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On the Asymptotic Behaviour of Solutions of Nonlinear Kinetic Equations (*).

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Summary. – *The asymptotic behaviour of solutions of the kinetic equations on the level of the Euler fluid dynamics is studied. In contrast to the previous studies in Sobolev spaces, the analysis is carried out in C^0 (with respect to the space variable) setting. Thus, one needs to use only few Hilbert and initial layer terms, and consequently a lower smoothness is required for the solution of the Euler system.*

1. – Introduction.

The asymptotic relationship between solutions of singularly perturbed Boltzmann equation (with Knudsen number ε as a small parameter) and of the system of compressible Euler equations was studied in [4], [13], [15], [24], [25].

The approach used in [4], [13], [15] as well as [16] and [17] was proposed by CAFLISCH and PAPANICOLAOU in [3], where a solution to the model Boltzmann equation was found in the form of a sum of a truncated power series in ε and of the remainder term.

This approach enabled to replace the singularly perturbed equation by a system of ε-independent equations and a weakly nonlinear equation for the remainder. Thus, the analysis has led to investigating the linearized form of the weakly nonlinear equation, and next to using the method of successive approximations.

However, the norm of the functional space has to be appropriate in order to deal with both the linear (L_2-norm) and nonlinear (L_∞-norm) terms. This has led to considering the Sobolev spaces with respect to the space variable x. On the other hand, to control the x-derivatives, one had to use a large number of terms in the truncated power series in ε (high number of Hilbert and initial layer terms). The approach needed a high smoothness of the solution of the Euler system and, therefore, it does not seem to be suitable for the usual asymptotic procedure.

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Moreover, use of Sobolev spaces is not suitable for the study of the validity of the Euler equations (cf. the discussion in [17]). This paper deals with the approach in C^0-space with respect to x-variable, by applying an idea suggested by HEINTZ ([11], [12]).

The space C^0 seems to be natural to study the asymptotic behaviour and in this setting it is possible to use only few Hilbert and initial layer terms and consequently to require a lower smoothness on the solution of the Euler system.

The crucial point of the present work is Lemma 8.2 which allows us to combine C^0-estimates with relevant L_2-properties of the linearization of the collision operator. This approach enables us to formulate a new theorem on the Euler limit and on the initial layer behaviour for the Boltzmann equations (Section 9). It can be also applied to other kinetic equations. In Sections 10 and 11 the Euler limit and initial layer behaviour for the Enskog and Povzner equations are investigated. However, to proceed with these equations, one has to control the x-derivative of the solution of the Boltzmann equation. In this case, the analysis is also realized in C^s-space ($s > 0$) with respect to x-variable.

The attempt can be also applied to the study of the asymptotic behaviour on the level of the Navier-Stokes fluid dynamics on the time interval on which a smooth solution of the Navier-Stokes system exists. This problem will be discussed in the forthcoming paper.

2. – The Boltzmann equation.

The Boltzmann equation, well-known and studied for more than 100 years, has a quite vast literature. The reader is referred to [6], [8], [9] and [23] for a review of physical and mathematical results. Let us only point out that referring to the analysis of existence of solutions, a very important result has been recently obtained by DIPERNA and LIONS [7]. Throughout the paper we assume that all functions are periodic with respect to the space variable x with fundamental domain $\Omega \subset R^d$ where $d = 1$, 2 or 3 (for details see [15]). Thus Ω can be treated as a d-dimensional torus. The dimensionless Boltzmann equation singularly perturbed by a small parameter $\varepsilon > 0$, representing the scale of the mean free path is studied. Consequently, the problem investigated is the following:

$$Df = \frac{1}{\varepsilon} J_0(f, f), \tag{2.1a}$$

$$f|_{t=0} = F, \tag{2.1b}$$

where D is the free-streaming operator:

$$D = \frac{\partial}{\partial t} + v \cdot \mathrm{grad}_x$$

and J_0 is a bilinear, symmetric collision operator.

$f = f(t, x, v)$ is an one-particle distribution function; $t \geq 0$ is time; $x \in \Omega$ is the position (on the d-dimensional torus) of a particle and $v \in R^3$ is its velocity. Usually, referring to the asymptotic analysis of the Boltzmann equation, the collision operator J_0 corresponding to Grad's cutoff hard potentials ([10]) is assumed (cf. [4], [5], [13] and [15]). However, up to now, the Enskog equation has no physically consistent formulation for softer potentials than the hard-spheres potential (see [2] for details).

Thus in the sequel the notations for the hard-spheres potential will be used, although whole mathematical analysis is also valid for Grad's cutoff potentials. Keeping this in mind, we can define

$$J_0(f_1, f_2) = \frac{1}{2}(J_0^+(f_1, f_2) + J_0^+(f_2, f_1) - J_0^-(f_1, f_2) - J_0^-(f_2, f_1)),$$

where

$$J_0^+(f_1, f_2)(x, v) = \int_{R^3}\int_{S^2} f_1(x, v_1')\, f_2(x, v')\, \Phi((v_1 - v)\cdot n)\, dn\, dv_1$$

and

$$J_0^-(f_1, f_2) = f_1 \cdot \nu_0(f_2),$$

$$\nu_0(f)(x, v) = \int_{R^3}\int_{S^2} f(x, v_1)\, \Phi((v_1 - v)\cdot n)\, dn\, dv_1 .$$

The standard notation has been used: a particle with the centre at $x \in \Omega$ and the velocity $v \in R^3$ collides with a particle with the centre at x and velocity v_1, and after the interaction the velocities are v' and v_1', respectively.

Note that as a function Φ we can use both

$$\Phi_1(y) = \max\{y, 0\}, \quad y \in R^1 \qquad \text{and} \qquad \Phi_2(y) = |y|, \quad y \in R^1,$$

provided that the collision operator J_0 is divided by 2 in the second case (cf. [6], II.3, p. 57). In the sequel the abbreviation BE for the Boltzmann equation, defined with both Φ_1 and Φ_2 as Φ will be used.

3. – The Enskog equation.

The BE model assumes the overall dimensions of particles can be neglected. In the case of dense gases, one has to replace this mass-point model by models which take into account the overall dimensions of particles. One of such an attempt leads to the Enskog equation model in which each particle is regarded as a hard sphere with a nonzero diameter. The reader is referred to the review [2] for a detailed analysis of the mathematical problems and results of various versions of the Enskog equation. Following the line of [16], one can formulate the problem for the Enskog equation in dimensionless form as follows

$$Df = \frac{1}{\varepsilon} J_\sigma(f, f) + \frac{\sigma}{\varepsilon^2} E_{\sigma,\varepsilon}(f; f, f), \tag{3.1a}$$

$$f|_{t=0} = F, \tag{3.1b}$$

where σ is a dimensionless parameter representing the scale of the hard-sphere diameter,

$$J_\sigma(f_1, f_2) = \frac{1}{2}(J_\sigma^+(f_1, f_2) + J_\sigma^+(f_2, f_1) - J_\sigma^-(f_1, f_2) - J_\sigma^-(f_2, f_1)),$$

$$J_\sigma^+(f_1, f_2)(x, v) = \int_{R^3} \int_{S^2} f_1(x + \sigma n, v_1')\, f_2(x, v')\, \Phi((v_1 - v)\cdot n)\, dn\, dv_1,$$

