Asymptotic Expansions Obtained by a Center Manifold Theorem (*).

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Sunto. – *Viene presentato un nuovo metodo per la determinazione degli sviluppi asintotici della soluzione esterna di sistemi di equazioni differenziali ordinarie singolarmente perturbati. Il metodo proposto, basato sulla teoria geometrica delle perturbazioni singolari e in particolare su un teorema di esistenza di varietà centrale, permette di ottenere le equazioni differenziali che definiscono le variabili « lente » senza la preventiva conoscenza dei corrispondenti sviluppi per le variabili « veloci ». Inoltre, se i sistemi vengono dati con condizioni iniziali, alcune formule che esprimono le corrette condizioni iniziali da assegnare alle equazioni differenziali trovate — formule già note nel « caso stabile » — vengono estese al « caso condizionalmente stabile »; il procedimento qui usato risulta anche più sintetico rispetto a quelli precedentemente proposti. Infine viene studiata un'applicazione ad una classe assai generale di equazioni derivanti dalla cinetica delle reazioni enzimatiche.*

1. – Introduction.

Consider the singularly perturbed system of ordinary differential equations

$$\varepsilon\dot{\xi} = F(\xi, \eta, \varepsilon)\,, \quad \dot{\eta} = G(\xi, \eta, \varepsilon)\,, \qquad (\dot{\ }) = d/dt \tag{1.1}$$

where $\xi, F \in R^\nu$, $\eta, G \in R^\mu$, $(\xi, \eta) \in \Omega \subset R^{\nu+\mu}$, $\varepsilon \in [0, \varepsilon_0) \subset R$, and $F, G \in C^{r+2}(\Omega \times [0, \varepsilon_0))$, $r \geq 0$.

It is known [8, Theorems 1 and 2] that, under suitable conditions—summarized in a simpler form in the next Section 2 which contains also basic notations and hypotheses—, the solution of (1.1) can be written as:

$$\begin{pmatrix} \xi(t, \varepsilon) \\ \eta(t, \varepsilon) \end{pmatrix} = \Gamma(t/\varepsilon, \varepsilon) + \gamma(t, \varepsilon)\,,$$

where Γ, γ are C^{r+1} in their arguments, and $|\Gamma(\tau, \varepsilon)| \leq C \exp\{-\delta\tau\}$, $C, \delta > 0$. $\Gamma(t/\varepsilon, \varepsilon)$ and $\gamma(t, \varepsilon)$ are called « inner solution » and « outer solution » of (1.1), respectively.

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In Section 3 of this paper we will present a new method of finding the coefficients $\gamma_i(t)$ of the smooth expansion $\gamma(t, \varepsilon) = \sum_{i=0}^{r} \gamma_i(t)\varepsilon^i + O(\varepsilon^{r+1})$. This method depends on the existence of a center manifold $\mathcal{C}$ for the system (1.1) such that $\gamma(t, \varepsilon)$ is the solution of (1.1) restricted to $\mathcal{C}$ [4, Theorem 9.1]. Even if $\mathcal{C}$ is, in general, not unique, the coefficients $\gamma_i(t)$ are unique [5, 8].

Differential equations defining the γ_i's are then obtained using approximations of $\mathcal{C}$, whose equations are computed in terms of F and G, provided that the jacobian of F is non-singular. Moreover, this procedure will also allow us to obtain the differential equations for the asymptotic expansions of the «slow variables» $\eta(t, \varepsilon)$ without involving the corresponding expansions of the «fast variables» $\xi(t, \varepsilon)$ (see formula (3.9) of the present paper). This can be helpful in many applications when one is mainly interested in the behaviour of the slowly varying components.

Previously developed results (see, for example, [8, 13]) gave the coefficients of the asymptotic expansions of the slow variables by means of the differential equations involving the coefficients of the asymptotic expansions of the fast variables too, these last being obtained by solving suitable algebraic equations.

Section 4 deals with a problem which is strictly related to the previous part: consider the Cauchy problem given by (1.1) with the initial condition, say $p(\varepsilon) = (\xi^0(\varepsilon), \eta^0(\varepsilon))$; it is quite natural to ask which is the correct initial condition to assign to the differential equation defining the i-th coefficient $\gamma_i(t)$ of the expansion of $\gamma(t, \varepsilon)$.

The answer is given by the formula (4.3) of this paper; the same formula has been also obtained in [13] for the stable case, i.e. when the jacobian of the so-called «boundary layer system» (the system (2.1) in the following) has all the eigenvalues with negative real part. We shall prove that the validity of those formulae can be extended to the «conditionally stable case», i.e. when the quoted jacobian has no eigenvalues with zero real part. Our proof, based on the existence of a center-stable manifold [4], is also simpler than the one given in [13].

Finally, Section 5 is concerned with an application of the results of the previous part to the differential equations describing the kinetics of a wide class of enzyme-catalyzed reaction systems for which biochemists are commonly interested in the temporal behaviour of the slow species. Usually, for these systems the zero-approximation $\gamma_0(t)$ is studied.

Here, for the very general case considered, we are able to write down explicitly the differential equations (with initial conditions) for the first approximation $\gamma_0(t) + \varepsilon\gamma_1(t)$ of the outer solution.

2. – Notations and hypotheses.

For a function F of several variables, $D_i^k F$ will denote the k-th derivative of F with respect to the i-th variable.

We make the following assumptions on (1.1):

(i) $F(\xi, \eta, 0) = 0$ has the solution $\xi = \varphi(\eta)$, for $\eta \in D \subset R^{\mu}$, D being a compact set.

(ii) There exists $\bar{\eta} \in \mathring{D}$ such that $G(\varphi(\bar{\eta}), \bar{\eta}, 0) = 0$, and $D_1 F(\varphi(\bar{\eta}), \bar{\eta}, 0)$ has no eigenvalues with zero real part.

From (ii) we deduce the existence of a compact neighbourhood $U \subset D$, such that $\bar{\eta} \in U$, and $D_1 F(\varphi(\eta), \eta, 0)$ has no eigenvalues with zero real part for any $\eta \in U$. Without loss of generality one can then suppose:

(iii) For any $\eta \in D$, $D_1 F(\varphi(\eta), \eta, 0)$ has no eigenvalues with zero real part.

From (iii) it follows that the boundary layer system:

$$\xi' = F(\xi, \eta, 0)\,, \qquad \xi' = d\xi/d\sigma\,,\ \sigma = t/\varepsilon\,, \tag{2.1}$$

has a hyperbolic equilibrium at $\varphi(\eta)$, for any $\eta \in D$, and then $\varphi(\eta)$ is an isolated root of the equation $F(\xi, \eta, 0) = 0$ in D. Let us summarize the previous arguments as follows:

H1) $F(\xi, \eta, 0) = 0$ has the isolated root $\xi = \varphi(\eta)$, $\eta \in D$; such a root is a hyperbolic fixed point of the boundary layer system (2.1), for any $\eta \in D$.

H2) There exists $\bar{\eta} \in \mathring{D}$ such that $G(\varphi(\bar{\eta}), \bar{\eta}, 0) = 0$, i.e. the degenerate system:

$$\dot{\eta} = G(\varphi(\eta), \eta, 0) \tag{2.2}$$

has a fixed point at $\bar{\eta} \in \mathring{D}$.

