Convergence to a Stationary State and Stability for Solutions of Quasilinear Parabolic Equations (*).

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Summary. – *In this paper some aspects of the asymptotic behavior of solutions of quasilinear (generally nonautonomous) parabolic equations are considered. Specifically a result of convergence to a stationary state is given and, under more restrictive conditions, some sharper descriptions of converging solutions are obtained. Finally a saddle point situation is examined. The employed techniques are abstract and inspired by the papers of Sobolevskii and Da Prato - Grisvard.*

0. – Introduction.

The theory of abstract quasilinear parabolic equations has been studied fairly completely from the point of view of the problem of the existence of solutions. I can mention the fundamental paper of SOBOLEVSKIĬ [18] and the more recent results of POTIER-FERRY [15] and LUNARDI [10].

A rather interesting subject is also the investigation of asymptotic properties of solutions; this problem has been studied rather extensively for semilinear equations (see in particular the book [7]), while not much has been done in the general quasilinear case (see [15], [16], [11]).

The aim of this paper is to extend to this case some results of stability of solutions and convergence to a stationary state which, as far as I known, have been proved only in more particular situations (see [7], ch. 5, [14], [12] for the semilinear case, [20], [13] for the linear case, [15] for the autonomous case).

The plan of the paper is the following: the first section contains some linear estimates which will be useful in the sequel. The second section is dedicated to the study of the convergence to a stationary state and contains some abstract results related to this problem and to the description of the behavior of solutions converging to the limit point. The third section contains some applications of the results of the previous one. The fourth section describes an abstract saddle point situation which is applied to a concrete example in the fifth and last section.

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1. – Some estimates for the linear case.

By now, X will be a complex Banach space with norm $\| \|$. The same symbol will be used for the norm in the space $\mathcal{L}(X)$. Also, C will be a generic positive constant depending on the assumptions made in each occasion. If A is a linear operator in X, $\varrho(A)$ and $\sigma(A)$ will be the resolvent set and the spectrum of A.

Now, let $\{A(t): t_0 \leqslant t \leqslant +\infty\}$ be a family of linear operators in X satisfying the following assumptions:

(A1) $\forall t \geqslant t_0$ $\varrho(A(t)) \supseteq \{\lambda \in \boldsymbol{C}: \operatorname{Re} \lambda \leqslant 0\}$, $\exists C_1 > 0$ such that

$$\|(\lambda - A(t))^{-1}\| \leqslant C_1(1 + |\lambda|)^{-1}$$

for $t_0 \leqslant t \leqslant +\infty$, $\operatorname{Re} \lambda \leqslant 0$.

(A2) $\forall t \geqslant t_0$, $D(A(t)) = X_1$ independent from t, with X_1 dense in X.

(A3) $\exists C_2 > 0$, $0 < \mu \leqslant 1$ such that

$$\|(A(t) - A(s))A(\tau)^{-1}\| \leqslant C_2 |t-s|^{\mu} \qquad \forall t, s, \tau \in [t_0, +\infty[\,.$$

(A4) $\|A(t) - A(\infty))A(\infty)^{-1}\| = \eta(t) \xrightarrow[t \to \infty]{} 0$.

under the stated conditions, the operators $-A(t)$ are infinitesimal generators of analytic semigroups $\{\exp(-sA(t)): s \geqslant 0\}$, such that

$$\|\exp(-sA(t))\| \leqslant C_3 \exp(-\delta_0 s), \qquad \|A(t)\exp(-sA(t))\| \leqslant C_3 s^{-1} \exp(-\delta_0 s)$$

for suitable constants C_3, $\delta_0 > 0$, depending on C_1 (see [19], [4]). These estimates allow to define the fractional powers $A(t)^{-\alpha}$, $\forall \alpha \geqslant 0$:

$$A(t)^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^{+\infty} s^{\alpha-1} \exp(-sA(t))\, ds, \qquad A(t)^0 = \text{identity}.$$

The operators $A(t)^{-\alpha}$ are injective and bounded. One defines

$$A(t)^{\alpha} = (A(t)^{-\alpha})^{-1}$$

$A(t)^{\alpha}$ is a closed densely defined operator and, if $\alpha < \beta$, $D(A^{\beta}) \subseteq D(A^{\alpha})$; the following interpolation estimate is available:

$$\|A(t)^{\beta} x\| \leqslant C(\alpha, \beta, \gamma) \|A(t)^{\gamma} x\|^{(\beta-\alpha)/(\gamma-\alpha)} \|A(t)^{\alpha} x\|^{(\gamma-\beta)/(\gamma-\alpha)}$$
$$\forall x \in D(A(t)^{\gamma}), \quad \text{if } \alpha < \beta < \gamma.$$

The constants $C(\alpha, \beta, \gamma)$ depend only on C_1 (see [4], part 2, 14).

We denote with $U(t, s)$ $(t_0 \leqslant s \leqslant t < \infty)$ the evolution operator generated by $\{A(t): t \geqslant t_0\}$.

We have the following estimates:

PROPOSITION 1.1. – *If* $0 \leqslant \gamma < \beta \leqslant 1$, $t_0 \leqslant \tau$, $t \leqslant +\infty$, *the conditions* (A1)-(A3) *are satisfied and* $\sup \|A(t)A(\tau)^{-1}\| < +\infty$, *one has*

$$\|A(t)^{\gamma} A(\tau)^{-\beta}\| \leqslant C_4(\gamma, \beta)$$

with $C_4(\gamma, \beta)$ *depending on* C_1, C_3, $\sup_{t,\tau} \|A(t)A(\tau)^{-1}\|$.

PROOF. – One has

$$\|A(t)^{\gamma} A(\tau)^{-\beta}\| = \|\Gamma(\beta)^{-1} \int_0^{+\infty} s^{\beta-1} A(t)^{\gamma} \exp(-sA(\tau))\, ds\| \leqslant$$
$$\leqslant \Gamma(\beta^{-1})\, C(0, \beta, 1) \int_0^{+\infty} s^{\beta-1} \|A(t) \exp(-sA(\tau))\|^{\gamma} \|\exp(-sA(\tau))\|^{1-\gamma} ds \leqslant$$
$$\leqslant \Gamma(\beta)^{-1} C(0, \beta, 1)\, C_4(1, 1)\, C_3 \int_0^{+\infty} s^{\beta-\gamma-1} \exp(-\delta_0 s)\, ds = C_4(\beta, \gamma)\,,$$
$$(C_4(1, 1) = \sup_{t,\tau} \|A(t)A(\tau)^{-1}\|)\,.$$

PROPOSITION 1.2. – *Assume the operators* $A(t)$ *satisfy* (A1)-(A3) *and moreover* $\|A(t)A(\tau)^{-1}\| \leqslant C_4(1, 1)$ *for* $t_0 \leqslant t$, $\tau \leqslant +\infty$.

Then, for $0 \leqslant \beta \leqslant \alpha < 1 + \mu$, $\delta < \delta_0 - (C_3 C_2 \Gamma(\mu))^{1/\mu}$ $\exists C_5 > 0$, *depending only on* C_1, C_2, μ, δ, C_3, β, α, $C_4(1, 1)$ *such that*

$$\|A(t)^{\alpha} U(t, \tau) A(\tau)^{-\beta}\| \leqslant C_5 (t - \tau)^{\beta - \alpha} \exp(-\delta(t - \tau))\,.$$

PROOF. – See [16], theorem 1.

PROPOSITION 1.3. – *Assume the conditions of proposition* 1.2 *are satisfied. Then,* $\forall \alpha \in [0, 1]$, $0 \leqslant \beta \leqslant \gamma < 1 + \mu$ *with* $0 \leqslant \gamma - \alpha \leqslant 1$, $\forall \theta < \beta$, $\delta < \delta_0 - (C_2 C_3 \Gamma(\mu))^{1/\mu}$, *there exists* C_6 *depending on* C_1, C_2, C_3, μ, δ, β, α, γ, θ, *satisfying*

$$\|A(\zeta)^{\alpha} [U(t + h, \tau) - U(t, \tau)] A(\tau)^{-\beta}\| \leqslant C_6 h^{\gamma - \alpha} (t - \tau)^{\beta - \gamma} \exp(-\delta(t - \tau))\,.$$

PROOF. – See [16], theorem 2.

PROPOSITION 1.4. – *Assume the conditions* (A1)-(A4) *are satisfied.*

Let $f: [t_0, +\infty[\to X$ *be such that* $\exists \alpha \in]0, 1[$, $N \in R^+$ *satisfying* $\|f(t) - f(\tau)\| \leqslant N|t - \tau|^{\alpha}$ $\forall t, \tau \geqslant t_0$.

Then, for $t_0 \leqslant \tau \leqslant t < +\infty$, $\int_{\tau}^{t} U(t, s) f(s)\, ds \in X_1$ *and*

$$\|A(t) \int_{\tau}^{t} U(t, s) f(s)\, ds\| \leqslant C_7 + C_8 \|f(t)\|\,,$$

with C_7 and C_8 depending only on C_1, C_2, μ, η, δ_0, N. Moreover, assume that there exists $f_0 \in X$ such that $\|f(t) - f_0\| \xrightarrow[t\to\infty]{} 0$. Then

$$\left\| A(t) \int_\tau^t U(t,s) f(s)\, ds \right\| \leqslant C_9 \sup_{s \geqslant \tau} \|f(s) - f_0\|^{\frac{1}{2}} + C_0 \|f(t)\| ,$$

with C_9 depending on C_1, C_2, μ, η, δ_0, N, α.

PROOF. – One has

$$A(t) \int_\tau^t U(t,s) f(s)\, ds = \int_\tau^t A(t)\, U(t,s)[f(s) - f(t)]\, ds + \\ + \int_\tau^t A(t)[U(t,s) - \exp(-(t-s)A(t))] f(t)\, ds + \big(1 - \exp(-(t-\tau)A(t))\big) f(t) .$$

From $\|A(t)\, U(t,s)\| \leqslant C(t-s)^{-1} \exp(-\delta(t-s))$ for $\delta < \delta_0$ (see [4], corollary of lemma 13.1), it follows

$$\left\| \int_\tau^t A(t)\, U(t,s)[f(s) - f(t)]\, ds \right\| \leqslant NC \int_\tau^t (t-s)^{\alpha-1} \exp(-\delta(t-s))\, ds \leqslant NC\delta^{-\alpha}\Gamma(\alpha) .$$

Moreover,

$$\|A(t)[U(t,s) - \exp(-(t-s)A(t))]\| \leqslant C(t-s)^{\mu/2-1} \exp(-\delta(t-s))$$

(see [4], lemma 13.1) and so

$$\left\| \int_\tau^t A(t)[U(t,s) - \exp(-(t-s)A(t))] f(t)\, ds \right\| \leqslant C\delta^{-\mu/2}\Gamma(\mu/2)\|f(t)\| .$$

Finally,

$$\|\big(1 - \exp(-(t-\tau)A(t))\big) f(t)\| \leqslant \big(1 + C_3 \exp(-\delta(t-\tau))\big)\|f(t)\| .$$

So, the first estimate is proved.

If $\|f(t) - f_0\| \xrightarrow[t\to\infty]{} 0$, we pose $\zeta(t) = \sup_{s\geqslant t} \|f(s) - f_0\|$.

