcal{O}_X; f)) \leqslant k-3 \qquad \text{so, by (2),}$$
$$\dim_P (X_{f(P)} \cap \operatorname{Sing}(X) \cap S_{k-1}(\mathcal{O}_{X_{f(P)}})) \leqslant k-3 \,.$$

Let $\tilde{A} \subset \boldsymbol{C}$ be the countable subset (see (I.2)) such that for each $c \in \boldsymbol{C} - \tilde{A}$ $\operatorname{Sing}(X_c) \subset X_c \cap \operatorname{Sing}(X)$ and let $A := \left(\bigcup_{k \geqslant 1} A_{k-1}\right) \cup \tilde{A} \subset \boldsymbol{C}$.

Then for each $k \geqslant 1$ and for each $P \in X$ such that $f(P) \in \boldsymbol{C} - A$ we have

$$\dim_P (\operatorname{Sing}(X_{f(P)}) \cap S_{k-1}(\mathcal{O}_{X_{f(P)}})) \leqslant k-3 \qquad \text{so, by [F]}$$

(2.27) the conclusion follows.

For X reduced the proof is analogous using the fact that a complex space X is reduced in a point x if and only if

$$\dim_x (\operatorname{Sing}(X) \cap S_k(\mathcal{O}_X)) \leqslant k-1 \qquad \text{for } k \geqslant 0.$$

(I.5) THEOREM. – *Let X be a normal [resp.: regular, reduced] complex space, $f\colon X \to Y$ a holomorphic map.*

Then there exists an analytically meagre subset $A \subset Y$ such that X_y is normal [resp.: regular, reduced] for each $y \in Y - A$.

Moreover, whenever f is proper, A is analytic of codimension $\geqslant 1$.

PROOF. – We shall prove the proposition for X normal complex space (since in the other case the argument is the same). Without restriction of generality we may assume Y reduced; moreover by [Fr] (IV.9) we may assume f flat, so by (0.3)a the set

$$N_f(X) := \{x \in X | X_{f(x)} \text{ is not normal in } x\}$$

is analytic in X, hence by (0.2) $A := f(N_f(X))$ is a countable union of locally analytic subsets of Y. We prove that $\operatorname{Int} A = \emptyset$ by induction on the dimension of Y.

If $\dim Y = 1$, this is an easy consequence of (I.4). So let $\dim Y = m > 1$ and let us assume that there exist $y \in Y$ and an open neighbourhood U_y of y in Y such that $f^{-1}(y')$ is not normal for each $y' \in U_y$. Let U be a smooth non empty open subset of U_y and, identifying U to an open subset of $\boldsymbol{C}^m$ to which it is biholomorphic, let $p\colon U \to \boldsymbol{C}$ the projection on one of the coordinate axes.

If we denote $g := \varphi \circ f\colon X \to \boldsymbol{C}$, by th. (I.4) there exists $c \in \boldsymbol{C}$ such that $g^{-1}(c) = f^{-1}(p^{-1}(c))$ is normal. So, if we consider $f|_{g^{-1}(c)}\colon g^{-1}(c) \to p^{-1}(c)$ since $p^{-1}(c)$ is a $(m-1)$-dimensional complex space, by inductive assumption there exists at least a point $z \in p^{-1}(c)$ such that $f^{-1}(z)$ is normal and we have a contraddiction.

(I.6) LEMMA (Y. T. Siu). – *Let X be a complex space and let $\mathcal{F}$ be an analytic coherent sheaf on X.*

Then there a locally finite family $(Y_i)_{i\in I}$ of irreducible analytic subsets of X such that for each

$$x \in X \qquad \operatorname{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = \{\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}\}\,,$$

where $\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}$ are the prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $Y_{i,x}$ with $x \in Y_i$.

PROOF. – This immediatly follows by [S] th. 4 taking as subsheaf of $\mathcal{F}$ the 0-sheaf.

(I.7) DEFINITION. – The analytic subset $(Y_i)_{i\in I}$ of lemma (I.6) are called *analytic subsets associated to the sheaf $\mathcal{F}$*. Since the family $(Y_i)_{i\in I}$ is locally finite, it is at most a countable one.