From the Implicit Function Theorem and H1) we get the existence of $\varphi(\eta, \varepsilon)$ such that $\varphi(\eta, 0) = \varphi(\eta)$ and $F(\varphi(\eta, \varepsilon), \eta, \varepsilon) \equiv 0$. Consider then the change of variables

$$\begin{cases} x = \xi - \varphi(\eta, \varepsilon)\,, \\ y = \eta - \bar{\eta}\,, \end{cases} \tag{2.3}$$

by means of which (1.1) and (2.2) read, respectively:

$$\begin{cases} \varepsilon\dot{x} = f(x, y, \varepsilon)\,, \\ \dot{y} \;= g(x, y, \varepsilon)\,, \end{cases} \tag{2.4}$$

and

$$\dot{y} = g(0, y, 0) \tag{2.5}$$

where:

$$f(x, y, \varepsilon) = F\big(x + \varphi(y + \bar{\eta}, \varepsilon), y + \bar{\eta}, \varepsilon\big) - \varepsilon D_1\varphi(y + \bar{\eta}, \varepsilon) G\big(x + \varphi(y + \bar{\eta}, \varepsilon), y + \bar{\eta}, \varepsilon\big)\,,$$

and

$$g(x, y, \varepsilon) = G(x + \varphi(y + \bar{\eta}, \varepsilon), y + \bar{\eta}, \varepsilon) .$$

We will also need the following auxiliary system

$$\text{(2.6)} \qquad \begin{cases} x' = f(x, y, \varepsilon), \\ y' = \varepsilon g(x, y, \varepsilon) , \\ \varepsilon' = 0 , \end{cases}$$

resulting from (2.4) by the time change $\sigma = t/\varepsilon$. The system (2.6) has the manifold of fixed points $E = \{(0, y, 0) \colon (0, y) \in \{0\} \times R^\mu \cap \Omega\}$. The Jacobian matrix of (2.6) evaluated at $(0, 0, 0) \in R^\nu \times R^\mu \times R$, is:

$$J(0, 0, 0) = \begin{bmatrix} D_1 f(0, 0, 0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

System (2.6) reads also:

$$\text{(2.7)} \qquad \begin{cases} x' = Ax + h(x, y, \varepsilon) , \\ y' = \varepsilon g(x, y, \varepsilon) , \\ \varepsilon' = 0 , \end{cases}$$

where

$$A := D_1 f(0, 0, 0) = D_1 F(\varphi(\bar{\eta}), \bar{\eta}, 0) , \qquad h(0, y, 0) \equiv 0 , \qquad D_j h(0, 0, 0) = 0$$

for $j = 1, 2, 3$, and $g(0, 0, 0) = 0$.

Let K be the image of D by the change of variables (2.3). From Theorem 9.1 in [4], we can deduce the existence of a C^{r+2}-center-manifold $\mathcal{C}$ for the system (2.6) near K (in the rest of this paper we shall simply say « a center manifold $\mathcal{C}$ »). Following J. CARR ([2], formula (1.3.6)), we can locally define $\mathcal{C}$ as the graph of a C^{r+2}-function $x = X(y, \varepsilon)$ such that:

$$\text{(2.8)} \qquad \mathcal{L}\{X(y, \varepsilon)\} := \varepsilon D_1 X(y, \varepsilon) g(X(y, \varepsilon), y, \varepsilon) - AX(y, \varepsilon) - h(X(y, \varepsilon), y, \varepsilon) \equiv 0 ,$$

and

$$X(y, 0) \equiv 0 , \qquad D_1 X(y, 0) \equiv 0 , \qquad D_2 X(0, 0) = 0 ,$$

for any y in a suitable compact set $K_1 \subset K$.

DEFINITION. – A C^1-function $u = u(y, \varepsilon)$, $(y, \varepsilon) \in \Omega_1 \times [0, \varepsilon_0) \subset R^\mu \times R$, $K \subset \Omega_1$, is a q-approximation of a center manifold $\mathcal{C}$ if:

$$\mathcal{L}\{u(y, \varepsilon)\} = O(\varepsilon^{q+1})$$

uniformly in $y \in K$.

For example a 0-approximation of a center manifold is $u(y, \varepsilon) \equiv 0$, since

$$\mathcal{L}\{0\} = -h(0, y, \varepsilon) = -\varepsilon D_3 h(0, y, \varepsilon^*) = O(\varepsilon)$$

uniformly in $y \in \Omega_1 \subset R^\mu$, ε^* being a suitable point in $[0, \varepsilon_0)$. Note also that $u(y, \varepsilon) \equiv 0$ is the so called «slow manifold» of the system (2.4).

3. – Asymptotic expansions by a q-approximation of a center manifold.

Firstly, we prove the following

THEOREM 1. – Let $u(y, \varepsilon)$ be a q-approximation of a center manifold $\mathcal{C}$. Then a (outer) solution of the system (2.4) on $\mathcal{C}$ satisfies:

$$\begin{cases} x(t, \varepsilon) = u\big(y(t, \varepsilon), \varepsilon\big) + O(\varepsilon^{q+1})\,, \\ \dot y(t, \varepsilon) = g\Big(u\big(y(t, \varepsilon), \varepsilon\big), y(t, \varepsilon), \varepsilon\Big) + O(\varepsilon^{q+1})\,. \end{cases} \tag{3.1}$$

Furthermore, if $\hat u(t, \varepsilon)$ is C^{r+1} with respect to ε, and $\hat u(t, \varepsilon) - u\big(y(t, \varepsilon), \varepsilon\big) = O(\varepsilon^{q+1})$, then the coefficients $\big(x_i(t), y_i(t)\big)$ of the Taylor expansions:

$$x(t, \varepsilon) = \sum_{i=0}^{q} x_i(t)\, \varepsilon^i + O(\varepsilon^{q+1})\,,$$

$$y(t, \varepsilon) = \sum_{i=0}^{q} y_i(t)\, \varepsilon^i + O(\varepsilon^{q+1})\,, \qquad q \leqslant r\,,$$

satisfy:

$$\begin{cases} x_i(t) = \dfrac{1}{i!}\dfrac{\partial^i}{\partial \varepsilon^i}\{\hat u(t, \varepsilon)\}\Big|_{\varepsilon=0}\,, \\ \dot y_i(t) = \dfrac{1}{i!}\dfrac{\partial^i}{\partial \varepsilon^i}\Big\{g\Big(\hat u(t, \varepsilon), \sum_{j=0}^{i} y_j(t)\,\varepsilon^j, \varepsilon\Big)\Big\}\Big|_{\varepsilon=0}\,, \end{cases} \tag{3.2}$$

for any $i \leqslant q$.

REMARKS. – (i) If we assume $u(y, \varepsilon) \in C^{r+1}(\Omega_1 \times [0, \varepsilon_0))$, the passage to the function $\hat{u}(t, \varepsilon)$ is obviously superfluous, and the thesis (3.2) still holds true with $\hat{u}$ replaced by u.

(ii) Formulae (3.1) and (3.2) hold true, generally, only on those finite time intervals $[0, T]$ such that the zero-approximation belongs to K for $t \in [0, T]$, since, in this case, $O(\varepsilon^{q+1})$ is surely uniform in t. In case the zero-approximation belongs to K for any $t \in [0, +\infty)$, the validity of (3.1) and (3.2), uniformly on $[0, +\infty)$, requires other assumptions. A remarkable case was studied in [8], where a suitable exponential dichotomy hypothesis on the degenerate system (2.5) is taken into account.

