# On Some Properties of the Groups $G(n, l)$ (*).

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**Sunto.** – *In* [1] *è stata costruita una famiglia di 3-varietà chiuse e connesse mediante l'uso di grafi 4-colorati sugli spigoli. Nella presente nota si studiano alcune proprietà algebriche dei gruppi fondamentali delle suddette 3-varietà, per i quali una presentazione è stata ottenuta in* [2]. *Il lavoro ha soprattutto un interesse algebrico, più che topologico, per le tecniche usate nelle dimostrazioni.*

**1.** – By $\Delta_n$ (resp. $N_n$) and $Z_n$ we will denote the set $\{0, 1, \ldots, n\}$ (resp. $\Delta_n - \{0\}$) and the ring of integers mod $n$ respectively.

In [1] edge-coloured graphs (named 3-*gems*, i.e. graph encoded 3-manifolds) are used to define a family of closed connected 3-manifolds which includes the lens spaces (see [3]).

For any two positive integers $n$ and $l$, there is such a manifold $M(n, l)$ which is proved to be homeomorphic to $M(l, n)$ by making use of topological considerations (see [4], corollary 1).

Let $G(n, l)$ be the fundamental group $\Pi_1(M(n, l))$ of $M(n, l)$.

Then the group $G(n, l)$ has the finite presentation (see [2])

$$(1)\qquad G(n, l) = \langle a_i (i \in \Delta_{n-1}) \colon a_0 a_1 \ldots a_{n-1} = 1\,, \quad a_i^{-1} a_{i-2}^{-1} a_{i-4}^{-1} \ldots a_{i-(l-3)}^{-1} a_{i-(l-2)} a_{i-(l-4)} \ldots a_{i-1} a_{i+1} = 1\,, \quad i \in \Delta_{n-1},\ \text{indices mod}\, n \rangle \qquad (l\ \text{odd})$$

and

$$(2)\qquad G(n, l) = \langle a_i\ (i \in \Delta_{n-1}) \colon a_0 a_1 \ldots a_{n-1} = 1, \quad a_i^{-1} a_{i-2}^{-1} a_{i-4}^{-1} \ldots a_{i-(l-2)}^{-1} a_{i-(l-3)} a_{i-(l-5)} \ldots a_{i-1} a_{i+1} = 1\,, \quad i \in \Delta_{n-1},\ \text{indices mod}\, n \rangle \qquad (l\ \text{even})\,.$$

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Obviously the group $G(1, n) \simeq G(n, 1)$ is trivial and $G(2, n) \simeq G(n, 2)$ is isomorphic to the cyclic group $Z_n$ for each positive integer $n$.

The aim of this paper is to study some algebraic properties of the groups $G(n, l)$ by using Tietze transformations and the extension theory of groups. The paper has mainly an algebraic interest, rather than a topological one, because of the techniques used in the proofs.

As general references for group theory see [5], [6], [7].

For presentations of groups in terms of generators and relations we refer to [8], [9], [10].

**2.** – Throughout this section the positive integers $n$ and $l$ will be assumed odd and coprime. By $\mathfrak{D}(n, l)$ we denote the Dyck's group of type $(n, l, 2)$ defined by the finite presentation $\langle u, v / u^n = v^l = (uv)^2 = 1\rangle$ (see [8], p. 54; [11], p. 196).

In order to prove the main result of this section we need the following simple lemma whose proof has been included to make the reading clear.

LEMMA 1. – *The centre $Z(\mathfrak{D}(n, l))$ of $\mathfrak{D}(n, l)$ is trivial.*

PROOF. – If $vu^i \in Z(\mathfrak{D}(n, l))$, then $vu^{i+1} = (vu^i)u = u(vu^i) = v^{-1}u^{i-1}$ since $uv = v^{-1}u^{-1}$, and thus $v^2 = u^{-2}$. If $n = 2p + 1$ $(p \geq 0)$, then $v^2 = u^{-2}$ implies that $u^n = (u^2)^p u = v^{-2p} u = 1$, whence $u = v^{2p}$. Thus we have a contradiction since $\mathfrak{D}(n, l)$ is not a finite cyclic group (see [8]). Therefore the centre of $\mathfrak{D}(n, l)$ must be a subgroup of $\langle u\rangle$. By replacing $u$ by $v$ in the above argument, we can analogously conclude that $Z(\mathfrak{D}(n, l))$ must be a subgroup of $\langle v\rangle$ too. Therefore the order of $Z(\mathfrak{D}(n, l))$ must be a divisor of both $n$ and $l$. Now the assumption $(n, l) = 1$ completes the proof. ∎

