A Model for Hysteresis of Distributed Systems (*).

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Summary. – *Memory effects of hysteresis type are taken into account as constitutive relations for parabolic and hyperbolic problems, also with free boundaries; existence results are proved. A scheme of construction of functionals representing hysteresis phenomena is presented and examples are given; in particular ferromagnetism is considered.*

Introduction.

In this paper we present an approach to the modelization of some hysteresis phenomena.

In § 1 we give an axiomatization of the essential properties in the case of two real variables connected by such a functional relation. For distributed systems, this is coupled with a partial differential equation; in § 2, 3 we prove existence results for some parabolic and hyperbolic problems, also with free boundaries.

Then in § 4 we introduce a procedure for the construction of examples of such « hysteresis functionals », by means of the resolution of a family of Cauchy problems for an ordinary differential equation. Some explicit examples are given in § 5.

Finally in § 6 we present a generalization of the mathematical model in order to fit it better to the phenomenology of ferromagnetism.

Mathematical studies about hysteresis have been conducted by KRASNOSEL'SKII and co-workers in recent years; we refer to [5] and [6] for surveys of results and for a large collection of references.

Some control problems exhibiting hysteresis arising in the theory of thermostats have been studied by GASHOFF and SPREKELS, cf. [3] and [4].

1. – Functionals representing memory.

Let the functions $v, z\colon [0, T] \to \boldsymbol{R}$ be related by a condition of the following type: $z(t)$ depends on the restriction of v to $[0, t]$ and on the initial value $z(0)$, i.e. formally

$$z(t) = f\big(v(\cdot)|_{[0,t]},\, z(0)\big) \qquad 0 < t < T\,. \tag{1.1}$$

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We are going to state this in a precise form. Let a couple $(S, \mathcal{F})$ be given such that

$$S\colon \boldsymbol{R}\times[0, T] \to \mathcal{P}(\boldsymbol{R}) \tag{1.2}$$

$$\begin{cases} \operatorname{Dom}(\mathcal{F}) = \{(v, t, \xi)\,|\,v \in C^0([0, T]),\ 0\leqslant t\leqslant T,\ \xi \in S(v(0), 0)\}; \\ \forall (v, t, \xi) \in \operatorname{Dom}(\mathcal{F}),\ \mathcal{F}(v, t, \xi) \in S(v(t), t) \end{cases} \tag{1.3}$$

$$\begin{cases} \forall v \in C^0([0, T]),\ \forall \xi \in S(v(0), 0),\ \text{the function } t \mapsto \mathcal{F}(v, t, \xi) \text{ is continuous} \\ \quad \text{in } [0, T] \end{cases} \tag{1.4}$$

$$\forall v \in C^0([0, T]),\ \forall \xi \in S(v(0), 0),\ \mathcal{F}(v, 0, \xi) = \xi \tag{1.5}$$

$$\begin{cases} \forall \bar{t} \in \,]0, T[,\ \forall v_1, v_2 \in C^0([0, T]) \text{ such that } v_1 = v_2 \text{ in } [0, \bar{t}], \\ \forall \xi \in S(v_1(0), 0),\ \mathcal{F}(v_1, \bar{t}, \xi) = \mathcal{F}(v_2, \bar{t}, \xi) \text{ (causality)}. \end{cases} \tag{1.6}$$

We assume $v \in C^0([0, T])$ and replace (1.1) by the more rigorous relation

$$z(t) = \mathcal{F}(v(\cdot), t, z(0)) \quad \text{for } 0 < t < T. \tag{1.7}$$

Actually, this last is more general than (1.1): it corresponds to a dependence of the form

$$z(t) = \tilde{f}(v(\cdot)|_{[0,t]}, t, z(0)) \quad \text{for } 0 < t < T \tag{1.8}$$

in which the constitutive law may change with time.

EXAMPLE 1 (trivial). – $\mathcal{F}$ reduces to a function.
Let $g \in C^0(\boldsymbol{R}\times[0, T])$ and set

$$\begin{cases} S(y, t) = \{g(y, t)\},\ \forall y \in \boldsymbol{R},\ \forall t \in [0, T] \\ \mathcal{F}(v, t, g(v(0), 0)) = g(v(t), t),\ \forall v \in C^0([0, T]),\ \forall t \in [0, T]. \end{cases} \tag{1.9}$$

(1.2), ..., (1.6) are trivially satisfied.

EXAMPLE 2. – Convolution.
Let $b \in C^0([0, T]^2)$, $g \in C^0(\boldsymbol{R})$ and set

$$\begin{cases} S(y, t) = \boldsymbol{R}, \quad \forall y \in \boldsymbol{R},\ \forall t \in [0, T] \\ \mathcal{F}(v, t, \xi) = \displaystyle\int_0^t b(t - \tau, t)\, g(v(\tau))\, d\tau + \xi,\ \forall v \in C^0([0, T]),\ \forall t \in [0, T],\ \forall \xi \in \boldsymbol{R}. \end{cases} \tag{1.10}$$

(1.2), ..., (1.6) hold.

This type of functional is used for the description of viscosity and many other memory effects (cf. [2], e.g.).

EXAMPLE 3. – Description of hysteresis.

This arises in ferromagnetism (in this case v and z are the modulus of the magnetic field H and of the induction field B, respectively), in plasticity (then v is the strain ε, z is the stress σ), etc.

Conditions (1.2), ..., (1.6) fit to the phenomenology of hysteresis, when jumps are smoothed; however cases with jumps are considered in problems (P2) and (P4) (cf. § 2 and § 3, respect.).

We can assume also the following property, which discriminates hysteresis from memory effects represented by convolution

$$(1.11)\quad \begin{cases} \forall(v, t, \xi) \in \mathrm{Dom}\,(\mathcal{F}),\ \forall s\colon [0, T] \to [0, T] \text{ monotone homeomorphism}\,, \\ \mathcal{F}(v, t, \xi) = \mathcal{F}\big(v\circ s^{-1}, s(t), \xi\big) \text{ (invariance for time-homeomorphism)}\,. \end{cases}$$

Throughout this paper, a functional fulfilling properties (1.3), ..., (1.6), (1.11) will be named « hysteresis functional ».

If moreover the constitutive law does not depend on time, then also the following condition holds

$$(1.12)\quad \begin{cases} \forall t', t'' \in [0, T] \text{ with } t' < t'',\ \forall v \in C^0([0, T]) \text{ such that } v(t) = \text{constant for} \\ t' \leqslant t \leqslant t'', \quad \forall \xi \in S\big(v(0), 0\big),\ \mathcal{F}(v, t, \xi) = \text{constant for } t' \leqslant t \leqslant t''\,. \end{cases}$$

Hysteresis can appear also in phase transitions; an example is given by supercooling and superheating in change of state. However these phenomena seem to require a different approach; an attempt in this sense is performed in [10].

Now consider the following property

$$(1.13)\quad \begin{cases} \forall \tau \in\,]0, T[,\ \exists (S^\tau, \mathcal{F}^\tau) \text{ fulfilling properties (1.2), ..., (1.6) (with } T \text{ replaced} \\ \qquad \text{by } T-\tau) \text{ and such that setting } \alpha_\lambda(\mu) = \lambda + \mu\ \ \forall \lambda, \mu \in \mathbf{R}, \\ S(\lambda, t) = S^\tau(\lambda, t-\tau)\,, \qquad \forall \lambda \in \mathbf{R},\ \forall t \in [\tau, T] \\ \mathcal{F}(v, t, \xi) = \mathcal{F}^\tau\big(v\circ\alpha_\tau, t-\tau, \mathcal{F}(v, \tau, \xi)\big),\ \forall (v, t, \xi) \in \mathrm{Dom}\,(\mathcal{F}) \text{ with } t > \tau \\ \hfill \text{(transition property)}\,. \end{cases}$$

This has the following meaning: for any $\tau \in\,]0, T[$, in order to evaluate $\mathcal{F}(v, t, \xi)$ for $t > \tau$, the information contained in $\mathcal{F}(v, \tau, \xi)$ can replace that given by ξ and by the evolution of v in $[0, \tau]$. Among other things this implies that $t = 0$ is not a privileged instant.

In general (1.13) is not fulfilled by the convolution functional.

The above properties can be easily extended to the case of vector valued variables $v, z\colon [0, T] \to \boldsymbol{R}^M (M > 1)$. However in this case the phenomenology of hysteresis is more complex. In ferromagnetism, for instance, hysteresis by rotation depends also on the rotational velocity (cf. [9], pag. 553, e.g.); hence it is not invariant by time-homeomorphism (cf, (1.11)). This phenomenon is due to viscosity; it can be neglected for low rotational velocities, hence in the approximation of quasi-stationarity.

2. – Parabolic problems.

Let an open bounded subset D of $\boldsymbol{R}^N$ $(N \geq 1)$ be occupied by a homogeneous system; set $Q = D \times]0, T[$.

We shall consider two variables $u, w\colon Q \to \boldsymbol{R}$ related by a condition of the form

$$w(x, t) = \mathcal{F}\big(u(x, \cdot), t, w(x, 0)\big) \qquad \text{for } (x, t) \in Q \tag{2.0}$$

(where $u(x, \cdot)$ denotes the restriction of u to $\{x\} \times [0, T]$).

This relation will be coupled with a partial differential equation of the form

$$\frac{\partial w}{\partial t} + Au = f \qquad \text{in } \mathcal{D}'(Q)$$

or

$$\begin{cases} \dfrac{\partial}{\partial t}(w + s) + Au = f & \text{in } \mathcal{D}'(Q) \\ s \in \operatorname{sgn}(w) & \text{in } Q \ (\operatorname{sgn} = \text{signum graph}) \end{cases}$$

(where A is a linear elliptic operator, f is a datum); suitable boundary conditions will be given.

Let $V \subset \mathcal{D}(D)$ be a dense and compact real Hilbert subspace of $L^2(D)$. Let $(S, \mathcal{F})$ fulfilling (1.2), ..., (1.6) be given. Let

$$\begin{cases} A\colon V \to V' \text{ be linear, bounded, symmetric and such that} \\ \exists \mu_1, \mu_2 > 0 \text{ constant: } \forall v \in V \ {}_{V'}\langle Av, v\rangle_V + \mu_1 \|v\|^2_{L^2(D)} \geq \mu_2 \|v\|^2_V \ \text{(coerciveness)} \end{cases} \tag{2.1}$$

(for instance, $V = H^1_0(D)$, $A = -\Delta - aI$ with $a > 0$; then take $\mu_1 > a$, $\mu_2 = \min(\mu_1 - a, 1)$).

$$u^0, w^0\colon D \to \boldsymbol{R} \text{ measurable; } w^0(x) \in S\big(u^0(x), 0\big) \text{ a.e. in } D, \ w^0 \in V' \tag{2.2}$$

$$f \in L^1(0, T; V'). \tag{2.3}$$

(P1) Find $u \in L^1(0, T; V)$ such that

$$u(x, \cdot) \in C^0([0, T]) \quad \text{a.e. in } D \tag{2.4}$$

$$u(x, 0) = u^0(x) \quad \text{a.e. in } D \tag{2.5}$$

and, setting

$$w(x, t) = \mathcal{F}(u(x, \cdot), t, w^0(x)), \quad \forall t \in [0, T], \text{ a.e. in } D, \tag{2.6}$$

$$w(\cdot, t) \text{ is measurable in } D\ \forall t \in [0, T],\ w \in W^{1,1}(0, T, V') \tag{2.7}$$

$$\frac{\partial w}{\partial t} + Au = f \quad \text{in } V', \quad \text{a.e. in }]0, T[\,. \tag{2.8}$$

REMARKS. – i) u and w are regarded as functions of t taking values in a space of functions of x, as well as functions of x taking values in a space of functions of t.

ii) By (1.5), (2.6) contains the initial condition $w(x, 0) = w^0(x)$ a.e. in D.