The first part of the proof of Theorem 1 depends essentially on the following

LEMMA 1. – Let B_1, B_2 be matrices with all eigenvalues with zero real part, and A be a matrix with eigenvalues with non-zero real part. Consider the system:

$$\left\{\begin{aligned} x' &= Ax + h(x, y_1, y_2)\,, \\ y_1' &= B_1 y_1 + B y_2 + g_1(x, y_1, y_2)\,, \\ y_2' &= B_2 y_2 + g_2(x, y_1, y_2)\,, \end{aligned}\right. \tag{3.3}$$

where h, g_1, g_2 represent higher order terms.

If $x = X(y_1, y_2)$ is a center manifold for the system (3.3) and $u(y_1, y_2)$ satisfies: $\mathfrak{L}\{u(y_1, y_2)\} = O(|y_2|^{q+1})$, uniformly in y_1 (here $\mathfrak{L}$ is defined in a way similar to (2.8)), then:

$$X(y_1, y_2) - u(y_1, y_2) = O(|y_2|^{q+1})$$

uniformly in y_1.

This Lemma is a simple generalization of Theorem 5, pag. 32, in [2]. So, we omit the proof.

PROOF OF THEOREM 1. – Since K is compact in $\mathcal{C}$, we can suppose, without loss of generality, that $\mathcal{C}$ is defined near K as the graph of a C^{r+2}-function $x = X(y, \varepsilon)$ such that the outer solution satisfies:

$$\begin{aligned} x(t, \varepsilon) &= X(y(t, \varepsilon), \varepsilon)\,, \\ \dot{y}(t, \varepsilon) &= g\big(X(y(t, \varepsilon), \varepsilon), y(t, \varepsilon), \varepsilon\big)\,. \end{aligned}$$

An easy application of Lemma 1 with $y_1 = y$ and $y_2 = \varepsilon$, gives:

$$X(y, \varepsilon) - u(y, \varepsilon) = O(\varepsilon^{q+1})\,,$$

uniformly in $y \in \Omega_1$. Then,

$$g(X(y, \varepsilon), y, \varepsilon) - g(u(y, \varepsilon), y, \varepsilon) = O(\varepsilon^{q+1})$$

and (3.1) is proved.

The assumptions on $\hat{u}(t, \varepsilon)$ also give, for any $k \leq q$,

$$g\big(X(y(t, \varepsilon), \varepsilon), y(t, \varepsilon), \varepsilon\big) - g\Big(\hat{u}(t, \varepsilon), \sum_{i=0}^{k} y_i(t)\, \varepsilon^i, \varepsilon\Big) = O(\varepsilon^{k+1}) .$$

Then the k-th derivative of the left-hand side of this equality (which now exists since $\hat{u}$ is C^r in ε) vanishes at $\varepsilon = 0$. The definition of $y_k(t)$, i.e.

$$y_k(t) = \frac{1}{k!} \frac{\partial^k}{\partial \varepsilon^k} y(t, \varepsilon)\Big|_{\varepsilon=0}$$

gives immediately the thesis of the Theorem.

The next result gives us as iterative method to construct, in practice, a q-approximation $u(y, \varepsilon)$ of a center manifold $\mathbb{C}$. This also increases the importance of the previous Theorem 1.

THEOREM 2. - Let $U_k(y)$ be the C^{r-k+2}-function in Ω_1, iteratively defined by:

$$D_1 f(0, y, 0)\, U_k(y) = \sum_{j=0}^{k-2} \binom{k}{j} (k-j)\, D U_{k-j-1}(y) \frac{\partial^j}{\partial \varepsilon^j} g\Big(\sum_{i=1}^{j} U_i(y) \frac{\varepsilon^i}{i!}, y, \varepsilon\Big)\Big|_{\varepsilon=0} + \\ - \frac{\partial^k}{\partial \varepsilon^k} f\Big(\sum_{i=1}^{k-1} U_i(y) \frac{\varepsilon^i}{i!}, y, \varepsilon\Big)\Big|_{\varepsilon=0}, \tag{3.4}$$

where $\sum^{\alpha} := 0$ if $\alpha \leq 0$. Then,

$$u(y, \varepsilon) := \sum_{k=1}^{q} U_k(y) \frac{\varepsilon^k}{k!} \tag{3.5}$$

is a q-approximation on Ω_1 of the center manifold $\mathbb{C}$ for any $q \leq r$. Moreover, setting:

$$\hat{U}_k(t, \varepsilon) = \sum_{h=0}^{q-k} \frac{\partial^h}{\partial \varepsilon^h} U_k\big(y(t, \varepsilon)\big)\Big|_{\varepsilon=0} \frac{\varepsilon^h}{h!},$$

then $\hat{U}_k(t, \varepsilon)$ is C^∞ with respect to ε, and $\hat{U}_k(t, \varepsilon) = U_k\big(y(t, \varepsilon)\big) + O(\varepsilon^{q-k+1})$. Finally, the function

$$\hat{u}(t, \varepsilon) := \sum_{k=1}^{q} \hat{U}_k(t, \varepsilon) \frac{\varepsilon^k}{k!}$$

is C^∞ with respect to ε, and $\hat{u}(t, \varepsilon) - u\big(y(t, \varepsilon), \varepsilon\big) = O(\varepsilon^{q+1})$.

PROOF. – We prove that $u(y, \varepsilon)$ defined by (3.5) and (3.4) is a q-approximation of $\mathcal{C}$ showing that

$$\mathcal{L}\{u(y, \varepsilon)\} = \mathcal{L}\left\{\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}\right\} = O(\varepsilon^{q+1}),$$

uniformly in $y \in \Omega_1$. This last condition explicitly writes:

$$\sum_{k=1}^{q} DU_k(y)\frac{\varepsilon^{k+1}}{k!}\, g\left(\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) - f\left(\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) = O(\varepsilon^{q+1}), \tag{3.6}$$

For any function φ of class C^{r+2} we firstly prove that

$$\varphi\left(\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) - D_1\varphi(0, y, 0)\, U_j(y)\frac{\varepsilon^j}{j!} - \varphi\left(\sum_{k=1}^{j-1} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) = O(\varepsilon^{j+1}), \tag{3.7}$$

for any $j \leqslant q$. In fact:

$$\left|\varphi\left(\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) - \varphi\left(\sum_{k=1}^{j} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right)\right| +$$

$$+ \left|\varphi\left(\sum_{k=1}^{j} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) - \varphi\left(\sum_{k=1}^{j-1} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right) - D_1\varphi(0, y, 0)\, U_j(y)\frac{\varepsilon^j}{j!}\right| \leqslant$$

$$\leqslant C(y)\,\varepsilon^{j+1} + \left|\int_0^{U_j(y)\varepsilon^j/j!} \left[D_1\varphi\left(\sum_{k=1}^{j-1} U_k(y)\frac{\varepsilon^k}{k!} + \eta, y, \varepsilon\right) - D_1\varphi(0, y, 0)\right] d\eta\right| \leqslant$$

$$\leqslant C(y)\,\varepsilon^{j+1} + \int_0^{U_j(y)\varepsilon^j/j!} [C_1(y)\,\varepsilon + C_2(y)\eta]\, d\eta \leqslant \hat{C}(y)\,\varepsilon^{j+1} = O(\varepsilon^{j+1}).$$