We have $\|f(s) - f(t)\| \leqslant (2N\zeta(\tau))^{\frac{1}{2}}(t-s)^{\alpha/2}$, from which

$$\left\| A(t) \int_\tau^t U(t,s)[f(s) - f(t)]\, ds \right\| \leqslant C\big(2N\zeta(\tau)\big)^{\frac{1}{2}} \int_\tau^t (t-s)^{\alpha/2-1} \exp(-\delta(t-s))\, ds \leqslant \\ \leqslant C\Gamma(\alpha/2)\, \delta^{-\alpha/2}\big(2N\zeta(\tau)\big)^{\frac{1}{2}} ,$$

and the second estimate follows.

2. – Abstract results of convergence and stability.

Now consider a real Banach space X. When it is necessary we shall identify X with its complexification $X_C = \{u_1 + iu_2: u_1, u_2 \in X\}$.

Let $-A$ be a linear operator in X, which is the infinitesimal generator of analytic semigroup $\{\exp(-tA): t \geq 0\}$, such that $\|\exp(-tA)\| \leq M_0 \exp(-\delta_0 t)$, with M_0, δ_0 positive.

We can use its fractional powers $A^\alpha (\alpha \in \boldsymbol{R})$, which are closed densely defined in X. If $\alpha \geq 0$, we pose $X_\alpha = D(A^\alpha)$ and $\|x\|_\alpha = \|A^\alpha x\|$, $\forall x \in X_\alpha$. Now, let $R \in]0, +\infty]$, such that $\forall u \in X_\alpha$ (for a fixed $\alpha \in [0, 1[$) with $\|u\|_\alpha < R$, $\forall t \in [0, +\infty]$ a linear operator $A(t, u)$ is defined in such a way that:

(B1) $D(A(t, u)) = D(A) = X_1$;

(B2) $\varrho(A(t, u)) \supseteq \{\lambda \in \boldsymbol{C}: \operatorname{Re}\lambda \leq 0\}$ and $\|(A(t, u) - \lambda)^{-1}\| \leq$
$$\leq \text{const}(\|u\|_\alpha)(1 + |\lambda|)^{-1},\ \|A(t, u)^{-1}\|_{\mathcal{L}(X, X_1)} \leq \text{const}(\|u\|_\alpha);$$

(here and in the following const $(r, s, ...)$ will mean a function depending on $r, s, ...$, which is increasing in each of its arguments)

(B3) $\forall s, t \in [0, +\infty]$, $\forall u, v \in B_R^\alpha$, $\forall w \in X_1$, $(B_R^\alpha = \{u \in X_\alpha: \|u\|_\alpha < R\})$
$$\|(A(t, u) - A(s, v))w\| \leq \text{const}(\|u\|_\alpha, \|v\|_\alpha)(|t - s|^\mu + \|u - v\|_\alpha)\|w\|_1;$$

(B4) $\|A(t, u)w - A(\infty, u)w\| \leq \text{const}(t; \|u\|_\alpha)\|w\|_1$, with $\lim_{t\to\infty} \text{const}(t, s) = 0$ $\forall s \in [0, R[$;

Let $f: [0, +\infty] \times B_R^\alpha \to X$, with

(F1) $\forall t \in [0, +\infty] u \to f(t, u)$ is of class C^1 from B_R^α (with the norm $\| \ \|_\alpha$) to X;

(F2) $\|f'_u(\infty, u)\|_{\mathcal{L}(X^\alpha, X)} \leq \text{const}(\|u\|_\alpha)$, $\forall u \in B_R^\alpha$;

(F3) $\|f(t, u) - f(s, u)\| + \|f'_u(t, u) - f'_u(s, u)\|_{\mathcal{L}(X^\alpha, X)} \leq \text{const}(\|u\|_\alpha)|t - s|^\mu$ $\forall u \in B_R^\alpha$, $s, t \in [0, +\infty[$;

(F4) $\|f(t, u) - f(\infty, u)\| + \|f'_u(t, u) - f'_u(\infty, u)\|_{\mathcal{L}(X^\alpha, X)} \leq \text{const}(t, \|u\|_\alpha)$, again converging to 0 (as $t \to +\infty$).

Consider now the problem:

$$\frac{du}{dt}(t) + A(t, u(t))u(t) = f(t, u(t)), \quad t \geq t_0 \geq 0 \quad u(t_0) = \bar{u} \in X_1 \cap B_R^\alpha. \tag{2.1}$$

A solution of (2.1) is, by definition, a mapping $u \in C([t_0, T[; X_1) \cap C^1([t_0, T[; X)$, with $t > t_0$, which satisfies (2.1) pointwise.

For the existence and the unicity of local solutions, see [4], part 2, 16 and [18].

Now we prove the following

PROPOSITION 2.1. – *Assume the conditions* (B1)-(B4), (F1)-(F4) *are satisfied and let u be the maximal solution of* (2.1), *for some* $\overline{u} \in X_1 \cap B_R^\alpha$. *Suppose that*

(*a*) *u is defined on* $[t_0, T[$ (*for some* $T \in]t_0, +\infty]$);

(*b*) $\sup_{[t_0, T[} \|u(t)\|_\beta < +\infty$ *for some* $\beta \in]\alpha, 1]$;

(*c*) $\sup_{[t_0, T[} \|u(t)\|_\alpha < R$.

Then, $T = +\infty$, $\sup_{[t_0, +\infty[} \|u(t)\|_1 < +\infty$ *and u is uniformly hölder continuous with values in* X_γ $\forall \gamma \in [0, 1[$.

More precisely, $\|u(t)\|_1 \leqslant C$, *depending on* $\sup_{[t_0, +\infty[} \|u(t)\|_\alpha$, $\sup_{[t_0, +\infty[} \|u(t)\|_\alpha$, $\sup_{[t_0, +\infty[} \|u(t)\|_\beta$ *and* $\|u(t) - u(s)\|_j \leqslant k|t - s|^\theta$ *with k and θ depending on* γ, $\sup_{[t_0, +\infty[} \|u(t)\|_\alpha$, $\sup_{[t_0, +\infty[} \|u(t)\|_\beta$.

PROOF. – From the proofs of theorems 16.1, 16.2 in [4] and from (B1)-[B4), (F1)-(F4), it follows that the integral equation

$$u(t) = U_u(t, t')u_0 + \int_{t'}^{t} U_u(t, s) f(s, u(s))\, ds$$

(with $U_u(t, s) =$ the evolution operator generated by $\{A(t, u(t))\}$) has a unique local solution u which is defined on an interval $[t', t' + h]$, with h depending only on $\|u_0\|_\alpha$ and $\|u_0\|_\beta$. Moreover, on this interval u is hölder-continuous with values in X_α, with constants which are again depending only on $\|u_0\|_\alpha$ and $\|u_0\|_\beta$. From this and from (*b*) and (*c*) one has $T = +\infty$ and the global hölder continuity of u on $[t_0, +\infty[$ with values in X_α.

Now, observe that this implies that $\|U_u(t, s)\| \leqslant M$, $\|A(t, u(t))\, U_u(t, s)\| \leqslant M(t - s)^{-1}$ if $t - s \leqslant h$, with M independent from s, t (this is a consequence of the theory developed in [4] part 2.3). So, from $u(t + h) = U_u(t + h, t) u(t) + \int_t^{t+h} U_u(t, s) f(s, u(s))\, ds$, one has

$$\|u(t + h)\|_1 \leqslant \|U_u(t + h, t) u(t)\|_1 + \left\| \int_t^{t+h} U_u(t, s) f(s, u(s))\, ds \right\|_1 .$$

$$\|U_u(t + h, t) u(t)\|_1 \leqslant$$
$$\leqslant \|AA(t + h, u(t + h))^{-1}\| \|A(t + h, u(t + h))\, U_u(t + h, t) A(t, u(t))^{-\gamma}\| \|A(t, u(t))^\gamma A^{-\beta}\| \|u(t)\|_\beta , \quad \text{with } 0 < \gamma < \beta .$$

One has, from [4], estimate 14.11,

$$\|A(t+h, u(t+h))\, U_u(t+h, t) A(t, u(t))^{-\gamma}\| \leqslant C h^{\gamma-1}.$$

From this and from (B2) it follows $\sup_{t \geqslant t_0} \| U_u(t+h, t) u(t)\|_1 < +\infty$. Moreover, with the same method and calculations similar to those of proposition 1.4, one draws

$$\sup_{[t', +\infty[} \left\| \int_t^{t+h} U_u(t, s) f(s, u(s))\, ds \right\|_1 < +\infty,$$

from which one has $\sup_{[t', +\infty[} \|u(t)\|_1 < +\infty$.

Finally, if

$$\alpha < \gamma < 1, \quad \|u(t) - u(\tau)\|_\gamma \leqslant$$
$$\leqslant \text{const}(\alpha, 1) \|u(t) - u(\tau)\|_1^{(\gamma-\alpha)/(1-\alpha)} \|u(t) - u(\tau)\|_\alpha^{(1-\gamma)/(1-\alpha)} \leqslant$$
$$\leqslant \text{const}(\alpha, 1) (2 \sup_{[t', +\infty[} \|u(t)\|_1)^{(\gamma-\alpha)/(1-\alpha)} \|u(t) - u(\tau)\|_\alpha^{(1-\gamma)/(1-\alpha)},$$

so that u is hölder continuous with values in X_γ.

PROPOSITION 2.2. – *Let* $u_0 \in B_R^\alpha$ *such that there exists* u *solution of* (2.1) *satisfying*:

(*a*) $\sup_{t \geqslant t} \|u(t)\|_\beta < +\infty$, *for some* $\beta \in]\alpha, 1]$;

(*b*) $\|u(t) - u_0\| \xrightarrow[t \to \infty]{} 0$.

Then,

(1) $u_0 \in X_1$, $A(\infty, u_0) u_0 = f(\infty, u_0)$;

(2) $\|u(t) - u_0\|_1 \xrightarrow[t \to \infty]{} 0$.

PROOF. – It follows easily from (*a*) and (*b*), by the usual interpolation inequality, that $\|u(t) - u_0\| \to 0$. By theorem 2.1, u is globally hölder continuous with values in X_α and bounded with values in X_1. This implies that the operators $A_u(t) = A(t, u(t))$ satisfy conditions like (A1)-(A4), with $A_u(\infty) = A(\infty, u_0)$. Moreover, $t \to f(t, u(t))$ is globally hölder continuous and $\|f(t, u(t)) - f(\infty, u_0)\| \xrightarrow[t \to \infty]{} 0$.

It follows from [20] that $u_0 \in X_1$,

$$A(\infty, u_0) u_0 = f(\infty, u_0), \qquad \frac{du}{dt}(t) \to 0.$$

Then

$$u(t) = A(t, u(t))^{-1} \left(f(t, u(t)) - \frac{du}{dt}(t) \right)$$

and from this the result follows easily.

REMARK 2.3. – The hypothesis (*a*) and (*b*) in proposition 2.2 can be replaced with:

(a') u is globally hölder continuous with values in X_α;

(b') $\|u(t) - u_0\|_\alpha \xrightarrow[t\to\infty]{} 0$,

and the conclusions are the same.

THEOREM 2.4. – *Assume the conditions* (B1)-(B4), (F1)-(F4) *are satisfied and*

(*a*) $f(\infty, 0) = 0$

(*b*) $\varrho(A(\infty, 0) - f'_u(\infty, 0)) \supseteq \{\lambda \in \mathbf{C}\colon \operatorname{Re}\lambda \leqslant 0\}$.

Then, $\forall \beta \in]\alpha, 1]$ *there exist* $T_0 \geqslant 0$, $\mu_0 > 0$, *such that the maximal solution of* (2.1) *is globally defined for every* $t_0 \geqslant T_0$, $\bar{u} \in X_1$ *with* $\|\bar{u}\|_\beta \leqslant \mu_0$ *and moreover,* $\|u(t)\|_1 \xrightarrow[t\to\infty]{} 0$.