(I.8) LEMMA. – *Let X be a complex space, let*

$$0 \to \mathcal{F} \xrightarrow[\alpha]{} \mathcal{G} \xrightarrow[\beta]{} \mathcal{H} \to 0$$

be an exact sequence of coherent analytic sheaves, and let $f: X \to \mathbf{C}$ *be a holomorphic function which is not constant on any irreducible component of* X.

Then there exists a countable subset $A \subset \mathbf{C}$ *such that for each* $c \in \mathbf{C} - A$ *the sequence*

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c} \quad \textit{is exact.}$$

PROOF. – Let $(Y_i)_{i\in I}$ and $(Z_j)_{j\in J}$ be the analytic subsets associated to the sheaves to the sheaves $\mathcal{G}/\mathcal{F}$ and $\mathcal{H}/\mathrm{Im}\,\beta$ respectively and let

$$A := \{c \in \mathbf{C} | X_c \supset Y_i \text{ for some } i\} \cup \{c \in \mathbf{C} | X_c \supset Z_j \text{ for some } j\}\,.$$

Let $c \in \mathbf{C} - A$ and let $x \in X_c$. Since f is not constant on each irreducible component of X, $\mathcal{O}_{X_c,x} \simeq \mathcal{O}_{X,x}/t\mathcal{O}_{X,x}$ where t is a regular element of $\mathcal{O}_{X,x}$. Moreover, since $c \in \mathbf{C} - A$, $t \notin \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{G}/\mathcal{F})_x \cup \mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{H}/\mathrm{Im}\,\beta)_x$ hence by [CGM] (I.1) the sequence

$$0 \to \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_c,x} \to \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_c,x} \to \mathcal{H}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_c,x}$$

is exact.

(I.9) LEMMA. – *Let* X *be a reduced complex space, let* Y *be a normal complex space and* $\pi: Y \to X$ *a finite modification* (1). *Then* $\pi: Y \to X$ *is the normalization of* X.

(I.10) PROPOSITION. – *Let* X *be a reduced complex space and let* $\pi: \tilde{X} \to X$ *be its normalization. Let* $f: X \to Y$ *be a holomorphic map and let* $\tilde{f} := f\circ\pi: \tilde{X} \to Y$.

Then there exists an analytically meagre subset $A \subset Y$ *such that for each* $y \in Y - A$ $\pi|_{\tilde{X}_y} \tilde{X}_y \to X_y$ *is the normalization of* X_y.

PROOF. – Let

$$A_1 := \{y \in Y | \tilde{X}_y \text{ is not normal}\}\,,$$

$$A_2 := \{y \in Y | N(X) \text{ contains an irreducible component of } X_y\}$$

(where $N(X)$ denotes the non-normal locus of X) and $A := A_1 \cup A_2$. Clearly for each $y \in Y - A$ $\pi|_{\tilde{X}_y}: \tilde{X}_y := \pi^{-1}(X_y) \to X_y$ is a finite modification and $\tilde{X}_y$ is normal, hence $\tilde{X}_y$ is the normalization of X_y (see: (I.9)). Since A_1 is analytically meagre by (1.5), we have only to prove that A_2 is analytically meagre too.

Let $P := \{x \in X | f \text{ is not flat in } x\}$ and let $P' := \{x \in N(X) | f|_{N(X)} \text{ is not flat in } x\}$. Then by [F] cor. p. 154 on $X - (P \cup P')$ the dimension formula holds both for f

---

(1) That is a proper, generically bijective holomorphic map.

and $f|_{N(X)}$, so, since codim $N(X) \geqslant 1$, $\dim X_y > \dim (X_y \cap N(X))$ for each $y \in f(X - (P \cup P'))$. Therefore $A_2 \subset Y - f(X - (P \cup P'))$, hence it is analytically meagre by [Fr] (IV.9).

(I.11) PROPOSITION. – *Let X be a maximal complex space and let $f: X \to \boldsymbol{C}$ be a holomorphic function.*

Then for each relatively compact open subset $U \subset X$ there exists a finite subset $A \subset \boldsymbol{C}$ such that $X_c \cap U$ is maximal for each $c \in \boldsymbol{C} - A$.