PROPOSITION 2. – *Let $Z(n, l)$ be the centre of $G(n, l)$. Then the factor group $G(n, l)/Z(n, l)$ is isomorphic to $\mathfrak{D}(n, l)$.*

COROLLARY 3. – *The group $I(n, l)$ of inner automorphisms of $G(n, l)$ is isomorphic to $\mathfrak{D}(n, l)$.*

PROOF. – Let us assume for $G(n, l)$ the finite presentation labelled (1). There is an automorphism $\sigma\colon G(n, l) \to G(n, l)$ defined by $\sigma(a_i) = a_{i+1}$ $(i \in \Delta_{n-1})$, where the indices are mod $n$. Obviously $\sigma^n$ is the identical automorphism of $G(n, l)$ and $a_i = \sigma^i(a_0)$ for each $i \in \Delta_{n-1}$. Each relation of $G(n, l)$ can be written as follows:

$$(3) \qquad \sigma^{i-(l-2)}(a_0)\sigma^{i-(l-4)}(a_0) \ldots \sigma^{i-1}(a_0)\sigma^{i+1}(a_0) = \\ = \sigma^{i-(l-3)}(a_0)\sigma^{i-(l-5)}(a_0) \ldots \sigma^{i-4}(a_0)\sigma^{i-2}(a_0)\sigma^{i}(a_0)\,.$$

By using the formula $a_i = \sigma^i(a_0)$, each relation obtained from (3) for $i \neq l - 1$ is easily proved to be a consequence of the relation (3) written for $i = l - 1$. Now let us consider the extension group $\dot{G}(n, l) = G(n, l)\langle\sigma\rangle$ with the product $(\sigma^i, g)(\sigma^j, h) = (\sigma^{i+j}, \sigma^j(g)h)$ (for the extension theory of Groups we refer to [6], p. 150; [7], vol. 2, p. 71, [5], p. 215).

A finite presentation for $\dot{G}(n, l)$ is

$$\dot{G}(n, l) = \langle a_0, \sigma\colon \sigma^n = 1,\ \sigma(a_0)\sigma^3(a_0)\ldots\sigma^{l-2}(a_0)\sigma^l(a_0) = \sigma^2(a_0)\sigma^4(a_0)\ldots\sigma^{l-5}(a_0)\sigma^{l-3}(a_0)\sigma^{l-1}(a_0)\rangle .$$

Since the commutator factor group of $G(n, l)$ is null (see [2]), then $G(n, l)$ is coincident with its commutator subgroup $G'(n, l)$. By setting $y = a_0^{-1}\sigma = (1, a_0^{-1})(\sigma, 1) = (\sigma, \sigma(a_0^{-1}))$, we prove that the second relation of $\dot{G}(n, l)$ is equivalent to $(y\sigma)^{(l-1)/2}y = (\sigma y)^{(l-1)/2}\sigma$. In fact, we have

$$y\sigma = a_0^{-1}\sigma\sigma = (\sigma, \sigma(a_0^{-1}))(\sigma, 1) = (\sigma^2, \sigma^2(a_0^{-1}))$$

$$(y\sigma)^2 = (\sigma^2, \sigma^2(a_0^{-1}))(\sigma^2, \sigma^2(a_0^{-1})) = (\sigma^4, \sigma^4(a_0^{-1})\sigma^2(a_0^{-1}))$$

$$(y\sigma)^3 = (\sigma^4, \sigma^4(a_0^{-1})\sigma^2(a_0^{-1}))(\sigma^2, \sigma^2(a_0^{-1})) = (\sigma^6, \sigma^6(a_0^{-1})\sigma^4(a_0^{-1})\sigma^2(a_0^{-1}))$$

$$\vdots$$

$$(y\sigma)^{(l-1)/2} = (\sigma^{l-1}, \sigma^{l-1}(a_0^{-1})\sigma^{l-3}(a_0^{-1})\ldots\sigma^4(a_0^{-1})\sigma^2(a_0^{-1}))$$