We introduce the set of piecewise linear continuous functions

$$\begin{cases} \tilde{C}^0([0, T]) = \{v \in C^0([0, T]) \mid \exists \{t_k\}_{k=0,\dots,m} \text{ with } t_0 = 0 < t_1 < \dots < t_m = T \\ \text{such that } v \text{ is linear in } [t_k, t_{k+1}], \quad \text{for } k = 0, \dots, m-1\}. \end{cases} \tag{2.9}$$

THEOREM 1. – Assume that (1.2), ..., (1.6), (1.12), (2.1), ..., (2.3) hold and moreover

$$\begin{cases} \exists c_i \colon [0, T] \to \mathbf{R}_+ \ (i = 1, 2) \colon \forall t \in [0, T],\ \forall y \in \mathbf{R},\ \forall z \in S(y, t), \\ |z| \leqslant c_1(t)|y| + c_2(t) \end{cases} \tag{2.10}$$

$$\begin{cases} \exists c_3, c_4, \lambda \text{ constant } (\lambda > 0) \colon \forall v \in \tilde{C}^0([0, T]),\ \forall \xi \in S(v(0), 0), \\ \|\mathcal{F}(v, \cdot, \xi)\|_{W^{\lambda,1}(0,T)} \leqslant c_3 \|v\|_{H^1(0,T)} + c_4 . \end{cases} \tag{2.11}$$

$$\begin{cases} \text{Let } \{v_n \in \tilde{C}^0([0, T])\}_{n \in \mathbf{N}},\ v \in C^0([0, T]) \text{ be such that } v_n(0) = v(0),\ \forall n \in \mathbf{N} \\ \text{and } v_n \to v \text{ in } C^0([0, T]) \text{ strong. Then } \forall \xi \in S(v(0), 0), \\ \mathcal{F}(v_n, \cdot, \xi) \to \mathcal{F}(v, \cdot, \xi) \text{ in } L^1(0, T) \text{ weak} \end{cases} \tag{2.12}$$

$$\begin{cases} \exists \alpha \colon \text{constant} > 0 \colon \forall t', t'' \in [0, T] \text{ with } t' < t'',\ \forall v_1, v_2 \in \tilde{C}^0([0, T]) \\ \text{with } v_1 = v_2 \text{ in } [0, t'],\ v_1 \text{ and } v_2 \text{ linear in } [t', t''],\ \forall \xi \in S(v_1(0), 0), \\ [\mathcal{F}(v_1, t'', \xi) - \mathcal{F}(v_2, t'', \xi)] \cdot [v_1(t'') - v_2(t'')] \geqslant \alpha [v_1(t'') - v_2(t'')]^2 . \end{cases} \tag{2.13}$$

$$(2.14)\quad \left\{\begin{array}{l} \text{Let } m \in \mathbf{N} \text{ and } t_0 = 0 < t_1 < \ldots < t_m = T;\ \forall y_0, \ldots, y_m \in \mathbf{R} \text{ let } y(\cdot) = \\ \pi(y_0, \ldots, y_m) \in C^0([0, T]) \text{ denote the function obtained interpolating li-} \\ \text{nearly } \{y(t_k) = y_k\}_{k=0,\ldots,m} \text{ in } [0, T]. \\ \text{Then } \forall y_0 \in \mathbf{R},\ \forall \xi \in S(y_0, 0),\ \forall t \in [0, T], \text{ the function } \mathbf{R}^m \to \mathbf{R}: (y_1, \ldots, \\ y_m) \mapsto \mathcal{F}(\pi(y_0, \ldots, y_m), t, \xi) \text{ is continuous} \end{array}\right.$$

$$(2.15)\quad \left\{\begin{array}{l} \forall v \in \tilde{C}^0([0, T]),\ \forall t \in [0, T], \text{ the function } S(v(0), 0) \to \mathbf{R}: \xi \to \mathcal{F}(v, t, \xi) \text{ is} \\ \text{continuous} \end{array}\right.$$

$$(2.16)\quad u^0 \in V$$

$$(2.17)\quad f = f_1 + f_2 \quad \text{with} \quad f_1 \in L^2(Q), \quad f_2 \in W^{1,1}(0, T; V').$$

Under these assumptions (P1) has at least one solution, such that moreover

$$(2.18)\quad \left\{\begin{array}{l} u \in H^1(0, T; L^2(D)) \cap L^\infty(0, T; V) \\ w \in W^{\lambda,1}(0, T; L^2(D)). \end{array}\right.$$

REMARKS:

i) (2.18) yields $u \in C^0_s([0, T]; V)$ (cf. [8], ch. 3, lemma 8.1).

ii) (2.13) gives a «parabolic character» to (2.8).

iii) (2.13) is not fulfilled by the convolution operator (1.10); on the other hand (2.10), ..., (2.15) are compatible with (1.11) and one-dimensional hysteresis phenomena (cf. §§ 4, 5).

iv) If $\mathcal{F}$ is a function, as in example 1 of § 1, then theorem 1 reduces to a well-known result (cf. [7], ch. 2, e.g.).

PROOF. – 1) *Approximation.* Let $m \in \mathbf{N}$, $k = T/m$.

$\underline{(\mathrm{P1})_m}$ Find $u^n_m \in V$ for $n = 1, \ldots, m$, such that, setting

$$(2.19)\quad \left\{\begin{array}{l} u_m(x, \cdot) \text{ is the function obtained interpolating linearly the values } u_m(x, \\ nk) = u^n_m(x) \text{ for } n = 0, \ldots, m\ (u^0_m(x) = u^0(x)) \text{ a.e. in } D \end{array}\right.$$

$$(2.20)\quad w^n_m(x) = \mathcal{F}(u_m(x, \cdot), nk, w^0(x)) \text{ a.e. for } x \in D, \text{ for } n = 1, \ldots, m,$$

then w^n_m is measurable in D, $w^n_m \in V'$ $\forall n$ and

$$(2.21)\quad \frac{w^n_m - w^{n-1}_m}{k} + A u^n_m = f^n_m \quad \text{in } V', \text{ for } n = 1, \ldots, m,$$

where

$$
(2.22)\qquad \begin{cases} f_m^n = f_{1m}^n + f_{2m}^n, \quad f_{1m}^n(x) = \dfrac{1}{k}\displaystyle\int_{(n-1)k}^{nk} f_1(x,t)\,dt \quad \text{a.e. in } D,\\ f_{2m}^n = f_2(nk) \quad \text{in } V', \quad w_m^0 = w^0 \quad \text{a.e. in } D. \end{cases}
$$

$\forall m \in \boldsymbol{N}$, we solve $(\mathrm{P1})_m$ step by step. Fix $n \in \{1, \dots, m\}$ and assume that the u_m^l's are known for $l = 0, \dots, n-1$; then by (1.6) $\mathcal{F}(u_m(x,\cdot), nk, w^0(x))$ depends only on $u_m^n(x)$ a.e. in D, i.e. (2.20) is of the form

$$
(2.23)\qquad w_m^n(x) = \Phi_m^n(u_m^0(x), \dots, u_m^n(x), w^0(x)) \equiv \Psi_m^n(u_m^n(x), x) \quad \text{a.e. in } D.
$$

By (2.14) and (2.15), $\Phi_m^n\colon \{(y_0, \dots, y^{n+1}) \in \boldsymbol{R}^{n+2}) \in \boldsymbol{R}^{n+2} | y^{n+1} \in S(y^0, 0)\} \to \boldsymbol{R}$ is continuous; therefore, as $u_m^0, \dots, u_m^n, w^0$ are measurable in D, also w_m^n is measurable in D.

By (2.10), $\forall v \in L^2(D)$, $\Psi_m^n(v) \in L^2(D)$; moreover by (2.13) the functional Ψ_m^n is strictly and cyclicly monotone. Therefore Ψ_m^n is the subdifferential of a lower semi-continuous, strictly convex functional $\Theta_m^n\colon L^2(D) \to \boldsymbol{R}$, i.e. $\Psi_m^n = \partial\Theta_u^n$.

Introduce the lower semi-continuous, strictly convex functional

$$
(2.24)\qquad J_m^n\colon V \to \boldsymbol{R}\colon v \mapsto \Theta_m^n(v) + \frac{k}{2}\,{}_{V'}\langle Av, v\rangle_V - \int_D w_m^{n-1} v\,dx - k\,{}_{V'}\langle f_m^n, v\rangle_V;
$$

by (2.2) and (2.13), if $k \leqslant \alpha/\mu_1$ then J_m^n is coercive; therefore it has a unique minimizing argument u_m^n, which is also the unique solution of $(\mathrm{P1})_m$.

2) *Estimates.* Multiply (2.21) against $u_m^n - u_m^{n-1}$ and sum for $n = 1, \dots, l$, for a generic $l \in \{1, \dots, m\}$. Notice that by (1.12) and (2.13)

$$
(2.25)\qquad \sum_{n=1}^{l} \int_D \frac{w_m^n - w_m^{n-1}}{k}\cdot(u_m^n - u_m^{n-1})\,dx \geqslant \alpha k \sum_{n=1}^{l} \left\| \frac{u_m^n - u_m^{n-1}}{k} \right\|_{L^2(D)}^2;
$$

by (2.2) $A + \mu_1 I$ is the subdifferential of a lower semi-continuous, convex and coercive functional $\Lambda\colon V \to \boldsymbol{R}$; therefore

$$
(2.26)\qquad \sum_{n=1}^{l} {}_{V'}\langle Au_m^n, u_m^n - u_m^{n-1}\rangle_V = \sum_{n=1}^{l} {}_{V'}\langle (A + \mu_1 I)u_m^n, u_m^n - u_m^{n-1}\rangle_V -
$$
$$
- \mu_1 \int_D u_m^n(u_m^n - u_m^{n-1})\,dx \geqslant \Lambda(u_m^n) - \Lambda(u_m^{n-1}) - \mu_1 \|u_m^n\|_{L^2(D)}^2 - \mu_1 \|u_m^n\|_{L^2(D)} \cdot \|u_m^{n-1}\|_{L^2(D)};
$$

furthermore

$$
(2.27)\qquad \sum_{n=1}^{l} \int_D f_{1m}^n(u_m^n - u_m^{n-1})\,dx \leqslant \left(\sum_{n=1}^{l} k \|f_{1m}^n\|_{L^2(D)}^2\right)^{\frac12} \cdot \left(\sum_{n=1}^{l} k \left\| \frac{u_m^n - u_m^{n-1}}{k} \right\|_{L^2(D)}^2\right)^{\frac12}
$$

$$(2.28)\quad \sum_{n=1}^{l} {}_{V'}\langle f_{2m}^n, u_m^n - u_m^{n-1}\rangle_V = {}_{V'}\langle f_{2m}^l, u_m^l\rangle_V - {}_{V'}\langle f_{2m}^1, u^0\rangle_V - \\ - \sum_{n=2}^{l} {}_{V'}\langle f_{2m}^n - f_{2m}^{n-1}, u_m^{n-1}\rangle_V \leq \text{const}\, \|f_{2m}\|_{W^{1,1}(0,T;V')}\cdot \max_{n=0,\dots,l} \|u_m^n\|_V$$

thus we get

$$(2.29)\quad \sum_{n=1}^{m} k \left\| \frac{u_m^n - u_m^{n-1}}{k} \right\|_{L^2(D)}^2 \leq \text{const}.$$

$$(2.30)\quad \max_{n=0,\dots,m} \|u_m^n\|_V \leq \text{const}\,.$$

(2.31) Let $w_m(x, t)$ be the function obtained interpolating linearly the values $w_m(x, nk) = w_m^n(x)$ for $n = 0, \dots, m$ a.e. in D; let $\hat{u}_m(x, t) = u_m^n(x)$ a.e. in D and $\hat{f}_m(t) = f_m^n$ in V' if $(n-1)k < t \leq nk$, for $n = 1, \dots, m$.