$$J_\sigma^-(f_1, f_2) = f_1 \cdot \nu_\sigma(f_2),$$

$$\nu_\sigma(f)(x, v) = \int_{R^3} \int_{S^2} f(x - \sigma n, v_1)\, \Phi((v_1 - v)\cdot n)\, dn\, dv_1,$$

$$E_{\sigma,\varepsilon}(f_1; f_2, f_3) = \frac{1}{2}(E_{\sigma,\varepsilon}^+(f_1; f_2, f_3) + E_{\sigma,\varepsilon}^+(f_1; f_3, f_2) - E_{\sigma,\varepsilon}^-(f_1; f_2, f_3) - E_{\sigma,\varepsilon}^-(f_1; f_3, f_2)),$$

$$E_{\sigma,\varepsilon}^+(f_1; f_2, f_3)(x, v) = \int_{R^3} \int_{S^2} \mathcal{Y}_{\sigma,\varepsilon}^+(f_1)\cdot f_2(x + \sigma n, v_1')\cdot f_3(x, v')\, \Phi((v_1 - v) n)\, dn\, dv_1,$$

and

$$E_{\sigma,\varepsilon}^-(f_1; f_2, f_3)(x, v) = f_3(x, v) \int_{R^3} \int_{S^2} \mathcal{Y}_{\sigma,\varepsilon}^-(f_1)\cdot f_2(x - \sigma n, v_1)\, \Phi((v_1 - v)\cdot n)\, dn\, dv_1.$$

The functionals $\mathcal{Y}_{\sigma,\varepsilon}^\pm$ are such that $1 + (\sigma/\varepsilon)\, \mathcal{Y}_{\sigma,\varepsilon}^\pm(f)$ represent the pair correlation functions due to the overall dimensions of particles (cf. [16] and [2]). According to physical theories available in the literature, the following assumption can be proposed

ASSUMPTION 3.1. –

(i) $0 < \sigma \leq \varepsilon^\mu \leq 1$,

(ii) $\mathcal{Y}_{\sigma,\varepsilon}^\pm(0) \equiv 0$,

(iii) $\forall f \in \boldsymbol{D}_{\sigma,\varepsilon}^*\colon |\mathcal{Y}_{\sigma,\varepsilon}^\pm(f)| \leq \mathcal{Y}^*$,

(iv) $\forall f_1, f_2 \in \boldsymbol{D}_{\sigma,\varepsilon}^*\colon |\mathcal{Y}_{\sigma,\varepsilon}^\pm(f_1) - \mathcal{Y}_{\sigma,\varepsilon}^\pm(f_2)| \leq l_* \sup_{\Omega} \left| \int_{R^3} (f_1 - f_2)\, dv \right|$,

where $\boldsymbol{D}_{\sigma,\varepsilon}^*$ is a set of physical consistency of the Enskog description (cf. [2] and [16]) and $\mathcal{Y}^*$, l^* are the constants independent of σ and ε.

In contrast to BE, the Enskog equation can be physically justified only when as Φ the function Φ_1 (introduced in the previous Section) is used. However, some proofs are valid only for $\Phi \equiv \Phi_2$ ([1] and [16]). The abbreviation EE_1 for the Enskog equation (3.1a) with Φ defined by Φ_1 and the abbreviation EE_2 for the Enskog equation with $\Phi \equiv \Phi_2$ will be used.

A simplified model can be obtained simply by putting $\mathcal{Y}^{\pm}_{\sigma,\varepsilon} \equiv 0$. This model is called the Boltzmann-Enskog model.

4. – The Povzner equation.

The Povzner equation has been introduced as opposed to the Enskog equation for purely mathematical purposes. This equation can be regarded as a modification of BE which differs from the classical version (BE) as it allows for a spatial «smearing» process of collisions. Such an idea has been used first by MORGENSTERN [19] and next, more generally, by POVZNER [22]. Although the latter has provided some consistent justification of this model, the Povzner equation is ignored by physicists.

Assuming that the smeared collision is possible if the distance between two particles is not greater than r, we can formulate the problem for the Povzner equation as follows:

$$Df = \frac{1}{\varepsilon} P_r(f, f), \tag{4.1a}$$

$$f|_{t=0} = F, \tag{4.1b}$$

where

$$P_r(f, f) = \frac{1}{r^3} \int_0^r \delta^3 J_\delta(f, f)\, d\delta,$$

with J_δ as introduced in the previous Section. The classical formulation of the Povzner equation leads to (4.1a) after a change of the variables $y \to x \pm \delta n$ where $x \in \Omega$ is fixed and $\delta \in R^1$, $n \in S^2$ (cf. [17]).

Analogously as in the previous Section by PE_1 we denote the Povzner equation with $\Phi \equiv \Phi_1$ and by PE_2 the Povzner equation with $\Phi \equiv \Phi_2$.

5. – The Euler system.

The following moments of the distribution function f, corresponding with fluid-dynamics will be called fluid-dynamic parameters of f:

$$\rho_f(t, x) = \int_{R^3} f(t, x, v)\, dv \tag{5.1a}$$

is the local density

$$u_f(t,x) = \frac{1}{\rho_f(t,x)} \int_{R^3} v f(t,x,v)\, dv \tag{5.1b}$$

is the macroscopic velocity vector and

$$T_f(t,x) = \frac{1}{3\rho_f(t,x)} \int_{R^3} |v|^2 f(t,x,v)\, dv - \rho_f(t,x)\, u_f^2(t,x) \tag{5.1c}$$

is the macroscopic temperature.

In this paper asymptotic relationships between the parameters ρ_f, u_f and T (where f is a solution of BE, EE_1, EE_2, PE_1, PE_2) and the corresponding quantities ρ, u and T obtained as a solution of the compressible Euler equations (in the paper called SCEE) are investigated. In the scope of this program a more general formulation is proposed, as a search of asymptotic relationships between solutions of the kinetic equations and the local Maxwellian M, whose fluid-dynamic parameters are ρ, u and T:

$$M(t,x,v) = \rho(t,x)(2\pi T(t,x))^{-3/2} \exp\left(-\frac{|v-u(t,x)|^2}{2T(t,x)}\right). \tag{5.2}$$

The analysis of such a problem needs existence theorems for both the kinetic equation and SCEE in a common time interval. The reader is referred to the book by MAJDA [18] for a review of results on existence of solutions for SCEE. The starting point is the following assumption

ASSUMPTION 5.1. – Let $t_0 \in]0, +\infty[$ and let (ρ_0, u_0, T_0) be such that a sufficienly smooth solution (ρ, u, T) of SCEE with the initial data (ρ_0, u_0, T_0) exists in the time interval $[0, t_0]$ and satisfies

$$\forall (t,x) \in [0,t_0] \times \Omega\colon \rho(t,x) \geq c_\rho > 0, \qquad T(t,x) \geq c_T > 0. \tag{5.3}$$

6. – Notations.