PROOF. – We put $\varrho(t, u) = f(t, u) - f'_u(t, 0)u$, $B(t, u) = A(t, u) - f'_u(t, 0)$.
Then (2.1) is equivalent to

$$\frac{du}{dt} + B(t, u(t))\,u(t) = f(t, 0) + \varrho(t, u(t)) \,, \qquad u(t_0) = \bar{u} \,.$$

Define $v(t) = A^\alpha u(t)$, $V_v(t, s)$ the evolution operator generated by $\{B(t, A^{-\alpha}v(t))\}$. It follows

$$v(t) = A^\alpha V_v(t, t_0)\bar{u} + \int_{t_0}^{t} A^\alpha V_v(t, s)[f(s, 0) + \varrho(s, A^{-\alpha}v(s))]\,ds \,. \tag{2.2}$$

Now fix $\theta < \beta - \alpha$. Define

$$S(t_0, \eta) = \{v \in C([t_0, +\infty[; X)\colon \|v(t) - v(\tau)\| \leqslant (t-\tau)^\theta,\ \|v(t)\| \leqslant \eta,\ \|v(t)\| \xrightarrow[t\to\infty]{} 0\} \,.$$

For $t_0 \geqslant 0$, $\eta > 0$, $S(t_0, \eta)$ is a closed subset of the Banach space of continuous and bounded functions from $[t_0, +\infty[$ to X.

Define, for $v \in S(t_0, \eta)$,

$$Tv(t) = A^\alpha V_v(t, t_0)\bar{u} + \int_{t_0}^{t} A^\alpha V_v(t, t)[f(s, 0) + \varrho(s, A^{-\alpha}v(s))]\,ds \,, \tag{2.3}$$

with $\bar{u} \in X_1$, $\|\bar{u}\|_\alpha < R$.

Assume $\|\exp(-t(A(\infty, 0) - f'_u(\infty, 0)))\| \leqslant M_0 \exp(-\delta_0 t)$, for some M_0, δ_0 positive. Then, for a fixed $\delta < \delta_0$, $\exists t(\delta) \geqslant 0$, $\eta(\delta) > 0$ such that

$$\|V_v(t, s)\| \leqslant C \exp(-\delta(t-s)) \,, \qquad \forall v \in S(t_0, \eta) \,,$$

for some constant C positive. This follows from the fact that, for any v less than θ and μ, for any $\varepsilon > 0$,

$$\sup_{t,\tau\geqslant t_0}\left(\|A(t, A^{-\alpha}v(t)) - A(\tau, A^{-\alpha}v(\tau))\|\right)/|t-\tau|^{v} < \varepsilon$$

if t_0 is sufficiently large and η is sufficiently small and so proposition 1.2 is available. Also, one verifies that $\|B(t, v(t))\, V_v(t, s)\| \leqslant C(t-s)^{-1}\exp[-\delta(t-s)]$, from which one draws

$$(2.4)\qquad \begin{cases} \|A^{\alpha} V_v(t, s)\| \leqslant C(t-s)^{-\alpha}\exp\left(-\delta(t-s)\right), \\ \|A^{\alpha} V_v(t, s)\bar{u}\| \leqslant C\exp\left(-\delta(t-s)\right)\|\bar{u}\|_{\beta}. \end{cases}$$

Further, a standard consequence of (F3)-(F4) and (a) is that, for t_0 sufficiently large, η sufficiently small,

$$\|f(s, 0) + \varrho(s, A^{-\alpha}v(s))\| \leqslant \varepsilon(1+\eta) \qquad (\forall v \in S(t_0, \eta))$$

for a fixed $\varepsilon > 0$. Therefore,

$$\|Tv(t)\| \leqslant C\exp\left(-\delta(t-t_0)\right)\|\bar{u}\|_{\beta} + C\varepsilon\int_{t_0}^{t}\exp\left(-\delta(t-s)\right)(t-s)^{-\alpha}\,ds(1+\eta),$$

from which one has that, for t_0 large enougy, η sufficiently small, $\|Tv(t)\| \leqslant \eta$ $\forall v \in S(t_0, \eta)$.

Moreover, for $t_0 \leqslant \tau \leqslant t$,

$$\begin{aligned}\|Tv(t) - Tv(\tau)\| \leqslant \|A^{\alpha}\left(V_v(t, t_0) - V_v(\tau, t_0)\right)\bar{u}\| &+ \\ + \left\|\int_{\tau}^{t} A^{\alpha} V_v(t, s)[f(s, 0) + \varrho(s, A^{-\alpha}v(s))]\,ds\right\| &+ \\ + \left\|\int_{t_0}^{t} A^{\alpha}[V_v(t, s) - V_v(\tau, s)][f(s, 0) + \varrho(s, A^{-\alpha}v(s))]\,ds\right\| &.\end{aligned}$$

One has, from poposition 1.3 and lemma 14.1 in [4]

$$\begin{aligned}&\|A^{\alpha}\left(V_v(t, t_0) - V_v(\tau, t_0)\right)\bar{u}\| \leqslant C(t-\tau)^{\theta}\exp\left(-\delta(\tau-t_0)\right)\|\bar{u}\|_{\beta},\\ &\left\|\int_{\tau}^{t} A^{\alpha} V_v(t, s)[f(s, 0) + \varrho(s, A^{-\alpha}v(s))]\,ds\right\| \leqslant C\varepsilon\int_{\tau}^{t}(t-s)^{-\alpha}\exp\left(-\delta(t-s)\right)ds\cdot \\ &\qquad\qquad \cdot(1+\eta) \leqslant C\varepsilon(t-\tau)^{1-\alpha}(1+\eta)\end{aligned}$$

Again from proposition 1.3, one has

$$\|A^{\alpha}[V_v(t, s) - V_v(\tau, s)]\| \leqslant C(t-\tau)^{\theta}(\tau-s)^{-\beta}\exp\left(-\delta(\tau-s)\right),$$

from which

$$\left\|\int_{t_0}^{t} A^{\alpha}[V_v(t,s)-V_v(\tau,s)][f(s,0)+\varrho(s,A^{-\alpha}v(s))]\,ds\right\| \leqslant C\varepsilon(1+\eta)(t-\tau)^{\theta}.$$

Finally, it is easily seen that $Tv(t) \xrightarrow[t\to\infty]{} 0$.

All these estimates imply that, for η sufficiently small, t_0 sufficiently large, if $\|\bar{u}\|_{\beta}$ is sufficiently little, $Tv \in S(t_0, \eta)$.

Now, if $v, w \in S(t_0, \eta)$, like in [4], proof of theorems 16.1, 16.2, one gets:

$$Tw(t)-Tv(t)=\int_{t_0}^{t} A^{\alpha}V_w(t,s)\{[B(s,A^{-\alpha}v(s))-B(s,A^{-\alpha}w(s))]A^{-\alpha}Tv(s)+ \\ +\varrho(s,A^{-\alpha}w(s))-\varrho(s,A^{-\alpha}v(s))]\}\,ds.$$

So,

$$\|Tw(t)-Tv(t)\| \leqslant \int_{t_0}^{t} \|A^{\alpha}V_w(t,s)\|\big(\|[B(s,A^{-\alpha}v(s))-B(s,A^{-\alpha}w(s))]A^{-1}\|\cdot \\ \cdot\|A^{1-\alpha}Tv(s)\|+\|\varrho(s,A^{-\alpha}w(s))-\varrho(s,A^{-\alpha}v(s))\|\big)\,ds.$$

One has

$$\|A^{\alpha}V_w(t,s)\| \leqslant C(t-s)^{-\alpha}\exp(-\delta(t-s)),$$

$$\|[B(s,A^{-\alpha}v(s))-B(s,A^{-\alpha}w(s))]A^{-1}\| \leqslant C\|v(s)-w(s)\|,$$

owing to the hypothesis (B1)-(B4), (F1)-(F4).

$$\|A^{1-\alpha}Tv(t)\| = \left\|AV_v(t,t_0)\bar{u}+A\int_{t_0}^{t} V_v(t,s)[f(s,0)+\varrho(s,A^{-\alpha}v(s))]\,ds\right\| \leqslant \\ \leqslant C(t-t_0)^{\gamma-1}\exp(-\delta(t-t_0))(\|\bar{u}\|_{\beta}+(1+\eta)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})$$

(by proposition 1.4, with $\gamma > 0$). Finally, for η sufficiently small,

$$\|\varrho(s,A^{-\alpha}w(s))-\varrho(s,A^{-\alpha}v(s))\| \leqslant \varepsilon\|w(s)-v(s)\|.$$

From this,

$$\|Tw(t)-Tv(t)\| \leqslant \text{const}(\varepsilon, \|\bar{u}\|_{\beta}) \sup_{t\geqslant t_0}\|w(t)-v(t)\|, \quad \text{with const}(\varepsilon, r) \xrightarrow[\varepsilon+r\to 0]{} 0.$$

So we have proved that there exist $T_0 \geqslant 0$, $\mu_0 > 0$, $\eta_0 > 0$, such that $\forall t_0 \geqslant T_0$, $\forall \bar{u} \in X_1$, with $\|\bar{u}\|_{\beta} \leqslant \mu_0$, one has that T has a unique fixed point in $S(t_0, \eta)$, for every $\eta \leqslant \eta_0$:

From this, the theorem follows easily, also taking remark 2.3 into account.

REMARK 2.5. – For the following, we need some more accurate estimates of solutions converging to 0.

So, let u be a solution converging to 0 in X_β ($\beta > \alpha$), with the hypothesis of theorem 2.4 satisfied. Fix $\delta < \delta_0$, with $\|\exp\big(-t(B(\infty, 0))\big)\| \leqslant M_0 \exp(-\delta_0 t)$. In the proof of theorem 2.4, we have seen that, if t_0 is sufficiently large and η is sufficiently small, for $v \in S(t_0, \eta)$, $\|A^\alpha V_v(t, s)\| \leqslant M \exp\big(-\delta(t-s)\big)(t-s)^{-\alpha}$. For a suitable t_0, $t \to A^\alpha u(t) \in S(t_0, \eta)$ and, defining $v(t) = A^\alpha u(t)$, one has $v(t) = Tv(t)$ (see (2.3)) with $\bar{u} = u(t_0)$. It follows (for $t \geqslant t_0$)

$$\|v(t)\| = \|Tv(t)\| \leqslant M' \exp\big(-\delta(t-t_0)\big)\|u(t_0)\|_\beta + \int_{t_0}^{t} M \exp\big(-\delta(t-s)\big)(t-s)^{-\alpha}$$

$$\Big(\|f(s, 0)\| + \big\|\varrho\big(s, A^{-\alpha}v(s)\big)\big\|\Big)\, ds \leqslant (\text{for suitable values of } \eta)$$

$$\leqslant M' \exp\big(-\delta(t-t_0)\big)\|u(t_0)\|_\beta + M\int_{t_0}^{t} \exp\big(-\delta(t-s)\big)(t-s)^{-\alpha}\|f(s, 0)\|\, ds + $$
$$+ M\varepsilon\int_{t_0}^{t} \exp\big(-\delta(t-s)\big)(t-s)^{-\alpha}\|v(s)\|\, ds\,,$$

so that, if we put

$$\varphi(t) = \exp[\delta t]\|v(t)\|\,, \qquad \Phi(t) = M' \exp(\delta t_0)\|u(t_0)\|_\beta + M\int_{t_0}^{t} \exp(\delta s)(t-s)^{-\alpha}\|f(s, 0)\|\, ds\,,$$

we have $\varphi(t) \leqslant \Phi(t) + M\varepsilon\int_{t_0}^{t}(t-s)^{-\alpha}\varphi(s)\, ds$, which implies (see [7], lemma 7.1.1) $\varphi(t) \leqslant \Phi(t) + \theta\int_{t_0}^{t} E'_{1-\alpha}\big(\theta(t-s)\big)\Phi(s)\, ds$, with

$$\theta = \big(M\varepsilon\Gamma(1-\alpha)\big)^{1/(1-\alpha)}\,, \qquad E_{1-\alpha}(z) = \sum_{n=0}^{\infty} z^{n(1-\alpha)}/\Gamma\big(n(1-\alpha)+1\big)\,, \qquad E'_{1-\alpha} = \frac{d}{dz}E_{1-\alpha}\,,$$

and so

$$\|u(t)\|_\alpha = \|v(t)\| \leqslant \exp(-\delta t)\Phi(t) + \theta \exp(-\delta t)\int_{t_0}^{t} E'_{1-\alpha}\big(\theta(t-s)\big)\Phi(s)\, ds\,,$$

for every $t \geqslant t_0$ sufficiently large.

