On the Distribution of Complex Numbers According to Their Transcendence Types (*).

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Summary. – *For each complex number its transcendence type is defined as a non-negative real number, which supplies a measure of its approximability by algebraic numbers. The distribution of complex numbers according to their transcendence types is studied and the existence of complex numbers with a given transcendence type is proved.*

Sunto. – *Per ogni numero complesso è definito il suo tipo di trascendenza come un numero reale non negativo che fornisce una misura della sua approssimabilità mediante numeri algebrici. Si studia la distribuzione dei numeri complessi in relazione al loro tipo di transcendenza e viene dimostrata l'esistenza di numeri complessi aventi tipo di trascendenza assegnato.*

1. – Introduction.

Let $\alpha \in \boldsymbol{C}$ and let E_α denote the set of non-negative real numbers τ for which there exists some positive constant $C = C(\alpha, \tau)$ such that

$$\operatorname{Log} |P(\alpha)| > - Ct(P)^\tau \tag{1}$$

holds for any polynomial P with integer coefficients and such that $P(\alpha) \neq 0$. By $t(P)$, as usually, we mean the size of P, i.e. the maximum between $\log H(P)$ and $\deg P$, where $H(P)$, the height of P, is the maximum of the absolute values of its coefficients. We define the transcendence type $\tau(\alpha)$ of α as the infimum of E_α (with $\tau(\alpha) = +\infty$ if $E_\alpha = \emptyset$). It is easy to see (see [7] p. 4.34) that the transcendence type $\tau(\alpha)$ can also be defined as the infimum of the exponents τ such that (1) holds only for irreducible polynomials P with integer coefficients and such that $P(\alpha) \neq 0$. The transcendence type also supplies a measure of the approximability of a complex number by algebraic numbers. For, the inequality

$$\operatorname{Log} |\alpha - \beta| > - t(\beta)^\tau \tag{2}$$

holds for any real number $\tau > \tau(\alpha)$ and for any algebraic number β, provided that $t(\beta)$ is a sufficiently large real number. Here $t(\beta)$ is the size of the minimal equation of β over $\boldsymbol{Z}$.

(*) Entrata in Redazione il 22 aprile 1987.
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For any τ, let A_τ be the set of all complex numbers with transcendence type τ. Of course $\alpha \in A_0$ if and only if α is a rational integer or a non-real algebraic integer of degree two over $\boldsymbol{Q}$. A_1 consists of all algebraic numbers which are not in A_0. Also $A_\tau = \emptyset$ for any $\tau \in (0, 1) \cup (1, 2)$ (see [7] p. 4.2). On the other hand, almost all complex numbers lie in A_2.

The situation is very similar to the case of the irrationality measures. The exponent τ in (1)–(2) plays the same role as the exponent λ in the inequality

$$\left|\alpha - \frac{p}{q}\right| > C|q|^{-\lambda}, \qquad p, q \in \boldsymbol{Z}. \tag{3}$$

The irrationality measure of an irrational number α is defined to be the infimum of the exponents λ for which (3) holds for any p, q and for a suitable $C = C(\alpha, \lambda) > 0$. By Roth's theorem, every irrational algebraic number α has irrationality measure 2, and by Khinchin's theorem almost all irrational numbers again have irrationality measure 2.

In the study of irrationality measures, a powerful tool is given by the theory of continued fractions, which provide an optimal estimate of the approximability by rational numbers. For example, the problem of the existence of real numbers with a fixed irrationality measure is solved by the use of continued fractions.

The corresponding problem for the transcendence measure seems to be more difficult, since algorithms similar to the continued fractions are lacking. The main purpose of the present paper is to prove the following theorem.

THEOREM 1. – For any real number $\tau \geqslant 2$, the set A_τ has the cardinality of the continuum.

In addition, in the final part of the paper we shall give some results about the topological properties of A_τ.

2. – The case $\tau \geqslant (3 + \sqrt{5})/2$.

We start from a simple case, in which we can exhibit some numbers which lie in A_τ.