$$(y\sigma)^{(l-1)/2}y = (y\sigma)^{(l-1)/2}(\sigma, \sigma(a_0^{-1})) = (\sigma^l, \sigma^l(a_0^{-1})\sigma^{l-2}(a_0^{-1})\ldots\sigma^5(a_0^{-1})\sigma^3(a_0^{-1})\sigma(a_0^{-1})) .$$

$$(\sigma y) = \sigma a_0^{-1}\sigma = (\sigma, 1)(\sigma, \sigma(a_0^{-1})) = (\sigma^2, \sigma(a_0^{-1}))$$

$$(\sigma y)^2 = (\sigma^2, \sigma(a_0^{-1}))(\sigma^2, \sigma(a_0^{-1})) = (\sigma^4, \sigma^3(a_0^{-1})\sigma(a_0^{-1}))$$

$$(\sigma y)^3 = (\sigma^4, \sigma^3(a_0^{-1})\sigma(a_0^{-1}))(\sigma^2, \sigma(a_0^{-1})) = (\sigma^6, \sigma^5(a_0^{-1})\sigma^3(a_0^{-1})\sigma(a_0^{-1}))$$

$$\vdots$$

$$(\sigma y)^{(l-1)/2} = (\sigma^{l-1}, \sigma^{l-2}(a_0^{-1})\sigma^{l-4}(a_0^{-1})\ldots\sigma^3(a_0^{-1})\sigma(a_0^{-1}))$$

$$(\sigma y)^{(l-1)/2}\sigma = (\sigma y)^{(l-1)/2}(\sigma, 1) = (\sigma^l, \sigma^{l-1}(a_0^{-1})\sigma^{l-3}(a_0^{-1})\ldots\sigma^4(a_0^{-1})\sigma^2(a_0^{-1})) .$$

The relation $(y\sigma)^{(l-1)/2}y = (\sigma y)^{(l-1)/2}\sigma$ holds iff we have

$$\sigma^l(a_0^{-1})\sigma^{l-2}(a_0^{-1})\ldots\sigma^5(a_0^{-1})\sigma^3(a_0^{-1})\sigma(a_0^{-1}) = \sigma^{l-1}(a_0^{-1})\sigma^{l-3}(a_0^{-1})\ldots\sigma^4(a_0^{-1})\sigma^2(a_0^{-1}) ,$$

that is the inverse of the second relation of the group $\dot{G}(n, l)$.

Since $a_0 a_1 \ldots a_{n-1} = 1$, we also prove that $y^n = 1$;

$$y^n = \underbrace{a_0^{-1}\sigma}_{1}\,\underbrace{a_0^{-1}\sigma}_{2}\ldots\underbrace{a_0^{-1}\sigma}_{n} = \underbrace{(\sigma, \sigma(a_0^{-1}))}_{1}\underbrace{(\sigma, \sigma(a_0^{-1}))}_{2}\ldots\underbrace{(\sigma, \sigma(a_0^{-1}))}_{n} =$$

$$= (\sigma^2, \sigma^2(a_0^{-1})\sigma(a_0^{-1}))\underbrace{(\sigma, \sigma(a_0^{-1}))}_{3}\ldots\underbrace{(\sigma, \sigma(a_0^{-1}))}_{n} =$$

$$= \big(\sigma^3, \sigma(\sigma^2(a_0^{-1})\sigma(a_0^{-1}))\,\sigma(a_0^{-1})\big)\underbrace{(\sigma, \sigma(a_0^{-1}))}_{4} \dots \underbrace{(\sigma, \sigma(a_0^{-1}))}_{n} =$$

$$= (\sigma^3, \sigma^3(a_0^{-1})\sigma^2(a_0^{-1})\sigma(a_0^{-1}))\underbrace{(\sigma, \sigma(a_0^{-1}))}_{4} \dots \underbrace{(\sigma, \sigma(a_0^{-1}))}_{n} =$$

$$= (\sigma^n, \sigma^n(a_0^{-1})\sigma^{n-1}(a_0^{-1}) \dots \sigma(a_0^{-1})) = (1, a_0^{-1}a_{n-1}^{-1} \dots a_1^{-1}) = (1, 1)\,.$$