REMARK 2.6. – If the conditions of theorem 2.4 are satisfied and, moreover, $f(t, 0) = 0$ $\forall t \geqslant 0$, an easy consequence of the theorem is a result of asymptotic stability of the null solution, which is a sort of nonautonomous version of theorem 2 in [15].

Under less general conditions, we can give a better description of the asymptotic behavior of solutions converging to 0.

THEOREM 2.7. – *Assume the conditions of theorem 2.4 are verified and* $A(t, u) = A(u)$, $f(t, u) = f(u)$ *(for* $t \geqslant 0$, $u \in B_R^\alpha$*). If* $B = -A(0) + f'(0)$, *suppose that* $\sigma(B) = \{\beta\} \cup \sigma_1$, *with* $\operatorname{Re}\beta > 0$, $\inf_{\sigma_1} \operatorname{Re}\lambda > \operatorname{Re}\beta$, B^{-1} *is compact in* X *and* β *is a simple* *eigenvalue of* B. *Moreover, if* $\varrho(u) = f(u) - f'(0)u$, *let* $\|\varrho(u)\| = 0(\|u\|_\alpha^{1+\nu})$, *for some* $\nu > 0$, *as* $\|u\|_\alpha \to 0$. *Then, if* u *is a solution of*

$$\frac{du}{dt} + A(u(t))u(t) = f(u(t)) \tag{2.5}$$

such that $\|u(t)\|_{\alpha'} \to 0$, *for some* $\alpha' > \alpha$,

$$u(t) = \exp(-\beta t)p + r(t), \quad \text{with } p \in \operatorname{Ker}(\beta - B), \quad \|r(t)\|_\gamma = o(\exp(-\beta t))$$

(for $t \to +\infty$*), for any* $\gamma < 1$.

To prove this result, we need the following

LEMMA 2.8. – *Let* B *satisfy the conditions of theorem 2.7,* $g\colon [t_0, +\infty[\to X$ *continuous,* $g(t) = O(\exp(-\beta' t))(t \to +\infty)$, *with* $\beta' > \operatorname{Re}\beta$.

If u *is a solution of*

$$\frac{du}{dt} + Bu = g(t), \quad \text{on } [t_0, +\infty[\tag{2.6}$$

$u(t) = \exp(-\beta t)p + r(t)$, *with* $p \in \ker(\beta - B)$, $\|r(t)\|_\gamma = o(\exp(-\beta t))(t \to +\infty)$, *for any* $\gamma < 1$.

PROOF. – We call P_1 and P_2 the projections onto $\ker(\beta - B)$ and $R(\beta - B)$ (respectively), such that $I = P_1 + P_2$. We put $Y_1 = \ker(\beta - B)$, $Y_2 = R(\beta - B)$ Y_1 and Y_2 are invariant with respect to B, $Y_1 \subseteq D(B^n)$, $\forall n \in N$.

We define $B_j = B|_{Y_j}$. $B_1 \in \mathcal{L}(X_1)$, $\sigma(B_1) = \{\beta\}$, $\sigma(B_2) = \sigma_1$.

If $u_j(t) = P_j u(t)$, one has

$$\frac{du_j}{dt}(t) + B_j u_j(t) = P_j g(t) .$$

As $B_1 u = \beta u$, $\forall u \in Y_1$,

$$u_1(t) = \exp(-\beta(t - t_0))u_1(t_0) + \int_{t_0}^{t} \exp(-\beta(t-s))P_1 g(s)\,ds =$$
$$= \exp(-\beta t)\Big(\exp(\beta t_0)u_1(t_0) + \int_{t_0}^{t} \exp(\beta s)P_1 g(s)\,ds\Big),$$

and, from this, $u_1(t) = \exp(-\beta t)p + r_1(t)$, with

$$p = \exp(\beta t_0)u_1(t_0) + \int_{t_0}^{+\infty} \exp(\beta s)P_1 g(s)\,ds \quad \text{and } \|r_1(t)\|_1 = o(\exp(-\beta t))$$

(remark that in Y_1 the norms $\| \ \|$ and $\| \ \|_1$ are equivalent). Moreover,

$$u_2(t) = \exp\big(-(t-t_0)B_2\big)u_2(t_0) + \int_{t_0}^{t} \exp\big(-(t-s)B_2\big)P_2 g(s)\,ds\,.$$

So, as $\sigma(B_2) = \sigma_1$,

$$\|\exp\big(-(t_1-t_0)B_2\big)u_2(t_0)\| \leqslant C\exp\big(-\beta''(t-t_0)\big)\|u_2(t_0)\|_1\,, \qquad \text{with } \beta'' > \operatorname{Re}\beta\,,$$

$$\Big\|\int_{t_0}^{t} \exp\big(-(t-s)B_2\big)P_2 g(s)\,ds\Big\|_\gamma \leqslant$$
$$\leqslant C\int_{t_0}^{t} (t-s)^{-\gamma}\exp\big(-\beta''(t-s)\big)\exp(-\beta' s)\,ds = o\big(\exp(-\beta t)\big)\,.$$

So, the lemma is proved.

PROOF OF THEOREM 2.7. – From (2.5) one has

$$\frac{du}{dt} + Bu(t) = \varrho\big(u(t)\big) + \big[A(0) - A\big(u(t)\big)\big]u(t)\,.$$

From remark 2.5, one has

$$\|u(t)\|_\alpha \leqslant M'\|u(t_0)\|_\beta \exp\big(-\delta(t-t_0)\big)\,E_{1-\alpha}\big(\theta(t-t_0)\big)\,.$$

From a suitable choice of ε (and of θ) and from the estimates of $E_{1-\alpha}$ one can find in [7], lemma 7.1.1, one draws $\|u(t)\|_\alpha \leqslant M(\delta)\exp(-\delta t)$ $\forall \delta < \operatorname{Re}\beta$. This implies that $\|\varrho(u(t))\| \leqslant C\exp\big(-\delta(1+v)t\big)$. If $\delta > \operatorname{Re}\beta/(1+v)$, one has $\delta(1+v) > \operatorname{Re}\beta$.

Now we need an estimate of $\|u(t)\|_1$. One has, with the notations of theorem 2.4,

$$\frac{du}{dt} + B\big(u(t)\big)u(t) = \varrho\big(u(t)\big)\,, \qquad \text{with } B(u) = A(u) - f'_u(0)\,,$$

so that, if $U(t,s)$ is the evolution operator generated by $\{B(u(t))\}$,

$$u(t) = U(t,t_0)u(t_0) + \int_{t_0}^{t} U(t,s)\,\varrho\big(u(s)\big)\,ds\,.$$

Now, $\|U(t,t_0)u(t_0)\|_1 \leqslant \|AB(u(t))^{-1}\|\,\|B(u(t))\,U(t,t_0)u(t_0)\|$, which implies (see [4], estimate (13.19))

$$\|U(t,t_0)u(t_0)\|_1 = O\big(\exp(-\theta t)\big)\,, \qquad \text{for some } \theta > 0\,.$$

Furthermore,

$$\left\|\int_{t_0}^{t} U(t,s)\,\varrho(u(s))\,ds\right\|_1 \leqslant \left\|\int_{t_0}^{t/2} U(t,s)\,\varrho(u(s)(\,ds\right\|_1 + \left\|\int_{t/2}^{t} U(t,s)\,\varrho(u(s))\,ds\right\|_1 .$$

$$\left\|\int_{t_0}^{t/2} U(t,s)\,\varrho(u(s))\,ds\right\|_1 \leqslant C\left\|\int_{t_0}^{t/2} B(u(t))\,U(t,s)\,\varrho(u(s))\,ds\right\| \leqslant$$
$$\leqslant C\int_{t_0}^{t/2} \exp\big(-\theta(t-s)\big)(t-s)^{-1}\exp\big(-\delta(1+v)s\big)\,ds \leqslant$$
$$\leqslant Ct^{-1}\int_{t_0}^{t/2} \exp\big(-\delta(t-s)\big)\exp\big(-\delta(1+v)s\big)\,ds = O\big(\exp(-\beta' t)\big)\,, \quad \text{for some } \beta' > 0\,.$$

Finally,

$$\left\|\int_{t/2}^{t} U(t,s)\,\varrho(u(s))\,ds\right\|_1 \leqslant C\left\|B(u(t))\int_{t/2}^{t} U(t,s)\,\varrho(u(s))\,ds\right\| \leqslant$$
$$\leqslant (\text{by proposition 1.4})\ C\exp\big(-\delta(1+v)t/4\big)\,.$$

It follows $\|u(t)\|_1 = O\big(\exp(-\theta t)\big)$, for some θ positive.
So

$$\big\|\big(A(0)-A(u(t))\big)u(t)\big\| \leqslant C\|A(0)-A(u(t))\|_{\mathcal{L}(X_1,X)}\exp[-\theta t] \leqslant$$
$$\leqslant (\text{owing to (B3)})\ C\|u(t)\|_\alpha \exp(-\theta t) \leqslant CM(\delta)\exp\big(-(\delta+\theta)t\big)\,,$$
$$\text{for any } \delta < \operatorname{Re}\beta\,.$$

So, we have proved that g satisfies the conditions of lemma 2.8 and the theorem follows.

THEOREM 2.9. – *Assume the conditions of theorem 2.4 are satisfied. Furthermore,* $f(t,0) = t^{-\varrho}f_1 + t^{-\varrho}f_2(t)$, *with* $\varrho > 0$, $f_2(t) \xrightarrow[t\to\infty]{} 0$. *Then, if u is a solution of*

$$\frac{du}{dt} + A\big(t, u(t)\big)u(t) = f\big(t, u(t)\big) \tag{2.7}$$

such that for some $\beta > \alpha \|u(t)\|_\beta \xrightarrow[t\to\infty]{} 0$

$$u(t) = t^{-\varrho}u_1 + r_1(t)\,,$$

with $u_1 \in X_1$, $\|r_1(t)\|_\gamma = o(t^{-\varrho})$, $\forall \gamma \in [0, 1[$.

For the proof, we need the following lemma:

LEMMA 2.10. – *Let $A(t)$ satisfy the conditions* (A1)-(A4); *if* $g\colon [t_0, +\infty[\to X$

is continuous, $\|g(t) - f_1\| \to 0$ *and*

$$u(t) = U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, s)g(s)\, ds ,$$

then $\|u(t) - A(\infty)^{-1}f_1\|_\gamma \xrightarrow[t\to\infty]{} 0 \ \forall\gamma \in [0, 1[$.