Throughout the paper, w_α is the following function

$$w_\alpha(v) = (1 + |v|^2)^{\alpha/2}, \qquad \alpha \in R^1.$$

Some functional spaces can be also defined. $L_2(R^3)$ denotes the Lebesgue space of measurable, real-valued functions, square integrable in R^3 with the norm $\|\cdot; L_2(R^3)\|$ and the inner product $(\cdot,\cdot)_{L_2(R^3)}$. Let $\omega_{R^3}, \omega_\Omega, \omega_{\Omega\times R^3}$ be strictly positive, smooth functions on R^3, Ω and $\Omega \times R^3$, respectively. $B_\infty(\omega_{R^3})$ denotes the space of continuous, real-valued functions on R^3 with the norm $\|f; B_\infty(\omega_{R^3})\| = \sup_{v\in R^3} |\omega_{R^3} f|$. Moreover,

$B_\infty^\alpha = B_\infty(w_\alpha)$ and $B_\infty^\alpha(\omega_{R^3}) = B_\infty(w_\alpha \cdot \omega_{R^3})$. $C^s(\Omega; \omega_\Omega)$ is the space of the functions which are continuous together with all their derivatives of orders $|\gamma| \leq s$ and equipped with the norm

$$\|f; C^s(\Omega; \omega_\Omega)\| = \sup_{\substack{0 \leq |\gamma| \leq s \\ x \in \Omega}} \left| \omega_\Omega \frac{\partial^{|\gamma|} f}{\partial x^\gamma} \right| .$$

Naturally, $C^s(\Omega) = C^s(\Omega; 1)$. $L_p(\Omega)$ is (for $1 < p < \infty$) the space of measurable, real valued functions whose p-th power is integrable on Ω and with the norm $\|f; L_p(\Omega)\|$. In addition, consider the following space consisting of functions on $\Omega \times R^3$:

1) $X_{\infty,\infty}^{\alpha,s}$ equipped with the norm

$$\|\cdot\|_{\infty,\infty}^{\alpha,s} = \|(\|\cdot; C^s(\Omega)\|); B_\infty^\alpha\| ,$$

2) $X_{\infty,p}^{\alpha}$ equipped with the norm

$$\|\cdot\|_{\infty,p}^{\alpha} = \|(\|\cdot; L_p(\Omega)\|); B_\infty^\alpha\| ,$$

3) $X_{2,\infty}^{;s}$ equipped with the norm

$$\|\cdot\|_{2,\infty}^{;s} = \|(\|\cdot; C^s(\Omega)\|); L_2(R^3)\| ,$$

4) $X_{2,p}$ equipped with the norm

$$\|\cdot\|_{2,p} = \|(\|\cdot; L_p(\Omega)\|); L_2(R^3)\| .$$

Now, let M be a local Maxwellian, whose fluid-dynamic parameters are ρ, u and T-as in Assumption 5.1. Let

$$M_0 = M|_{t=0} , \tag{6.1}$$

i.e. M_0 is a local Maxwellian with ρ_0, u_0 and T_0 as the fluid-dynamic parameters. Let M_+ be a global Maxwellian (i.e. Maxwellian, whose parameters are constant) such that for all $t \in [0, t_0]$, $x \in \Omega$, $v \in R^3$ and $\alpha \in R^1$ we have

$$w_\alpha(v) M(t, x, v) \leq c_\alpha M_+(v) \tag{6.2}$$

where the constant c_α depends only on α.

Let $Y_0^{\alpha,s}$ and $Y_+^{\alpha,s}$ be the spaces equipped with the norms

$$N_0^{\alpha,s}\{\cdot\} = \|(\|\cdot; C^s(\Omega; M_0^{-1/2})\|); B_\infty^\alpha\| ,$$

$$N_+^{\alpha,s}\{\cdot\} = \|(\|\cdot; C^s(\Omega); B_\infty^\alpha(M_+^{-1/2})\| .$$

Finally, recall the idea of (extrinsic or free-streaming) trajectories (cf. [23]). For t and v fixed, let $\gamma_{(t,v)}$ be the translation on Ω defined by

$$\gamma_{(t,v)} x = x + tv \quad (\text{mod period}) . \tag{6.3}$$

The (extrinsic) trajectory in $R^1 \times \Omega \times R^3$ is the line defined parametrically:

$$t \mapsto (t, \gamma_{(t,v)} x, v) . \tag{6.4}$$

Now let Γ_t be one-parameter family of operators

$$(\Gamma_t f)(x, v) = f(\gamma_{(t,v)} x, v) . \tag{6.5}$$

$f^{\#}$ is the function considered along the (extrinsic) trajectories, i.e.

$$f^{\#}(t) = \Gamma_t f(t) . \tag{6.6}$$

Moreover, the following (ε and σ-depending) two-parameters family of operators can be introduced

$$U_{\varepsilon,\sigma}(t, t') f = (\Gamma_{t'-t} f) \exp\left(-\frac{1}{\varepsilon} \int_{t'}^{t} \Gamma_{t''-t} \nu_\sigma(M)(t'')\, dt'' \right) . \tag{6.7}$$

For $\sigma = 0$ one writes simply U_ε instead of $U_{\varepsilon,0}$.

7. – Hilbert and initial layer expansions.

The solution $f(t)$ to BE has been searched in [15] in the form of the following sum

$$f(t) = \sum_{j=0}^{a} \varepsilon^j h_j(t) + \sum_{j=0}^{a} \varepsilon^j l_j\left(\frac{t}{\varepsilon}\right) + \varepsilon^b z(t) , \tag{7.1}$$

where h_j are Hilbert expansion terms, l_j-initial layer terms and z is a remainder term. The reader is referred to [15] for details; here are only recalled some preliminary results. We can start from the following assumption on initial data F (in (2.1*b*), (3.1*b*) and (4.1*b*)).

ASSUMPTION 7.1. – Let F be decomposed into the hydrodynamic and the non-hydrodynamic parts as follows:

$$F = M_0 + G \tag{7.2}$$

where M_0 is a local Maxwellian whose fluid-dynamic parameters are ρ_0, u_0 and T_0-as in Assumption 5.1 (cf. (6.1)), and G is a function with null fluid-dynamic parameters i.e.

$$\int \psi_i G\, dv = 0 \qquad (i = 0, \dots, 4) \tag{7.3}$$

with

$$\psi_0(v) = 1 , \qquad \psi_i(v) = v_i \ \ (i = 1, 2, 3) , \qquad \psi_4(v) = v^2 , \tag{7.4}$$

and such that

$$G \in Y_0^{\alpha, s} \tag{7.5}$$

with α and s large enough and

$$N_0^{4,0}\{G\} \leqslant \theta \tag{7.6}$$

where θ is a critical constant independent of ε.

In [15] it has been shown that if Assumption 7.1 is satisfied then the following is true:

7.*A*) There exist Hilbert expansion terms $h_0, h_1, \ldots, h_a$ sufficiently smooth with respect to $t \in [0, t_0]$ and $x \in \Omega$ such that

$$h_0 = M \tag{7.7a}$$

and

$$\left| w_\alpha \frac{\partial^{k+|\gamma|}}{\partial t^k \partial x^\gamma} h_j \right| \leqslant \mathrm{const}\, M^{1/2} \tag{7.7b}$$

for all $\alpha \geqslant 0,\ k \geqslant 0,\ |\gamma| \geqslant 0\ t \in [0, t_0],\ x \in \Omega$ and $v \in R^3\ j = 1, \ldots, a$.

7.*B*) there exist initial layer terms $l_0, \ldots, l_a$ and numbers $\alpha_j, s_j, \delta_j > 0$ such that

$$l_j \in C^1([0, +\infty[; Y_0^{\alpha_j, s_j}) \tag{7.8a}$$

and

$$\sup_{\tau \geqslant 0} N_0^{\alpha_j, s_j}\{\exp(\delta_j \tau)\, l_j(\tau)\} \leqslant \mathrm{const} \tag{7.8b}$$

for $j = 0, \ldots, a$.