(2.21) becomes

$$(2.32)\quad \frac{\partial w_m}{\partial t} + A\hat{u}_m = \hat{f}_m \quad \text{in } V', \quad \text{a.e. in }]0, T[$$

and (2.29), (2.30) yield

$$(2.33)\quad \|u_m\|_{H^1(0,T;L^2(D))\cap L^\infty(0,T;V)} \leq \text{const}\,,$$

$$(2.34)\quad \|\hat{u}_m\|_{L^\infty(0,T;V)} \leq \text{const}\,,$$

by (2.32), (2.34) we have

$$(2.35)\quad \|w_m\|_{H^1(0,T;V')} \leq \text{const}$$

and by (2.11), (2.33)

$$(2.36)\quad \|\mathcal{F}(u_m(x,\cdot), t, w^0(x))\|_{W^{\lambda,1}(0,T;L^1(D))} \leq \text{const}\,.$$

3) *Limit.* By the above a priori estimates there exist u, w such that, possibly taking subsequences,

$$(2.37)\quad u_m \to u \quad \text{in } H^1(0,T;L^2(D)) \cap L^\infty(0,T;V) \text{ weak star}$$

$$(2.38)\quad \hat{u}_m \to u \quad \text{in } L^\infty(0,T;V) \text{ weak star}$$

$$(2.39)\quad w_m \to w \quad \text{in } H^1(0,T;V') \text{ weak}\,.$$

Taking $m \to \infty$ in (2.32) we get (2.8). By (2.37) we have

$$u_m(x, \cdot) \to u(x, \cdot) \text{ in } C^0([0, T]) \text{ strong, a.e. in } D \tag{2.40}$$

then by (2.12)

$$\mathcal{F}(u_m(x, \cdot), t, w^0(x)) \to \mathcal{F}(u(x, \cdot), t, w^0(x)) \quad \text{in } L^1(0, T) \text{ weak, a.e. in } D \tag{2.41}$$

thus by (2.36) and Lemma 1 (see below) we get

$$\mathcal{F}(u_m(x, \cdot), t, w^0(x)) \to \mathcal{F}(u(x, \cdot), t, w^0(x)) \quad \text{in } L^1(0, T; L^2(D)) \text{ weak}. \tag{2.42}$$

By (2.20) and Lemma 2 (see below) we have

$$\begin{aligned}\|w_m(x, t) - \mathcal{F}(u_m(x, \cdot), t, w^0(x))\|_{L^1(0,T;L^2(D))} &\leqslant \\ &\leqslant Ck^\lambda \|\mathcal{F}(u_m(x, \cdot), t, w^0(x)\|_{W^{\lambda,1}(0,T;L^2(D))}\end{aligned} \tag{2.43}$$

with C positive constant; the last two formulas and (2.36) yield

$$w_m \to \mathcal{F}(u(x, \cdot), t, w^0(x)) \quad \text{in } L^1(0, T; L^2(D)) \text{ weak} \tag{2.44}$$

then by (2.39) we get (2.6). ■

LEMMA 1. – Let $\{v_n\}_{n\in N}$ be such that $\|v_n\|_{L^1(0,T;L^2(D))} \leqslant \text{const}$, $v_n(x, \cdot) \to v(x, \cdot)$ in $L^1(0, T)$ weak a.e. in D. Then $v_n \to v$ in $L^1(0, T; L^2(D))$ weak.

PROOF. – Fix $\varphi \in L^\infty(0, T)$, set $U_n(x) = \int_0^T v_n(x, t)\varphi(t)\, dt$, $U(x) = \int_0^T v(x, t)\varphi(t)\, dt$; thus we have $\|U_n\|_{L^2(D)} \leqslant \text{const}$, $U_n \to U$ a.e. in D; this yields (cf. [7] pag. 13) $U_n \to U$ in $L^2(D)$ weak; i.e.

$$\begin{aligned}\forall \psi \in L^2(D), \quad \iint_Q v_n(x, t)\varphi(t)\psi(x)\, dx\, dt = \int_D U_n(x)\varphi(x)\, dx &\to \\ \to \int_D U(x)\psi(x)\, dx = \iint_Q v(x, t)\varphi(t)\psi(x)\, dx\, dt&,\end{aligned}$$

whence the thesis, as $\{\varphi(t)\psi(x) | \varphi \in L^\infty(0, T),\ \psi \in L^2(D)\}$ is dense in $L^\infty(0, T; L^2(D)) = \left(L^1(0, T; L^2(D))\right)'$. ■

LEMMA 2. – Let X be a reflexive Banach space; let $r > 0$, $1 \leqslant p < \infty$. Then there exists a constant C such that for any $f \in W^{r,p}(0, T; X)$, denoting by f_m the piecewise linear function obtained interpolating f in $t = 0, k, \ldots, mk = T$,

$$\|f - f_m\|_{L^p(0,T;X)} \leqslant Ck^r \|f\|_{W^{r,p}(0,T;X)} \,. \tag{2.45}$$

For the proof, cf. [1], ch. 3, e.g..

REMARKS. - i) We emphasize the importance of using a time-discretization approximation in order to prove the existence result.

Assumptions are formulated using piecewise linear continuous functions in view of this approach. But, whereas the generalization of (2.11), (2.12), (2.14) and (2.15) to continuous functions v would not be limitative, the extension of (2.13) presents some difficulties.

The monotonicity condition

$$\begin{cases} \forall v_1, v_2 \in C^0([0,T]) \text{ with } v_1(0) = v_2(0),\ \forall \xi \in S(v_1(0), 0)\,, \\ \int_0^T [\mathcal{F}(v_1, t, \xi) - \mathcal{F}(v_2, t, \xi)] \cdot [v_1(t) - v_2(t)]\, dt \geqslant 0 \end{cases} \tag{2.46}$$

and even the order preservation property.

$$\begin{cases} \forall v_1, v_2 \in C^0([0,T]) \text{ with } v_1(0) = v_2(0) \text{ and } v_1 \geqslant v_2 \text{ in } [0,T]\,, \\ \forall \xi \in S(v_1(0), 0),\ \forall t \in [0,T],\ \mathcal{F}(v_1, t, \xi) \geqslant \mathcal{F}(v_2, t, \xi) \end{cases} \tag{2.47}$$

are not compatible with some important hysteresis phenomena like as ferromagnetism.

What in general seems right to require is a property of the form

$$\begin{cases} \forall t', t'' \in [0,T] \text{ with } t' < t'',\ \forall v_1, v_2 \in C^0([0,T]) \text{ such that } v_1 = v_2 \text{ in } [0,t']. \\ v_1 \geqslant v_2 \text{ and both are monotone in } [t', t''],\ \forall \xi \in S(v_1(0), 0), \\ \quad \mathcal{F}(v_1, t'', \xi) \geqslant \mathcal{F}(v_2, t'', \xi) \end{cases} \tag{2.48}$$

(or also $\mathcal{F}(v_1, t'', \xi) - \mathcal{F}(v_2, t'', \xi) \geqslant \alpha[v_1(t'') - v_2(t'')]$, with α positive constant); indeed (2.13) is the reduction of this property to v_1 and v_2 piecewise linear (what is sufficient for the proof of the existence result when a time-discretization approximation is used).

ii) Uniqueness of solution of (P1) is an open question. By the above considerations it does not seem immediate.

iii) Numerical resolution of problems $(\mathrm{P1})_m$'s can be performed by standard methods.

vi) In the above developments no assumtpion has been used concerning the direction of rotation of hysteresis loops; therefore these apply to both cases in which w lags or anticipates w.r.t. u.

Now we take into account a free boundary parabolic problem. Let $G\colon \boldsymbol{R} \to \mathcal{P}(\boldsymbol{R})$ be a maximal monotone graph with bounded range:

$$G(\boldsymbol{R}) = \bigcup_{\xi \in \boldsymbol{R}} G(\xi)\colon \text{ bounded}\,. \tag{2.49}$$

Let

$$s^0\colon D \to \boldsymbol{R} \quad \text{measurable}\,, \qquad s^0(x) \in G(\boldsymbol{R}) \quad \text{a.e. in } D; \tag{2.50}$$

assume that (1.2), ..., (1.6), (2.1), ..., (2.3) hold and set

$$p_0 = w^0 + s^0 \in V'\,. \tag{2.51}$$

(P2) Find $u \in L^1(0, T; V)$, $s \in L^\infty(Q)$ such that (2.4), (2.5) hold and, setting (2.6), $w(\cdot, t)$ is measurable in D a.e. for $t \in \,]0, T[$, $w + s \in W^{1,1}(0, T; V')$,

$$s \in G(w) \quad \text{a.e. in } Q \tag{2.52}$$

$$\frac{\partial}{\partial t}(w + s) + Au = f \quad \text{in } V'\,, \quad \text{a.e. in }]0, T[\tag{2.53}$$

$$(w + s)|_{t=0} = p^0 \text{ in } V'\,. \tag{2.54}$$

REMARK. – If G is a proper graph, then (P2) is the weak formulation of a free boundary problem.

For example, let G be the signum graph, $V = H^1_0(D)$, $A = -\Delta$ and set

$$\mathcal{L} = \{(x, t) \in Q \,|\, w(x, t) = 0\}; \tag{2.55}$$

assume that $\mathcal{L}$ has no interior points and that it is smooth enough; then formally (2.53) corresponds to

$$\frac{\partial w}{\partial t} - \Delta u = f \quad \text{in } Q \setminus \mathcal{L} \tag{2.56}$$

$$2\nu_t + [\![\nabla u]\!] \cdot \nu_x = 0 \quad \text{on } \mathcal{L} \tag{2.57}$$

(where $(\nu_x, \nu_t) = (\nu_{x_1}, \ldots, \nu_{x_N}, \nu_t)$ is normal to $\mathcal{L}$ and directed in the sense of increasing w, $[\![\cdot]\!]$ denotes the jump across $\mathcal{L}$).