LEMMA 1. – Let $\alpha \in \boldsymbol{R}$. If there exists a sequence of integers p_n, q_n satisfying

$$gcd(p_n, q_n) = 1$$

$$0 < q_n \uparrow + \infty$$

$$\log q_{n+1} \leqslant c_1 (\log q_n)^d \qquad (d \geqslant 1)$$

$$-c_3 (\log q_n)^\lambda \leqslant \log |\alpha - p_n/q_n| \leqslant -c_2 (\log q_n)^\lambda \qquad (\lambda > 2)$$

for some constants c_1, c_2, c_3, then $\lambda \leqslant \tau(\alpha) \leqslant \max(\lambda, 1 + d/(\lambda - 1))$.

PROOF. – We may verify the condition on the transcendence type of α only for irreducible polynomials (see [7] p. 4.34).

Let P be an irreducible polynomial of $\boldsymbol{Z}[x]$ with $t(P)$ large. Let $n \in \boldsymbol{N}$ such that

$$\log q_n \leqslant t(P) < \log q_{n+1} . \tag{1}$$

If $P(p_n/q_n) = 0$ then $P(x) = q_n x - p_n$ and so $\log |P(\alpha)| \geqslant - c_4 t(P)^\lambda$. Similarly, if $P(p_{n+1}/q_{n+1}) = 0$ we have $\log |P(\alpha)| \geqslant - c_5 t(P)^\lambda$. Thus we may assume $P(p_n/q_n) \neq 0$ and $P(p_{n+1}/q_{n+1}) \neq 0$.

We distinguish two cases:

Case 1: assume that

$$t(P) < \frac{c_2 (\log q_n)^{\lambda-1}}{2[2 + \log(|\alpha| + 1)]} .$$

Then, using also inequalities (1)

$$1 \leqslant q_n^{t(P)} |P(p_n/q_n)| \leqslant |\alpha - p_n/q_n| \cdot \exp\big([2 + \log(|\alpha| + 1)]\, t(P) \log q_n\big) + \\ + |P(\alpha)| \cdot \exp\big(t(P) \log q_n\big) \leqslant \exp\left(-\frac{c_2}{2} (\log q_n)^\lambda\right) + |P(\alpha)| \cdot \exp\big(t(P)^2\big) .$$

Hence

$$\log |P(\alpha)| \geqslant - \tfrac{1}{2} t(P)^2 \geqslant - \tfrac{1}{2} t(P)^\lambda$$

Case 2: assume now

$$t(P) \geqslant \frac{c_2 (\log q_n)^{\lambda-1}}{2[2 + \log(|\alpha| + 1)]} .$$

We have

$$1 \leqslant q_{n+1}^{t(P)} |P(p_{n+1}/q_{n+1})| \leqslant |\alpha - p_{n+1}/q_{n+1}| \exp\big([2 + \log(|\alpha| + 1)]\, t(P) \log q_{n+1}\big) + \\ + |P(\alpha)| \exp\big(t(P) \log q_{n+1}\big) \leqslant \tfrac{1}{2} + |P(\alpha)| \cdot \exp\big(c_6 t(P)^{1 + d/(\lambda - 1)}\big) .$$

Hence

$$\log |P(\alpha)| \geqslant - c_7 t(P)^{1 + d/(\lambda-1)} . \qquad \text{Q.E.D.}$$

PROPOSITION 2. – Let $(3 + \sqrt{5})/2 \leqslant \tau < + \infty$. The map

$$\varphi \colon \{0, 1\}^{\boldsymbol{N}} \to A_\tau , \qquad f \to \sum_{k=1}^{+\infty} 2^{-2^{[\tau^k]} - f(k)}$$

is well-defined and one-to-one.

PROOF. – Let $f \in \{0, 1\}^N$ and for any integer $n \geqslant 2$ define p_n, q_n as follows:

$$p_n = \sum_{k=1}^{n} 2^{2^{[\tau^n]}+f(n)-2^{[\tau^k]}-f(k)}, \quad q_n = 2^{2^{[\tau^n]}+f(n)}.$$

The hypotheses of lemma 1 are satisfied with $\lambda = d = \tau$ (note that $2 \nmid p_n$, whence $gcd(p_n, q_n) = 1$). Hence

$$\tau \leqslant \tau(\varphi(f)) \leqslant \max\left(\tau, 1 + \frac{\tau}{\tau - 1}\right) = \tau$$

and $\varphi(f) \in A_\tau$.