7.*C*) BE with initial data F is equivalent to the following nonlinear equation for the remainder z:

$$Dz = \frac{2}{\varepsilon} J_0(M, z) + \frac{2}{\varepsilon} J_0(l_0, z) + 2 \sum_{j=1}^{a} \varepsilon^{j-1} J_0(h_j + l_j, z) + \varepsilon^{b-1} J_0(z, z) + \varepsilon^{a-b} \mathcal{O}\!\!\!\mathcal{l} \tag{7.9a}$$

with initial data

$$z|_{t=0} = 0 \tag{7.9b}$$

where $\mathcal{O}\!\!\!\mathcal{l}$ is a complicated term «regular» when $\varepsilon \downarrow 0$ and depending on the Hilbert terms as well as on the initial layer terms.

The nonlinear and nonhomogeneous terms are multiplied by numbers ε^{b-1} and ε^{a-b}. Therefore, for a and b being properly chosen, the equation (7.9*a*) is weakly nonlinear. Thus, its classical approach is based on the analysis of its linearized form.

8. – Linearized equation for the remainder.

To deal with the linearized equation for the remainder the Grad's idea can be applied consisting in that the operator $J_0(M,\cdot)$ is symmetrized to get the non-positive in $L_2(R^3)$ operator

$$L_0 = M^{-1/2} J_0(M, M^{1/2}\cdot).$$

Moreover, the main mathematical difficulties can be overcome when the equation (7.9) is replaced by the following system of equations:

$$\text{(8.1}a) \qquad Dz_0 = \frac{1}{\varepsilon} L_0 z_0 + \frac{1}{\varepsilon} \sum_{i=1,2} \chi M^{-1/2} \mathcal{K}_0 z_i ,$$

$$\text{(8.1}b) \qquad Dz_1 + \frac{1}{\varepsilon} \nu_0(M)\cdot z_1 = -M_+^{-1/2}(DM^{1/2}) z_0 +$$

$$+ \frac{1}{\varepsilon}(1-\chi) M_+^{-1/2} \mathcal{K}_0 z_1 + \frac{2}{\varepsilon} M_+^{-1/2} J_0(l_0, M_+^{1/2} z_1) + A(M^{1/2} z_0 + M_+^{1/2} z_1 + M_+^{1/2} z_2) + \mathcal{Q},$$

$$\text{(8.1}c) \qquad Dz_2 + \frac{1}{\varepsilon}\nu_0(M)\cdot z_2 = \frac{1}{\varepsilon}(1-\chi) M_+^{-1/2} \mathcal{K}_0 z_2 + \frac{2}{\varepsilon} M_+^{-1/2} J_0(l_0, M_+^{1/2} z_2 + M^{1/2} z_0)$$

with initial data

$$\text{(8.1}d) \qquad z_0|_{t=0} = z_1|_{t=0} = z_2|_{t=0} = 0 .$$

$\chi = \chi(v)$ is a characteristic function of the ball of the radius $\varkappa$ with center at the origin in R^3, where $\varkappa$ must be propertly chosen. The operator $\mathcal{K}_0$ is given by

$$\mathcal{K}_0 z = J_0^+(M, M_+^{1/2} z) + J_0^+(M_+^{1/2} z, M) - M\cdot \nu_0(M_+^{1/2} z).$$

Moreover,

$$Az = 2M_+^{-1/2} \sum_{j=1}^{a} \varepsilon^{j-1} J_0(h_j + l_j, z).$$

The term $\mathcal{Q}$ corresponds to the sum of the nonlinear and nonhomogeneous terms, and is assumed to be known. Now, it is easy to see that after decomposition

$$z = M^{1/2} z_0 + M_+^{1/2} z_1 + M_+^{1/2} z_2 ,$$

the system (8.1) is equivalent to the linearized equation for the remainder.

The following integral version of the system (8.1)

$$\text{(8.2)} \qquad z_i(t) = \int_0^t U_\varepsilon(t,t') Z_i[z_0, z_1, z_2, \mathcal{Q}; \varepsilon](t')\, dt' , \qquad i = 0,1,2,$$

can now be analyzed, where Z_1 and Z_2 are the right-hand side of equations (8.1*b*) and (8.1*c*), respectively, and Z_0 is defined as follows

$$Z_0 = \frac{1}{\varepsilon} K_0 z_0 + \frac{1}{\varepsilon} \sum_{i=1,2} \chi M^{-1/2} \mathcal{K}_0 z_i ,$$

with

$$K_0 = M^{-1/2} (J_0^+ (M, M^{-1/2} \cdot) + J_0^+ (M^{-1/2} \cdot, M) - M \nu_0 (M^{-1/2} \cdot)) .$$

Using similar estimations as in [15] the following lemma can be proved.

LEMMA 8.1. – If $\varepsilon > 0$ is sufficiently small then the solution $(z_0(t), z_1(t), z_2(t))$ of the linear system (8.2) satisfies the following a priori estimation:

$$\sup_{[0,t]} \|z_0\|_{\infty,\infty}^{\beta,k} \leq \text{const} \sup_{[0,t]} \|z_0\|_{2,\infty}^{k} + c_\varkappa \sum_{i=1,2} \sup_{[0,t]} \|z_i\|_{\infty,\infty}^{0,k} , \tag{8.3a}$$

$$\sup_{[0,t]} \|z_1\|_{\infty,\infty}^{\beta,k} \leq \text{const} \cdot \varepsilon \cdot \sup_{[0,t]} \|z_0\|_{2,\infty}^{k} + \text{const} \cdot \varepsilon \cdot \sup_{[0,t]} \|\mathcal{A}\|_{\infty,\infty}^{\beta-1,k} , \tag{8.3b}$$

$$\|z_2(t)\|_{\infty,\infty}^{\beta,k} \leq \text{const} \cdot \theta \cdot \exp\left(-\frac{t}{\varepsilon}\delta_*\right) \sup_{[0,t]} \|z_0\|_{2,\infty}^{k} + c_\varkappa \cdot \varepsilon \cdot \sup_{[0,t]} \|\mathcal{A}\|_{\infty,\infty}^{-1,k} \tag{8.3c}$$

for all $t \in]0, t_0]$, $\beta \geq 0$, $k \geq 0$, where $0 < \delta_* < \delta_0$, δ_0 has been defined in (7.8*b*) and $c_\varkappa$ is a constant depending on $\varkappa$. ■

The following lemma is the crucial point in our analysis.

LEMMA 8.2. – If $\varepsilon > 0$ is sufficiently small, then the solution $(z_0(t), z_1(t), z_2(t))$ of the system (8.2) satisfies the following a priori estimation

$$\sup_{[0,t]} \|z_0\|_{2,\infty}^{0} \leq \frac{\text{const}}{\varepsilon^{d/2}} \sup_{[0,t]} \|z_0\|_{2,2} + \text{const} \sum_{i=1,2} \sup_{[0,t]} \|z_i\|_{\infty,\infty}^{0,0} , \tag{8.4}$$

for all $t \in]0, t_0]$, where d is the space dimension.