THEOREM 2. – Assume that (2.49), (2.50) and the hypotheses of theorem 1 hold and that

$$\begin{cases} \text{if } \{v_m(x, t)\}_{m \in N} \text{ is bounded in } \tilde{C}^0([0, T]; V), \\ \text{then } \{\mathcal{F}(v_m(x, \cdot), t, w^0(x))\}_{m \in N} \text{ is bounded in } L^1(0, T, V). \end{cases} \tag{2.58}$$

Then (P2) has at least one solution, such that moreover (2.18) holds.

PROOF. - Let $m \in \boldsymbol{N}$, $k = T/m$.

$\underline{(\mathrm{P2})_m}$ Find $u_m^n \in V$, $s_m^n \in L^\infty(D)$ for $n = 1, \dots, m$, such that —setting (2.19) and (2.20)—w_m^n is measurable in D $\forall n$,

$$s_m^n \in G(w_m^n) \quad \text{a.e. in } D,\ \forall n \tag{2.59}$$

and, setting $p_m^n = w_m^n + s_m^n$ for $n = 1, \dots, m$, $p_m^0 = p^0$ a.e. in D, then $p_m^n \in V'$ and

$$\frac{p_m^n - p_m^{n-1}}{k} + Au_m^n = f_m^n \quad \text{in } V', \quad \text{for } n = 1, \dots, m, \tag{2.60}$$

with the position (2.22).

$\forall m \in \boldsymbol{N}$, existence and uniqueness of solution of $(\mathrm{P2})_m$ and a priori estimates (2.33), (2.34) can be shown as for $(\mathrm{P1})_m$.

Let $s_m(x, t)$ be the function obtained interpolating linearly the values $s_m(x, nk) = s_m^n(x)$ for $n = 0, \dots, m$ a.e. in D; set $p_m = w_m + s_m$ and use notations (2.31). (2.60) becomes

$$\frac{\partial p_m}{\partial t} + A\hat{u}_m = \hat{f}_m \quad \text{in } V', \quad \text{a.e. in }]0, T[\tag{2.61}$$

and then by (2.34)

$$\|p_m\|_{H^1(0,T;V')} \leqslant \text{const}; \tag{2.62}$$

this last and (2.59) yield

$$\|w_m\|_{L^\infty(0,T;V')} \leqslant \text{const}. \tag{2.63}$$

Thus we get the existence of u, w, s such that possibly taking subsequences (2.37), (2.38) hold and setting $p = w + s$

$$w_m \to w \quad \text{in } L^\infty(0, T; V') \text{ weak star} \tag{2.64}$$

$$s_m \to s \quad \text{in } L^\infty(Q) \text{ weak star} \tag{2.65}$$

$$p_m \to p \quad \text{in } H^1(0, T; V') \text{ weak}. \tag{2.66}$$

Taking $m \to \infty$ in (2.61) we get (2.53).

In order to prove (2.6), notice that (2.33) and (2.58) yield

$$\|\mathcal{F}(u_m(x, \cdot), t, w^0(x)\|_{L^1(0,T;V)} \leqslant \text{const} \tag{2.67}$$

therefore, using also (2.36) and (2.42),

$$\mathcal{F}(u_m(x,\cdot),t,w^0(x)) \to \mathcal{F}(u(x,\cdot),t,w^0(x) \quad \text{in } L^1(0,T;L^2(D)) \text{ strong} \tag{2.68}$$

then, by (2.43),

$$w_m \to \mathcal{F}(u(x,\cdot),t,w^0(x)) \quad \text{in } L^1(0,T;L^2(D)) \text{ strong} \tag{2.69}$$

and this yields (2.6). Moreover, as s is the subdifferential of a lower semi-continuous convex functional R, we have

$$\iint_Q [R(w_m) - R(v)]\,dx\,dt \leqslant \iint_Q s_m(w_m - v)\,dx\,dt\,, \qquad \forall v \in L^1(Q)\,; \tag{2.70}$$

taking $m \to \infty$, by (2.65) and (2.69) we get

$$\iint_Q [R(w) - R(v)]\,dx\,dt \leqslant \iint_Q s(w - v)\,dx\,dt\,, \qquad \forall v \in L^1(Q) \tag{2.71}$$

i.e. (2.52). ■

Also for (P2) uniqueness of solution is an open question.

REMARK. – Let (2.0) be coupled with an equation of the form

$$\frac{\partial u}{\partial t} + Au + w = f \quad \text{in } \mathcal{D}'(Q)\,, \tag{2.72}$$

in which the «hysteresis functional» is not in the principal part. After derivation w.r.t. t this reduces to

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + w\right) + A\,\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t} \quad \text{in } \mathcal{D}'(Q)\,; \tag{2.73}$$

for this the procedure of the next section can be used.

3. – Hyperbolic problems.

We are going to study the case in which the relation (2.0) between $u, w: Q \to \boldsymbol{R}$ is coupled with a partial differential equation of the form

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + w\right) + Au = f \quad \text{in } \mathcal{D}'(Q)$$

or

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial t} + w + s\right) + Au = f & \text{in } \mathcal{D}'(Q) \\ s \in \operatorname{sgn}(w) & \text{in } Q \text{ (sgn = signum graph)} \end{cases}$$

(where A is a linear elliptic operator, f is a datum); suitable initial and boundary conditions will be given.

Assume that (1.2), ..., (1.6), (2.1), ..., (2.3) hold and that

$$\omega^0 \in V' . \tag{3.1}$$

(P3) Find $u \in L^1(0, T; V)$ such that

$$u(x, \cdot) \in C^0([0, T]) \quad \text{a.e. in } D \tag{3.2}$$

$$u(x, 0) = u^0(x) \quad \text{a.e. in } D \tag{3.3}$$

and, setting

$$w(x, t) = \mathcal{F}\big(u(x, \cdot), t, w^0(x)\big) \quad \forall t \in [0, T], \text{ a.e. in } D, \tag{3.4}$$

$$w(\cdot, t) \text{ is measurable in } D\ \forall t \in [0, T], \quad \frac{\partial u}{\partial t} + w \in W^{1,1}(0, T; V') \tag{3.5}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + w\right) + Au = f \quad \text{in } V', \quad \text{a.e. in }]0, T[\tag{3.6}$$

$$\left[\frac{\partial u}{\partial t} + w\right]_{t=0} = \omega^0 \quad \text{in } V' . \tag{3.7}$$

REMARK. – By (1.5), (3.4) contains the initial condition $w(x, 0) = w^0(x)$ a.e. in D.

THEOREM 3. – Under the same assumptions as in Theorem 1, if moreover $z^0 = \omega^0 - w^0 \in L^2(D)$, then (P3) has at least one solution such that moreover

$$\begin{cases} u \in W^{1,\infty}(0, T; L^2(D)) \cap L^\infty(0, T; V) , \\ w \in W^{\lambda,1}(0, T; L^2(D)) . \end{cases} \tag{3.8}$$

REMARK. – (3.8) yields $u \in C^0_s([0, T]; V)$ (cf. [8], ch. 3, lemma 8.1).

PROOF. – 1) *Approximation.* As an approached problem $(\mathrm{P3})_m$, consider $(\mathrm{P1})_m$

with (2.21) replaced by the following equation

$$\frac{z_m^{n'} - z_m^{n-1}}{k} + \frac{w_m^n - w_m^{n-1}}{k} + Au_m^n = f_m^n \quad \text{in } V', \quad \text{for } n = 1, \dots, m \tag{3.9}$$

where $z_m^0 = z^0$, $\quad z_m^{n'} = \dfrac{u_m^n - u_m^{n-1}}{k} \; (n = 1, \dots, m) \quad$ a.e. in D.

$\forall m$, existence and uniqueness of solution can be shown as for $(\mathrm{P1})_m$.

2) *Estimates*. Multiply (3.9) against $u_m^n - u_m^{n-1}$ and sum for $n = 1, \dots, l$, for a generic $l \in \{1, \dots, m\}$. Notice that

$$\sum_{n=1}^{l} \int_D \frac{z_m^n - z_m^{n-1}}{k} \cdot (u_m^n - u_m^{n-1})\, dx = \tag{3.10}$$
$$= \sum_{n=1}^{l} \int_D (z_m^n - z_m^{n-1}) z_m^n \, dx \geqslant \tfrac{1}{2} \sum_{n=1}^{l} \left(\|z_m^n\|^2_{L^2(D)} - \|z_m^{n-1}\|^2_{L^2(D)} \right) = \tfrac{1}{2} \|z_m^l\|^2_{L^2(D)} - \tfrac{1}{2} \|z^0\|^2_{L^2(D)};$$

using also (2.25), ..., (2.28) we get

$$\max_{n=0,\dots,m} \|z_m^n\|_{L^2(D)} \leqslant \text{const}, \tag{3.11}$$

$$\max_{n=0,\dots,m} \|u_m^n\|_V \leqslant \text{const}. \tag{3.12}$$

Let $w_m, u_m, \hat{u}_m, \hat{f}_m$ be defined as in (2.19), (2.31). Let z_m be the function obtained interpolating linearly the values $z_m(x, nk) = z_m^n(x)$ for $n = 0, \dots, m$, a.e. in D; let $\hat{z}^m(x, t) = z_m^n(x)$ a.e. in D if $(n-1)k < t \leqslant nk$, for $n = 1, \dots, m$.

(3.9) becomes

$$\frac{\partial}{\partial t}(z_m + w_m) + A\hat{u}_m = \hat{f}_m \quad \text{in } V', \quad \text{a.e. in }]0, T[. \tag{3.13}$$

(3.11) and (3.12) yield

$$\|z_m\|_{L^\infty(0,T;\,L^2(D))} \leqslant \text{const} \tag{3.14}$$

$$\|\hat{z}_m\|_{L^\infty(0,T;\,L^2(D))} \leqslant \text{const} \tag{3.15}$$

$$\|u_m\|_{W^{1,\infty}(0,T;\,L^2(D)) \cap L^\infty(0,T;\,V)} \leqslant \text{const} \tag{3.16}$$

$$\|\hat{u}_m\|_{L^\infty(0,T;\,V)} \leqslant \text{const} \tag{3.17}$$

and by (3.13), (3.17)

$$\|z_m + w_m\|_{H^1(0,T;\,V')} \leqslant \text{const} \tag{3.18}$$

whence by (3.14)

$$\|w_m\|_{L^\infty(0,T;V')} \leqslant \text{const}\,. \tag{3.19}$$

3) *Limit.* By the above estimates, there exist u, z, w such that possibly taking subsequences

$$z_m \to z \quad \text{in } L^\infty(0, T; L^2(D)) \text{ weak star} \tag{3.20}$$

$$z_m \to z \quad \text{in } L^\infty(0, T; L^2(D)) \text{ weak star} \tag{3.21}$$

$$u_m \to u \quad \text{in } W^{1,\infty}(0, T; L^2(D)) \cap L^\infty(0, T; V) \text{ weak star} \tag{3.22}$$

$$\hat{u}_m \to u \quad \text{in } L^\infty(0, T; V) \text{ weak star} \tag{3.23}$$

$$w_m \to w \quad \text{in } L^\infty(0, T; V') \text{ weak star} \tag{3.24}$$

as $\forall m \in N$ $\hat{z}_m = \partial u_m/\partial t$ a.e. in Q, we get $z = \partial u/\partial t$ a.e. in Q, thus recalling (3.18)

$$z_m + w_m \to \frac{\partial u}{\partial t} + w \quad \text{in } H^1(0, T; V') \text{ weak} \tag{3.25}$$

and taking $m \to \infty$ in (3.13) we get (3.6), as well as (3.7).
(3.4) can be deduced as in the proof of theorem 1. ∎

Also for (P3) uniqueness of solution is an open question.
Assume that (1.2), ..., (1.6), (2.1), ..., (2.3), (3.1) hold.