Finally, if $f, g \in \{0, 1\}^N$ and $f \neq g$, let $k_0 = \min\{k \in N: f(k) \neq g(k)\}$. Then

$$|\varphi(f) - \varphi(g)| \geqslant \tfrac{1}{2} 2^{-2^{[\tau^{k_0}]}} - 2 \sum_{k=k_0+1}^{+\infty} 2^{-2^{[\tau^k]}} > 0. \quad \text{Q.E.D.}$$

COROLLARY 1. – If $\tau \geqslant (3 + \sqrt{5})/2$ then $\operatorname{Card}(A_\tau) = c$

PROOF. – If $\tau \in \left[(3 + \sqrt{5})/2, +\infty\right)$ we apply proposition 2. If $\tau = +\infty$ we need only to consider the map ψ:

$$\psi: \{0, 1\}^N \to A_{+\infty}, \quad f \to \sum_{k=1}^{+\infty} 2^{-2k! - f(k)}. \quad \text{Q.E.D.}$$

3. – The general case.

For the proof of theorem 1 we will show that for each $\tau > 2$ there exists a sequence (α_n) of algebraic numbers, with $t(\alpha_n) \leqslant \exp[(\tau - 1)^n]$ and

$$-B \exp[\tau(\tau - 1)^n] \leqslant \log|\alpha_n - \alpha_{n+1}| < -A \exp[\tau(\tau - 1)^n].$$

Then the number $\xi = \lim_n \alpha_n$ must lie in A_τ by theorem 2 of [5] or theorem 2 of [1]. We need some lemmas:

LEMMA 2. – Let $\alpha \in \boldsymbol{C}$, $|\alpha| < 1$. $\forall h, d \in \boldsymbol{R}$ with $\min(h, d) \geqslant 12$ $\exists P \in \boldsymbol{Z}[x] - \{0\}$ with $\log H(P) \leqslant h$ and $\deg P \leqslant d$ such that

$$|P(\alpha)| < \exp(-\tfrac{1}{4} dh)$$

(see [7] p. 1.35)

LEMMA 3. – Let $\alpha, \beta \in \overline{\boldsymbol{Q}}$ with $\alpha \neq \beta$. If $H(\alpha)$ denotes the height of the minimal equation of α over $\boldsymbol{Z}$, then

$$\log|\alpha - \beta| \geqslant -\deg\alpha \deg\beta - \log H(\alpha)\deg\beta - \log H(\beta)\deg\alpha$$

(see [7] p. 1.30).

LEMMA 4. – Let $P, Q \in \mathbf{Z}[x]$. If $P|Q$ then $\log H(P) \leqslant \log H(Q) + \deg Q$ (see [7] p. 4.11).

LEMMA 5. – Let $P \in \mathbf{Z}[x] - \{0\}$ and $\xi \in \mathbf{C}$. If $P'(\xi) \neq 0$ then

$$\min_{P(\alpha)=0} |\xi - \alpha| \leqslant \frac{\deg P |P(\xi)|}{|P'(\xi)|}.$$

PROOF. – We can assume $P(\xi) \neq 0$. Suppose

$$P(x) = a \prod_{i=1}^{\deg P} (x - \alpha_i)$$

with $0 < |\xi - \alpha_1| \leqslant \ldots \leqslant |\xi - \alpha_d|$. We have

$$|P'(\xi)| = \left| P(\xi) \sum_{i=1}^{\deg P} \frac{1}{\xi - \alpha_i} \right| \leqslant \frac{\deg P |P(\xi)|}{|\xi - \alpha_1|} \qquad \text{Q.E.D.}$$

LEMMA 6. – Let $\alpha \in \overline{\mathbf{Q}}$ with $H(\alpha) \deg \alpha \geqslant 2$. Assume that $\exists Q \in \mathbf{Z}[x] - \{0\}$ with $-\infty < \log |Q(\alpha)| < 6 \deg \alpha (\log H(\alpha) + \log \deg \alpha)$. Then $\exists \beta \in \overline{\mathbf{Q}}$ with

$$\deg \beta \leqslant \max(\deg \alpha, \deg Q)$$

$$\log H(\beta) \leqslant \max(\deg \alpha, \deg Q) + \max(\log H(\alpha), \log H(Q)) + \log 2$$

such that

$$-\infty < \log |\alpha - \beta| < \log [2 \max(\deg \alpha, \deg Q)] + \tfrac{1}{2} \log |Q(\alpha)|$$