PROOF. – Assume that $(z_0(t), z_1(t), z_2(t))$ is a solution of the system of equations (8.2). Thus

$$z_0 = \mathcal{P} z_0 + \sum_{i=1,2} \mathcal{R} z_i \tag{8.5a}$$

where

$$\mathcal{P} z(t) = \frac{1}{\varepsilon} \int_0^t U_\varepsilon(t, t') K_0 z(t') \, dt'$$

and

$$\mathcal{R}z(t) = \frac{1}{\varepsilon}\int_0^t U_\varepsilon(t,t')\,\chi M^{-1/2}\mathcal{K}_0 z(t')\,dt'.$$

χ has been introduced in this Section as a characteristic function of the ball of radius $\varkappa$ (with $\varkappa$-large, but independent of ε). In the proof, χ (or $1-\chi$) denote both the characteristic function and the operator defined as multiplication by the function χ (or $1-\chi$).

From (8.5*a*) it follows

$$z_0 = \chi\mathcal{P}\chi z_0 + \chi\mathcal{P}(1-\chi)z_0 + (1-\chi)\mathcal{P}z_0 + \sum_{i=1,2}\mathcal{R}z_i. \tag{8.5b}$$

Then

$$z_0 = (\chi\mathcal{P})^2\chi z_0 + (\chi\mathcal{P})^2(1-\chi)z_0 + \chi\mathcal{P}(1-\chi)z_0 + \\ +(1-\chi)\mathcal{P}z_0 + \sum_{i=1,2}\chi\mathcal{P}\mathcal{R}z_i + \sum_{i=1,2}\mathcal{R}z_i. \tag{8.5c}$$

Next

$$z_0 = (\chi\mathcal{P})^4\chi z_0 + \sum_{j=1}^{4}(\chi\mathcal{P})^j(1-\chi)z_0 + (1-\chi)\mathcal{P}z_0 + \sum_{i=1,2}\Big(\sum_{j=1,2,3}(\chi\mathcal{P})^j\mathcal{R}z_i + \mathcal{R}z_i\Big). \tag{8.5d}$$

Applying estimations the same as in [14] and [15], one has

$$\sup_{[0,t]}\|(1-\chi)\mathcal{P}z_0\|_{2,\infty}^{\cdot,0} \leqslant \text{const}\sup_{[0,t]}\|(1-\chi)\mathcal{P}z_0\|_{\infty,\infty}^{2,0} \leqslant \frac{\text{const}}{1+\varkappa}\sup_{[0,t]}\|z_0\|_{\infty,\infty}^{1,0}, \tag{8.6a}$$

and

$$\sup_{[0,t]}\|\chi\mathcal{P}(1-\chi)z_0\|_{2,\infty}^{\cdot,0} \leqslant \text{const}\sup_{[0,t]}\|\mathcal{P}(1-\chi)z_0\|_{\infty,\infty}^{2,0} \leqslant \\ \leqslant \text{const}\sup_{[0,t]}\|(1-\chi)z_0\|_{\infty,\infty}^{1,0} \leqslant \frac{\text{const}}{1+\varkappa}\sup_{[0,t]}\|z_0\|_{\infty,\infty}^{2,0}. \tag{8.7a}$$

In the same way

$$\sup_{[0,t]}\|(\chi\mathcal{P})^j(1-\chi)z_0\| \leqslant \frac{\text{const}}{1+\varkappa}\sup_{[0,t]}\|z_0\|_{\infty,\infty}^{1,0}, \qquad \text{for } j=2,3,4. \tag{8.7b}$$

By (8.3*a*) one has

$$\sup_{[0,t]}\|(1-\chi)\mathcal{P}z_0\|_{2,\infty}^{\cdot,0} \leqslant \frac{\text{const}}{1+\varkappa}\sup_{[0,t]}\|z_0\|_{2,\infty}^{\cdot,0} + c_\varkappa\sum_{i=1,2}\sup_{[0,t]}\|z_i\|_{\infty,\infty}^{0,0}, \tag{8.6b}$$

and

$$\sup_{[0,t]} \|(\chi\mathcal{P})^j(1-\chi)\, z_0\|^{0}_{2,\infty} \leqslant \frac{\text{const}}{1+\varkappa} \sup_{[0,t]} \|z_0\|^{0}_{2,\infty} + c_\varkappa \sum_{i=1,2} \sup_{[0,t]} \|z_i\|^{0,0}_{\infty,\infty}, \tag{8.7c}$$

for $j = 1, 2, 3, .$

Moreover, (cf. [14] and [15]),

$$\sup_{[0,t]} \|(\chi\mathcal{P})^j \mathcal{R} z\|^{0}_{2,\infty} \leqslant c_\varkappa \sup_{[0,t]} \|z\|^{0,0}_{\infty,\infty}. \tag{8.8}$$

Consider now the operator $(\chi\mathcal{P})^2\chi$:

$$(\chi\mathcal{P})^2 \chi z(t, x, v) = \frac{1}{\varepsilon^2} \int_0^t \int_0^{t'} \int_{R^3} \int_{R^3} \exp\left(-\frac{1}{\varepsilon}\int_{t'}^{t} \nu_0(M)(t'', x-(t-t')\,v, v)\, dt'\right) \cdot$$

$$\cdot \exp\left(-\frac{1}{\varepsilon}\int_{t''}^{t'} \nu_0(M)(t''', x-(t-t')\,v-(t'-t'')\,\xi, \xi)\, dt'''\right) \chi(v)\, k_0(t', x-(t-t')\,v, v, \xi) \cdot$$

$$\cdot \chi(\xi)\, k_0(t'', x-(t-t')\,v-(t'-t'')\,\xi, \xi, v_1)\, \chi(v_1) \cdot$$

$$\cdot z(t'', x-(t-t')\,v-(t'-t'')\,\xi, v_1)\, d\xi\, dv_1\, dt''\, dt',$$

where k_0 is the kernel of the operator K_0 (cf. [21], (2.9)). By (6.2) and the Grad's estimations ([21], (2.13) and (2.16)) one obtains

$$|(\chi\mathcal{P})^2 \chi z(t, x, v)| \leqslant \frac{\text{const}}{\varepsilon^2} \int_0^t \int_0^{t'} \int_{R^3} \int_{R^3} \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right) \cdot$$

$$\cdot \exp\left(-\frac{\nu_*}{\varepsilon}(t'-t'')\right) \chi(v)\, k_{++}(v, \xi)\, \chi(\xi)\, k_{++}(\xi, v_1) \cdot$$

$$\cdot \chi(v_1)\, |z(t'', x-(t-t')\,v-(t'-t'')\,\xi, v_1)|\, d\xi\, dv_1\, dt''\, dt',$$

where $\nu_* > 0$ is a constant and

$$k_{++}(v, \xi) = |v-\xi|^{-1} \exp\left(-c|v-\xi|^2 - c\frac{(v^2-\xi^2)^2}{|v-\xi|^2}\right)$$

with a constant $c > 0$.

Changing the variables $\xi \mapsto x_1 = x - (t-t')v - (t'-t'')\xi$ one gets

$$|(\chi\mathcal{P})^2\chi z(t,x,v)| \leqslant \frac{c_\varkappa}{\varepsilon^2}\int_0^t\int_0^{t'}\int_{R^3}\int_{\Omega} \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right)\exp\left(-\frac{\nu_*}{\varepsilon}(t'-t'')\right)\cdot$$

$$\cdot (t'-t'')^{-d}\chi(v)\,k_{++}(v,\xi)\,\chi(\xi)\,k_{++}(\xi,v_1)\,\chi(v_1)|z(t'',x_1,v_1)|\,dx_1\,dv_1\,dt''\,dt',$$

where now $\xi = (x_1 - x + (t-t')v)/(t''-t')$ and $c_\varkappa$ is a constant depending on $\varkappa$. For simplicity, the notation for the case $d=3$ has been used. When $d=1$ or 2, it is necessary to integrate additionally over R^{3-d}.