PROOF. – Without restricting generality (see: [Fr] (IV.9) and [F] cor. p. 154) we may assume that f is not constant on any irreducible component of X. Let π: $\tilde{X} \to X$ be the normalization of X and let $R := (\tilde{X} \times_X \tilde{X})_{\text{red}}$. By definition of maximalization (and using the same notations as in (0.4)), the sequence of coherent analytic sheaves over X

$$0 \to \mathcal{O}_X \xrightarrow[\pi^*]{} \pi_* \mathcal{O}_{\tilde{X}} \xrightarrow[(g_1 - g_2)^*]{} \pi'_* \mathcal{O}_R \quad \text{is exact.}$$

Let us denote $\tilde{f} := f \circ \pi : \tilde{X} \to \boldsymbol{C}$ and $h := f \circ \pi' : R \to \boldsymbol{C}$. Since f is not constant on any irreducible component of X, $\tilde{f}$ and h are not constant on any irreducible component of $\tilde{X}$ and R respectively.

Let U be a relatively compact open subset of X, let $\tilde{U} := \pi^{-1}(U)$, $\overline{U} := \pi'^{-1}(U)$. Since π and hence π' are proper maps, $\tilde{U}$ and $\overline{U}$ are relatively compact open subsets of $\tilde{X}$ and R respectively.

Let $(Y_i)_{i \in I}$ and $(\overline{Y}_j)_{j \in J}$ be the analytic subsets associated to the sheaves $(\pi_* \mathcal{O}_{\tilde{X}})/\mathcal{O}_X$ and $\pi'_* \mathcal{O}_R / \operatorname{Im} (g_1 - g_2)^*$ respectively (see (I.6) and (I.7)) and let

$$A_1 := \{c \in \boldsymbol{C} | X_c \supset Y_i \text{ for some } i \text{ such that } Y_i \cap U \neq \emptyset\}$$

$$A_2 := \{c \in \boldsymbol{C} | X_c \supset \overline{Y}_j \text{ for some } j \text{ such that } \overline{Y}_j \cap U \neq \emptyset\} .$$

Since $(Y_i)_{i \in I}$ and $(\overline{Y}_j)_{j \in J}$ are locally finite families, A_1 and A_2 are finite.

Let $Z := \{z \in R | \operatorname{grad} h(z) = 0\}$; $W := \{y \in \tilde{X} | \operatorname{grad} \tilde{f}(z) = 0\}$; $\bar{Z}_1 := Z - \operatorname{Sing} Z$, $\bar{Z}_2 := \operatorname{Sing} Z - \operatorname{Sing} (\operatorname{Sing} Z), \ldots$; $\tilde{W}_1 := W - \operatorname{Sing} W$, $\tilde{W}_2 := \operatorname{Sing} W - \operatorname{Sing} (\operatorname{Sing} W), \ldots$; and let $(\bar{Z}_i^{(\nu)})_{\nu \in \Lambda_i}$ [resp.: $(\overline{W}_j^{(\gamma)})_{\gamma \in \Gamma_j}$] be the connected components of $\bar{Z}_i$ [resp.: $\tilde{W}_j$]. Let $\{\bar{Z}_\lambda\}_{\lambda \in \Lambda} = \{\bar{Z}_i^{(\nu)}\}_{i = 1, \ldots, \dim Z + 1; \nu \in \Lambda_i}$, $\{\overline{W}_\mu\}_{\mu \in \Gamma} = \{\overline{W}_j^{(\gamma)}\}_{j = 1, \ldots, \dim W + 1; \gamma \in \Gamma_j}$. Clearly

$$A_3 := \{c \in \boldsymbol{C} | \operatorname{Sing} (R_c \cap \overline{U}) \not\subset \operatorname{Sing} (\overline{U}) \cap R_c\} = \\ = \{c \in \boldsymbol{C} | R_c \supset \bar{Z}_\lambda \text{ for some } \lambda \text{ such that } \bar{Z}_\lambda \cap \overline{U} \neq \emptyset\}$$

and

$$A_4 := \{c \in \boldsymbol{C} | \operatorname{Sing} (\tilde{X}_c \cap \tilde{U}) \not\subset \operatorname{Sing} (\tilde{U}) \cap \tilde{X}_c\} = \\ = \{c \in \boldsymbol{C} | \tilde{X}_c \supset \overline{W}_\mu \text{ for some } \mu \text{ such that } \overline{W}_\mu \cap \tilde{U} \neq \emptyset\}$$

(see: (I.2)), so A_3 and A_4 are finite.