PROOF. – First of all, $\|U(t, t_0)u(t_0)\|_1 \xrightarrow[t\to\infty]{} 0$. Now,

$$A^\gamma \int_{t_0}^{t} U(t, s)g(s)\, ds = \int_{t_0}^{t} A^\gamma U(t, s)[g(s) - f_1]\, ds + A^\gamma A(t)^{-1} \int_{t_0}^{t} A(t)[U(t, s) - \\ - \exp\big(-(t-s)A(t)\big)] f_1\, ds + A^\gamma A(t)^{-1}\big(1 - \exp\big(-(t-t_0)A(t)\big)\big) f_1 \xrightarrow[t\to\infty]{} A^\gamma A(\infty)^{-1} f_1$$

(this can be proved using the estimates of lemma 13.1 in [4]).

PROOF OF THEOREM 2.9. – By theorem 2.2, $\|u(t)\|_\gamma \to 0$, $\forall\gamma \in [0, 1]$. Define $v(t) = t^\varrho u(t)$. Then,

$$\frac{dv}{dt} + \big(A(t, u(t)) - f'_u(t, 0) - \varrho t^{-1}\big)v(t) = f_1 + f_2(t) + t^\varrho \varrho\big(t, u(t)\big) .$$

To apply lemma 2.10, with $A(t) = A\big(t, u(t)\big) - f'_u(t, 0) - \varrho t^{-1}$, and so prove the theorem, we have to show that $\|t^\varrho \varrho\big(t, u(t)\big)\| \xrightarrow[t\to\infty]{} 0$. By remark 2.5,

$$\|u(t)\|_\alpha \leqslant \exp(-\delta t)\Phi(t) + \theta \exp(-\delta t) \int_{t_0}^{t} E'_{1-\alpha}\big(\theta(t-s)\big)\Phi(s)\, ds ,$$

with

$$\Phi(t) = M' \exp(\delta t_0)\|u(t_0)\|_\beta + M \int_{t_0}^{t} \exp(\delta s)(t-s)^{-\alpha}\|f(s, 0)\|\, ds .$$

One has

$$\Phi(t) \leqslant M' \exp(\delta t_0)\|u(t_0)\|_\beta + C \int_{t_0}^{t} \exp(\delta s)(t-s)^{-\alpha} s^{-\varrho}\, ds ,$$

so that

$$\exp(-\delta t)\Phi(t) \leqslant M' \exp\big(-\delta(t-t_0)\big)\|u(t_0)\|_\beta + C \int_{t_0}^{t} \exp\big(-\delta(t-s)\big)(t-s)^{-\alpha} s^{-\varrho}\, ds ,$$

$$\int_{t_0}^{t} \exp\big(-\delta(t-s)\big((t-s)^{-\alpha} s^{-\varrho}\, ds = t^{-\varrho} \int_{0}^{t-t_0} \exp(-\delta s) s^{-\alpha}(1 - s/t)^{-\varrho}\, ds .$$

One has

$$\int_{0}^{t-t_0} \exp(-\delta s) s^{-\alpha}(1 - s/t)^{-\varrho}\, ds \leqslant \int_{0}^{t/2} \exp(-\delta s) s^{-\alpha}(1 - s/t)^{-\varrho}\, ds + \\ + \int_{t/2}^{t} \exp(-\delta s) s^{-\alpha}(1 - s/t)^{-\varrho}\, ds \leqslant 2^\varrho \int_{0}^{+\infty} \exp(-\delta s) s^{-\alpha}\, ds + 2^{\alpha+\varrho-1} \exp(-\delta t/2)\, t^{1-\alpha} .$$

So, $\exp(-\delta t)\Phi(t) = O(t^{-\varrho})$.

Moreover, $E'_{1-\alpha}(\theta(t-s)) \leqslant C(\theta(t-s))^{-\alpha} \exp(\theta(t-s))$ (see [7] th. 7.1.1), so that

$$\exp(-\delta r)\int_{t_0}^{t} E'_{1-\alpha}(\theta(t-s))\,\Phi(s)\,ds \leqslant C\int_{t_0}^{t}(t-s)^{-\alpha}\exp\big((\theta-\delta)(t-s)\big)\,s^{-\varrho}\,ds\,.$$

As θ can be arbitrarily small, one can prove as before that

$$\exp[-\delta t]\int_{t_0}^{t} E'_{1-\alpha}(\theta(t-s))\,\Phi(s)\,ds = O(t^{-\varrho})(t\to+\infty)\,.$$

Therefore, $\|u(t)\|_\alpha = O(t^{-\varrho})(t\to+\infty)$.

From (F1)-(F4), one has $\|\varrho(t,u)\| \leqslant \psi(\|u\|_\alpha)\|u\|_\alpha$, with $\psi(\tau) \xrightarrow[\tau\to 0]{} 0$. It follows $\|t^\varrho \varrho(t,u(t))\| \leqslant t^\varrho \psi(\|u\|_\alpha)\|u(t)\|_\alpha \leqslant C\psi(\|u(t)\|_\alpha) \to 0$. So, lemma 2.10 can be applied.

3. – Some examples and applications.

In this section we want to apply the results of the previous one to the study of the asymptotic behavior of the solutions of the problem

$$(3.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{i,j=1}^{n} a_{ij}(t,x,u,Du)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j} = f(t,x,u,Du)\,, & t\geqslant t_0\,,\ x\in\Omega \\ u(t,x) = 0\,, \quad \text{for } t\geqslant t_0\,,\ x\in\partial\Omega & \\ u(t_0,x) = u_0(x)\,, \quad \text{for } x\in\Omega & \end{cases}$$

Here

$$Du(x) = \left(\frac{\partial u}{\partial x_1}(x),\dots,\frac{\partial u}{\partial x_n}(x)\right),\qquad u_0\in W^{2m,p}(\Omega)\cap W_0^{1,p}(\Omega)\,,\qquad \text{with } 1<p<+\infty\,.$$

We assume that Ω is a bounded, regular domain in R^n with smooth boundary $\partial\Omega$.

Moreover, for $1\leqslant i,j\leqslant n$, we have the mappings $a_{ij}\colon [0,+\infty]\times\bar\Omega\times I_R\times I^n_{R'}\to \boldsymbol{R}$, with $R'\in\,]0,+\infty]$, $I_{R'} = \,]-R',R'[$, $I^n_{R'} = \{z\in\boldsymbol{R}^n\ |z|<R'\}$.

The following conditions are requested to be satisfied:

(C1) $\sup\limits_{t,x,u,p}|a_{ij}(t,x,u,p)| < +\infty$;

(C2) $\exists v>0$ such that $\sum\limits_{i,j=1}^{n} a_{ij}(t,x,u,p)\xi_i\xi_j \geqslant v|\xi|^2$, $\forall \xi\in\boldsymbol{R}^n$;

(C3) $|a_{ij}(t,x,u,p) - a_{ij}(s,y,v,q)| \leqslant A_1\big(|t-s|^\mu + |x-y|^\mu + |u-v| + |p-q|\big)$, with $A_1>0$, $\mu\in\,]0,1]$;

(C4) $|a_{ij}(t,x,u,p) - a_{ij}(\infty,x,u,p)| \xrightarrow[t\to\infty]{} 0$, uniformly in x,u,p; f is defined on $[0,+\infty]\times\bar\Omega_{R'}\times I^n_{R'}$ with values in $\boldsymbol{R}$ and:

(G1) $(u, p) \to f(t, x, u, p) \in C^1(I_{R'} \times I^n_{R'})$ $\forall (t, x) \in [0, +\infty] \times \bar{\Omega}$;

(G2) $|D_{(u,p)} f(t, x, u, p)| \leqslant A_2$, with $A_2 > 0$, $\forall t \in [0, +\infty]$, $x \in \bar{\Omega}$;

(G3) $|D_{(u,p)} f(t, x, u, p) - D_{(v,q)} f(s, y, v, q)| \leqslant A_3(|t-s|^\mu + |x-y|^\mu + |u-v| + |p-q|)$, for $s, t \in [0, +\infty[$, $x, y \in \bar{\Omega}$, $u, v \in I_{R'}$, $p, q \in I^n_{R'}$;

(G4) $|f(t, x, u, p) - f(s, y, u, p)| \leqslant A_4(|t-s|^\mu + |x-y|^\mu)$ for $s, t \in [0, +\infty[$, $x, y \in \bar{\Omega}$, $u, v \in I_{R'}$, $p, q \in I^n_{R'}$;

(G5) $|f(t, x, u, p) - f(\infty, x, u, p)| + |D_{(u,p)} f(t, x, u, p) - D_{(u,p)} f(\infty, x, u, p)| \xrightarrow[t\to\infty]{} 0$, uniformly with respect to x, u, p.

We pose $X = L^p(\Omega)$, $D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, $Au = -\Delta u$. If Y and Z are a couple of compatible Banach spaces, we write $(Y, Z)_{\theta,q}$ $(0 < \theta < 1,\ q \in [1, +\infty])$ to indicate the real interpolation space with indexes θ, q (see, for example, [2]). One has, for

$$0 < s_1 < \alpha < s_2 < 1\,,\quad (X, D(A))_{s_2,q} \hookrightarrow (X, D(A))_{\alpha,1} \hookrightarrow D(A^\alpha) \hookrightarrow (X, D(A))_{\alpha,\infty} \hookrightarrow (X, D(A))_{s_1,q}$$

(see also [15], pages 318-319). Assume $1/2p < s_1 < \alpha < s_2$, with $s_1, s_2 \neq \frac{1}{2}$.

In this case, $(X, D(A))_{s_j,p} = \{u \in W^{2s_j,p}(\Omega) : u|_{\partial\Omega} = 0\}$ (see [5], th. 7.5, [6], 1.10). If $p = 2$, $\alpha > \frac{1}{4}$, A is self adjoint and $D(A^\alpha) = \{u \in H^{2\alpha}(\Omega) : u|_{\partial\Omega} = 0\}$ (see [9], vol. 1, ch. 1, th. 10.1). Further, if $\alpha > \frac{1}{2}$ and $p \neq 2$ or $\alpha \geqslant \frac{1}{2}$ and $p = 2$, $D(A^\alpha) \hookrightarrow W^{1,p}(\Omega)$.

If $p > n$, $\alpha > \frac{1}{2} + n/2p$, $D(A^\alpha) \hookrightarrow C^{1,\gamma}(\bar{\Omega})$ for some $\gamma \in]0, 1[$ and the imbedding is compact.

So, there exists $R > 0$, such that $\|u\|_\alpha \leqslant R$ implies $\|u\|_{C^1(\bar{\Omega})} \leqslant R'$. Define $B^\alpha_R = \{u \in D(A^\alpha) : \|u\|_\alpha < R\}$. For $u \in B^\alpha_R$, let

$$\text{(3.2)}\qquad \begin{cases} D(A(t, u)) = D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \\ A(t, u)v = -\sum_{i,j=1}^n a_{ij}(t, x, u, Du) \dfrac{\partial^2 v}{\partial x_i \partial x_j}\,. \end{cases}$$

We have

LEMMA 3.1. – *Let* $a_{ij}, b_j, c \in C^{0,\gamma}(\bar{\Omega})$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n)$ *and assume* $c(x) \geqslant 0$, $\forall x \in \bar{\Omega}$, $\sum_{i,j}^n a_{ij}(x)\xi_i\xi_j \geqslant v|\xi^2|$, *for some* $v > 0$, $\forall \xi \in \mathbf{R}^n$. *For* $1 < p < +\infty$, *define*

$$D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\,, \qquad A_p u = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x) \frac{\partial u}{\partial x_j} + c(x) u\,.$$

Then, in $L^p(\Omega)$ $\varrho(A_p) \supseteq \{z \in \mathbf{C} : \operatorname{Re} z \leqslant 0\}$.