Next

$$|(\chi\mathcal{P})^2\chi z(t,x,v)| \leqslant \frac{c_\varkappa}{\varepsilon^2}\int_0^t \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right)\cdot$$

$$\cdot\int_0^{t'}\exp\left(-\frac{\nu_*}{\varepsilon}(t'-t'')\right)(t'-t'')^{-d}\int_{R^3}\left(\int_{\Omega}(\chi(v)\,k_{++}(v,\xi)\,\chi(\xi)\,k_{++}(\xi,v_1)\,\chi(v_1)\right)^{(p-1)/p}\cdot$$

$$\cdot\|z(t'',\cdot,v_1);\,L_p(\Omega)\|\,dv_1\,dt''\,dt'.$$

We can change variables $x_1 \mapsto \xi = (x_1 - x + (t-t')v)/(t''-t')$ and obtain

$$|(\chi\mathcal{P})^2\chi z(t,x,v)| \leqslant \frac{c_\varkappa}{\varepsilon^2}\int_0^t \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right)\cdot$$

$$\cdot\int_0^{t'}\exp\left(-\frac{\nu_*}{\varepsilon}(t'-t'')\right)(t'-t'')^{-d/p}\int_{R^3}\left(\int_{R^3}\chi(v)\,k_{++}^{p/(p-1)}(v,\xi)\,\chi(\xi)\,k_{++}^{p/(p-1)}(\xi,v_1)\,\chi(v_1)\,d\xi\right)^{(p-1)/p}\cdot$$

$$\|z(t'',\cdot,v_1);\,L_p(\Omega)\|\,dv_1\,dt''dt'.$$

Then

$$\|(\chi\mathcal{P})^2\chi z(t)\|_{2,\infty}^0 \leqslant \frac{c_\varkappa}{\varepsilon^2}\int_0^t \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right)\int_0^{t'}\exp\left(-\frac{\nu_*}{\varepsilon}(t'-t')\right)(t'-t'')^{-d/p}\cdot$$

$$\cdot\left\{\int_{R^3}\left[\int_{R^3}\left(\int_{R^3}\chi(v)k_{++}^{p/(p-1)}(v,\xi)\,\chi(\xi)\,k_{++}^{p/(p-1)}(\xi,v_1)\,\chi(v_1)\,d\xi\right)^{(p-1)/p}\|z(t'',\cdot,v_1);L_p(\Omega)\|\,dv_1\right]^2\cdot dv\right\}^{1/2}\cdot$$

$$\cdot\,dt''\,dt'.$$

Assuming that $p > d$, by a straightforward calculation it follows

$$\sup_{[0,t]} \|(\chi\mathscr{P})^2 \chi z\|_{2,\infty}^{0} \leqslant \frac{c_\varkappa}{\varepsilon^{d/p}} \sup_{[0,t]} \|z\|_{2,p} \cdot \int_{R^3}\int_{R^3} \left(\int_{R^3} \chi(v)\, k_{++}^{p/(p-1)}(v,\xi)\, \chi(\xi)\, k_{++}^{p/(p-1)}(\xi, v_1)\, \chi(v_1)\, d\xi \right)^{2((p-1)/p)} dv_1\, dv\,.$$

By the definition of k_{++} one has

$$\sup_{[0,t]} \|(\chi\mathscr{P})^2 \chi z\|_{2,\infty}^{0} \leqslant \frac{c_\varkappa}{\varepsilon^{d/p}} \sup_{[0,t]} \|z\|_{2,p} \sup_{v, v_1 \in R^3} \int_{\mathscr{B}^3(\varkappa)} |\xi - v|^{-p/(p-1)} \cdot |\xi - v_1|^{-p/(p-1)}\, d\xi\,,$$

where $\mathscr{B}^3(\varkappa)$ is the ball of radius $\varkappa$ in R^3.

Thus if $p > 3$ then

$$\sup_{[0,t]} \|(\chi\mathscr{P})^2 \chi z\|_{2,\infty}^{0} \leqslant \frac{c_\varkappa}{\varepsilon^{d/p}} \sup_{[0,t]} \|z\|_{2,p}\,. \tag{8.9}$$

Now estimate $\|(\chi\mathscr{P})^2 \chi z(t)\|_{2,p}$. Like previously one has

$$\|(\chi\mathscr{P})^2 \chi z(t,\cdot,v); L_p(\Omega)\| \leqslant \frac{c_\varkappa}{\varepsilon^2} \int_0^t \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right) \int_0^{t'} \exp\left(-\frac{\nu_*}{\varepsilon}(t'-t'')\right)(t'-t'')^{-d} \cdot \int_{R^3} \left(\int_\Omega \left(\int_\Omega \chi(v)\, k_{++}(v,\xi)\, \chi(\xi)\, k_{++}(\xi, v_1)\, \chi(v_1) |z(t'', x_1, v_1)|\, dx_1 \right)^p dx \right)^{1/p} dv_1\, dt'\, dt''\,,$$

with $\xi = (x_1 - x + (t-t')v)/(t'-t'')$ where the notation for the case $d = 3$ has been used for simplicity. Young's theorem can now be applied:

$$\|(\chi\mathscr{P})^2 \chi z(t,\cdot,v); L_p(\Omega)\| \leqslant \frac{c_\varkappa}{\varepsilon^2} \int_0^t \exp\left(-\frac{\nu_*}{\varepsilon}(t-t')\right) \int_0^{t'} \exp\left(-\frac{\nu_*}{\varepsilon}(t'-t'')\right)(t'-t'')^{-d} \cdot \int_{R^3} \left(\int_\Omega \chi(v)\, k_{++}^{2p/(2+p)}(v,\eta)\, \chi(\eta)\, k_{++}^{2p/(2+p)}(\eta, v_1)\, \chi(v_1)\, dx \right)^{(2+p)/2p} \cdot \|z(t'',\cdot,v_1); L_2(\Omega)\|\, dv_1\, dt'\, dt''\,,$$

with

$$\eta = \frac{x + (t-t')v}{t'-t''}.$$

The substitution $x \to \eta = (x + (t - t')v)/(t' - t'')$ yields

$$\|(\chi\mathcal{P})^2 \chi z(t, \cdot, v); L_p(\Omega)\| \leqslant \frac{c_\varkappa}{\varepsilon^2} \int_0^t \exp\left(-\frac{\nu_*}{\varepsilon}(t - t')\right) \int_0^{t'} \exp\left(-\frac{\nu_*}{\varepsilon}(t' - t'')\right) \cdot$$

$$\cdot (t' - t'')^{-d((p-2)/2p)} \int_{R^3} \left(\int_{R^3} \chi(v) k_{++}^{2p/(2+p)}(v, \eta) \chi(\eta) k_{++}^{2p/(2+p)}(\eta, v_1) \chi(v_1) \, d\eta \right)^{(2+p)/2p} \cdot$$

$$\cdot \|z(t'', \cdot, v_1); L_2(\Omega)\| \, dv_1 \, dt' \, dt'' .$$

Thus, as before

$$\sup_{[0,t]} \|(\chi\mathcal{P})^2 \chi z\|_{2,p} \leqslant c_\varkappa \varepsilon^{-d((p-2)/2p)} \sup_{[0,t]} \|z\|_{2,2}, \tag{8.10}$$

provided that $3 < p < 6$. Now choosing p such that $3 < p < 6$ by (8.9) and (8.10) one obtains

$$\sup_{[0,t]} \|(\chi\mathcal{P})^4 \chi z\|_{2,\infty}^{;0} \leqslant c_\varkappa \varepsilon^{-d/2} \sup_{[0,t]} \|z\|_{2,2} . \tag{8.11}$$