From (3.7) we obtain the following equalities, for any $h < j \leqslant q \leqslant r$,

$$\frac{\partial^j}{\partial\varepsilon^j}\varphi\left(\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right)\bigg|_{\varepsilon=0} = D_1\varphi(0, y, 0)\, U_j(y) + \frac{\partial^j}{\partial\varepsilon^j}\varphi\left(\sum_{k=1}^{j-1} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right)\bigg|_{\varepsilon=0}, \tag{3.8a}$$

$$\frac{\partial^h}{\partial\varepsilon^h}\varphi\left(\sum_{k=1}^{q} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right)\bigg|_{\varepsilon=0} = \frac{\partial^h}{\partial\varepsilon^h}\varphi\left(\sum_{k=1}^{h} U_k(y)\frac{\varepsilon^k}{k!}, y, \varepsilon\right)\bigg|_{\varepsilon=0}. \tag{3.8b}$$

Conditions on the $U_k(y)$ for (3.6) holds true, are now found imposing that the j-th derivative with respect to ε of the left-hand side of (3.6) vanishes at $\varepsilon = 0$, for any $j = 1, \ldots, q$. Taking into account (3.5) and (3.7), this is equivalent to:

$$\begin{aligned}\frac{\partial^j}{\partial\varepsilon^j}\left\{\sum_{k=1}^{q} DU_k(y)\,\frac{\varepsilon^{k+1}}{k!}\,g\left(\sum_{k=1}^{q} U_k(y)\,\frac{\varepsilon^k}{k!},\, y,\, \varepsilon\right)\right\}\Bigg|_{\varepsilon=0} - \frac{\partial^j}{\partial\varepsilon^j} f\left(\sum_{k=1}^{q} U_k(y)\,\frac{\varepsilon^k}{k!},\, y,\, \varepsilon\right)\Bigg|_{\varepsilon=0} = \\ = \frac{\partial^j}{\partial\varepsilon^j}\left\{\sum_{k=1}^{j-1} DU_k(y)\,\frac{\varepsilon^{k+1}}{k!}\,g\left(\sum_{k=1}^{q} U_k(y)\,\frac{\varepsilon^k}{k!},\, y,\, \varepsilon\right)\right\}\Bigg|_{\varepsilon=0} + \\ - D_1 f(0, y, 0)\, U_j(y) - \frac{\partial^j}{\partial\varepsilon^j} f\left(\sum_{k=1}^{j-1} U_k(y)\,\frac{\varepsilon^k}{k!},\, y,\, \varepsilon\right)\Bigg|_{\varepsilon=0} = \\ = \sum_{h=0}^{j-2}\binom{j}{h}(j-h)\,DU_{j-h-1}(y)\,\frac{\partial^h}{\partial\varepsilon^h}\,g\left(\sum_{k=1}^{h} U_k(y)\,\frac{\varepsilon^k}{k!},\, y,\, \varepsilon\right)\Bigg|_{\varepsilon=0} + \\ - D_1 f(0, y, 0)\, U_j(y) - \frac{\partial^j}{\partial\varepsilon^j} f\left(\sum_{k=1}^{j-1} U_k(y)\,\frac{\varepsilon^k}{k!},\, y,\, \varepsilon\right)\Bigg|_{\varepsilon=0},\end{aligned}$$

for $j \leq q$. This proves the first part of the Theorem. The last part is an easy computation and follows from the definition of $\hat{U}_k(t, \varepsilon)$.

REMARKS. – (i) The results of both Theorem 1 and 2 can be used in order to find the differential equation that the ε^i-component $y_i(t)$ of $y(t, \varepsilon)$ has to satisfy. Let $u(y, \varepsilon)$, $\hat{u}(t, \varepsilon)$ be as in Theorem 2. For the computation of (3.2) we only need the l-th derivative of $\hat{u}(t, \varepsilon)$ at $\varepsilon = 0$, $0 \leq l \leq i$. Then we have:

$$\begin{aligned}\frac{\partial^l}{\partial\varepsilon^l}\left(\sum_{k=1}^{q}\hat{U}_k(t, \varepsilon)\,\frac{\varepsilon^k}{k!}\right)\Bigg|_{\varepsilon=0} = \sum_{k=1}^{l}\binom{l}{k}\frac{\partial^{l-k}}{\partial\varepsilon^{l-k}}\,\hat{U}_k(t, \varepsilon)\Big|_{\varepsilon=0} = \\ = \sum_{k=1}^{l}\binom{l}{k}\frac{\partial^{l-k}}{\partial\varepsilon^{l-k}}\,U_k(y(t, \varepsilon))\Big|_{\varepsilon=0} = \sum_{k=1}^{l}\binom{l}{k}\frac{\partial^{l-k}}{\partial\varepsilon^{l-k}}\,U_k\left(\sum_{j=0}^{l-1} y_j(t)\,\varepsilon^j\right)\Bigg|_{\varepsilon=0}.\end{aligned}$$

This last quantity is exactly the same as the l-th derivative, with respect to ε, evaluated at $\varepsilon = 0$, of $u\left(\sum_{j=0}^{l-1} y_j(t)\,\varepsilon^j, \varepsilon\right)$ when the $U_k(y)$ are considered as C^r-functions. We can give then the following *formal* equations for $(x_i(t), y_i(t))$:

$$\text{(3.9)}\qquad \left\{\begin{aligned} \dot{y}_i(t) &= \frac{1}{i!}\,\frac{\partial^i}{\partial\varepsilon^i}\left\{g\left(u\left(\sum_{k=0}^{i-1} y_k(t)\,\varepsilon^k, \varepsilon\right), \sum_{k=0}^{i} y_k(t)\,\varepsilon^k, \varepsilon\right)\right\}\Bigg|_{\varepsilon=0}, \\ x_i(t) &= \frac{1}{i!}\,\frac{\partial^i}{\partial\varepsilon^i}\left\{u\left(\sum_{k=0}^{i-1} y_k(t)\,\varepsilon^k, \varepsilon\right)\right\}\Bigg|_{\varepsilon=0}. \end{aligned}\right.$$

where $u(y, \varepsilon) = \sum_{k=1}^{q} U_k(y)(\varepsilon^k/k!)$ is considered as a C^r-function in (y, ε). This gives no problem since in the above expressions only the derivatives of correct orders occur.

(ii) Observe that (3.9) define the equations for $(x_i(t), y_i(t))$ only by means of the slow components $y_0(t), y_1(t), \ldots, y_{i-1}(t)$. Previous conventional methods (see, for example [8] and its references) need to find also the equations which define the fast components $x_0(t)$, $x_1(t)$, ..., $x_{i-1}(t)$.

As an example of application of the above results, let us write explicitly the differential equations defining the first approximation $y_0(t) + \varepsilon y_1(t)$. From (3.8) one easily find:

$$\dot{y}_0(t) = g(0, y_0(t), 0)$$

and, using also (3.5):

$$\dot{y}_1(t) = D_1 g(0, y_0(t), 0)\, U_1(y_0(t)) + D_2 g(0, y_0(t), 0)\, y_1(t) + D_3 g(0, y_0(t), 0) \tag{3.10}$$

where, from (3.4), one has to put: $U_1(y) = -[D_1 f(0, y, 0)]^{-1} D_3 f(0, y, 0)$.

4. – Computation of initial conditions.