Thus another finite presentation for $\dot{G}(n, l)$ is derived:

$$\dot{G}(n, l) = \langle \sigma, y\colon \sigma^n = y^n = 1,\ (y\sigma)^{(l-1)/2}y = (\sigma y)^{(l-1)/2}\sigma \rangle\,.$$

Let $\tau$ be the automorphism of $\dot{G}(n, l)$ defined by $\tau(\sigma) = y$ and $\tau(y) = \sigma$. Let $\ddot{G}(n, l)$ be the extension group $\dot{G}(n, l)\langle\tau\rangle = \big(G(n, l)\langle\sigma\rangle\big)(\langle\tau\rangle)$.

Obviously $\tau^2 = 1$ and $[\ddot{G}(n, l)\colon G(n, l)]$ divides $2n$. In $\ddot{G}(n, l)$ the relation $(y\sigma)^{(l-1)/2}y = (\sigma y)^{(l-1)/2}\sigma$ is proved to be equivalent to $(\sigma\tau)^l = (\tau\sigma)^l$. In fact, we have

$$(\sigma\tau) = (1, \sigma)(\tau, 1) = (\tau, \tau(\sigma))$$

$$(\sigma\tau)^2 = (\tau, \tau(\sigma))(\tau, \tau(\sigma)) = (\tau^2, \tau^2(\sigma)\tau(\sigma)) = (1, \sigma\tau(\sigma))$$

$$(\sigma\tau)^3 = (1, \sigma\tau(\sigma))(\tau, \tau(\sigma)) = (\tau, \tau(\sigma)\sigma\tau(\sigma))$$

$$\vdots$$

$$(\sigma\tau)^l = (\tau, \underbrace{\tau(\sigma)\sigma}_{1} \dots \underbrace{\tau(\sigma)\sigma}_{(l-1)/2}\tau(\sigma))$$

$$(\tau\sigma) = (\tau, 1)(1, \sigma) = (\tau, \sigma)$$

$$(\tau\sigma)^2 = (\tau, \sigma)(\tau, \sigma) = (\tau^2, \tau(\sigma)\sigma) = (1, \tau(\sigma)\sigma)$$

$$(\tau\sigma)^3 = (1, \tau(\sigma)\sigma)(\tau, \sigma) = (\tau, \sigma\tau(\sigma)\sigma)$$

$$\vdots$$

$$(\tau\sigma)^l = (\tau, \underbrace{\sigma\tau(\sigma)}_{1} \dots \underbrace{\sigma\tau(\sigma)}_{(l-1)/2}\sigma)$$

Since $\tau(\sigma) = y$, the formula

$$\underbrace{\tau(\sigma)\sigma}_{1} \dots \underbrace{\tau(\sigma)\sigma}_{(l-1)/2}\tau(\sigma) = \underbrace{\sigma\tau(\sigma)}_{1} \dots \underbrace{\sigma\tau(\sigma)}_{(l-1)/2}\sigma$$

is equivalent to $(y\sigma)^{(l-1)/2}y = (\sigma y)^{(l-1)/2}\sigma$.

Thus the group $\ddot{G}(n, l)$ admits the following presentation:

$$\ddot{G}(n, l) = \langle \sigma, \tau\colon \sigma^n = \tau^2 = 1,\ (\sigma\tau)^l = (\tau\sigma)^l \rangle\,.$$

Since $(\sigma\tau)^l$ commutes with the two generators $\sigma, \tau$ of $\ddot{G}(n, l)$, the cyclic subgroup $B = \langle(\sigma\tau)^l\rangle$ is central in $\ddot{G}(n, l)$:

$$(\sigma\tau)^l\sigma = \underbrace{\sigma\tau}_{1} \dots \underbrace{\sigma\tau}_{l}\, \sigma = \sigma \underbrace{\tau\sigma}_{1} \dots \underbrace{\tau\sigma}_{l} = \sigma(\tau\sigma)^l$$

$$(\sigma\tau)^l\tau = \underbrace{\sigma\tau}_{1} \dots \underbrace{\sigma\tau}_{l}\, \tau = \underbrace{\sigma\tau}_{1} \dots \underbrace{\sigma\tau}_{l-1}\, \sigma = \tau^2 \underbrace{\sigma\tau}_{1} \dots \underbrace{\sigma\tau}_{l-1}\, \sigma = \tau \underbrace{(\tau\sigma)}_{1} \dots \underbrace{(\tau\sigma)}_{l} = \tau(\tau\sigma)^l.$$

We obtain that $[\ddot{G}(n, l): \ddot{G}'(n, l)] = 2n$ as the commutator factor group

$$\ddot{G}(n, l)/\ddot{G}'(n, l) = \langle\sigma, \tau: \sigma^n = \tau^2 = 1, \sigma\tau = \tau\sigma\rangle$$

is isomorphic with the direct product $Z_n \times Z_2$.