At the end, we can choose $\varkappa$ sufficiently large and by the estimations (8.6*b*), (8.7*c*) and (8.11), applied to the eq. (8.5*d*) we can obtain a priori estimation (8.4). Thus, the proof of Lemma 8.2 is completed. ∎

Lemmas 8.1 and 8.2 imply the following lemma

LEMMA 8.3. – If $\varepsilon > 0$ is sufficiently small then the solution $(z_0(t), z_1(t), z_2(t))$ of the system (8.2) satisfies the following a priori estimation

$$\sup_{[0,t]} \|z_0\|_{\infty,\infty}^{\beta;0} \leqslant \frac{\text{const}}{\varepsilon^{d/2}} \sup_{[0,t]} \|z_0\|_{2,2} + \text{const}\, \varepsilon \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{-1,0}, \tag{8.12a}$$

$$\sup_{[0,t]} \|z_1\|_{\infty,\infty}^{\beta;0} \leqslant \text{const}\, \varepsilon^{1-d/2} \sup_{[0,t]} \|z_0\|_{2,2} + \text{const}\, \varepsilon \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{\beta-1,0}, \tag{8.12b}$$

$$\|z_2(t)\|_{\infty,\infty}^{\beta;0} \leqslant \frac{\text{const}}{\varepsilon^{d/2}} \theta \exp\left(-\frac{t}{\varepsilon}\delta_*\right) \sup_{[0,t]} \|z_0\|_{2,2} + \text{const}\, \varepsilon \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{-1,0}, \tag{8.12c}$$

for all $t \in]0, t_0]$ and $\beta \geqslant 0$ where the constants denoted by «const» can depend on $\varkappa$.

PROOF. – Inequalities (8.4), (8.3*b*) and (8.3*c*) imply

$$\sup_{[0,t]} \|z_0\|_{2,\infty}^{;0} \leqslant \frac{\text{const}}{\varepsilon^{d/2}} \sup_{[0,t]} \|z_0\|_{2,2} + \text{const} \cdot (\varepsilon + \theta) \sup_{[0,t]} \|z_0\|_{2,\infty}^{;0} + \text{const}\, \varepsilon \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{-1,0} .$$

Since θ and ε are small enough

$$\sup_{[0,t]} \|z_0\|_{2,\infty}^{1,0} \leqslant \frac{\text{const}}{\varepsilon^{d/2}} \sup_{[0,t]} \|z_0\|_{2,2} + \text{const}\,\varepsilon \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{-1,0}. \tag{8.13}$$

Now (8.3) together with (8.13) give (8.12) and the proof is completed. ■

Now, the main lemmas can be formulated

LEMMA 8.4. – Let $\mathfrak{a}^\# \in L_\infty([0, t_0]; X_{\infty,\infty}^{\beta-1,0})$ for some $\beta \geqslant 2$. If ε is sufficiently small then a unique solution (z_0, z_1, z_2) of the linear system (8.2) exists in $(L_\infty([0, t_0]; X_{\infty,\infty}^{\beta,0}))^3$. Moreover,

$$z_i^\# \in C^0([0, t_0]; X_{\infty,\infty}^{\beta-1,0}), \qquad \frac{d}{dt} z_i^\# \in L_\infty([0, t_0]; X_{\infty,\infty}^{\beta-2,0})$$

for $i = 0, 1, 2$ and the following estimations are satisfied

$$\sup_{[0,t]} \|z_i\|_{\infty,\infty}^{\beta,0} \leqslant \frac{\text{const}}{\varepsilon^{d/2}} \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{\beta-1,0}, \tag{8.14}$$

where the constant denoted by «const» depends (exponentially) on t_0.

PROOF. – Using the same arguments as in [15] (cf. [15, (5.36)]) one can prove that

$$\sup_{[0,t]} \|z_0\|_{2,2} \leqslant c_{t_0} \sup_{[0,t]} \|\mathfrak{a}\|_{\infty,\infty}^{-1,0}. \tag{8.15}$$

Then (8.14) follows by (8.12) and (8.15).

Existence and smoothness of the unique solution can be proven in the same way as in [14]. ■

Differentiating system (8.2) formally with respect to x, we can obtain the system for the derivatives of z_0, z_1, z_2. Now, requiring higher smoothness properties in Assumptions 5.1 and 7.1 the analogous estimations as in [15] (cf. [15, (5.39)]) can be applied to see that with each differentiation one power of ε is lost. Thus we have

LEMMA 8.5. – Let $\mathfrak{a}^\# \in L_\infty([0, t_0]; X_{\infty,\infty}^{\beta-1,k})$ for some $\beta \geqslant 2$ and $k \geqslant 0$. If ε is sufficiently small then a unique solution (z_0, z_1, z_2) of the linear system (8.2) exists in $(L_\infty([0, t_0]; X_{\infty,\infty}^{\beta,k}))^3$. The solution satisfies

$$z_i^\# \in C^0([0, t_0]; X_{\infty,\infty}^{\beta-1,k}), \qquad \frac{d}{dt} z_i^\# \in L_\infty([0, t_0]; X_{\infty,\infty}^{\beta-2,k})$$

for $i = 0, 1, 2$, and

$$\sup_{[0,t_0]} \|z_i\|_{\infty,\infty}^{\beta,k} \leqslant \frac{\text{const}}{\varepsilon^{d/2+k}} \sup_{[0,t_0]} \|\mathfrak{a}\|_{\infty,\infty}^{\beta-1,k}. \tag{8.16}$$

Moreover, if in addition $k \geq 2$ and $\mathcal{a} \in C^0([0, t_0]; X^{\beta-2, k-1}_{\infty, \infty})$ then

$$z_i \in C^0([0, t_0]; X^{\beta-1, k-1}_{\infty, \infty}) \cap C^1([0, t_0]; X^{\beta-2, k-2}_{\infty, \infty}). \quad \blacksquare$$

9. – The Euler limit for BE.

If a and b in (7.1) are such that

$$b_* \equiv b - 1 - \frac{d}{2} > k, \tag{9.1a}$$

and

$$a_* \equiv a - b - \frac{d}{2} \geq k - b_*, \tag{9.1b}$$

with $k = 0, 1, 2, \ldots$ then Lemma 8.5 can be used to construct a solution of equation (7.9) by the successive approximation method. In this way the Euler limit theorem for BE is obtained

THEOREM 9.1. – Let $k \geq 0$ and let Assumptions 5.1 and 7.1 be satisfied with α and s depending on k (cf. (7.5)). If $0 < \varepsilon \leq \varepsilon_0$, where ε_0 is a critical value depending on t_0, then a solution f_B of BE with initial data F exists in $L_\infty([0, t_0]; Y^{\beta, k}_+)$ and

$$f^\#_B \in C^0([0, t_0]; Y^{\beta, k}_+), \tag{9.2a}$$

$$\frac{d}{dt} f^\#_B \in L_\infty([0, t_0]; Y^{\beta, k}_+), \tag{9.2b}$$

$$\sup_{t \in [0, t_0]} N^{\beta, k}_+ \left\{ f_B(t) - M(t) - l_0\left(\frac{t}{\varepsilon}\right)\right\} \leq c_{t_0} \cdot \varepsilon, \tag{9.3}$$

for all $\beta \geq 0$. Moreover, if $k \geq 2$ then

$$f_B \in C^0([0, t_0]; Y^{\beta, k-1}_+) \cap C^1([0, t_0]; Y^{\beta, k-2}_+). \tag{9.4}$$

In Theorem 9.1 the asymptotic relationship between BE and SCEE as well as the initial layer effect (cf. (9.3)) are defined. A smaller number of Hilbert and initial layer terms is used as compared to Theorem 3 of [15]. In fact, by (9.1), it is enough to take

$$b > 1 + \frac{d}{2}$$

and

$$a = 1 + d.$$

However to prove that the solution f_B is a strong one, we have to take

$$b > 3 + \frac{d}{2}, \qquad a = 3 + d .$$

10. – The Euler limit for EE_1 and PE_1.