Let $N(X)$ be the non-normal locus of X and let

$A_5 := \{c \in \boldsymbol{C} | N(X)$ contains an irreducible component of X_c, whose intersection with U is non empty$\}$.

For each $k \geq 0$ let $T_k := \operatorname{Sing} R \cap S_k(\mathcal{O}_R; h) \subset R$, $S_k := \operatorname{Sing} \tilde{X} \cap S_k(\mathcal{O}_{\tilde{X}}; \tilde{f}) \subset \tilde{X}$. Since h and $\tilde{f}$ are not constant on any irreducible components of R and $\tilde{X}$ respectively, for each $k \geq 0$ we have $T_k = \operatorname{Sing} R \cap S_{k+1}(\mathcal{O}_R)$, $S_k = \operatorname{Sing} \tilde{X} \cap S_{k+1}(\mathcal{O}_{\tilde{X}})$, so for each k $T_k \subset T_{k+1}$ [resp.: $S_k \subset S_{k+1}$] and $\forall k \geq \dim R + 1$ [resp.: $\forall k \geq \dim X + 1$] we have $T_k = \operatorname{Sing} R$ [resp.: $S_k = \operatorname{Sing} \tilde{X}$]. We denote $\forall k = 0, \ldots, \dim R + 1$

$B_k := \{c \in \boldsymbol{C} | R_c$ contains an irreducible component of T_k, whose intersection with $\bar{U}$ is non empty$\}$

and $\forall k' = 0, \ldots, \dim X + 1$.

$C_k := \{c \in \boldsymbol{C} | \tilde{X}_c$ contains an irreducible component of $S_{k'}$, whose intersection with $\tilde{U}$ is non empty$\}$

Obviously each B_k [resp.: $C_{k'}$] is finite.

Let $A := \left(\bigcup_{i=1}^{5} A_i\right) \cup \left(\bigcup_{k=0}^{\dim R+1} B_k\right) \cup \left(\bigcup_{k'=0}^{\dim \tilde{X}+1} C_{k'}\right)$ and let $c \in \boldsymbol{C} - A$.

Since $c \notin A_1 \cup A_2$, by the proof of (I.8) the sequence

$$0 \to \mathcal{O}_{X_c} \to (\pi_* \mathcal{O}_{\tilde{X}}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c} \to (\pi'_* \mathcal{O}_R) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c}$$

is exact over U.

Let us observe that for each $c \in \boldsymbol{C}$

$$(\pi_* \mathcal{O}_{\tilde{X}}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c} \simeq \pi_* \mathcal{O}_{\tilde{X}_c} \quad \text{and} \quad (\pi'_* \mathcal{O}_R) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_c} \simeq \pi'_* \mathcal{O}_{R_c}.$$

Since $c \notin A_3 \cup \left(\bigcup_{k=0}^{\dim R+1} B_k\right)$ the complex space $R_c \cap \bar{U}$ is reduced (see the proof of th. (1.4)), so by [CGM] (1.2) $\pi'_* \mathcal{O}_{R_c} \simeq \pi'_* \mathcal{O}_{(\tilde{X}_c \times_{X_c} \tilde{X}_c)_{\mathrm{red}}}$ over U.

Moreover since $c \notin A_4 \cup A_5 \cup \left(\bigcup_{k'=1}^{\dim X+1} C_{k'}\right)$, by (I.10) $\tilde{X}_c \cap \tilde{U}$ is the normalization of $X_c \cap U$.

So, for each $c \in A$ the sequence

$$0 \to \mathcal{O}_{X_c} \to \pi_* \mathcal{O}_{\tilde{X}_c} \to \pi'_* \mathcal{O}_{(\tilde{X}_c \times_{X_c} \tilde{X}_c)_{\mathrm{red}}}$$

is exact over U and $\tilde{X}_c \cap \tilde{U}$ is the normalization of $X_c \cap U$, hence $X_c \cap U$ is maximal.

(I.12) THEOREM. – *Let X be a maximal complex space and let $f\colon X \to \boldsymbol{C}$ be a holomorphic function.*

Then there exists a countable subset $A \subset \boldsymbol{C}$ such that X_c is maximal for each $c \in \boldsymbol{C} - A$.