Further $\ddot{G}(n, l) \simeq \ddot{G}'(n, l)\langle\sigma\tau\rangle$ and

$$o\big((\sigma\tau)^l\big) = o(\sigma\tau)/\big(o(\sigma\tau), l\big) = 2n/(n, l) = 2n\,.$$

Since $G(n, l) = G'(n, l) \subseteq \ddot{G}'(n, l)$, there is an epimorphism

$$\ddot{G}(n, l)/G(n, l) \twoheadrightarrow \ddot{G}(n, l)/\ddot{G}'(n, l);$$

therefore it follows that $[\ddot{G}(n, l): G(n, l)] \geqslant [\ddot{G}(n, l): \ddot{G}'(n, l)] = 2n$. Finally we have $G(n, l) = \ddot{G}'(n, l)$ as $[\ddot{G}(n, l): G(n, l)]$ divides $2n$. Since $l$ is odd, i.e. $l = 2p + 1$ for some positive integer $p$, we have $B = \langle(\sigma\tau)^l\rangle = \langle\sigma^l\tau^l\rangle = \langle\sigma^l\tau^{2p+1}\rangle = \langle\sigma^l\tau\rangle = \langle\sigma\tau\rangle$ in the abelian group $\ddot{G}(n, l)/\ddot{G}'(n, l)$.

Then

$$\ddot{G}(n, l) \simeq \ddot{G}'(n, l)\langle\sigma\tau\rangle = \ddot{G}'(n, l)B = G(n, l)B\,.$$

The factor group $\ddot{G}(n, l)/B$ has the following finite presentation

$$\ddot{G}(n, l)/B = \langle\sigma, \tau: \sigma^n = \tau^2 = 1, (\sigma\tau)^l = 1\rangle\,.$$

By setting $v = \sigma\tau$ and $u = \sigma^{-1}$, it follows that the group

$$\ddot{G}(n, l)/B = \langle u, v: u^n = v^l = 1, (uv)^2 = 1\rangle$$

is isomorphic with the Dyck's group $\mathfrak{D}(n, l)$ of type $(n, l, 2)$. Thus the subgroup $B$ is the centre $Z(\ddot{G}(n, l))$ of $\ddot{G}(n, l)$ since $\mathfrak{D}(n, l)$ has trivial centre and $B$ is central in $\ddot{G}(n, l)$. Further the sequence of isomorphic groups

$$\mathfrak{D}(n, l) \simeq \ddot{G}(n, l)/B \simeq \big(G(n, l)B\big)/B \simeq G(n, l)/\big(G(n, l) \cap B\big) =$$
$$= G(n, l)/\Big(G(n, l) \cap Z\big(\ddot{G}(n, l)\big)\Big)$$

implies that $Z(n, l) \subseteq G(n, l) \cap Z(\ddot{G}(n, l))$. Since $G(n, l) = \ddot{G}'(n, l) \subseteq \ddot{G}(n, l)$, we also have $G(n, l) \cap Z(\ddot{G}(n, l)) = \ddot{G}'(n, l) \cap Z(\ddot{G}(n, l)) \subseteq Z(\ddot{G}'(n, l)) \subseteq Z(n, l)$ so that the proof is completed. ■

COROLLARY 4:

1) *The group* $G(n, l)$ *is infinite whenever* $n > 5$, $l > 1$ *or* $n > 1$, $l > 5$.

2) $G(n, l)$ *is isomorphic to* $G(n', l')$ *iff* $\{n, l\} = \{n', l'\}$.

3) $G(3, 5) \simeq G(5, 3)$ *is the special linear homogeneous group* $SL(2, 5)$.