In this Section we will find a formula giving the initial conditions to associate with the equations (3.2) when the original problem (2.4) has the initial condition $p(\varepsilon) = (x^0(\varepsilon), y^0(\varepsilon))$. The formula is the same as the one given in [13, pag. 29] where only the «stable case» was considered, i.e. the case where the Jacobian $D_1 f(0, y, 0)$ has all eigenvalues with negative real part. Here the validity of the formula (4.3) is extended to the more general case where $D_1 f(0, y, 0)$ has no eigenvalues with zero real part («conditionally stable case»), provided that $p(\varepsilon)$ is suitably chosen. Thus consider the Cauchy problem:

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon)\,, & x(0) = x^0(\varepsilon)\,, \\ \dot{y} \;= g(x, y, \varepsilon)\,, & y(0) = y^0(\varepsilon)\,. \end{cases} \tag{4.1}$$

Setting $\sigma = t/\varepsilon$, (4.1) can be written as:

$$\begin{cases} x' = f(x, y, \varepsilon)\,, & x(0) = x^0(\varepsilon)\,, \\ y' = \varepsilon g(x, y, \varepsilon)\,, & y(0) = y^0(\varepsilon)\,, \\ \varepsilon' = 0\,, & \end{cases} \tag{4.2}$$

where the last equation reflects the fact that ε is a parameter.

Let $\big(x(\sigma, \varepsilon, p(\varepsilon)), y(\sigma, \varepsilon, p(\varepsilon))\big)$ be the solution to (4.2), and $(x^q(\sigma, \varepsilon), y^q(\sigma, \varepsilon))$ be its q-truncation, that is:

$$x^q(\sigma, \varepsilon) = \sum_{i=0}^{q} x_i(\sigma)\, \varepsilon^i\,,$$

$$y^q(\sigma, \varepsilon) = \sum_{i=0}^{q} y_i(\sigma)\, \varepsilon^i,$$

where $x_k(\sigma) = (1/k!)(\partial^k/\partial\varepsilon^k)x(\sigma, \varepsilon, p(\varepsilon))|_{\varepsilon=0}$ etc. By $x = u(y, \varepsilon)$ we will denote again a q-approximation of the center manifold $\mathcal{C}$, for example the one constructed in Theorem 2. The problem of finding a «suitable» initial condition for the outer solution consists in the determination of a point $(x^*(\varepsilon), y^*(\varepsilon), \varepsilon) \in \mathcal{C}$ such that the difference between the solution to (3.1) with initial conditions $(x^*(\varepsilon), y^*(\varepsilon))$ and the solution to (4.1) is bounded above. in modulus, by $C \exp\{-\delta t/\varepsilon\}$, $C, \delta > 0$. Obviously, having the equation of $\mathcal{C}$, we only need $y^*(\varepsilon)$. Moreover, since we are interested in the construction of the $y_i(t)$, $0 \leqslant i \leqslant q$, we only need the ε^i-component of $y^*(\varepsilon)$ in its expansion in power of ε:

$$y^*(\varepsilon) = \sum_{i=0}^{q} y_i^* \varepsilon^i + O(\varepsilon^{q+1}) .$$

For a set $S \subset R^{\nu+\mu}$, let

$$A^+(S) := \{p^0 = (x^0, y^0) \colon (x(\sigma, \varepsilon, p^0), y(\sigma, \varepsilon, p^0)) \in S, \text{ for any } \sigma \geqslant 0\} .$$

$A^+(S)$ is the positively invariant subset of S. Let $\mathcal{C}^s$ be the C^{r+2}-center-stable manifold near K, whose existence has been established in [4]. We shall assume the following:

$$\text{(H)} \qquad p(\varepsilon) = (x^0(\varepsilon), y^0(\varepsilon)) \in A^+(\mathcal{C}^s) .$$

For example, if $y = 0$ is *exponentially stable* for the degenerate system and $D_1 f(0, y, 0)$ has all eigenvalues with negative real part, then (H) is certainly satisfied by any point belonging to the domain of influence of the equilibrium of the boundary layer system.

Let $(\bar{x}(\sigma, \varepsilon, p^*(\varepsilon)), \bar{y}(\sigma, \varepsilon, p^*(\varepsilon)))$ be the solution to (4.2) with $p^*(\varepsilon) \in \mathcal{C}$.

The aim of this Section is to show the following:

THEOREM 3. – Suppose (H) hold. Then there exists $p^*(\varepsilon) = (x^*(\varepsilon), y^*(\varepsilon))$ such that:

$$|x(\sigma, \varepsilon, p(\varepsilon)) - \bar{x}(\sigma, \varepsilon, p^*(\varepsilon))| + |y(\sigma, \varepsilon, p(\varepsilon)) - \bar{y}(\sigma, \varepsilon, p^*(\varepsilon))| \leqslant C \exp\{-\delta\sigma\} , \quad C, \delta > 0 .$$

Furthermore, setting

$$y^*(\varepsilon) = \sum_{k=0}^{q} y_k^* \varepsilon^k + O(\varepsilon^{q+1}) \quad \text{and} \quad x^*(\varepsilon) = u\Big(\sum_{k=0}^{q} y_k^* \varepsilon^k\Big) + O(\varepsilon^{q+1}) ,$$

we have:

$$\text{(4.3)} \qquad \begin{cases} y_0^* = y^0(0) , \\ y_k^* = \dfrac{1}{k!}\Bigg[\displaystyle\int_0^{+\infty} \frac{\partial^{k-1}}{\partial\varepsilon^{k-1}}\Big\{g(x^{k-1}(\sigma, \varepsilon), y^{k-1}(\sigma, \varepsilon), \varepsilon) + \\ \qquad -g\Big(u\Big(\bar{y}\Big(\sigma, \varepsilon, \displaystyle\sum_{i=0}^{k-1} y_i^* \varepsilon^i\Big), \varepsilon\Big), \bar{y}\Big(\sigma, \varepsilon, \displaystyle\sum_{i=0}^{k-1} y_i^* \varepsilon^i\Big), \varepsilon\Big)\Big\}\Big|_{\varepsilon=0} d\sigma + \\ \qquad + D^k y^0(\varepsilon)\Big|_{\varepsilon=0}\Bigg], \quad \text{for } k \geqslant 1 . \end{cases}$$

REMARKS. – (i) For the computation of the integral in (4.3) we have to evaluate the derivatives $D_1^j u$, $0 \leqslant j \leqslant q-1$, and a priori it is not guaranteed their existence. Nevertheless, we can change u by an $\hat{u}$ as in Theorem 1. Taking $u(y, \varepsilon)$ as in Theorem 2, we may apply the arguments of the Remark (i) following that Theorem, hence we may suppose that all derivatives of u we need, exist.

(ii) In (4.3) we may change $x^{k-1}(\sigma, \varepsilon)$, $y^{k-1}(\sigma, \varepsilon)$ by $x(\sigma, \varepsilon, p(\varepsilon))$, $y(\sigma, \varepsilon, p(\varepsilon))$. We prefer the form (4.3) because it emphasizes the fact that x_k^*, y_k^* depend only on the $(k-1)$-truncation of the inner solution. Differential equations for these can be found taking the derivatives with respect to ε of the equation (4.2) and evaluating the result at $\varepsilon = 0$. Furthermore, having the equation $x = X(y, \varepsilon)$ of $\mathcal{C}$, we may obviously change u by X in Theorem 3.

For simplicity we will suppose that $\mathcal{C}$ has the equation $x = X(y, \varepsilon)$ for $y \in K$, $\varepsilon \in [0, \varepsilon_0)$.