(P4) Find $u \in L^1(0, T; V)$, $s \in L^\infty(Q)$ such that, setting (3.4), then (3.2), (3.3), (3.5) hold and

$$s \in \operatorname{sgn}(w) \quad \text{a.e. in } Q \tag{3.24}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + w + s\right) + Au = f \quad \text{in } V', \quad \text{a.e. in }]0, T[\tag{3.25}$$

$$\left(\frac{\partial u}{\partial t} + w + s\right)\Big|_{t=0} = \omega^0 \quad \text{in } V'\,. \tag{3.26}$$

REMARK. – Analogously to (P2), also (P4) is the weak formulation of a free boundary problem, with (2.56) and (2.57) replaced by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial w}{\partial t} - \Delta u = f \qquad \text{in } Q \setminus \mathcal{L}\,, \tag{3.27}$$

$$\left(\left[\!\left[\frac{\partial u}{\partial t}\right]\!\right] + 2\right)\nu_t + [\![\nabla u]\!] \cdot \vec{\nu}_x = 0 \qquad \text{on } \mathcal{L}\,. \tag{3.28}$$

THEOREM 4. – Under the assumptions of theorems 2 and 3, (P4) has at least one solution such that moreover (3.8) holds.

Proof follows the procedures used for theorems 2 and 3.

Also for (P4) uniqueness of solution is an open question.

Generalizations. – i) Theorems 1, ..., 4 can be easily extended to the case in which $u(x,t)$, $w(x,t) \in \boldsymbol{R}^M$ $(M > 1)$. In particular existence results can be proved if in (2.13) strict monotonicity is replaced by strict cyclic monotonicity.

ii) Similar results can be obtained for other boundary conditions, for instance of type of Neumann, of Dirichlet non-homogeneous, or of mixed type.

iii) If D is occupied by a non-homogeneous system, we may consider a couple $(S_x, \mathcal{F}_x)$ depending on x, a.e. for $x \in D$. More precisely let $(\tilde{S}, \tilde{\mathcal{F}})$ be given such that

$$\tilde{S}\colon \boldsymbol{R} \times Q \to \mathcal{P}(\boldsymbol{R}) \tag{3.29}$$

$$\begin{cases} \operatorname{Dom}(\tilde{\mathcal{F}}) = \{(v, x, t, \xi) | v \in C^0([0, T]),\ x \in D,\ t \in [0, T],\ \xi \in \tilde{S}(v(0), x, t\}; \\ \quad \forall (v, x, t, \xi) \in \operatorname{Dom}(\tilde{\mathcal{F}}),\ \tilde{\mathcal{F}}(v, x, t, \xi) \in \tilde{S}(v(t), x, t)\,. \end{cases} \tag{3.30}$$

Set, a.e. for $x \in D$,

$$\begin{cases} S_x(y, t) = \tilde{S}(y, x, t)\,, \qquad \forall (y, t) \in \boldsymbol{R} \times [0, T]\,, \\ \quad \mathcal{F}_x(v, t, \xi) = \tilde{\mathcal{F}}(v, x, t, \xi),\ \forall (v, t, \xi) \text{ such that } v \in C^0([0, T])\,, \end{cases} \tag{3.31}$$

We assume $(\tilde{S}, \tilde{\mathcal{F}})$ to be such that the corresponding $(S_x, \mathcal{F}_x)$'s fulfill properties (1.2), ..., (1.6), a.e. for $x \in D$ and that $\mathcal{F}$ is globally measurable.

Extension of problems (P1), ..., (P4) is immediate. Existence theorems 1, ..., 4 hold also in this case if properties (2.10), ..., (2.13) are assumed to hold uniformly w.r.t. x (possibly with the exception of a set with vanishing measure).

4. – A scheme of construction of hysteresis functionals.

Let (S, G) be such that

$$S\colon \boldsymbol{R} \to \mathcal{P}(\boldsymbol{R}) \setminus \{\emptyset\}; \qquad \forall y \in \boldsymbol{R},\ S(y) \text{ is closed} \tag{4.1}$$

(for the sake of simplicity, we assume S to be time-independent)

$$\begin{cases} \operatorname{Dom}(G) = \{(\zeta, \xi, \varrho) | \zeta \in \boldsymbol{R},\ \xi \in S(\zeta),\ \varrho \in \boldsymbol{R}\}; \\ \forall (\zeta, \xi, \varrho) \in \operatorname{Dom}(G)\,, \quad G(\zeta, \xi, \varrho) \in S(\varrho) \end{cases} \tag{4.2}$$

$$\forall\zeta\in\boldsymbol{R},\ \forall\xi\in S(\zeta),\ \varrho\mapsto G(\zeta,\xi,\varrho)\text{ is continuous in }\boldsymbol{R} \tag{4.3}$$

$$\forall\zeta\in\boldsymbol{R},\ \forall\xi\in S(\zeta),\ G(\zeta,\xi,\zeta)=\xi \tag{4.4}$$

$$\begin{cases}\forall\zeta\in\boldsymbol{R},\ \forall\xi\in S(\zeta),\ \forall\varrho,\bar\varrho\in\boldsymbol{R}\text{ with }\zeta\leqslant\varrho<\bar\varrho\text{ or }\bar\varrho<\varrho\leqslant\zeta\,,\\ G(\zeta,\xi,\bar\varrho)=G\big(\varrho,G(\zeta,\xi,\varrho),\bar\varrho\big)\ \text{(transition property for }G)\,.\end{cases} \tag{4.5}$$

Examples will be given in the next section.

At first we define $\mathcal{F}$ recursively for $v\in\tilde{C}^0([0,T])$ (defined in (2.9))

$$\left|\begin{array}{l}\forall v\in\tilde{C}^0([0,T]),\ \forall\xi\in S\big(v(0)\big),\ \mathcal{F}(v,0,\xi)=\xi;\\ \forall\{t_k\}_{k=0,\ldots,m}\text{ as in (2.9)},\ \forall k\in\{0,\ldots,m-1\},\ \forall t\in\,]t_k,t_{k+1}],\\ \quad\mathcal{F}(v,t,\xi)=G\big(v(t_k),\mathcal{F}(v,t_k,\xi),v(t)\big);\end{array}\right. \tag{4.6}$$

by (4.5), this definition is independent of the subdivision $\{t_k\}_{k=0,\ldots,m}$.

We require G to be such that $\mathcal{F}$ defined in (4.6) fulfills the following crucial property

$$\begin{cases}\forall\zeta\in\boldsymbol{R},\ \forall\xi\in S(\zeta),\ \{v\in\tilde{C}^0([0,T])\,|\,v(0)=\zeta\}\to C^0\big([0,T]\big):\\ \quad v\mapsto\mathcal{F}(v,\ldots,\xi)\text{ is uniformly continuous w.r.t. the maximum norm;}\end{cases} \tag{4.7}$$

this implies that if $\{v_n\in\tilde{C}^0([0,T])\}_{n\in N}$ is a Cauchy sequence w.r.t. the maximum norm and if $v_n(0)=v_1(0)\ \forall n\in\boldsymbol{N}$, then $\forall\xi\in S\big(v_1(0)\big)$ also $\mathcal{F}(v_n,\cdot,\xi)$ is a Cauchy sequence w.r.t. the same norm. This allows to extend $\mathcal{F}$ univocally as follows

$$\left\{\begin{array}{l}\forall v\in C^0\big([0,T]\big)\,,\quad\text{let }v_n\in\tilde{C}^0([0,T])\text{ be such that }v_n(0)=v(0)\ \ \forall n\in\boldsymbol{N},\\ v_n\to v\text{ in }C^0([0,T])\text{ strong; }\forall t\in[0,T],\ \forall\xi\in S\big(v(0)\big),\ \mathcal{F}(v,t,\xi)=\\ =\lim\limits_{n\to\infty}\mathcal{F}(v_n,t,\xi);\end{array}\right. \tag{4.8}$$

(1.3), ..., (1.6) are fulfilled, as well as (1.11), ..., (1.13) $\big($with $(S^\tau,\mathcal{F}^\tau)=(S,\mathcal{F})$, $\forall\tau\in\,]0,T[\big)$ and (2.12), (2.14).

If (2.10) holds, then

$$\forall v\in C^0([0,T]),\ \forall\xi\in S\big(v(0)\big),\ \|\mathcal{F}(v,\cdot,\xi)\|_{L^2(0,T)}\leqslant c_1\|v\|_{L^2(0,T)}+c_2; \tag{4.9}$$

if

$$\forall\zeta\in\boldsymbol{R},\ \forall\xi\in S(\zeta),\ \varrho\mapsto G(\zeta,\xi,\varrho)\text{ is Lipschitz-continuous, uniformly w.r.t. }\zeta,\xi \tag{4.10}$$

then, denoting by L the Lipschitz constant,

$$(4.11)\qquad \forall v\in \tilde{C}^0([0,T]),\ \forall\xi\in S(v(0)),\ \left\|\frac{\partial}{\partial t}\mathcal{F}(v,t,\xi)\right\|_{L^2(0,T)}\leqslant L\left\|\frac{dv}{dt}\right\|_{L^2(0,T)};$$

therefore, if both (4.9) and (4.11) hold, then

$$(4.12)\qquad \forall v\in \tilde{C}^0([0,T]),\ \forall\xi\in S(v(0)),\ \|\mathcal{F}(v,t,\xi)\|_{H^1(0,T)}\leqslant \max(c_1,L)\cdot\|v\|_{H^1(0,T)}+c_2$$

which is stronger than (2.11). If

$$(4.13)\qquad \begin{cases}\exists\alpha \text{ constant}>0\colon \forall\zeta\in\boldsymbol{R},\ \forall\xi\in S(\zeta),\ \forall\varrho_1,\varrho_2\in\boldsymbol{R}\\ [G(\zeta,\xi,\varrho_1)-G(\zeta,\xi,\varrho_2)]\cdot(\varrho_1-\varrho_2)\geqslant\alpha(\varrho_1-\varrho_2)^2\end{cases}$$

then (2.13) holds. If

$$(4.14)\qquad \forall(\zeta,\varrho)\in\boldsymbol{R}^2,\ \text{the function } \boldsymbol{R}\to\boldsymbol{R}\colon \xi\mapsto G(\zeta,\xi,\varrho) \text{ is continuous},$$

then (2.15) holds.

A class of functions G fulfilling the above properties can be defined implicitly by a family of Cauchy problems for an ordinary differential equation as follows.

Let S be as in (4.1); set $\mathcal{S}=\{(\zeta,\xi)\in\boldsymbol{R}^2|\xi\in S(\zeta)\}$.