4) *There is an epimorphism of* $G(n, l)$ *onto the alternating group* $A_5$ *of degree* 5 *iff* $n \equiv 0 \pmod 5$, $l \equiv 0 \pmod 3$ *or* $n \equiv 0 \pmod 3$, $l \equiv 0 \pmod 5$.

PROOF:

1) $G(n, l)$ admits $\mathfrak{D}(n, l)$ as factor group and $\mathfrak{D}(n, l)$ is infinite whenever $(l-2)(n-2) \geqslant 4$ (see [8], p. 54).

2) $G(n, l) \simeq G(n', l')$ gives $G(n, l)/Z(n, l) \simeq G(n', l')/Z(n', l')$ iff $\mathfrak{D}(n, l) \simeq \mathfrak{D}(n', l')$ iff $\{n, l\} = \{n', l'\}$.

3) The group $G(3, 5)(\simeq G(5, 3))$ has the presentation $\langle X, Y / X^3 = Y^2 = (X^{-1}Y)^5 \rangle$ (see [12], case $n = 5$, $k = 1$, p. 221). By setting $R = Y^{-1}X$ and $S = X^{-1}$, the group $G(3, 5) \simeq \langle R, S \colon R^5 = S^3 = (RS)^2 \rangle$ is isomorphic to the binary polyhedral group $\langle 5, 3, 2 \rangle$ (see [8], p. 68), which is $SL(2, 5)$ as proved in [13], p. 80.

4) Let $S_5$ be the symmetric group on $N_5$. We can suppose that $n \equiv 0 \pmod 5$ and $l \equiv 0 \pmod 3$ without loss of generality. Since $\mathfrak{D}(n, l) = \langle u, v / u^n = v^l = (uv)^2 = 1 \rangle$ is a factor group of $G(n, l)$, it suffices to construct an epimorphism of $\mathfrak{D}(n, l)$ onto $A_5$. Let us consider the correspondences $u \to p_1$ and $v \to p_2$, where $p_1$ (resp. $p_2$) is the cycle of order five (resp. three) $\langle 12345 \rangle$ (resp. $\langle 143 \rangle$). By making use of the substitution product, we have $(p_1 \cdot p_2)^2 = (\langle 12 \rangle \langle 45 \rangle)^2 =$ identity and $p_1^n = p_1^{5\xi} = p_2^l = p_2^{3\eta} =$ identity for some integers $\xi, \eta$. Thus there is an epimorphism of $\mathfrak{D}(n, l)$ onto the subgroup $H$ of $S_5$ generated by $p_1$ and $p_2$. Now the cycles $p_1, p_2$ belong to $A_5$ since they are even permutations. Furthermore the order of $H$ must be divided by the integers 5, 3 and 2, whence by 30. Since $A_5$ has no subgroups of order 30, it follows that $H = A_5$ as requested

For the converse implication, we first note that if there is an epimorphism $\varphi$ of $G(n, l)$ onto a group $K$ with trivial centre ($K = A_5$ for example), then there also exists an epimorphism of $\mathfrak{D}(n, l)$ onto $K$. In fact, the centre $Z(n, l)$ of $G(n, l)$ must be a subgroup of $\operatorname{Ker}\varphi$ since the centre of $K$ is trivial. Therefore we have

$$K \simeq G(n, l)/\operatorname{Ker}\varphi \simeq G(n, l)/Z(n, l)/\operatorname{Ker}\varphi/Z(n, l) \simeq \mathfrak{D}(n, l)/\operatorname{Ker}\varphi/Z(n, l)$$

(use prop. 2), whence $\mathfrak{D}(n, l)$ admits $K$ as factor group. Let $\psi$ be an epimorphism of $\mathfrak{D}(n, l)$ onto $K$. If $K = A_5$, $\psi(u) = S_1$ and $\psi(v) = S_2$, then the order of $S_1$ (resp. $S_2$) must be a divisor of both $n$ and 60 (resp. $l$ and 60). Since $n$, $l$ are odd and coprime, the proof of statement 4) is completed. ∎

Recently D. L. JOHNSON and R. M. THOMAS [12] have obtained an independent proof that the groups $G(n, 3)$ are pairwise non-isomorphic whenever $n$ is odd and coprime to 3.