Let $\zeta = x - X(y, \varepsilon)$. In the new coordinates (ζ, y), (4.2) reads:

$$\tag{4.4} \begin{cases} \zeta' = \varphi(\zeta, y, \varepsilon)\,, & \zeta(0) := x^0(\varepsilon) - X(y^0(\varepsilon), \varepsilon) := \zeta^0(\varepsilon)\,, \\ y' = \varepsilon\psi(\zeta, y, \varepsilon)\,, & y(0) = y^0(\varepsilon)\,, \\ \varepsilon' = 0\,, & \varepsilon(0) = \varepsilon\,. \end{cases}$$

In these coordinates our center manifold has the simple equation $\zeta = 0$ (near K), and then, from its invariance, it follows $\varphi(0, y, \varepsilon) \equiv 0$.

The equations (4.4) restricted to $\mathcal{C}$ can be written:

$$\tag{4.5} y' = \varepsilon\psi(0, y, \varepsilon)$$

(here we do not consider initial conditions). The invariant manifolds:

$$\mathcal{F}(0, y, 0) = \{(\zeta, y, 0) \in \mathcal{C}^s \colon \zeta \in R^\nu,\ (\zeta, y) \in \Omega\}$$

can be extended [4] to a C^{r+1}-invariant family of C^{r+2}-submanifolds of $\mathcal{C}^s$: $\mathcal{F}(0, y, \varepsilon) \subset \subset \Omega \times [0, \varepsilon_0)$ such that if $p^0(\varepsilon) := (\zeta^0(\varepsilon), y^0(\varepsilon), \varepsilon) \in \mathcal{F}(0, y^*(\varepsilon), \varepsilon)$, then

$$\big(\zeta(\sigma, \varepsilon, p^0(\varepsilon)), y(\sigma, \varepsilon, p^0(\varepsilon)), \varepsilon\big) \in \mathcal{F}\big(0, \bar{y}(\sigma, \varepsilon, y^*(\varepsilon)), \varepsilon\big)$$

for any $\sigma \geqslant 0$ such that $\big(\zeta(\sigma, \varepsilon, p^0(\varepsilon)), y(\sigma, \varepsilon, p^0(\varepsilon)), \varepsilon\big) \in \mathcal{C}^s$. Furthermore the distance between $\big(\zeta(\sigma, \varepsilon, p^0(\varepsilon)), y(\sigma, \varepsilon, p^0(\varepsilon))\big)$ and $\big(0, \bar{y}(\sigma, \varepsilon, y^*(\varepsilon))\big)$ is bounded above by $C \exp\{-\delta\sigma\}$ [4]. Here $y(\sigma, \varepsilon, y^*(\varepsilon))$ is the solution to (4.5) with the initial condition $y(0) = y^*(\varepsilon)$. From $\psi(0, y, \varepsilon) = g(X(y, \varepsilon), y, \varepsilon)$ we get that $\bar{y}(t/\varepsilon, \varepsilon, y^*(\varepsilon))$ is the y-component of the outer solution.

In the following we shall write $\mathcal{F}(y^*(\varepsilon), \varepsilon)$, $\zeta(\sigma, \varepsilon)$ etc. instead of $\mathcal{F}(0, y^*(\varepsilon), \varepsilon)$, $\zeta(\sigma, \varepsilon, p_0(\varepsilon))$ etc.; furthermore, unless otherwise specified, we shall refer to systems (4.4) and (4.5).

From (H) it follows: $(\zeta(\sigma, \varepsilon), y(\sigma, \varepsilon), \varepsilon) \in \mathcal{F}(\bar{y}(\sigma, \varepsilon, y^*(\varepsilon)), \varepsilon)$ for any $\sigma \geq 0$, if

$$(\zeta^0(\varepsilon), y^0(\varepsilon), \varepsilon) \in \mathcal{F}(y^*(\varepsilon), \varepsilon) . \tag{4.6}$$

So, from the properties of $\mathcal{F}$ and the fact that $\mathcal{F}(y^*(\varepsilon), \varepsilon)$ intersects transversally C only in $(0, y^*(\varepsilon), \varepsilon)$, the first part of Theorem 3 follows easily.

Now, the problem is to find $y^*(\varepsilon)$ in terms of $(\zeta^0(\varepsilon), y^0(\varepsilon))$ in such a way that (4.6) holds.

Owing to the transversality of $\mathcal{F}(y^*, 0)$ to $\zeta = 0$, we can write the local equation of $\mathcal{F}(y^*, \varepsilon)$, for small ε. Let $(\zeta, y, \varepsilon) \in \mathrm{C}^s$; then

$$(\zeta, y, \varepsilon) \in \mathcal{F}(y^*, \varepsilon) \Leftrightarrow y = y^* + \varepsilon Y(\zeta, \varepsilon) , \tag{4.7}$$

where $Y(\zeta, \varepsilon)$ is C^r with respect to ε, and C^{r+1} with respect to ζ, $Y(0, \varepsilon) \equiv 0$ because of $(0, y^*, \varepsilon) \in \mathcal{F}(y^*, \varepsilon)$. Condition (4.6) then becomes

$$y(\sigma, \varepsilon) = \bar{y}(\sigma, \varepsilon, y^*(\varepsilon)) + \varepsilon Y(\zeta(\sigma, \varepsilon), \varepsilon) \quad \text{if} \quad y^0(\varepsilon) = y^*(\varepsilon) + \varepsilon Y(\zeta^0(\varepsilon), \varepsilon). \tag{4.8}$$

Setting $\varepsilon = 0$ in (4.8) we see that $y_0^* := y^*(0) = y^0(0)$ (and this is independent of the equation of $\mathcal{F}(y^*, \varepsilon)$ because of the above stated properties of $Y(\zeta, \varepsilon)$). This means that the initial condition we have to give to the zero-approximation is exactly $y^0(0)$.

In order to clarify the proof of Theorem 3 we first show the way to compute y_1^*. Taking the derivative with respect to ε of (4.8) and evaluating at $\varepsilon = 0$, we obtain

$$y_1(\sigma) = D_3\bar{y}(\sigma, 0, y^0(0))\, Dy^*(0) + D_2\bar{y}(\sigma, 0, y^0(0)) + Y(\zeta(\sigma, 0), 0) . \tag{4.9}$$

Obviously (4.9) holds only for those σ such that $(\zeta(\sigma, \varepsilon), y(\sigma, \varepsilon))$ belongs to a suitable neighbourhood of some point of K. Nevertheless, if we suppose σ sufficiently large, then $(\zeta(\sigma, \varepsilon), y(\sigma, \varepsilon))$ is sufficiently close to K, and it belongs to one of the neighbourhoods where $\mathcal{F}(y^*, \varepsilon)$ has an equation like in (4.7). So, it will be sufficient to show that every $Y_i(\zeta(\sigma, 0), 0)$ decays to zero exponentially, as $\sigma \to 0$. In fact, supposing this is true, from (4.9) it follows:

$$\lim_{\sigma \to \infty} \{y_1(\sigma) - D_3\bar{y}(\sigma, 0, y^0(0))\, y_1^* - D_2\bar{y}(\sigma, 0, y^0(0))\} = 0 , \tag{4.10}$$

and then the thesis will result from:

1) $\bar{y}(\sigma, 0, y) = y \Rightarrow D_3\bar{y}(\sigma, 0, y^0(0)) = \mathrm{Id}$;

2) from $y_1(\sigma) = D_2 y(\sigma, \varepsilon)|_{\varepsilon=0}$ we get

$$y_1'(\sigma) = \psi(\zeta(\sigma, 0), y(\sigma, 0), 0) = g(x(\sigma, 0), y^0(0), 0) \; ; \tag{4.11}$$

3) from the definition of $\bar{y}(\sigma, \varepsilon, y^0(0))$ we have

$$\begin{cases} D_2 \bar{y}(\sigma, 0, y^0(0))' = \psi(0, y^0(0), 0) = g(0, y^0(0), 0) \, , \\ D_2 \bar{y}(0, 0, y^0(0)) = 0 \, . \end{cases} \tag{4.12}$$

In fact, from (4.10), we have

$$y_1^* = \lim_{\sigma \to \infty} \{y_1(\sigma) - D_2 \bar{y}(\sigma, 0, y^0(0))\} = \int_0^{+\infty} \{y_1'(\sigma) - D_2 \bar{y}(\sigma, 0, y^0(0))'\} \, d\sigma + \\ + y_1^0 = \int_0^{+\infty} \{g(x(\sigma, 0), y^0(0), 0) - g(0, y^0(0), 0)\} \, d\sigma + y_1^0 \, ,$$

which is exactly (4.3) for $k = 1$.