Let $g_l, g_r\colon \mathcal{S}\to\boldsymbol{R}$ (not necessarily continuous) be such that $\forall(\zeta,\xi)\in\mathcal{S}$ the following problem has one and only one solution $\varrho\mapsto\omega(\varrho)=G(\zeta,\xi,\varrho)$ piecewise of class C^1:

$$(4.15)\qquad \begin{cases}\dfrac{d\omega}{d\varrho}(\varrho)|_{(\text{left})}\left(\equiv\lim\limits_{h\to0^+}\dfrac{\omega(\varrho)-\omega(\varrho-h)}{h}\right)=g_l(\varrho,\omega(\varrho))\,, & \forall\varrho\leqslant\zeta\\ \dfrac{d\omega}{d\varrho}(\varrho)|_{(\text{right})}\left(\equiv\lim\limits_{h\to0^+}\dfrac{\omega(\varrho+h)-\omega(\varrho)}{h}\right)=g_r(\varrho,\omega(\varrho))\,, & \forall\varrho\geqslant\zeta\\ \omega(\zeta)=\xi\,. \end{cases}$$

In this case (4.2), ..., (4.5) hold. Set (4.6); (4.7) must be checked case by case.

For a moment assume that it holds, so that it is possible to extend $\mathcal{F}$ as in (4.8). Fix $v\in\tilde{C}^0([0,T])$, $\xi\in S(v(0))$ and set $f(t)=\mathcal{F}(v,t,\xi)$, $\forall t\in[0,T]$; we have

$$(4.16)\qquad \begin{cases}\dfrac{df}{dt}(t)=g_r(v(t),f(t))\left[\dfrac{dv}{dt}(t)\right]^+-g_l(v(t),f(t))\left[\dfrac{dv}{dt}(t)\right]^- & \text{a.e. in }]0,T[\,;\\ f(0)=\xi\end{cases}$$

(formally: $df(t)=g_r(t,f(t))[dv(t)]^+-g_l(t,f(t))[dv(t)]^-$, for t increasing, i.e. $dt>0$).

This is the differential form of the « hysteresis relation »; notice that it requires more regularity for v than the integral form $f(t) = \mathcal{F}(v, t, \xi)$.

It is possible to modify each one of the above variational problems—say (P1)—replacing (2.6) by

$$\left\{\begin{array}{ll} w \in H^1(0, T; L^2(D)) & \\ \dfrac{\partial w}{\partial t} = g_r(u, w)\cdot\left(\dfrac{\partial u}{\partial t}\right)^+ - g_l(u, w)\cdot\left(\dfrac{\partial u}{\partial t}\right)^- & \text{a.e. in } Q \\ w(x, 0) = w^0(x) & \text{a.e. in } D \end{array}\right. \tag{4.17}$$

which a priori is unrelated with the corresponding (2.6).

This problem can be approximated by the $(\mathrm{P1})_m$'s of § 2, as the differential and integral forms of the « hysteresis relation » are equivalent for piecewise linear continuous functions; therefore the same a priori estimates hold. But difficulties arise in the limit procedure (in particular for proving that $(\partial u_m/\partial t)^+ \to (\partial u/\partial t)^+$ in $L^2(Q)$ weak).

Analogous considerations hold for (P2), ..., (P4).

If $|g_l|$, $|g_r| \leqslant$ constant, then (4.10) is fulfilled; thus if also (2.10) holds, then (4.12) is satisfied. If $g_l, g_r \geqslant \alpha$: constant > 0, then (4.13) is fulfilled. If g_l, g_r are continuous, then (4.14) holds.

In conclusion, sufficient conditions can be given so that the assumptions of theorems 1 and 2 are satisfied for $\mathcal{F}$ constructed according to the procedures ketched in this section, i.e. $(g_l, g_r) \mapsto G \mapsto \mathcal{F}$.

In principle this construction can be extended to the vector case. But it does not seem easy to define G by a family of Cauchy problems as in (4.15).

5. – Examples of hysteresis functionals.

We are going to specify some possible choices of g_l, g_r: $S \to \boldsymbol{R}$; if these are suitable, G and $\mathcal{F}$ will be derived using the procedure of § 4.

EXAMPLE 1. – Let $0 < \alpha < \beta$, $q_1 < q_2$ (all constants). Set

$$S(\varrho) = [\beta(\varrho - q_2), \beta(\varrho - q_1)]\,, \qquad \forall \varrho \in \boldsymbol{R}$$

$$\left\{\begin{array}{lll} g_r(\varrho, \omega) = g_l(\varrho, \omega) = \alpha & \text{if} & \beta(\varrho - q_2) < \omega < \beta(\varrho - q_1) \\ g_l(\varrho, \omega) = \alpha,\ g_r(\varrho, \omega) = \beta & \text{if} & \omega = \beta(\varrho - q_2) \\ g_l(\varrho, \omega) = \beta,\ g_r(\varrho, \omega) = \alpha & \text{if} & \omega = \beta(\varrho - q_1)\,. \end{array}\right. \tag{5.1}$$

In this situation, $\forall \zeta \in \boldsymbol{R}$, $\forall \xi \in S(\zeta)$, (4.15) has one and only one solution; moreover the construction of § 4 $(g_l, g_r) \mapsto G \mapsto \mathcal{F}$ can be carried out: as far as (4.7) is concerned, $\mathcal{F}$ is even Lipschitz-continuous with Lipschitz constant β.

This procedure yields a functional $\mathcal{F}$ fulfilling (1.2), ..., (1.6), (2.10), ..., (2.14) as well as (2.47), (2.72).

Fig. 1 represents the graph of the function $\varrho \mapsto G(\zeta, \xi, \varrho)$ for any (ζ, ξ) of the closed segment $[A, B]$.

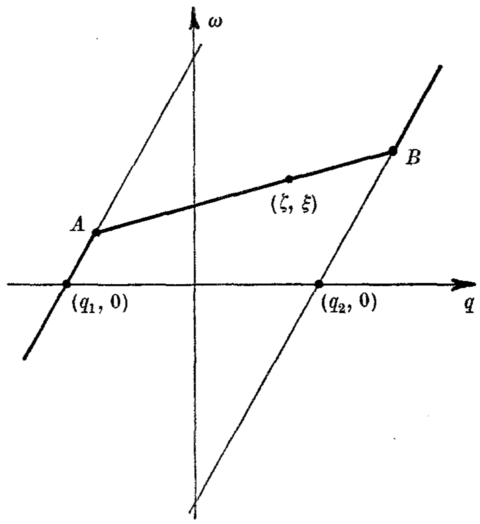

Figure 1

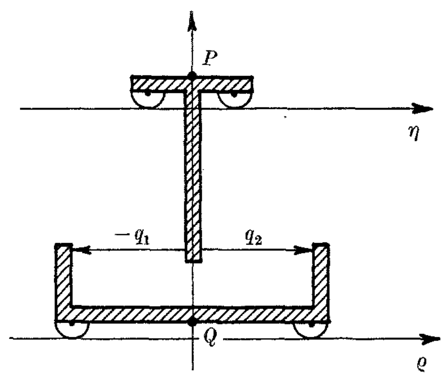

Figure 2

In the device drawn in fig. 2, the truck above carries along the one below, due to the shaft coming down and to the two ends of the lower car; the lack of bilateral contact causes the delay. Inertia is neglected. ϱ(η respect.) is the coordinate of P (Q respect.) and here $\alpha = 1$, $\beta = 2$; $\varrho(t)$ and $\omega(t) = \varrho(t) + \eta(t)$ are related by the functional $\mathcal{F}$ corresponding to g_l and g_r as in (5.1).

EXAMPLE 2. – See fig. 3

$$
(5.2)\quad \left\{\begin{array}{ll}
\text{in } \mathcal{T}\colon g_l = g_r = \alpha & \text{on } a_i\colon g_l = g_r = \delta_i \quad (i = 1, 2) \\
\text{on }]A, B[\colon g_l = \alpha,\ g_r = \beta & \text{on }]B, D[\colon g_l = \alpha,\ g_r = \gamma \\
\text{on }]D, E[\colon g_l = \beta,\ g_r = \alpha & \text{on }]E, A[\colon g_l = \gamma,\ g_r = \alpha \\
g_l(A) = \delta_1,\ g_r(A) = \beta & g_l(B) = \alpha,\ g_r(B) = \gamma \\
g_l(D) = \beta,\ \ g_r(D) = \delta_2 & g_l(E) = \gamma,\ g_r(E) = \alpha\,.
\end{array}\right.
$$

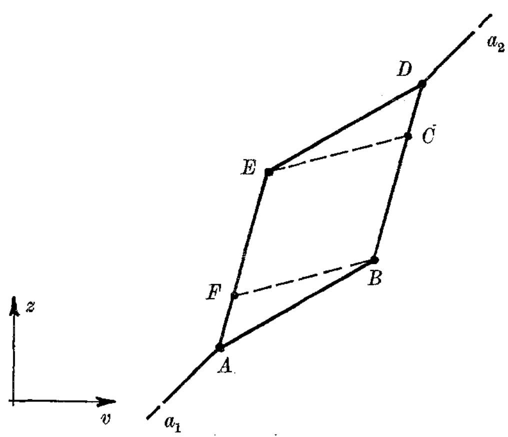

Figure 3

δ_i: slope of a_i $(i = 1, 2)$ α: slope of $[F, B]$, $[E, C]$
β: slope of $[A, B]$, $[E, D]$ γ: slope of $[A, E]$, $[B, D]$
with $0 < \alpha \leqslant \beta < \gamma < \infty$; $0 < \delta_i < \infty$ $(i = 1, 2)$
$\mathcal{T}$ = open parallelogram $ABDE$; $\mathcal{S} = \bar{\mathcal{T}} \cup a_1 \cup a_2$

Also in this case the construction of § 4 $(g_l, g_r) \mapsto G \mapsto \mathcal{F}$ can be carried out; the properties of $\mathcal{F}$ are similar to those of the preceding example.

A different model for the situation corresponding to $\alpha = \beta$, $\gamma = +\infty$ is introduced in [10].