In this Section we follow the methods of [17] and [16]. A solution f_{E1} of EE_1 is searched in the form

$$f_{E1} = f_B + \sigma z , \tag{10.1}$$

where f_B is the solution of BE given by Theorem 9.1 and the «remainder» z satisfies the following nonlinear equation

$$Dz = \frac{2}{\varepsilon} J_\sigma(f_B, z) + \frac{\sigma}{\varepsilon} J_\sigma(z, z) + \frac{2\sigma}{\varepsilon^2} E_{\sigma,\varepsilon}(f_B + \sigma z, f_B, z) + \tag{10.2a}$$

$$+ \frac{1}{\varepsilon^2} E_{\sigma,\varepsilon}(f_B + \sigma z; f_B, f_B) + \frac{\sigma^2}{\varepsilon^2} E_{\sigma,\varepsilon}(f_B + \sigma z;\, z, z) + \frac{1}{\varepsilon} \hat{J}_\sigma^{(1)}(f_B) ,$$

with initial data

$$z|_{t=0} = 0 , \tag{10.2b}$$

where

$$\hat{J}_\sigma^{(1)}(f) = \sigma^{-1}(J_\sigma(f, f) - J_0(f, f)) .$$

By virtue of Theorem 9.1, one can consider f_B to be sufficiently smooth with respect to x-variable and thus

$$\sup_{[0, t_0]} N_+^{\beta - 1, 0}\{\hat{J}_\sigma^{(1)}(f_B)\} \leq c_{t_0} \tag{10.3}$$

(cf. estimations in [15] and [16]) where the constant c_{t_0} is independent of σ and ε but depends on t_0. The methods from Section 8 cannot be used to study the problem (10.2), because the symmetrized version of the operator $J_\sigma(M, \cdot)$, (i.e. the operator $L_\sigma = M^{-1/2} J_\sigma(M, M^{1/2} \cdot)$ for the case $\Phi \equiv \Phi_1$) does not have $L_2(R^3)$-properties similar to those of L_0. Thus, the theory given in [15] cannot be applied. However, under rather restrictive assumption on relationship between σ and ε, the following theorem can be proved using the ideas of [17]:

THEOREM 10.1. – Let the conditions of Theorem 9.1 with $k \geq 1$ and let Assumption 3.1 with $\mu = 1$ be satisfied. Moreover, let

$$0 < \sigma \exp\left(\frac{ct_0}{\varepsilon}\right) < c , \tag{10.4}$$

where the constants denoted by c are properly chosen. Then a solution f_{E1} of EE_1 with initial data F exists in $L_\infty([0, t_0]; Y_+^{\beta, k-1})$, and

$$f_{E1}^{\#} \in C^0([0, t_0]; Y_+^{\beta, k-1}), \tag{10.5a}$$

$$\frac{d}{dt} f_{E1}^{\#} \in L_\infty([0, t_0]; Y_+^{\beta, k-1}), \tag{10.5b}$$

$$\sup_{[0, t_0]} N_+^{\beta, k-1}\{f_{E1} - f_B\} \leq c_{t_0} \sigma^{1/2}. \tag{10.6}$$

Moreover, if $k \geq 3$ then

$$f_{E1} \in C^0([0, t_0]; Y_+^{\beta, k-2}) \cap C^1([0, t_0]; Y_+^{\beta, k-3}). \tag{10.7}$$

PROOF. – Under Assumption (10.4) with c properly chosen, the equation (10.2) becomes weakly nonlinear. In fact, the nonlinear term is multiplied by a small number and nonhomogeneous term has no singularities with respect to σ. Thus, by Assumption 3.1 the existence (and uniqueness) can be proved by the approximation method, as it was done previously. The solution z is of the order of $\varepsilon^{-1-d/2-k}$ (ε is considered to be small but fixed). Thus, by (10.1) and again by (10.4) the relationship (10.6) can be established. ∎

REMARK 10.1. – By (9.3) and (10.4) the inequality (10.6) leads to

$$\sup_{t \in [0, t_0]} N_+^{\beta, k-1}\left\{f_{E1}(t) - M(t) - l_0\left(\frac{t}{\varepsilon}\right)\right\} \leq c_{t_0} \varepsilon, \tag{10.8}$$

giving the asymptotic relationship between EE_1 and SCEE.

REMARK 10.2. – Existence of the solution f_{P1} can be proved exactly in the same way (cf. [17]). The formulation of the existence theorem is similar to the one for EE_1. Moreover,

$$\sup_{t \in [0, t_0]} N_+^{\beta, k-1}\left\{f_{P1}(t) - M(t) - l_0\left(\frac{t}{\varepsilon}\right)\right\} \leq c_{t_0} \varepsilon. \tag{10.9}$$

11. – The Euler limit for EE_2 and PE_2..

In contrast to the case $\Phi \equiv \Phi_1$, in the case $\Phi \equiv \Phi_2$ we can use L_2-estimates of the operator $L_\sigma = M^{-1/2} J_\sigma(M, M^{1/2} \cdot)$ (cf. Lemma 6.1 in [16]) and apply the methods from Section 8. Thus, following [16], one has

THEOREM 11.1. – Let the condition of Theorem 9.1 with $k \geq 1$ and let Assumption 3.1 with $\mu \geq 6 + d + 2(k-1)$ be satisfied. Then a solution f_{E2} of EE_2 with initial data F

exists in $L_\infty([0, t_0]; Y_+^{\beta, k-1})$ and

$$f_{E2}^{\#} \in C^0([0, t_0]; Y_+^{\beta, k-1}), \tag{11.1a}$$

$$\frac{d}{dt} f_{E2}^{\#} \in L_\infty([0, t_0]; Y_+^{\beta, k-1}), \tag{11.1b}$$

$$\sup_{t \in [0, t_0]} N_+^{\beta, k-1} \left\{ f_{E2}(t) - M(t) - l_0\left(\frac{t}{\varepsilon}\right) \right\} \leq c_{t_0} \varepsilon . \tag{11.2}$$

Moreover, if $k \geq 3$ then

$$f_{E2} \in C^0([0, t_0]; Y_+^{\beta, k-2}) \cap C^1([0, t_0]; Y_+^{\beta, k-3}). \quad \blacksquare \tag{11.3}$$

In the same way the theorem for PE_2 can be formulated. In this case it is enough to take $\mu \geq 4 + d + 2(k-1)$.

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