PROOF. – If $|Q'(\alpha)| \geqslant \sqrt{|Q(\alpha)|}$ then the assertion directly follows from lemma 5. Assume $|Q'(\alpha)| < \sqrt{|Q(\alpha)|}$. $\tilde{Q} = Q + P$, where P is the minimal equation of α over $\mathbf{Z}$. Then P and P' are coprime. By using their resultant we find that

$$|P'(\alpha)| \geqslant [H(\alpha) \deg \alpha]^{-2 \deg \alpha}.$$

On the other hand

$$|Q'(\alpha)| < [H(\alpha) \deg \alpha]^{-3 \deg \alpha} < \tfrac{1}{2} [H(\alpha) \deg \alpha]^{-2 \deg \alpha}$$

hence

$$|\tilde{Q}'(\alpha)| \geqslant \tfrac{1}{2} [H(\alpha) \deg \alpha]^{-2 \deg \alpha} \geqslant \tfrac{1}{2} |\tilde{Q}(\alpha)|^{\frac{1}{3}}.$$

Using lemma 5 again (with $P = \tilde{Q}$), we complete the proof. Q.E.D.

We are now ready to prove a proposition which will allow us to define the sequence (α_n) inductively.

PROPOSITION 3. – Let $t, s, k \in \mathbf{R}$ with $k \geq 1$, $s \geq 1$, $t \geq 8\cdot 10^4 ks$.
Let $\alpha \in \overline{\mathbf{Q}}$, $|\alpha| < \frac{1}{2}$, with

$$s/k < \deg\alpha \leq s \qquad \log H(\alpha) \leq ks$$

then $\exists \gamma \in \overline{\mathbf{Q}}$ with

$$t/(2\cdot 10^4 k) < \deg\gamma \leq t \qquad \log H(\gamma) \leq 2kt$$

such that

$$-4kst < \log|\gamma - \alpha| < -\tfrac{1}{15}st .$$

PROOF. – Lemma 2 asserts the existence of a polynomial $Q \in \mathbf{Z}[x] - \{0\}$ with

$$\deg Q \leq s/k , \qquad \log H(Q) \leq kt$$

which satisfies

$$-\infty < \log|Q(\alpha)| \leq -\tfrac{1}{4}st .$$

By lemma 6 we can find an algebraic number β with

(2) $$\deg\beta \leq s , \qquad \log H(\beta) \leq s + kt + \log 2$$

satisfying

(3) $$-\infty < \log|\beta - \alpha| \leq \log 2 + \log s - \tfrac{1}{8}st < -\tfrac{1}{10}st .$$

Now lemma 3 shows that $st/10 \leq s^2 + ks^2 + s\log H(\beta)$, thus

(4) $$\log H(\beta) \geq t/10 - (k+1)s > t/15 .$$

Using lemma 3 again, we have:

(5) $$\log|\beta - \alpha| > -3kst .$$

We have lifted the height of α; we now want to lift its degree. As before, we can choose a polynomial $Q \in \mathbf{Z}[x] - \{0\}$ with $t(Q) \leq t/30$ satisfying $\log|Q(\beta)| \leq -t^2/3600$. Inequality (4) and lemma 4 ensure that $Q(\beta) \neq 0$; hence, using (2), lemma 6 yields an algebraic number γ with

(6) $$\deg\gamma \leq t/30 , \qquad \log H(\gamma) \leq 2kt$$

which satisfies

$$\log|\gamma - \beta| < -t^2/8000 . \tag{8}$$

Again using lemma 3 with inequalities (2), (7) and (8), we find

$$t^2/8000 \leqslant 2kt(s + \deg\gamma) .$$

Thus

$$\deg\gamma > t/(20\,000\,k)$$

Finally, (3), (4) and (8) show that

$$\log|\gamma - \alpha| < -\tfrac{1}{15} st , \quad \log|\gamma - \alpha| > -4kst . \qquad \text{Q.E.D.}$$