EXAMPLE 3. – See fig. 4

$$
(5.3)\quad \left\{\begin{array}{ll}
\text{in } \mathcal{T}_1 \cup]A, D[\colon g_l = \beta,\ g_r = \alpha & \text{in } \mathcal{T}_2 \cup]B, C[\colon g_l = \alpha,\ g_r = \beta \\
\text{on }]A, B[\cup]C, D[\colon \quad g_l = g_r = \alpha & \\
\text{on }]B, D[\colon g_l = g_r = \beta & \text{on } a_i\colon g_l = g_l = \delta_i \quad (i = 1, 2) \\
g_l(A) = \delta_1,\ g_r(A) = \alpha & g_c(B) = \alpha,\ g_r(B) = \beta \\
g_l(C) = \alpha,\ \ g_r(C) = \delta_2 & g_l(D) = \beta,\ g_r(D) = \alpha\,.
\end{array}\right.
$$

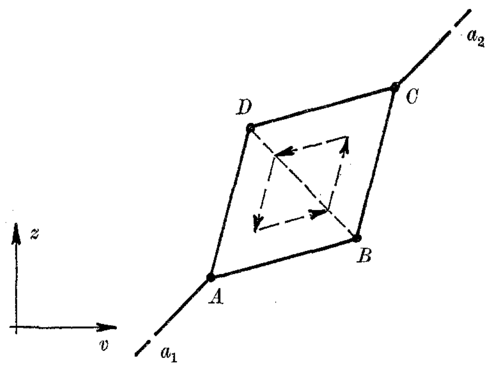

Figure 4

δ_i: slope of a_i $(i = 1, 2)$
α: slope of $[A, B]$, $[D, C]$ $\qquad$ β: slope of $[A, D]$, $[B, C]$

with $0 < \alpha < \beta < \infty$, $0 < \delta_i < \infty$ $(i = 1, 2)$
$\mathcal{T}_1$ (resp. $\mathcal{T}_2$) = open triangle ABD (resp. BCD)
$\mathcal{T} \equiv \mathcal{T}_1 \cup \mathcal{T}_2 \cup \,]B, D[$ $\qquad$ $\mathcal{S} = \overline{\mathcal{T}} \cup a_1 \cup a_2$

Also in this case (4.15) has a unique solution; here we cannot carry out the construction of § 4 $(g_l, g_r) \mapsto G \mapsto \mathcal{F}$, since (4.7) does not hold, as we are going to show.

Set

$$\alpha(\tau) = \begin{cases} 1 & \text{if} \quad 0 \leqslant \tau < 1 \\ -1 & \text{if} \quad 1 \leqslant \tau < 2 \end{cases}$$

and extend it periodically to the whole $\boldsymbol{R}^+$; $\forall n \in \boldsymbol{N}$, set $\varphi_n(0) = 0$, $\varphi_n'(t) = \alpha(nt)$ for $t \geqslant 0$. The graph of $\varphi_n \in \tilde{C}^0([0, T])$ is drawn in fig. 5 a.

Let $(\zeta, \xi) \in \mathcal{T}_2$ (e.g.), set $v_n(t) = \zeta + \varphi_n(t)$, for $0 \leqslant t \leqslant T$. It is easy to check that if T is large enough then $\big(v_n(T), \mathcal{F}(v_n, T, \xi)\big) \in \,]C, D[$ (see fig. 5 b); thus as $n \to \infty$, v_n tends in $C^0([0, T])$ strong to the function identically equal to ζ, but $\mathcal{F}(v_n, \cdot, \xi)$ does not tend to the function identically equal to ξ.

Therefore (4.7) does not hold in this case.

As a matter of fact, defining $\mathcal{F}(v, t, \xi)$ for $v \in \tilde{C}^0([0, T])$ as in (4.6), here we have (cf. (4.7)).

$$(5.4) \qquad \begin{cases} \forall \zeta \in \boldsymbol{R},\ \forall \xi \in S(\zeta),\ \{v \in \tilde{C}^0([0, T]) \,|\, v(0) = \zeta\} \to C^0([0, T]):\ v \mapsto \mathcal{F}(v, \cdot, \xi) \\ \quad \text{is uniformly continuous w.r.t. the variation norm} \end{cases}$$

(we remind that $\forall v \in \tilde{C}^0([0, T])$ the variation is defined as follows: if $\{t_k\}_{k=0,\dots,m}$

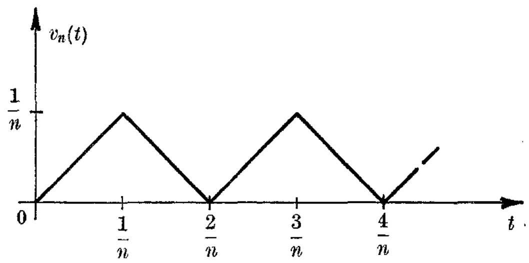

Figure 5*a*

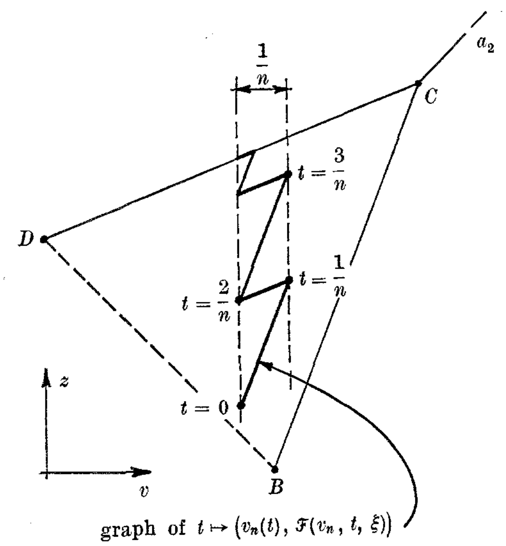

Figure 5*b*

are as in (2.9), then $\mathcal{V}(v) = \sum_{k=1}^{n} |v_k - v_{k-1}|\big)$; this allows to extend the definition of $\mathcal{F}(v, t, \xi)$ to all $v \in C^0([0, T]) \cap BV(0, T)$ $(BV(0, T)$ being the space of functions with bounded variation endowed with the norm $\mathcal{V})$. However we cannot extend the proof of the above theorems to this $\mathcal{F}$, as we do not have estimates for $u(x, \cdot)$ in a compact subspace of $L^2(D\colon BV(0, T))$.

EXAMPLE 4. – See Fig. 6.

The path $ABCDA$ represents the hysteresis cycle of a one dimensional magnetic body (v, z corresponding to the fields H, B respect.). $ABCDA$ is covered by the « state point » (v, z) in the anticlockwise versus.

Consider the system of oriented curves obtained starting from a generic point of the branch ABC (CDA, resp.) and letting v to decrease to v_A (to increase to v_C, resp.),

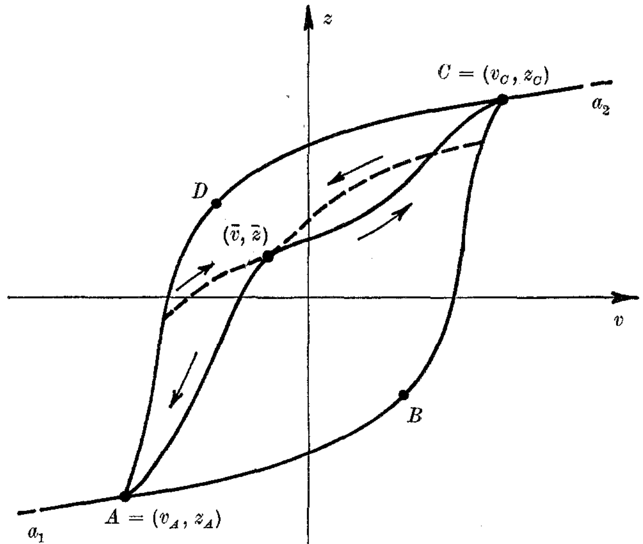

Figure 6

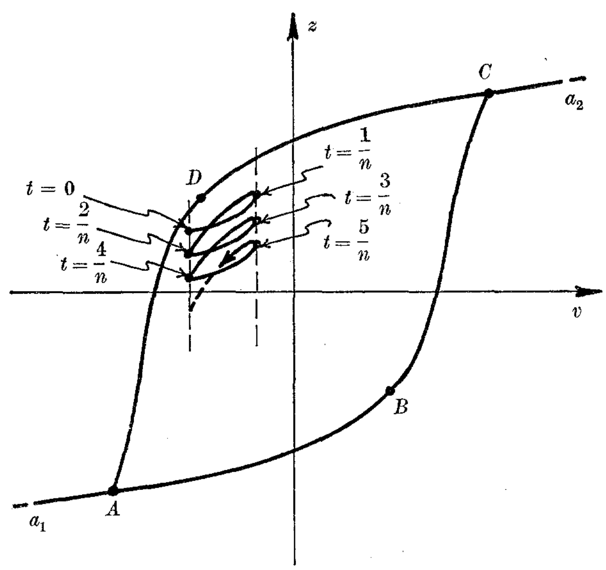

Figure 7

these curves are assumed to be of class C^1 and with a slope uniformly bounded inferiorly by $l > 0$, superiorly by $L < +\infty$.

For any point $(v, z) \in S$, there are two oriented curves passing through it; the two oriented branches getting out of $(\bar{v}, \bar{z})$ define a curve $\varrho \mapsto \omega(\varrho) = G(\bar{v}, \bar{z}, \varrho)$ (see fig. 6).

Notice that (2.47) cannot be expected to hold in general.

Let $\mathcal{F}$ be defined by (4.6); for this functional it is possible to express considerations analogous to those presented for the preceding example: however small oscillations of $v(t)$ around a value ζ can move the «state point» $(v, z)(t) = \big(v(t), \mathcal{F}(v, t, \xi)\big)\big(\xi \in S(v(0))\big)$ far from its initial value (see fig. 7).

Thus also in this case (4.7) does not hold.

In the next section we shall deal with a modification of the last two models in order to cope with the difficulties encountered here.

6. – A generalization of the model.

«Transition property» (1.13) is not consistent with the phenomenology of hysteresis for ferromagnetic materials, even in the case in which the direction of $H(t)$ and $B(t)$ does not change with time. For instance for a virgin material, hence with $H(0) = 0$, $B(0) = 0$, if H changes monotonically then the «state point» $\big(H(t), B(t)\big) \in \boldsymbol{R}^2$ moves along the so-called first magnetization curve; if after a generic path $\big(H(t), B(t)\big)$ comes back to $(0, 0)$, then in general it will no longer describe the first magnetization curve (cf. [9], pag. 548, e.g.). Moreover small oscillations of $H(t)$ cannot cause large changes of $B(t)$.

We are going to show how the difficulties encountered in the examples 3, 4 of § 5 may be overcome by the introduction of something like an «internal parameter».

Instead of considering the complicate pattern of the hysteresis cycle of ferromagnetism, we prefer to introduce this procedure in the simpler situation of example 3 of § 5.