PROOF. – If $\gamma' \geqslant \gamma$, define in $C^{0,\gamma'}(\bar{\Omega})$ the operator $B_{\gamma'}$:

$$D(B_{\gamma'}) = \{u \in C^{2,\gamma'}(\bar{\Omega}) | u|_{\partial\Omega} = 0\}\ , \quad B_{\gamma'}u = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u\ .$$

By virtue of the maximum principle, Krein-Rutman theory of positive operators and Schauder estimates (see for a brief survey [17], in particular pages 21-22), $\varrho(B_{\gamma'}) \supseteq \{z \in \boldsymbol{C}\colon \operatorname{Re} z \leqslant 0\}$.

By [1], $\varrho(A_p) \neq \emptyset$ and $(A_p - \mu)^{-1}$ is compact in $L^p(\Omega)$ $\forall \mu \in \varrho(A_p)$. Assume $\operatorname{Re}\lambda \leqslant 0$, $\lambda u - A_p u = 0$. Then, if $\mu \in \varrho(A_p)$, $\operatorname{Re}\mu \leqslant 0$, $\mu u - A_p u = (\mu - \lambda)u$. If $2p > n$, u is hölder continuous, for some $\gamma' \leqslant \gamma$. Then there exists a unique $v \in D(B_{\gamma'})$, such that $\mu v - B_{\gamma'}v = (\mu - \lambda)u$. On the other hand, u is the unique solution of $\mu v_1 - A_p v_1 = (\mu - \lambda)u$. So, $v = u$ and $u \in C^{2,\gamma'}(\bar{\Omega})$. It follows $\lambda u - B_{\gamma'}u = 0$, that is, $u = 0$. If $2p \leqslant n$, but $3p > n$, there exists q such that $2q > n$ and $u \in L^q(\Omega)$. Take $\mu \in \varrho(A_p) \cap \varrho(A_q)$ with $\operatorname{Re}\mu \leqslant 0$. One can verify as before that u is the unique solution of $\mu v - A_q v = (\mu - \lambda)u$. Iterating this proceeding one has the result.

LEMMA 3.2. – *The operators* $A(t, u)$ *defined in* (3.2) *satisfy the conditions* (B1)-(B4) *for any* $X = L^p(\Omega)$ *and for any* α *such that* $p > n$, $\alpha > \frac{1}{2} + n/2p$.

PROOF. – From [1] (in particular th. 2.1) one has that $\exists C_0 > 0$, $\theta_0 \in]0, \pi/2[$, such that $|\lambda| \geqslant C_0$, $|\operatorname{Arg}\lambda| > \theta_0$ imply $\lambda \in \varrho(A(t, u))$,

$$\|(A(t, u) - \lambda)^{-1}\| \leqslant \text{const}\,(\|u\|_\alpha)(1 + |\lambda|)^{-1}\,. \tag{3.3}$$

By lemma 3.1, $\varrho(A(t, u)) \supseteq \{\lambda \in \boldsymbol{C}\colon \operatorname{Re}\lambda \leqslant 0\}$. We want to prove that (3.3) is true for every λ, with $\operatorname{Re}\lambda \geqslant 0$. In fact, if it were not so, there would exist $\varrho \in]0, R[$, $\forall v \in \boldsymbol{N}$ $t_v \in [0, +\infty]$, $u_v \in X_\alpha$, with $\|u_v\|_\alpha \leqslant \varrho$, $\lambda_v \in \boldsymbol{C}$ with $\operatorname{Re}\lambda_v \leqslant 0$, $v_v \in D(A)$, such that $\|v_v\| = 1$ and

$$\|A(t_v, u_v)v_v - \lambda_v v_v\| < (1 + |\lambda_v|)v^{-1}\,. \tag{3.4}$$

The sequence (λ_v) is clearly bounded. So one can assume $\lambda_v \to \lambda_0 \in \boldsymbol{C}$, $t_v \to t_0 \in [0, +\infty]$, $u_v \to u_0$ in $C^{1,\gamma'}(\bar{\Omega})$, for some $\gamma' \in]0, 1[$. If we define

$$D(B) = D(A)\,, \qquad Bv = -\sum_{i,j=1}^{n} a_{ij}(t_0, x, u_0(x), Du_0(x))\frac{\partial^2 u}{\partial x_i \partial x_j}\,,$$

we have that $\varrho(B) \supseteq \{\lambda \in \boldsymbol{C}\colon \operatorname{Re}\lambda \leqslant 0\}$. From $\|A(t_v, u_v)v_v - \lambda_v v_v\| \xrightarrow[v\to\infty]{} 0$, it follows $\lambda_0 \in \sigma(B_0)$ and this is a contradiction.

The remaining part of the theorem has a trivial proof.

Now, define:

$$(3.5)\qquad \begin{cases} f\colon [0, +\infty]\times B^\alpha_R \to X \\ f[t, u)(x) = f(t, x, u(x), Du(x)) \,. \end{cases}$$

The proof of the following lemma is easy:

LEMMA 3.3. – *Assume $p > n$, $\alpha > \frac{1}{2} + n/2p$ and the conditions* (G1)-(G5) *satisfied. If f is defined as in* (3.5), *it satisfies the conditions* (F1)-(F4).

Now we apply the abstract results of section 2, always assuming $p > n$, $\alpha > \frac{1}{2} + n/2p$.

THEOREM 3.4. – *Let u be the maximal solution of* (3.1) *(which implies that $t \to u(t, \cdot) \in C([t_0, T[; W^{2,p}(\Omega)) \cap C^1([t_0, T[; L^p(\Omega))$, for some $T > t_0$) and assume the conditions* (C1)-(C4), (F1)-(F5) *are satisfied, $\sup_{t\geqslant t_0} \|u(t, \cdot)\|_{W^{s,p}(\Omega)} < +\infty$ for some $s > 2\alpha$ and there exists $u_0 \in L^p(\Omega)$ such that $\|u(t, \cdot) - u_0\|_{L^p(\Omega)} \to 0$.*
Then

$$u_0 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\,, \qquad \|u_0\|_\alpha < R\,, \qquad \sum_{i,j=1}^{n} a_{ij}(\infty, x, u_0(x), Du_0(x))\,\frac{\partial^2 u}{\partial x_i\,\partial x_j} =$$
$$= f(\infty, x, u_0, Du_0) \quad \text{and} \quad \|u(t, \cdot) - u_0\|_{W^{2,p}(\Omega)} \xrightarrow[t\to\infty]{} 0\,.$$

Moreover, $t \to u(t, \cdot)$ is globally hölder continuous with values in $W^{s,p}(\Omega)$, $\forall s < 2$.

PROOF. – It follows from propositions 2.1 and 2.2

THEOREM 3.5. – *Assume the conditions* (C1)-(C4), (G1)-(G5) *are satisfied. Further,*

(*a*) $f(\infty, x, 0, 0) = 0 \quad \forall x \in \Omega$;

(*b*) $\dfrac{\partial f}{\partial u}(\infty, x, 0, 0) \geqslant 0 \quad \forall x \in \Omega$.

Then, $\forall s > 2\alpha$, there exists $T_0 \geqslant 0$, $\mu_0 > 0$, such that the maximal solution u of (3.1) *is globally defined and $\|u(t, \cdot)\|_{W^{2,p}(\Omega)} \to 0$ if $\|u_0\|_{W^{s,p}(\Omega)} \leqslant \mu_0$, $t_0 \geqslant T_0$.*

PROOF. – It follows from lemma 3.1 and theorem 2.4.

THEOREM 3.6. – *Assume $a_{ij}(t, x, u, p) = a_{ij}(x, u, p)$, $f(t, x, u, p) = f(x, u, p)$, and the conditions* (C1)-(C4), (G1)-(G5) *are satisfied. Let:*

(*a*) $f(x, 0, 0) = 0$

(*b*) $\dfrac{\partial f}{\partial u}(x, 0, 0) \geqslant 0 \quad \forall x \in \bar{\Omega}$.

Define

$$D(B) = D(A)\,, \qquad Bu = -\sum_{i,j=1}^{n} a_{i,j}(x, 0)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{j=1}^{n}\frac{\partial f}{\partial p_j}(x, 0, 0)\frac{\partial u}{\partial x_j} + \frac{\partial f}{\partial u}(x, 0, 0)u\,.$$

Call $\beta = \inf\{\operatorname{Re}\lambda\colon \lambda \in \boldsymbol{C},\ \lambda \in \sigma(B)\}$. *Then*:

(1) β *is a positive eigenvalue*;

(2) β *is a simple eigenvalue*;

(3) *If* u *is a solution of* (3.1) *converging to* 0 *in* $W^{s,p}(\Omega)$, *for some* $s > 2\alpha$,

$$u(t, x) = \exp(-\beta t)p(x) + r(t, x)\,,$$

with $p \in D(B)$, $(\beta - B)p = 0$, $\exp(\beta t)\|r(t, \cdot)\|_{W^{\sigma,p}(\Omega)} \xrightarrow[t\to\infty]{} 0$, $\forall \sigma < 2$.

PROOF. – (1), (2) are consequences of [8], th. 6.3, easily extendible (using for example the method of lemma 3.1) to the operator defined in a Sobolev space.

Using (G3), one can verify that theorem 2.7 is applicable and so the result follows.

THEOREM 3.7. – *Consider the problem* (3.1) *and assume* $f(t, x, 0, 0) = t^{-\varrho} f_1(x) + t^{-\varrho} f_2(t, x)$ *with* $\varrho > 0$, $f_1 \in L^p(\Omega)$, $\|f_2(t, \cdot)\|_{L^p(\Omega)} \xrightarrow[t\to\infty]{} 0$.

If $p > n$, $s > 1 + n/p$ *and* $\|u(t, \cdot)\|_{W^{s,p}(\Omega)} \xrightarrow[t\to\infty]{} 0$, *then*

$$u(t, x) = t^{-\varrho} u_1(x) + t^{-\varrho} r_1(t, x)$$

with $u_1, r_1(t, \cdot) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\|r_1(t, \cdot)\|_{W^{\sigma,p}(\Omega)} \to 0$, $\forall \sigma < 2$.

PROOF. – It follows from theorem 2.9.

4. – Saddle points.

In all the previous cases, the operator $A(\infty, 0) + f'_u(\infty, 0)$ had a spectrum contained in theright halfplane of $\boldsymbol{C}$, so that the semigroup generated by it had an exponential decay. Now, we consider the case when this is no more necessarily true. To this aim, we adopt a different functional approach.

We recall some definitions one can find in [3].

Let $\theta \in]0, 1[$; assume $-A$ is the infinitesimal generator of an analytic semigroup in X. We define:

$$D_\theta(A) = \{x \in X\colon \lim_{t\to\infty} t^\theta A(A + t)^{-1}x = 0\}\,.$$

If $x \in D_\theta(A)$, we pose

$$\|x\|_\theta = \|x\| + \sup_{t \geq t'} \|t^\theta A(A+t)^{-1}x\| ,$$

if t' is such that $[t', +\infty[\subseteq \varrho(-A)$. $D_{1+\theta}(A)$ will be $\{x \in D(A): Ax \in D_\theta(A)\}$ and $\|x\|_{1+\theta} = \|x\| + \|Ax\|_\theta$.