**3.** – In this section we completely compute the commutator factor group $G(n, l)/G'(n, l)$ of $G(n, l)$ for any two positive integers $n$ and $l$. Note that $G(n, l)/G'(n, l)$ represents the first integral homology group of $M(n, l)$ and therefore throughout this section it will be denoted by $H_1(M(n, l))$. The following proposition implies as a direct consequence that the first Betti number of $M(n, l)$ is always even (possibly null).

PROPOSITION 5. – *The commutator factor group* $H_1(M(n, l))$ *of* $G(n, l)$ *is given by the following table*:

$$l \text{ odd} \Rightarrow \begin{cases} n \text{ even} \Rightarrow H_1(M(n, l)) = \begin{cases} Z_l & \text{if } (n, l) = 1 \\ \underbrace{Z \times \ldots \times Z}_{(n,l)-1 \text{ times}} \times Z_{l/(n,l)} & \text{if } (n, l) > 1 \end{cases} \\ n \text{ odd} \Rightarrow H_1(M(n, l)) = \begin{cases} 0 & \text{if } (n, l) = 1 \\ \underbrace{Z_2 \times \ldots \times Z_2}_{(n,l)-1 \text{ times}} & \text{if } (n, l) > 1 \end{cases} \end{cases}$$

$$l \text{ even} \Rightarrow \begin{cases} n \text{ odd} \Rightarrow H_1(M(n, l)) = \begin{cases} Z_n & \text{if } (n, l) = 1 \\ \underbrace{Z \times \ldots \times Z}_{(n,l)-1 \text{ times}} \times Z_{n/(n,l)} & \text{if } (n, l) > 1 \end{cases} \\ n \text{ even} \Rightarrow H_1(M(n, l)) = \begin{cases} Z_{nl/2} & \text{if } (n, l) = 2 \\ \underbrace{Z \times \ldots \times Z}_{(n,l)-2 \text{ times}} \times Z_{2nl/(n,l)^2} & \text{if } (n, l) > 2 \end{cases} \end{cases}$$

PROOF:

I) $l$ *odd*.

Let us assume the finite presentation labelled (1) for the fundamental group $G(n, l)$ of $M(n, l)$. In the commutator factor group $H_1(M(n, l))$ of $G(n, l)$, the relation $a_i a_{i+l} = 1$ $(i \in \Delta_{n-1})$ holds (see [2]).

By induction on $k \in Z$, it follows that $a_i^{-1} = a_{i+l+2lk}$. Thus the group $H_1(M(n, l))$ has a finite presentation whose generators only are $a_1, a_2, \ldots, a_l$. Further such a presentation admits the following relations

$$a_i = a_{i+2lk} = a^{-1}_{i+l+2lk} \qquad (i \in N_l) \tag{4}$$

$$a_i a_{i+1} = a_{i+1} a_i \tag{5}$$

$$a_1 a_2^{-1} a_3 a_4^{-1} \ldots a_{l-3}^{-1} a_{l-2} a_{l-1}^{-1} a_l = 1 \tag{6}$$

I.1) $l$ odd, $n$ even and $(n, l) = 1$. There exists an integer $k$ (resp. $h$) such that $i + 1 + l + 2lk \equiv i \pmod n$ (resp. $i + 2lh \equiv i + 2 \pmod n$). Then the relations $a_i = a_{i+1}^{-1}$ and $a_i = a_{i+2}$ hold for each $i \in N_l$, $i$ odd. Further we have $a_1 = a_i$ (resp. $a_1 = a_i^{-1}$) for $i \in N_l$, $i$ odd (resp. $i$ even). By making use of these formulas, the relation (6) becomes $(a_1)^l = 1$; therefore

$$H_1(M(n, l)) = < a_1 \colon a_i^l = 1 > \simeq Z_l \,.$$

I.2) $l$ odd, $n$ even and $(n, l) > 1$. There exists an integer $k$ such that $i \equiv i + (n, l) + 2lk + l \pmod n$. For each integer $h$, it follows that $i + l + 2lh \not\equiv i \pmod n$, $i \not\equiv j + 2lh \pmod n$, $i \not\equiv j + l + 2lh \pmod n$ and $i \not\equiv j \pmod n$ whenever $1 \leqslant i < j \leqslant (n, l)$.