PROOF. – It follows by (I.11) by taking a countable covering $\{U_i\}_{i\in \boldsymbol{N}}$ of X, where U_i is a relatively compact open subset of X.

(I.13) PROPOSITION. – *Let X be a maximal complex space and let $f\colon X \to \boldsymbol{C}$ be a holomorphic function.*

Then the subset

$$M_f(X) := \{x \in X | X_{f(x)} \text{ is not maximal in } x\}$$

is analytic in X.

PROOF. – For each $c \in \boldsymbol{C}$ let $M(X_c)$ denote the non-maximal locus of X_c, so we have $M_f(X) = \bigcup_{c\in C} M(X_c)$.

Let $\{U_i\}_{i\in \boldsymbol{N}}$ be an open covering of X, where each U_i is a relatively compact open subset of X. By (I.11) for each $i \in \boldsymbol{N}$ there exists a finite subset $A^{U_i} \subset \boldsymbol{C}$ such that $M(X_c) \cap U_i = \emptyset$ for each $c \in \boldsymbol{C} - A^{U_i}$, hence $M_f(X) \cap U_i = \bigcup_{c\in A^{U_i}} (M(X_c) \cap U_i)$. Therefore, since $M(X_c)$ is analytic in X_c (see: [F], p. 124) $M_f(X) \cap U_i$ is analytic in U_i and the conclusion follows.

(I.14) COROLLARY. – *Let X be a maximal complex space, let Y be a 1-dimensional reduced complex space and let $f\colon X \to Y$ be a homolorphic map.*

$$M_f(X) := \{x \in X | X_{f(x)} \text{ is not maximal in } x\}$$

is analytic in X.

PROOF. – With the same notation as in (I.13) we have

$$M_f(X) = \Big(\bigcup_{f(x)\in \operatorname{Reg} Y} M(X_{f(x)})\Big) \cup \Big(\bigcup_{f(x)\in \operatorname{Sing} Y} M(X_{f(x)})\Big)$$

where $\bigcup_{f(x)\in \operatorname{Reg} Y} M(X_{f(x)})$ is an analytic subset of X by (I.13).

Moreover, since for any compact subset $K \subset Y$ the set $K \cap \operatorname{Sin} Y$ is finite, then for any relatively compact open subset $U \subset X$ the set $A^U := \{y \in \operatorname{Sing} Y | M(X_y) \cap \cap U \neq \emptyset\}$ is finite. Therefore, with the same argument as in (I.13), also $\bigcup_{f(x)\in \operatorname{Sing} Y} M(X_{f(x)})$ is analytic in X, and the conclusion follows.

II. – Bertini type theorems.

In this section, by using Sard type theorems of section I and a theorem of Bănică on the fibers of a flat morphism (see: (0.3)a), we prove Bertini type theorems for regular, normal and reduced complex spaces.

(II.1) LEMMA. – *Let X be a complex space, let V be a finite dimensional vector space of holomorphic functions on X, and let $F\colon X\times V\to \boldsymbol{C}\times V$ be the holomorphic map defined by $(x,f)\mapsto (f(x),f)$.*

Then F is flat in every point $(x_0,f_0)\in X\times V$ such that f_0 is flat in x_0.

PROOF. – Let us assume $\dim_{\boldsymbol{C}} V=r$ and let $(\xi_1,\dots,\xi_r)$ be the coordinates in V with respect to a fixed basis $g_1,\dots,g_r$ and let z be the coordinate in $\boldsymbol{C}$. Then $\boldsymbol{C}\times V\simeq \boldsymbol{C}^{r+1}$ with coordinates $(z,\xi_1,\dots,\xi_r)$ and, up to a translation, we may assume that $(f_0(x_0),f_0)$ is the origin of $\boldsymbol{C}^{r+1}$.