The following result is a simple elaboration of theorem 3.1 in [3]:

PROPOSITION 4.1. – *Let $-A$ the infinitesimal generator of an analytic semigroup and assume $\varrho(A) \supseteq \{z \in \mathbf{C}: \operatorname{Re} z \leq 0\}$.*

Let $f: [t_0, +\infty[\to D_\theta(A)$ continuously ($t_0 \geq 0$). We pose

$$u(t) = \int_{t_0}^{t} \exp(-(t-s)A)\, f(s)\, ds .$$

Then:

(1) *if f is bounded, u is continuous and bounded with values in $D_{1+\theta}(A)$;*

(2) *if $\|f(t)\|_\theta \xrightarrow[t\to\infty]{} 0$, $\|u(t)\|_{1+\theta} \xrightarrow[t\to\infty]{} 0$.*

PROOF. – From [3], th. 3.1, we know that u is continuous with values in $D_{1+\theta}(A)$. One has

$$\|u(t)\| = \left\|\int_{t_0}^{t} \exp(-(t-s)A)\, f(s)\, ds\right\| \leq \int_{t_0}^{t} M \exp(-\delta(t-s)) \|f(s)\|\, ds = I(t)$$

(for some $M, \delta > 0$). It is easily seen that I is bounded if f is bounded and converges to 0 if f converges to 0.

If Γ is the clockwise oriented boundary of $\{z \in \mathbf{C}: |\operatorname{Arg} z| \leq \theta_0\}$ (for a suitable $\theta_0 \in]0, \pi/2[$), one has

$$u(t) = (2\pi i)^{-1} \int_{t_0}^{t} \left(\int_\Gamma \exp(-\lambda(t-s)(A-\lambda)^{-1} d\lambda\right) f(s)\, ds =$$
$$= (2\pi)^{-1} \int_{t_0}^{t} \exp -\delta(t-s) \left(\int_\Gamma \exp(-(\lambda-\delta)(t-s))(A-\lambda)^{-1} d\lambda\right) f(s)\, ds .$$

If δ is chosen sufficiently small,

$$\int_{\Gamma^\delta} \exp(-(\lambda-\delta)(t-s))(A-\lambda)^{-1} d\lambda = \int_{\Gamma-\delta} \exp(-\lambda(t-s))(A-\lambda-\delta)^{-1} d\lambda =$$
$$= (\text{by Cauchy's theorem}) \int_\Gamma \exp(-\lambda(t-s))(A-\lambda-\delta)^{-1} d\lambda .$$

So, $u(t) = (2\pi i)^{-1}\int_{t_0}^{t} \exp(-\delta(t-s))\left(\int_{\Gamma} \exp(-\lambda(t-s))(A-\lambda-\delta)^{-1}\,d\lambda\right) f(s)\,ds$.

$$Au(t) = \delta u(t) + (A-\delta)u(t) = \delta u(t) + (2\pi i)^{-1}\int_{t_0}^{t} \exp(-\delta(t-s)) \cdot$$
$$\cdot \left(\int_{\Gamma} \exp(-\lambda(t-s))(A-\delta)(A-\lambda-\delta)^{-1}\,d\lambda\right) f(s)\,ds\,.$$

Therefore, we have the following estimate:

$$\|Au(t)\| \leqslant \delta\|u(t)\| +$$
$$+ C\int_{t_0}^{t} \exp(-\delta(t-s))\left(\int_{\Gamma} \exp(-(t-s)\operatorname{Re}\lambda)\|(A-\delta)(A-\lambda-\delta)^{-1} f(s)\|\,|d\lambda|\right) ds\,.$$

By virtue of [3], lemma 3.2, $\|(A-\delta)(A-\lambda-\delta)^{-1}f(s)\| \leqslant C|\lambda|^{-\theta}\|f(s)\|_\theta$, so that

$$\|Au(t)\| \leqslant \delta\|u(t)\| + C\int_{t_0}^{t} \exp(-\delta(t-s))\left(\int_{\Gamma} \exp(-(t-s)\operatorname{Re}\lambda)|\lambda|^{-\theta}\|f(s)\|_\theta|d\lambda|\right) ds\,.$$

One has $\int_{\Gamma} \exp(-(t-s)\operatorname{Re}\lambda)|\lambda|^{-\theta}|d\lambda| \leqslant C(t-s)^{\theta-1}$, so that

$$\|Au(t)\| \leqslant \delta\|u(t)\| + C\int_{t_0}^{t} \exp(-\delta(t-s))(t-s)^{\theta-1}\|f(s)\|_\theta\,ds\,,$$

which is uniformly bounded in t if f is bounded and tends to 0 if $\|f(s)\|_\theta \xrightarrow[s\to\infty]{} 0$. Finally, for $\xi \geqslant 1$,

$$\|A(A+\xi)^{-1}Au(t)\| = \left\|(2\pi i)^{-1}\int_{\Gamma}\left(\int_{t_0}^{t} \exp(-\lambda(t-s))\,\lambda(\lambda+\xi)^{-1}A(A-\lambda)^{-1}f(s)\,ds\right)d\lambda\right\| \leqslant$$
$$\leqslant C\int_{0}^{+\infty}\left(\int_{t_0}^{t} \exp(-r\cos(\theta_0)(t-s))\,r^{1-\theta}|r\exp(i\theta_0)+\xi|^{-1}\|f(s)\|_\theta\,ds\right)dr\,.$$

If

$$\sup_{s\geqslant t_0}\|f(s)\|_\theta < +\infty\,, \qquad \int_{t_0}^{t} \exp(-t\cos(\theta_0)(t-s))\|f(s)\|_\theta\,ds \leqslant Cr^{-1}$$
$$\|A(A+\xi)^{-1}Au(t)\| \leqslant C\int_{0}^{+\infty} r^{-\theta}|r\exp(i\theta_0)+\xi|^{-1}\,dr \leqslant C\xi^{-\theta}$$

and so (1) is proved, also taking into account the fact that $D_\theta(A)$ is a closed subspace of $\{u\in X\colon \sup_{\xi\geqslant 1}\|\xi^\theta A(A+\xi)^{-1}u\| < +\infty\}$.

If $\|f(t)\|_\theta \xrightarrow[t\to\infty]{} 0$, we remark that

$$\int_{t_0}^{t} \exp(-r\cos(\theta_0)(t-s))\|f(s)\|_\theta\,ds \leqslant$$
$$\leqslant \int_{t_0}^{T(\varepsilon)} \exp(-r\cos(\theta_0)(t-s))\|f(s)\|_\theta\,ds + \varepsilon(r\cos(\theta_0))^{-1}\ \left(\text{if } \|f(s)\|_\theta \leqslant \varepsilon \text{ for } s \geqslant T(\varepsilon)\right).$$

This gives

$$\|A(A+\xi)^{-1}Au(t)\| \leqslant C\int_0^{+\infty}\left(\int_{t_0}^{T(\varepsilon)}\exp(-r\cos(\theta_0)(t-s))\, r^{-\theta}|r\exp(i\theta_0)+\xi|^{-1}\|f(s)\|_\theta\, ds\right)dr + C\varepsilon r^{-\theta}\int_0^{+\infty}|r\exp(i\theta_0)+\xi|^{-1}\,dr\,.$$

The second term can be majorized by $C\,\varepsilon\,\xi^{-\theta}$, the first by

$$C\int_0^{\infty+} r^{-\theta}|r\exp(i\theta_0)+\xi|^{-1}\exp(-r\cos(\theta_0))(t-T(\varepsilon))\,dr = C\xi^{-\theta}\int_0^{+\infty} r^{-\theta}|r\exp(i\theta_0)+1|^{-1}\exp\big(-r\xi\cos(\theta_0)(t-T(\varepsilon))\big)\,dr\,.$$

It is easily seen that the integral tends to 0 as $t\to\infty$, uniformly with respect to $\xi\geqslant 1$. So, the proposition is proved.

Now let $-B$ be the infinitesimal generator of an analytic semigroup in X, such that

(H) $\sigma(B)\cap\{z\in \boldsymbol{C}\colon \operatorname{Re} z=0\}=\emptyset$.

Define

$$\sigma_1=\sigma(B)\cap\{z\in\boldsymbol{C}\colon \operatorname{Re} z<0\}\,,\qquad \sigma_2=\sigma(B)\cap\{z\in\boldsymbol{C}\colon \operatorname{Re} z>0\}\,.$$

Let Γ be a counterclockwise oriented closed path, contained in $\varrho(B)\cap\{z\in\boldsymbol{C}.$ $\operatorname{Re} z<0\}$, which turns around σ_1. Define

$$P_1=(2\pi i)^{-1}\int_\Gamma(z-B)^{-1}\,dz\,,\qquad P_2=1-P_1$$

P_1 and P_2 are projections. If $X_j=P_j(X)$, X_j is invariant with respect to B, $X_1\subseteq\bigcap_{k=1}^{\infty}D(B^k)$. We call B_j the part of B in X_j. One has $B_1\in\mathcal{L}(X_1)$, $\sigma(B_1)=\sigma_1$, $\sigma(B_2)=\sigma_2$. $-B_2$ is the infinitesimal generator of an analytic semigroup $\{\exp(-tB_2)\colon t\geqslant 0\}$ in X_2, such that $\|\exp(-tB_2)\|\leqslant M\exp(-\delta t)$, with M and δ positive. Moreover, $\exp(-tB_j)x=\exp(-tB)x$, if $x\in X_j$: Finally,

$$P_j\exp(-tB)=\exp(-tB)P_j=\exp(-tB_j)P_j\,.$$

One has:

LEMMA 4.2. – *Let* $x\in X$. *Then* $\exp(-tB)x\xrightarrow[t\to\infty]{}0$ *if and only if* $x\in X_2$. *Moreover, if* $x\in X_2\cap D_{1+\theta}(B)$, $\|\exp(-tB)x\|_{1+\theta}\xrightarrow[t\to\infty]{}0$.

PROOF. – If $x\in X$, $x=P_1x+P_2x$ and

$$\exp(-tB)x=\exp(-tB_1)P_1x+\exp(-tB_2)P_2x\,.$$

One has $\exp(-tB_2)P_2x \xrightarrow[t\to\infty]{} 0$. As $\sigma(B_1)=\sigma_1$, $\|\exp(tB_1)\| \leqslant M\exp(-\delta t)$, for $\delta \geqslant 0$, $M \geqslant 0$.

So $\exp(-tB)x \to 0$ iff $\exp(-tB_1)P_1x \to 0$. This implies that

$$\exp(tB_1)(\exp(-tB_1)P_1x) \to 0\,,$$

that is, $P_1x = 0$.