Let g_l, g_r be as in (5.3); fix $k \in \,]\alpha, \beta[$. $\forall(\varrho, \omega) \in \mathcal{S}$, $\forall p \in [-1, 1]$, set

$$(6.1) \qquad j_l(\varrho, \omega, p) = \begin{cases} k & \text{if } (\varrho, \omega) \in \mathcal{T} \cup \,]B, C[\, \cup \,]CD[\text{ and } p \neq -1 \\ g_l(\varrho, \omega) & \text{otherwise} \end{cases}$$

$$(6.2) \qquad j_r(\varrho, \omega, p) = \begin{cases} k & \text{if } (\varrho, \omega) \in \mathcal{T} \cup \,]A, B[\, \cup \,]D, A[\text{ and } p \neq 1 \\ g_r(\varrho, \omega) & \text{otherwise}\,. \end{cases}$$

Fix a $\lambda > 0$; $\forall p \in [-1, 1]$, $\forall \mu \in \boldsymbol{R}$, set

$$(6.3) \qquad \gamma(p, \mu) = \begin{cases} \min\,(p + \lambda\mu, 1) & \text{if } \mu \geqslant 0 \\ \max\,(p + \lambda\mu, -1) & \text{if } \mu \leqslant 0 \end{cases}$$

thus $\gamma(p, \mu) \in [-1, 1]$; notice that the following transition property holds

$$(6.4) \qquad \forall p \in [-1, 1],\ \forall \mu_1, \mu_2 \in \boldsymbol{R} \text{ with } \mu_1 \cdot \mu_2 > 0,\ \gamma(p, \mu_1 + \mu_2) = \gamma(\gamma(p, \mu_1), \mu_2)\,.$$

Set

$$(6.5)\qquad \begin{cases} \forall(\varrho,\omega)\in\mathcal{S},\ \forall p\in[-1,1],\ \forall\zeta\in\mathbf{R}\,, \\ \tilde{g}_i(\varrho,\omega,p,\zeta)\equiv j_i(\varrho,\omega,\gamma(p,\varrho-\zeta)) \qquad (i=l,r) \end{cases}$$

$\forall(\zeta,\xi)\in\mathcal{S}$, $\forall p\in[-1,1]$, let $\varrho\mapsto\omega(\varrho)=\tilde{G}(\zeta,\xi,\varrho,p)$ be the solution (existing and unique) of problem (4.15) with $g_i(\varrho,\omega(\varrho))$ replaced by $\tilde{g}_i(\varrho,\omega(\varrho),p,\zeta)$ $(i=l,r)$; that is

$$(6.6)\qquad \frac{d\omega}{d\varrho}(\varrho)\Big|_{(\text{left})}=\begin{cases} k & \text{if } (\varrho,\omega(\varrho))\in\mathcal{T}\cup\,]B,C[\,\cup\,]C,D[\text{ and} \\ & \begin{cases}\varrho\leqslant\zeta\\ p+\lambda(\varrho-\zeta)>-1\end{cases} \quad \left(\text{i.e. } \zeta-\dfrac{1+p}{\lambda}<\varrho\leqslant\zeta\right) \\ g_l(\varrho,\omega(\varrho)) & \text{if } (\varrho,\omega(\varrho))\notin\mathcal{T}\cup\,]B,C[\,\cup\,]C,D[\\ & \qquad\qquad \text{or } \varrho\leqslant\zeta-\dfrac{1+p}{\lambda} \end{cases}$$

$$(6.7)\qquad \frac{d\omega}{d\varrho}(\varrho)\Big|_{(\text{right})}=\begin{cases} k & \text{if } (\varrho,\omega(\varrho))\in\mathcal{T}\cup\,]A,B[\,\cup\,]D,A[\text{ and} \\ & \begin{cases}\varrho\geqslant\zeta\\ p+\lambda(\varrho-\zeta)<1\end{cases} \quad \left(\text{i.e. } \zeta\leqslant\varrho<\zeta+\dfrac{1-p}{\lambda}\right) \\ g_r(p,\omega(\varrho)) & \text{if } (\varrho,\omega(\varrho))\notin\mathcal{T}\cup\,]A,B[\,\cup\,]D,A[\\ & \qquad\qquad \text{or } \varrho\geqslant\zeta+\dfrac{1-p}{\lambda} \end{cases}$$

$$(6.8)\qquad \omega(\zeta)=\xi\,.$$

$\forall p\in[-1,1]$, the function $(\zeta,\xi,\varrho)\mapsto G(\zeta,\xi,\varrho)=\tilde{G}(\zeta,\xi,\varrho,p)$ fulfills properties (4.2), ..., (4.4). We have also (cf. (4.5))

$$(6.9)\qquad \begin{cases} \forall\zeta\in\mathbf{R},\ \forall\xi\in S(\zeta),\ \forall p\in[-1,1],\ \forall\varrho,\bar{\varrho}\in\mathbf{R} \text{ with } \zeta\leqslant\varrho<\bar{\varrho} \text{ or } \bar{\varrho}<\varrho\leqslant\zeta, \\ \tilde{G}(\zeta,\xi,\bar{\varrho},p)=\tilde{G}(\varrho,\tilde{G}(\zeta,\xi,\varrho,p),\bar{\varrho},\gamma(p,\varrho-\zeta)) \quad (\text{transition property})\,. \end{cases}$$

We define $\mathcal{H}$ (the evolution of the internal parameter) and $\tilde{\mathcal{F}}$ (the «hysteresis functional») at first in $\tilde{C}^0([0,T])$:

$$(6.10)\qquad \begin{cases} \forall v\in\tilde{C}^0([0,T]),\ \forall\eta\in[-1,1],\ \mathcal{H}(v,0,\eta)=\eta;\ \forall\{t_k\}_{k=0,\dots,m} \text{ as in (2.9)}, \\ \forall k\in\{0,\dots,m-1\},\ \forall t\in\,]t_k,t_{k+1}]\,,\ \mathcal{H}(v,t,\eta)=\gamma(\mathcal{H}(v,t_k,\eta),v(t)-v(t_k)) \end{cases}$$

$$(6.11)\qquad \begin{cases} \forall v\in\tilde{C}^0([0,T]),\ \forall\xi\in S(v(0)),\ \forall\eta\in[-1,1],\ \tilde{\mathcal{F}}(v,0,\xi,\eta)=\xi \\ \forall\{t_k\}_{k=0,\dots,m} \text{ as in (2.9)},\ \forall k\in\{0,\dots,m-1\},\ \forall t\in\,]t_k,t_{k+1}], \\ \qquad \tilde{\mathcal{F}}(v,t,\xi,\eta)=\tilde{G}(v(t_k),\tilde{\mathcal{F}}(v,t_k,\xi,\eta),v(t),\mathcal{H}(v,t_k,\eta)) \end{cases}$$

both definitions are independent of the subdivision $\{t_k\}_{k=0,\dots,m}$, as a consequence of transition properties (6.4), (6.9).

The following crucial properties of Lipschitz-continuity w.r.t. v are fulfilled

$$\text{(6.12)} \quad \begin{cases} \forall v_1, v_2 \in \tilde{C}^0([0,T]) \text{ with } v_1(0) = v_2(0),\ \forall \eta \in [-1,1], \\ \max\limits_{[0,T]} |\mathcal{K}(v_1, \cdot, \eta) - \mathcal{K}(v_2, \cdot, \eta)| \leqslant \lambda \max\limits_{[0,T]} |v_1 - v_2| \end{cases}$$

$$\text{(6.13)} \quad \begin{cases} \forall v_1, v_2 \in \tilde{C}^0([0,T]) \text{ with } v_1(0) = v_2(0) \text{ and } |v_1 - v_2| \leqslant \dfrac{2}{\lambda},\ \forall \xi \in S(v_1(0)), \\ \forall \eta \in [-1,1], \quad \text{setting } C = \max(\beta, \delta_1, \delta_2), \\ \max\limits_{[0,T]} |\mathcal{F}(v_1, \cdot, \xi, \eta) - \mathcal{F}(v_2, \cdot, \xi, \eta)| \leqslant C \max\limits_{[0,T]} |v_1 - v_2| \end{cases}$$

this last is due to reversibility for small oscillations. With reference to continuum mechanics, hysteresis corresponds to a plastic behavior for large deformations: the « internal parameter » $p(t) = \mathcal{K}(v, t, \eta)$ introduces elasticity for small oscillations.

(6.12) and (6.13) allow to extend $\mathcal{K}$ and $\mathcal{F}$ to all $v \in C^0([0,T])$ by continuity, analogously to (4.8).

Setting $\Sigma(l) = [-1, 1]\ \forall l \in \boldsymbol{R}$, the couple $(\Sigma, \mathcal{K})$ fulfills properties (1.2), ..., (1.6) and also (1.11), ..., (1.13).

Fix $\eta \in [-1, 1]$ and set $(v, t, \xi) \mapsto \mathcal{F}(v, t, \xi) = \tilde{\tilde{\mathcal{F}}}(v, t, \xi, \eta)$; then $(S, \mathcal{F})$ fulfills properties (1.1), ..., (1.6) as well as (1.11), (1.12), (2.10), ..., (2.13). Moreover (cf. (1.13))

$$\text{(6.14)} \quad \begin{cases} \forall \tau \in]0, T[,\ \forall (v, t, \xi, \eta) \in \operatorname{Dom}(\tilde{\tilde{\mathcal{F}}}) \text{ with } t > \tau, \\ \tilde{\tilde{\mathcal{F}}}(v, t, \xi, \eta) = \tilde{\tilde{\mathcal{F}}}\big(v \circ \alpha_\tau, \alpha_{-\tau}(t), \tilde{\tilde{\mathcal{F}}}(v, t, \xi, \eta), \mathcal{K}(v, \tau, \eta)\big) \text{ (transition property)}. \end{cases}$$

As far as (2.15) is concerned, we can repeat the considerations expressed for example 1 of § 5.

An « internal parameter » $\mathcal{K}(v, t, \eta)$ can be introduced analogously for the example 4 of § 5, corresponding to ferromagnetism. Notice that reversibility for small oscillations of H corresponds to phenomenology (cfr. .9], pag. 549, e.g.); at microscopic level this can be interpreted as the presence of an elastic part in the movement of Bloch walls at any reversal in the direction.

Instead of a constant k we can also choose a function $k\colon \mathrm{S} \to \boldsymbol{R}$ such that

$$\text{(6.15)} \quad \forall (\varrho, \omega) \in \mathrm{S}, \qquad (0 < l \leqslant) \min_{i=l,r} g_i(\varrho, \omega) \leqslant k(\varrho\ \omega) \leqslant \max_{i=l,r} g_i(\varrho, \omega) (\leqslant L < \infty);$$

further developments are similar to the ones above.

Therefore in particular we get a transition property in the form of (6.14), which is consistent with the phenomenology of the hysteresis cycle.

Added in proofs. – Remark concerning theorem 1 of § 2.

In (2.12) replace convergence in $L^1(0, T)$ weak by convergence in $C^0([0, T])$ strong. This implies (2.14). Moreover in this case (2.40) entails

$$\text{(I)} \qquad \mathcal{F}(u_m(x, \cdot), t, w^0(x)) \to \mathcal{F}(u(x, \cdot), t, w^0(x)) \quad \text{in } C^0([0, T]) \text{ strong, a.e. in D}$$

and then by (2.20) also

$$\text{(II)} \qquad w_m \to w = \mathcal{F}(u(x, \cdot), t, w^0(x)) \quad \text{in } C^0([0, T]) \text{ strong, a.e. in D}.$$

Now in (2.10) let c_i be independent of t, for $i = 1, 2$. Then by (2.33)

$$\text{(III)} \qquad \begin{cases} \|\mathcal{F}(u_m(x, \cdot), t, w^0(x))\|_{L^\infty(0, T; L^2(\mathrm{D}))} \leqslant \text{const}, \\ \|w_m\|_{L^\infty(0, T; L^2(\mathrm{D}))} \leqslant \text{const}, \end{cases}$$

and, possibly extracting further subsequences, in (I) and (II) convergences are in $L^\infty(0, T; L^2(\mathrm{D}))$ weak star, too.

In conclusion, this strengthening of (2.10), (2.12) allows to avoid assumptions (2.11) and (2.14) in theorem 1, the regularity result $w \in W^{\lambda,1}(0, T; L^2(\mathrm{D}))$ being replaced by $w \in L^\infty(0, T; L^2(\mathrm{D}))$. Similar considerations hold for theorems 2, 3 and 4.

Notice that the functionals $\mathcal{F}$ constructed in § 4 and 5 fulfill also these stronger properties.

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