By [F] cor. p. 154 we have to prove that the images of the germs of the coordinate functions $(z,\xi_1,\dots,\xi_r)$ at the origin through the local homomorphism

$$\tilde{F}:=\tilde{F}_{(x_0,f_0)}\colon \mathcal{O}_{\boldsymbol{C}^{r+1},0}\to \mathcal{O}_{X\times V,(x_0,f_0)}$$

are a $\mathcal{O}_{X\times V,(x_0,f_0)}$-regular sequence. We can observe that if $x\in X$, $f\in V$ and $f=\sum_{i=1}^{n}\lambda_i g_i$ we have: $F(x,f)=\left(\sum_{i=1}^{n}\lambda_i g_i(x),\lambda_1,\dots,\lambda_r\right)$, so it is easy to verify that

$$\mathcal{O}_{X\times V,(x_0,f_0)}/\tilde{F}(\xi_1)\simeq \mathcal{O}_{X,x_0}\hat{\otimes}_{\boldsymbol{C}}\boldsymbol{C}\{\xi_2,\dots,\xi_r\},\ \dots,\ \mathcal{O}_{X\times V,(x_0,f_0)}/\big(\tilde{F}(\xi_1),\dots,\tilde{F}(\xi_r)\big)\simeq \mathcal{O}_{X,x_0}.$$

Moreover, since f_0 is flat in x_0, again by [F] cor. p. 154, the germ of z is not a zero-divisor in $\mathcal{O}_{X,x_0}$, so the conclusion follows.

(II.2) LEMMA. – *With the same notations as in lemma* (II.1) *let $(x_0,f_0)\in X\times V$ be such that $f_0(x_0)=0$ but f_0 is not identically zero on any irreducible component of X.*

Then, if $F^{-1}(0,f_0)$ is normal [resp.: a manifold, reduced], for every compact subset $K\subset X$ there exist $\varepsilon\in\boldsymbol{R}^+$ and an open neighbourhood $\mathcal{V}_K(f_0)$ of f_0 in V such that, for each $(c,f)\in\boldsymbol{C}\times V$ with $|c|<\varepsilon$ and $f\in\mathcal{V}_K(f_0)$, $F^{-1}(c,f)$ is normal [resp.: a manifold, reduced] at each point of $F^{-1}(c,f)\cap(K\times V)$.

PROOF. – Since f_0 is not identically zero on any irreducible component of X, by (II.1) F is flat in each point of $F^{-1}(0,f_0)$, hence by [Fr] (IV.9) there exists an open subset $U\subset X\times V$ such that $U\supset F^{-1}(0,f_0)$ and F is flat over U. So by (0.3)a) the set

$$N:=\{(x,f)\in U\,|\,F^{-1}(F(x,f))\ \text{is normal in}\ (x,f)\}$$

$$\big[\text{resp.: } S:=\{(x,f)\in U\,|\,F^{-1}(F(x,f))\ \text{is a manifold in}\ (x,f)\},$$

$$R:=\{(x,f)\in U\,|\,F^{-1}(F(x,f))\ \text{is reduced in}\ (x,f)\}\big]$$

is open in U (hence in $X\times V$) and contains $F^{-1}(0,f_0)$.

Let K be a compact subset of X. Since the map $F|_{K\times V}: K\times V \to \boldsymbol{C}\times V$ is proper and $N \cap (K\times V)$ [resp.: $S \cap (K\times V)$, $R \cap (K\times V)$] is an open subset of $K\times V$ containing $F^{-1}(0, f_0) \cap (K\times V)$, then

$$W := \boldsymbol{C}\times V - F(X\times V - N \cap (K\times V))$$

$$[\text{resp.: } W' := \boldsymbol{C}\times V - F(X\times V - S \cap (K\times V)) \,,$$

$$W'' := \boldsymbol{C}\times V - F(X\times V - R \cap (K\times V))]$$

is an open subset of $\boldsymbol{C}\times V$ containing $(0, f_0)$ and such that

$$F^{-1}(c, f) \cap (K\times V) \subset N \cap (K\times V)$$

$$[\text{resp.: } F^{-1}(c, f) \cap (K\times V) \subset S \cap (K\times V) \,,$$

$$F^{-1}(c, f) \cap (K\times V) \subset R \cap (K\times V)]$$

for each $(c, f) \in W$ [resp.: $(c, f) \in W'$, $(c, f) \in W''$].

Hence the conclusion follows.

(II.3) REMARK. – We recall that a subset M of a complete metric space V is said to be *fat* if there exists a countable family $\{U_i\}_{i\in N}$ of dense open subsets U_i of V such that $M \supset \bigcap_{i\in N} U_i$.

Clearly a countable intersection of fat subsets of V is fat; moreover by Baire's theorem every fat subset of a complete metric space is dense. We recall also that the complement of a fat subset is « *maigre* » according to [Bo], § 1, n. 16.