Finally, if

$$x \in X_2 \cap D_{1+\theta}(B)\,, \quad \|B\exp(-tB)x\| = \|\exp(-tB_2)B_2x\| \leqslant M\exp(-\delta t)\|B_2x\| \xrightarrow[t\to\infty]{} 0\,.$$

If ξ is sufficiently large,

$$\begin{aligned}\|\xi^\theta B(B+\xi)^{-1}B\exp(-tB)x\| &= \|\xi^\theta B_2(B_2+\xi)^{-1}B_2\exp(-tB_2)x\| = \\ &= \|\exp(-tB_2)(\xi^\theta B_2(B_2+\xi)^{-1}B_2x)\| \leqslant M\exp(-\delta t)\|\xi^\theta B(B+\xi)^{-1}Bx\| \leqslant \\ &\leqslant M\exp(-\delta t)\|x\|_{1+\theta} \xrightarrow[t\to\infty]{} 0\,.\end{aligned}$$

The main result of this section is the following

THEOREM 4.3. – *Let B satisfy* (H), f: $[0, +\infty[\times D_{1+\theta}(B) \to D_\theta(B)$ *such that*

(*a*) $f(t,0)=0$ $\forall t \geqslant 0$, $x \to f(t,x)$ *is differentiable* $\forall t \geqslant 0$, $(t,x) \to f'_x(t,x)$ *is continuous with values in* $\mathcal{L}(D_{1+\theta}(B), D_\theta(B))$ *and bounded on bounded subsets of* $D_{1+\theta}(B)$ *uniformly with respect to t.*

(*b*) $\|f'_x(t,x)\|_{\mathcal{L}(D_{1+\theta}(B),D_\theta(B))} \xrightarrow[\|x\|_{1+\theta}\to 0]{} 0$, *uniformly with respect to t.*

(*c*) $-B + f'_x(t,x)$ *is the restriction to* $D_{1+\theta}(B)$ *of the infinitesimal generator of analytic semigroup in X with domain* $D(B)$ *and* $D_{1+\theta}(B - f'_x(t,u)) = D_{1+\theta}(B)$ (*identifying an operator with its restriction*). *For* $x \in D_{1+\theta}(B)$, *we call* $z(t,t_0,x)$ *the maximal solution of*

$$\frac{dz}{dt} + Bz(t) = f(t, z(t))\,, \qquad z(t_0) = x$$

(*for its existence and unicity see* [3], th. 4.1).

Let $\varrho > 0$. *We pose, for* $t_0 \geqslant 0$,

$$S_{\varrho,\sigma} = \{x \in D_{1+\theta}(B)\colon \|z(t,t_0,x)\|_{1+\theta} \leqslant \varrho\} \qquad \forall t \geqslant t_0\,,$$

$$\|P_2x\| \leqslant \sigma\,, \qquad \lim_{t\to\infty}\|z(t,t_0,x)\|_{1+\theta} = 0\,, \qquad B_\sigma = \{x \in D_{1+\theta}(B) \cap X_2\colon \|x\|_{1+\theta} \leqslant \sigma\}\,.$$

Then, there exist $\sigma, \varrho > 0$ *such that* P_2 (*restricted to* $S_{\varrho,\sigma}$) *is a homeomorphism between* $S_{\varrho,\sigma}$ *and* B_σ (*with the topology induced by* $D_{1+\theta}(B)$).

PROOF. – Let $x \in S_{\varrho,\sigma}$, for $\varrho, \sigma > 0$. We put $z(t) = z(t, t_0, x)$. One has

$$z(t) = \exp\big(-(t-t_0)B\big)x + \int_{t_0}^{t} \exp\big(-(t-s)B\big) f(s, z(s))\, ds\,.$$

If

$$z_j(t) = P_j z(t)\,, \qquad z_j(t) = \exp\big(-(t-t_0)B_j\big)P_j x + \int_{t_0}^{t} \exp\big(-(t-s)B_j\big)P_j f(s, z(s))\, ds\,.$$

For $j = 1$, applying $\exp\big((t-t_0)B_1\big)$, one has

$$P_1 x = \exp\big((t-t_0)B_1\big)z_1(t) - \int_{t_0}^{t} \exp\big((s-t_0)B_1\big)P_1 f(s, z(s))\, s\, ds\,,$$

and from this,

$$P_1 x = -\int_{t_0}^{+\infty} \exp\big((s-t_0)B_1\big)P_1 f(s, z(s))\, ds\,.$$

It follows

$$z(t) = \exp\big(-(t-t_0)B_2\big)P_2 x + \int_{t_0}^{t} \exp\big(-(t-s)B_2\big)P_2 f(s, z(s))\, ds + \\ - \int_{t}^{+\infty} \exp\big((s-t)B_1\big)P_1 f(s, z(s))\, ds\,.$$

For $\varrho > 0$, define

$$Y_\varrho = \{u \in C([t_0, +\infty[; D_{1+\theta}(B)) \colon \|u(t)\|_{1+\theta} \leqslant \varrho,\ \|u(t)\|_{1+\theta} \xrightarrow[t\to\infty]{} 0\}$$

$$Tz(t) = \exp\big(-(t-t_0)B_2\big)a + \int_{t_0}^{t} \exp\big(-(t-s)B_2\big)P_2 f(s, z(s))\, ds - \\ - \int_{t}^{+\infty} \exp\big((s-t)B_1\big)P_1 f(s, z(s))\, ds\,,$$

with $a \in B_\sigma$ $(\sigma > 0)$,

$$Y = \{u \in C([t_0, +\infty[; D_{1+\theta}(B)) \colon \|u(t)\|_{1+\theta} \to 0\}\,.$$

By virtue of proposition 4.1, $T(Y_\varrho^{1}) \subseteq Y$. It is not difficult to verify, applying again prop. 4.1 and (*b*), that

$$\|Tz(t)\|_{1+\theta} \leqslant M \exp\big(-\delta(t-t_0)\big) + \mathrm{const}\,(\varrho)\,\varrho\,,$$

with $\mathrm{const}\,(\varrho) \xrightarrow[\varrho\to 0]{} 0$. Furthermore, if $z, v \in Y_\varrho$,

$$\|Tz(t) - Tv(t)\|_{1+\theta} \leqslant C \sup_{\substack{s \geqslant t_0 \\ \|w\|_{1+\theta} \geqslant \varrho}} \|f'_x(s, w)\|_{\mathcal{L}(D_{1+\theta}(B), D_\theta(B))} \|z(s) - v(s)\|_{1+\theta}\,.$$

Using again (*b*), one concludes that, if $\varrho \leqslant \varrho_0$ (ϱ_0 sufficiently small), $\|Tz - Tv\|_Y \leqslant \leqslant \frac{1}{2}\|z - v\|_X$; the whole discussion implies that, if $\varrho \leqslant \varrho_0$, $\sigma \leqslant \sigma(\varrho)$, $\forall a \in B_\sigma$, T has a unique fixed point φ_a in Y. Consider now $P_2\colon S_{\varrho,\sigma} \to B_\sigma$. P_2 is continuous. It is also a bijection. In fact, let $a \in B_\sigma$. Then $x \in S_{\varrho,\sigma}$ is such that $P_2 x = a$ iff $Tz(t, t_0, x) = z(t, t_0, x)$. But there exists a unique fixed point of T and this implies that P_2 is a bijection. Also, one has $P_2^{-1} a = a - \int_{t_0}^{+\infty} \exp\left((s - t_0)B_1\right) P_1 f(s, \varphi_a(s))\, ds$.

It is easily proved that, if

$$a, b \in B_\sigma\,, \qquad \|\varphi_a(t) - \varphi_b(t)\|_{1+\theta} \leqslant M \exp\left(-\delta(t - t_0)\right)\|a - b\|_{1+\theta} + \tfrac{1}{2}\|\varphi_a - \varphi_b\|_Y\,,$$

and from this the continuity of $a \to \varphi_a$ and of P_2^{-1} follows easily. Therefore, the result is completely proved.

REMARK 4.4. – $S_{\varrho,\sigma}$ and B_σ are tangent in 0 in the following sense: if $x \in S_{\varrho,\sigma}$

$$\lim_{\|x\|_{1+\theta} \to 0} \|P_2 x - x\|_{1+\theta} / \|x\|_{1+\theta} = 0\,.$$

This can be verified, remarking that, if $\|a\|_{1+\theta} \leqslant \varrho'$, $\|b\|_{1+\theta} \leqslant \varrho'$, with $\varrho' \leqslant \varrho$, $\|\varphi_a - \varphi_b\|_Y \leqslant \mathrm{const}\,(\varrho')\|a - b\|_{1+\theta}$ and $\mathrm{const}\,(\varrho') \xrightarrow[\varrho' \to 0]{} 0$.

5. – An example.

Consider the problem (with $I = [0, 1]$)

$$(5.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - a\left(u, \dfrac{\partial u}{\partial x}\right)\dfrac{\partial^2 u}{\partial x^2} = g(u)\,, \qquad x \in I\,,\ t \geqslant 0\,. \\ u(t, 0) = u(t, 1) = 0 \\ u(0, x) = u_0(x) \in H^2(I) \cap H_0^1(I) \end{cases}$$

with the following conditions:

(*a*) $a \in C^2(\mathbf{R}^2)$, $g \in C^1(\mathbf{R})$;

(*b*) $g(0) = 0$;

(*c*) $a(u, p) > 0 \ \forall (u, p) \in \mathbf{R}^2$.

We pose $D(B) = H^2(I) \cap H_0^1(I)$, $Bv = -a(0, 0)v'' + g'(0)v$.

We shall use the following facts:

LEMMA 5.1 (*see* [3], *proposition* 6.2 *and* 6.3). – *If* $\theta < \frac{1}{4}$,

$$D(B) = h_2^{2\theta}(I) = \{u \in L^2(I)\colon t^{-2\theta}\|u(t + \cdot) - u\|_{L^2(I \cap (I - t))} \xrightarrow[t \to 0]{} 0\,.$$

LEMMA 5.2. – *Let* $0<\theta<\frac{1}{4}$, $u\in D_\theta(B)$, $v\in H^1(I)$. *Then* $uv\in D_\theta(I)$ *and* $\|uv\|_\theta\leqslant\leqslant C\|u\|_{H^1(I)}\|v\|_\theta$ *with* C *independent from* u *and* v.

PROOF. – It is an easy consequence of Lemma 5.1.

The proof of the following lemma is trivial:

LEMMA 5.3. – *Assume* $\theta<\frac{1}{4}$. *Then* $D_{1+\theta}(B)=\{u\in H^2(I)\cap H_0^1(I)\colon u''\in h_2^{2\theta}(I)\}$.

Now define

$$f\colon D_{1+\theta}(B)\to D_\theta(B)\,,\quad f(u)(x)=[a(u(x),u'(x))-a(0,0)]u''+g(u(x))-g'(0)u(x)\,.$$

We have:

LEMMA 5.4 – f *satisfies the conditions* (*a*), (*b*), (*c*) *of theorem* 4.3.

PROOF. – It is a standard computation, using lemmata 5.2 and 5.3.
So we have:

THEOREM 5.5. – *Let* $g'(0)+a(0,0)k^2\pi^2\neq 0$ $\forall k\in N$. *Then theorem* 4.4 *is applicable if* $0<\theta<\frac{1}{4}$. *In this case* $X_2=\{u\in L^2(I)\colon \int_0^1 u(t)\sin(r\pi t)\,dt=0\ \forall r\leqslant j\}$, *with* j *such that* $g'(0)+a(0,0)j^2\pi^2<0<g'(0)+a(0,0)(j+1)^2\pi^2$.

PROOF. – One has $\sigma(B)=\{g'(0)+a(0,0)k^2\pi^2\colon k\in Z\}$, with corresponding eigenvectors $\{\sin(k\pi t)\colon k\in Z\}$. As B is self-adjoint, P_1 (see its definition in section 4) is an orthogonal projection and, as the eigenvalues of B are simple, it is easily seen that X_1 is the vector space generated by $\{\sin(r\pi t)\colon r=1,\dots,j\}$. From this the result follows.

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Note. – After sending the paper, the author became aware of the preprint: G. DA PRATO - A. LUNARDI, *Stability, instability and central manifold theorem for fully nonlinear autonomous parabolic equations in Banach space* (Scuola Normale Superiore Pisa, June 1985), which also contains the results of section 4.