The fact that $|Y(\zeta(\sigma, 0), 0)| \leqslant C \exp\{-\delta\sigma\}$ follows easily from $Y(\zeta, \varepsilon) \in C^r$, and $|\zeta(\sigma, 0)| \leqslant C \exp\{-\delta\sigma\}$, this last inequality resulting from the fact that on the center-stable manifold $\mathfrak{C}^s \subset \Omega$ the Jacobian $D_1 f(x, y, 0)$, $y \in K$, has all eigenvalues with strictly negative real part. With these arguments in mind we can now prove Theorem 3.

PROOF OF THEOREM 3. – We shall show that, for $k \geqslant 1$:

$$\lim_{\sigma \to \infty} \left\{ y_k(\sigma) - \frac{1}{k!} \frac{\partial^k}{\partial \varepsilon^k} \bar{y}\left(\sigma, \varepsilon, \sum_{i=0}^{k} y_i^* \varepsilon^i\right)\bigg|_{\varepsilon=0} \right\} = 0 \, . \tag{4.13}$$

This is sufficient to prove (4.3), since

$$y_k'(\sigma) = \frac{1}{k!} \frac{\partial^k}{\partial \varepsilon^k} y(\sigma, \varepsilon)\bigg|_{\varepsilon=0} = \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial \varepsilon^{k-1}} g(x(\sigma, \varepsilon), y(\sigma, \varepsilon), \varepsilon)\bigg|_{\varepsilon=0}$$

and

$$\frac{\partial^k}{\partial \varepsilon^k} \bar{y}\left(\sigma, \varepsilon, \sum_{i=0}^{k} y_i^* \varepsilon^i\right)\bigg|_{\varepsilon=0} = k! \, y_k^* + \\ + \int_0^{\sigma} \frac{\partial^{k-1}}{\partial \varepsilon^{k-1}} \left\{ g\left(u\left(\bar{y}\left(\tau, \varepsilon, \sum_{i=0}^{k-1} y_i^* \varepsilon^i\right), \varepsilon\right), \bar{y}\left(\tau, \varepsilon, \sum_{i=0}^{k-1} y_i^* \varepsilon^i\right), \varepsilon\right)\right\}\bigg|_{\varepsilon=0} d\tau$$

(see also (3.8)). Taking the derivative, with respect to ε of (4.8) and evaluating it at $\varepsilon = 0$, we get:

$$y_k(\sigma) - \frac{1}{k!} \frac{\partial^k}{\partial \varepsilon^k} \bar{y}\left(\sigma, \varepsilon, \sum_{i=0}^{k} y_i^* \varepsilon^i\right)\bigg|_{\varepsilon=0} = \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial \varepsilon^{k-1}} Y(\zeta(\sigma, \varepsilon), \varepsilon)\bigg|_{\varepsilon=0} ,$$

so it is sufficient to show that the right hand side of this last equality tends to zero exponentially fast. Because of the smootheness of $Y(\zeta, \varepsilon)$ and $\zeta(\sigma, \varepsilon)$ this becomes a consequence of

$$|D_2^j \zeta(\sigma, 0)| \leqslant C \exp(-\delta\sigma), \qquad C, \delta > 0, \qquad \text{for any } j = 0, 1, \ldots, k-1.$$

Since $p^0(\varepsilon) \in A^+(\mathbb{C}^s)$, we may suppose that $D_1 f(0, y^0(0), 0)$ has all eigenvalues with negative real part; then $|\zeta(\sigma, 0)| \leqslant C \exp\{-\delta\sigma\}$, where δ is any positive number less than $-\operatorname{Re}\lambda$, $\lambda \in \operatorname{Spec} D_1 f(0, y^0(0), 0)$. Suppose, first, $j = 1$. Taking the derivative of (4.4) and evaluating it at $\varepsilon = 0$, we get:

$$D_2\zeta(\sigma, 0)' = D_1\varphi(\sigma, 0) D_2\zeta(\sigma, 0) + D_2\varphi(\sigma, 0) y_1(\sigma) + D_3\varphi(\sigma, 0), \tag{4.14}$$

where, for simplicity, $D_i\varphi(\sigma, 0) := D_i\varphi(\zeta(\sigma, 0), y(\sigma, 0), 0)$ etc.

From $\varphi(0, y, \varepsilon) \equiv 0$ and $|\zeta(\sigma, 0)| \leqslant C \exp\{-\delta\sigma\}$ we have $|D_i\varphi(\sigma, 0)| \leqslant C \exp\{-\delta\sigma\}$ when $i = 2, 3$. Moreover, from (4.11) it follows that $y_1(\sigma) = D_2 y(\sigma, 0)$ is of bounded growth, since $\zeta(\sigma, 0) \to 0$ as $\sigma \to +\infty$. Now, observe that

$$D_1\varphi(\sigma, 0) = D_1\varphi(0, y^0(0), 0) + [D_1\varphi(\zeta(\sigma, 0), y^0(0), 0) - D_1\varphi(0, y^0(0), 0)]$$

and then, from the roughness of the exponential dichotomy (see [3, 6, 11]), we obtain that the system:

$$z' = D_1\varphi(\sigma, 0) z$$

has a fundamental matrix satisfying: $|\Phi(\sigma)| \leqslant C \exp\{-\delta'\sigma\}$, where δ' is a positive number less than δ. Let $A(\sigma) := D_1\varphi(\sigma, 0)$; then (4.14) becomes:

$$D_2\zeta(\sigma, 0)' = A(\sigma) D_2\zeta(\sigma, 0) + \varrho(\sigma), \tag{4.15}$$

where $|\varrho(\sigma)| \leqslant C \exp\{-\delta\sigma\}$. An easy application of the variation of constants formula gives:

$$|D_2\zeta(\sigma, 0)| \leqslant C \exp\{-\delta'\sigma\},$$

and this is exactly what we wanted. Finally, taking the second derivative with respect to ε of (4.4) and evaluating it at $\varepsilon = 0$, we have:

$$y_2'(\sigma) = \frac{1}{2}\frac{\partial}{\partial\varepsilon}\psi(\zeta(\sigma, \varepsilon), y(\sigma, \varepsilon)\ \varepsilon)\Big|_{\varepsilon=0} = \chi(\zeta(\sigma, 0), D_2\zeta(\sigma, 0), y^0(0), y_1(\sigma))$$

and then the growth of $y_2(\sigma)$ is at most as $y_1(\sigma)$, i.e. as σ.