(II.4) REMARK. – We recall that if L is a holomorphic line bundle on a complex space X and $s \in \Gamma(X, L)$ a holomorphic section, then the zero-set $Z := \{s = 0\}$ can be provided in a natural way with a cmplex structure as follows: the structural sheaf of Z is defined by the exact sequence

$$\mathfrak{L}^* \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

where $\mathfrak{L}$ is the sheaf of the germs of holomorphic sections of L.

In the following we always consider the zero-sets of holomorphic sections with this natural structure.

(II.5) THEOREM. – *Let X be a normal [resp.: regular, reduced] complex space L a holomorphic line bundle on X and $V \subset \Gamma(X, L)$ a finite dimensional linear subspace which generates L. Then there exists a fat subset $M \subset V$ such that for each $s \in M$ the zero-set $\{s = 0\}$ is a normal [resp.: regular, reduced] complex space.*

PROOF. – Let $\{K_i\}_{i\in N}$ be a countable covering of X such that $\forall i \in \boldsymbol{N}\ K_i$ is compact and it is contained in an open subset U_i of X such that there exists $F_i \in V$ which is never zero on U_i. For each $i \in \boldsymbol{N}$ let $M_i := \{s \in V | Z := \{s = 0\}$ be a 1-codimensional analytic subset of X, which is normal [resp.: regular, reduced] at every point of $Z \cap K_i\}$.

I Step: *M_i is open in V.*

Let $f \in M_i$. Since L is trivial on U_i, with respect to this trivialization, f is a holomorphic function on U_i, which is not identically zero on any irreducible component of U_i; then by lemma (II.2) there exists an open neighbourhood $\mathcal{V}_{K_i}(f)$ of f in V such that $\mathcal{V}_{K_i}(f) \subset M_i$.

II Step: *M_i is dense in V.*

Let $g \in V$. By our assumptions g/F_i is a holomorphic function on U_i, then by (I.4) there exists an arbitrarily small $c \in \boldsymbol{C}$ such that the complex space $\{g/F_i = c\} \cap \cap U_i$ is normal [resp.: regular, reduced], hence $g - cF_i \in M_i$.

Now let $M := \bigcap_{i\in N} M_i$ and by remark (II.3) the conclusion follows.

(II.6) COROLLARY. – *Let $X \subset \boldsymbol{C}^n$ be a normal [resp.: regular, reduced] locally analytic subset. Then there exists a fat subset M of the space $\mathcal{H}$ of all hyperplanes in $\boldsymbol{C}^n$ such that for every $H \in M$, $X \cap H$ is normal [resp.: regular, reduced].*

(II.7) COROLLARY. – *Let X be a normal [resp.: regular, reduced] compact complex space, L a holomorphic line bundle on X and $V \subset \Gamma(X, L)$ a linear subspace which generates L and such that every $s \in V - \{0\}$ is not identically zero on any irreducible component of X.*

Then there exists a proper algebraic subset $A \subset V$ such that for each $s \in V - A$ the zero-set $\{s = 0\}$ is normal [resp.: regular, reduced].

PROOF. – Let $\{U_i\}_{i=1,\dots,r}$ be an open covering of X on which L is trivial. With respect to this trivialization of L, every $f \in V$ is a holomorphic function on each U_i, so for each $i = 1, \dots, r$ we can consider the holomorphic map $F_i\colon U_i \times V \to \boldsymbol{C} \times V$ defined by $F_i(x, f) := (f(x), f)$ and we have

$$\begin{aligned}\tilde{A}_i := \{(x, f) \in U_i \times (V - \{0\}) | f(x) = 0 \text{ and } \{f = 0\} \text{ is}\\ \text{not normal [resp.: regular, reduced] in } x\} =\\ = F_i^{-1}(\{0\} \times (V - \{0\})) \cap \{(x, f) \in U_i \times (V - \{0\}) | F_i^{-1}(F_i(x, f)) \text{ is}\\ \text{not normal [resp.: regular, reduced] in } (x, f)\}.\end{aligned}$$

Since every $f \in V - \{0\}$ is not identically zero on any irreducible component of X, then F_i is flat on $U_i \times (V - \{0\})$, hence by $(0.3)a$ $\tilde{A}_i$ is an analytic subset of $U_i \times \times (V - \{0\})$. Therefore $\tilde{A} := \bigcup_{i=1}^{r} \tilde{A}_i$ is analytic in $X \times (V - \{0\})$, so, as the canonical projection $p: X \times (V - \{0\}) \to V - \{0\}$ is proper, $A' := p(\tilde{A})$ is a proper analytic subset of $V - \{0\}$. Moreover, A' is obviously a cone, hence it is algebraic in $V - \{0\}$, therefore $A := A' \cup \{0\}$ is a proper algebraic subset of V.