COROLLARY 2. – Let $d > 1$. Put $c_0 = 20\log 10/d(d-1)$ and, for each $n \in N$,

$$k_n = 10^{5n} \qquad s_n = \exp(c_0 d^n) .$$

Then for each $n \in N$ and $f\colon N \to \{0, 1\}$, there exists $\alpha_n^f \in \overline{Q}$ such that:

I) $t(\alpha_n^f) \leqslant k_n^{f(n)+1} s_n$

II) $-4k_n^{1+f(n)} k_{n+1}^{f(n+1)} s_n^{d+1} < \log|\alpha_n^f - \alpha_{n+1}^f| < -\frac{1}{15} k_n^{f(n)} k_{n+1}^{f(n+1)} s_n^{d+1}$

III) $f|_{[1,n]} = g|_{[1,n]} \Rightarrow \alpha_k^f = \alpha_k^g$ for $k = 1, \ldots, n$.

PROOF. – We repeatedly apply proposition 3 with the choices

$$k = k_n , \quad s = s_n k_n^{f(n)} , \quad t = s_{n+1} k_{n+1}^{f(n+1)} \qquad \text{Q.E.D.}$$

PROOF OF THEOREM 1. – For each $f\colon N \to \{0, 1\}$ with $f(1) = 0$ we put $\xi_k = \lim_n \alpha_n^f$ where the α_n^f are as in corollary 2. From II) of corollary 2 we find

$$\log|\xi_f - \alpha_n^f| < -\tfrac{1}{20} k_n^{f(n)} k_{n+1}^{f(n+1)} s_n^{d+1} , \quad \log|\xi_f - \alpha_n^f| > -5k_n^{1+f(n)} k_{n+1}^{f(n+1)} s_n^{d+1} . \tag{9}$$

If P_n is the minimal polynomial of α_n^f, it is easy to see that

$$\log|P_n(\xi_f)| < -\tfrac{1}{25} s_n^{d+1} , \quad \log|P_n(\xi_f)| > -10^{16}\, 10^{15n} s_n^{d+1}$$

(use (9) and see [7] p. 4.40).

Thus, by theorem 2 of [*P*] or theorem 2 of [*A*], ξ_f lies in A_{d+1}: We have proved the map

$$\varphi_d\colon \Big\{f\colon N \to \{0, 1\} \text{ with } f(1) = 0\Big\} \to A_{d+1} , \quad f \to \xi_f$$

is well-defined for each $d > 1$. We claim that φ_d is one-to-one. For let $f, g\colon N \to \{0, 1\}$ with $f(1) = g(1) = 0$, and assume $f \neq g$. Then, if $n_0 = \min\{n \in N\colon f(n) \neq g(n)\} - 1$ and $f(n_0 + 1) = 1$, $g(n_0 + 1) = 0$, from (9) and III) of corollary 2 we obtain:

$$|\xi_f - \xi_g| \geqslant |\xi_g - \alpha^g_{n_0}| - |\xi_f - \alpha^f_{n_0}| \geqslant \exp(-5k_{n_0}^{1+f(n_0)} s_{n_0}^{1+d}) - \exp(-\tfrac{1}{20} k_{n_0}^{f(n_0)} k_{n_0+1} s_{n_0}^{1+d}) > 0 .$$

Thus, if $\tau > 2$, Card $(A_\tau) = c$. Finally, we have claimed in the introduction that almost all complex numbers lie in A_2. Hence Card $(A_2) = c$. Q.E.D.

Let $A \subset \boldsymbol{C}$, and let Λ be a subset of A. We say that Λ is algebraically independent if every finite subset of Λ is algebraically independent. If Λ is a maximal algegraically independent subset of A, which exists by Zorn's lemma, then

$$\Lambda \subset A \subset \bigcup_{\{\lambda_1 \dots \lambda_n\} \subset \Lambda} \; \bigcup_{P \in \boldsymbol{Z}[x_0 \dots x_n] \setminus \{0\}} \{\alpha \in \boldsymbol{C}\colon P(\alpha, \lambda_1, \dots, \lambda_n) = 0\}$$

hence

$$\text{Card}(\Lambda) \leqslant \text{Card}(A) \leqslant \max(\aleph_0, \text{Card}(\Lambda)) .$$

This proves:

COROLLARY 3. – For each $\tau \geqslant 2$ there exists an algebraically independent set $\Lambda \subset A_\tau$ with Card $(\Lambda) = c$.