We may then use induction to show that, for any $j = 1, \ldots, k$:

(1) $D_2^{j-1}\zeta(\sigma, 0)$ satisfies an equation like (4.15) and then:

$$|D_2^{j-1}\zeta(\sigma, 0)| \leqslant C \exp\{-\delta'\sigma\};$$

(2) $y_j(\sigma)$ cannot grow faster than σ^j, as $\sigma \to +\infty$.

Finally, if $\mathcal{C}$ and $\mathcal{F}(y^*, \varepsilon)$ have not a single equation around K, we may cover K with a finite number of neighbourhoods where they have an equation. We need then to consider a finite number of equations like (4.4) and relationships like (4.8). Nevertheless the previous arguments show that

$$\left|\frac{\partial^{k-1}}{\partial\varepsilon^{k-1}} Y_m(\zeta_m(\sigma, \varepsilon), \varepsilon)\Big|_{\varepsilon=0}\right| \leqslant C \exp\{-\delta'\sigma\},$$

and then (4.13) is still valid. This proves completely the Theorem.

We conclude this Section observing that from (4.13) other formulae for y_k^* could be obtained. Let us give for example an alternative formula for y_1^*. In this case (4.13) writes (see also (4.10)):

$$y_1^* = \lim_{\sigma\to\infty}\{y_1(0) - D_2\bar{y}(\sigma, 0, y^0(0))\}. \tag{4.16}$$

But from (4.12) it follows:

$$D_2\bar{y}(\sigma, 0, y^0(0)) = g(0, y^0(0), 0)\,\sigma;$$

moreover,

$$\sigma y_1'(\sigma) = \frac{\partial}{\partial\varepsilon}\{\sigma\varepsilon g(x(\sigma, \varepsilon), y(\sigma, \varepsilon), \varepsilon)\}\Big|_{\varepsilon=0} = g(x(\sigma, 0), y^0(0), 0)\,\sigma.$$

Since $x(\sigma, 0) \to 0$ as $\sigma \to +\infty$, we may combine these two last equations to obtain:

$$|D_2\bar{y}(\sigma, 0, y^0(0)) - \sigma y_1'(\sigma)| \leqslant C|\sigma| \exp\{-\delta\sigma\} \to 0, \quad \text{as } \sigma \to +\infty.$$

Then, from (4.16):

$$y_1^* = \lim_{\sigma\to\infty}\{y_1(\sigma) - \sigma y_1'(\sigma)\}$$

(see also [7, 13]).

5. – 1^{st} approximation for a class of enzyme reaction systems.

It is known (see for example [1, 7, 12]) that the temporal evolution of a great number of enzyme reaction systems is described by differential equations whose adimensional form can be written as

$$(5.1)\qquad \begin{cases} \varepsilon \dfrac{d\xi}{dt} = A_0(s)\xi + a_0(s)\,, & \xi(0) = \xi^0\,, \\ \dfrac{ds}{dt} = B_0(s)\xi + b_0(s) + v(s)\,, & s(0) = s^0\,, \end{cases}$$

where $\xi \in \Sigma = \left\{(\xi_1, \ldots, \xi_\nu) \in \overline{R_+^\nu}\colon \sum_{j=1}^{\nu} \xi_j \leqslant 1\right\}$ is the vector of the independent enzyme concentrations (the fast variables) and $s \in \overline{R_+^\mu}$ is the vector of the independent ligand concentrations (the slow variables). $A_0(s)$, $a_0(s)$, $B_0(s)$, $b_0(s)$ describe the internal kinetics; $v(s)$ describes the input/output exchanges.

Let D be an open domain containing a compact set $K \subset \overline{R_+^\mu}$. In [1, 10] it is proved that, if $K \subset R_+^\mu$, then $A_0(s)$ is an invertible matrix with all eigenvalues with negative real part for any s belonging to a neighbourhood of K (usually the same conclusion holds even if $K \subset \overline{R_+^\mu}$).

Suppose that the degenerate system

$$(5.2)\qquad \frac{ds}{dt} = B_0(s)\varphi(s) + b_0(s) + v(s)\,, \qquad \varphi(s) = -A_0(s)^{-1}a_0(s)$$

has a fixed point $\bar{s} \in K$. In [1] it has been proved that, under widely satisfied hypotheses, system (5.2) takes the simple form

$$(5.3)\qquad \frac{ds}{dt} = \alpha V(s) + v(s)\,,$$

where $V(s)$ is a *scalar* function, $V\colon \overline{R_+^\mu} \to R$, and $\alpha \in R^\mu$ is a constant vector.

In this Section we will give a convenient expression for the component $s_1(t)$ of the first approximation $s_0(t) + \varepsilon s_1(t)$, $s_0(t)$ being the solution to the degenerate problem (5.3) (or (5.2)). By the change of variables (2.3) we can transform the given system (5.1) into the form (2.4), where now it is:

$$(5.4)\qquad \begin{cases} f(x, y, \varepsilon) = A_0(y + \bar{s})x - \varepsilon D\varphi(y + \bar{s})[\alpha V(y + \bar{s}) + v(y + \bar{s})]\,, \\ g(x, y, \varepsilon) = \alpha V(y + \bar{s}) + v(y + \bar{s})\,. \end{cases}$$

The differential equation (3.10) for $s_1(t)$ reads now:

$$(5.5)\qquad \dot{s}_1(t) = M\big(s_0(t)\big)\, s_1(t) + N\big(s_0(t)\big)\,,$$

where

$$(5.6)\qquad \begin{cases} M(s) = D[\alpha V(s) + v(s)]\,, \\ N(s) = B_0(s)A_0(s)^{-1}D\varphi(s)[\alpha V(s) + v(s)]\,. \end{cases}$$

The scalar function $V(s)$, determined by the internal mechanism, is the so-called « pseudo-steady-state velocity »; there exist standard rules to compute it; for example the King and Altman graphical rules [9]. The other terms in (5.6), such as $B_0(s)$ and $v(s)$ are explicitly given, a priori, in (5.1), and have generally a simple form. To determine now the initial condition associated with (5.5), we must consider the solution of the boundary layer system

$$\frac{dx}{d\sigma} = A_0(y + \bar{s})x\,, \qquad x(0) = \xi^0 - \varphi(s^0) := x^0\,,$$

$$\frac{dy}{d\sigma} = 0\,, \qquad y(0) = s^0 - \bar{s}\,,$$

that is

$$\frac{dx}{d\sigma} = A_0(s^0)\,x\,, \qquad x(0) = x^0$$

whose solution is $x(\sigma, 0) = \exp\{A_0(s^0)\sigma\}x^0$. Since the initial conditions ξ^0 and s^0 are independent of ε, we have $s_1^* = y_1^*$; (4.3) with $k = 1$, gives then

$$s_1^* = \int_0^{+\infty} B_0(s^0)x(\sigma, 0)\,d\sigma = -B_0(s^0)A_0(s^0)^{-1}x^0 = -B_0(s^0)A_0(s^0)\big(\xi^0 - \varphi(s^0)\big)\,. \tag{5.7}$$

In many cases it is possible to choose the independent enzyme species in such a way that $\xi^0 = 0$. Then (5.7) reads:

$$s_1^* = B_0(s^0)A_0(s^0)^{-1}\varphi(s^0) = -B_0(s^0)[A_0(s^0)^{-1}]^2 a_0(s^0)\,.$$

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