(II.8) COROLLARY. – *Let $X \subseteq \boldsymbol{P}_n(\boldsymbol{C})$ be a normal [resp.: regular, reduced] complex subvariety.*

Then the general hyperplane section of X is a normal [resp.: regular, reduced] variety.

PROOF. – This is a consequence of (II.7), if we take

$$L := \mathcal{O}_X(1) = \mathcal{O}_{\boldsymbol{P}^n}(1) \otimes_{\mathcal{O}_{\boldsymbol{P}^n}} \mathcal{O}_X \quad \text{and} \quad V := \operatorname{Im}\left(\Gamma(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1)) \to \Gamma(X, \mathcal{O}_X(1))\right).$$

(II.9) REMARK. – The proof of theorem (II.5) and corollaries (II.6), (II.7), (II.8) can be applied to complex spaces which have a property $\mathcal{T}$ for which a Sard type theorem (S') (analogue to (I.4) and (I.12)) and a resut similar to $(0.3)a$ hold. So, these Bertini type theorems can perhaps be extended to Gorenstein complex spaces (for which C. BĂNICĂ and M. STOIA [BS] proved the analogues of $(0.3)a$, b and to maximal complex spaces (for which the Sard type theorem (I.12) holds).

REFERENCES

[AAL] W. ADKINS - A. ANDREOTTI - J. V. LEAHY, *Weakly normal complex spaces*, Atti Acc. Naz. Lincei (1980).

[AN] A. ANDREOTTI - F. NORGUET, *La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique*, Ann. Sc. Norm. Sup. Pisa, **21** (1967), pp. 31-82.

[B] C. BĂNICĂ, *Le lieu réduit et le lieu normal d'un morphisme*, Proceedings of the Romanian-Finnish Seminar on Complex Analysis, Bucharest, 1976, Lect. Notes Math. 743, Springer-Verlag, 1979, pp. 389-398.

[BS] C. BĂNICĂ - M. STOIA, *Gorenstein points of a flat morphism of complex spaces*, Revue Roumaine Math. Pures Appl., **26**, no. 5 (1981), pp. 687-690.

[Bo] N. BOURBAKI, *Topologie générale.* (Fascicule de résultats.) Hermann, Paris, 1964.

[CGM] C. CUMINO - S. GRECO - M. MANARESI, *Bertini theorems for weak normality*, to appear on Comp. Math.

[F] G. FISCHER, *Complex Analytic Geometry*, Lect. Notes Math. 538, Springer-Verlag, 1976.

[Fl] H. FLENNER, *Die Sätze von Bertini für lokale Ringe*, Math. Ann., **229** (1977), pp. 97-111.

[Fr] J. FRISCH, *Points de platitude d'un morphisme d'espaces analytiques complexes*, Inv. Math., **4** (1967), pp. 118-138.

[K] W. E. KUAN, *A note on a generic hyperplane section of an algebraic variety*, Can. J. Math., **22**, no. 5 (1970), pp. 1047-1053.

[M] J. W. MILNOR, *Topology from the differentiable view-point*, University Press of Virginia, 1965.

[R] R. REMMERT, *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann., **133** (1957), pp. 328-370.

[Se] A. SEIDENBERG, *The hyperplane sections of normal varieties*, Trans. Amer. Math. Soc., **50** (1941), pp. 357-386.

[S] Y. T. SIU, *Noether-Lasker decomposition of coherent analytic subsheaves*, Trans. Amer. Math. Soc., **135** (1969), pp. 375-385.

[ST] Y. T. SIU - G. TRAUTMANN *Gap sheaves and extensions of coherent analytic subsheaves* Lect. Notes Math. 172, Springer-Verlag, 1971.