4. – Geometric properties.

We claimed in the introduction that $\boldsymbol{C} - A_2$ is a negligible set. Moreover we can prove:

THEOREM 2. – $\boldsymbol{C} - A_2$ has d-dimensional Hausdorff measure (1) zero for any $d > 0$.

PROOF. – For each $k, n \in \boldsymbol{N}$ let

$$\Lambda_n = \{P \in \boldsymbol{Z}[x] \text{ irreducible: } [t(P)] = n\}$$

$$\Omega_{n,k} = \bigcup_{P \in \Lambda_n} \{\alpha \in \boldsymbol{C}\colon |P(\alpha)| < \exp(-kn^2)\}$$

and define the set $\Omega_k = \bigcup_{n \geqslant 1} \Omega_{n,k}$. Then $\boldsymbol{C} - A_2 \subset \bigcap_k \Omega_k$. For any $d, \varepsilon > 0$ choose an integer $\boldsymbol{k}$ with

$$\boldsymbol{k} \geqslant \max(10, 2/d + (2 + 4/d)\log(1/\varepsilon), 10/d + 4) .$$

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(1) I take the opportunity to thank Dr. Venturini for suggesting the possibility of using Hausdorff measure instead of Lebesgue measure in this context.

For all $n \in \boldsymbol{N}$, $P \in A_n$, $z \in \boldsymbol{C}$, we have:

$$\log |P(z)| \geqslant -3n^2 + \log \min_{P(\alpha)=0} |z - \alpha|$$

(see [7] p. 4.40).

Thus, if $\alpha_1 \ldots \alpha_t$ are the roots of P,

$$|z - \alpha| \geqslant \exp(-\boldsymbol{k}n^2/2) \quad i = 1 \ldots t \Rightarrow \log |P()z| \geqslant -\boldsymbol{k}n^2 .$$

Then, if $B(\alpha, \varrho) = \{z \in \boldsymbol{C} \colon |z - \alpha| < \varrho\}$,

$$\Omega_k \subset \bigcup_{n \geqslant 1} \bigcup_{P \in A_n} \bigcup_{P(\alpha)=0} B(\alpha, \exp(-\boldsymbol{k}n^2/2))$$

and

$$\operatorname{diam}\big(B(\alpha, \exp(-\boldsymbol{k}n^2/2))\big) < \varepsilon$$

$$\sum_{n \geqslant 1} \sum_{P \in A_n} \sum_{P(\alpha)=0} \Big(\operatorname{diam}\big(B(\alpha, \exp(-\boldsymbol{k}n^2/2))\big)\Big)^d < \varepsilon .$$

This proves that $\boldsymbol{C} - A_2$ has d-dimensional Hausdorff measure zero. Q.E.D.

COROLLARY 4. – $\boldsymbol{C} - A_2$ is totally disconnected and therefore A_τ is totally disconnected for any $\tau > 2$.

PROOF. – See [3] Corollary 2.10.12, p. 176.

COROLLARY 5. – A_2 is arcwise connected.

PROOF. – See [4] Theorem IV 4, p. 48 and Theorem VII 3, p. 104. Finally we have:

THEOREM 3. – Let $2 \leqslant \tau < +\infty$. Then A_τ is a dense set of first category.

PROOF. – For the first statement, note that for any $\alpha \in A_\tau$ $A_\tau \supseteq \alpha + \overline{\boldsymbol{Q}}$.
For the second, given any $\varepsilon > 0$ define, for each $n \in \boldsymbol{N}$,

$$\Omega_n = \big\{\alpha \in \boldsymbol{C} \colon \forall P \in \boldsymbol{Z}[x] - \{0\}\, |P(\alpha)| \geqslant \exp\big(-nt(P)^{\tau+\varepsilon}\big)\big\} .$$

Then Ω_n is a closed subset of $\boldsymbol{C}$ with $\mathring{\Omega}_n = \emptyset$ (since $\Omega_n \cap \overline{\boldsymbol{Q}} = \emptyset$) and $A_\tau \subset \bigcup_{n \in \boldsymbol{N}} \Omega_n$.
Q.E.D.

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