Then we have $a_i = a^{-1}_{i+(n,l)+l} = a_{i+2(n,l)+l} = \ldots = a^{-1}_{i+2l}$ for each $i \in N_{(n,l)}$. Since $i + l + 2lk \equiv i + (n, l) \pmod n$, the relation (4) gives $a_i = a^{-1}_{i+(n,l)}$. Then the formula (6) becomes $(a_1 a_2 \ldots a_{(n,l)})^{l/(n,l)} = 1$. By setting $z = a_1 a_2 \ldots a_{(n,l)}$, we obtain

$$H_1(M(n, l)) = \langle a_1, a_2, \ldots, a_{(n,l)-1}, z / z^{l/(n,l)} = 1, \quad a_i a_j = a_j a_i, \, z a_i = a_i z > \simeq \underbrace{Z \times \ldots \times Z}_{(n,l)-1 \text{ times}} \times Z_{l/(n,l)} .$$

I.3) $l$ odd, $n$ odd and $(n, l) = 1$. This case is a consequence of a statement proved in [2].

I.4) $l$ odd, $n$ odd and $(n, l) > 1$. There exists an integer $k$ (resp. $h$) such that $i + l + 2lk \equiv i \pmod n$ (resp. $i \equiv i + (n, l) + 2lh \pmod n$) for each $i \in N_l$. Thus we have $a_i = a_i^{-1}$ and $a_i = a_{i+(n,l)}$ for each $i \in N_l$. The relation $a_i = a_{i+p(n,l)}$ holds for $i \in N_{(n,l)}$ and $p \in N_{l/(n,l)}$. Since $i \not\equiv j + 2lk \pmod n$, $i \not\equiv j + l + 2lk \pmod n$ and $i \not\equiv j \pmod n$ whenever $1 \leqslant i < j \leqslant (n, l)$, the relation (6) becomes $(a_1 a_2 \ldots a_{(n,l)})^{l/(n,l)} = 1$.

The integer $l/(n, l)$ is odd as $l$ is odd; since $a_i^2 = 1$, it follows that $a_1 a_2 \ldots a_{(n,l)} = 1$. Finally we have

$$H_1(M(n, l)) = \langle a_1, a_2, \ldots, a_{(n,l)-1} \colon a_i^2 = 1 \;\; (i \in N_{(n,l)-1}), \; a_i a_j = a_j a_i \rangle ,$$

$$\text{i.e. } H_1(M(n, l)) \simeq \underbrace{Z_2 \times \ldots \times Z_2}_{(n,l)-1 \text{ times}} .$$

II) $l$ *even.*

Assume for $G(n, l)$ the finite presentation labelled (2).
Let us consider two consecutive relations of $G(n, l)$.

$$a_i^{-1}a_{i-2}^{-1}a_{i-4}^{-1}\dots a_{i-(l-2)}^{-1}a_{i-(l-3)}a_{i-(l-5)}\dots a_{i-1}a_{i+1}=1$$
$$a_{i+1}^{-1}a_{i-1}^{-1}a_{i-3}^{-1}\dots a_{i-(l-3)}^{-1}a_{i-(l-4)}a_{i-(l-6)}\dots a_i a_{i+2}=1\,.$$

By pairwise multiplication and simplification in the abelian group $H_1(M(n, l))$, we get $a_i = a_{i+l}$. By induction on $k \in Z$, it follows that $a_i = a_{i+lk}$ for each $i \in \Delta_{n-1}$.

II.1) $l$ even, $n$ odd and $(n, l) = 1$. There exists an integer $k$ such that $i + lk \equiv i + 1 \pmod n$, i.e. $a_i = a_{i+1}$ for each $i \in \Delta_{n-1}$. In this case the relation

$$a_i^{-1}a_{i-2}^{-1}a_{i-4}^{-1}\dots a_{i-(l-2)}^{-1}a_{i-(l-3)}a_{i-(l-5)}\dots a_{i-1}a_{i+1}=1$$

is an identity for each $i \in \Delta_{n-1}$. The relation $a_0 a_1 \dots a_{n-1} = 1$ becomes $a_1^n = 1$. Thus we have $H_1(M(n, l)) = \langle a_1 : a_1^n = 1\rangle \simeq Z_n$.

II.2) $l$ even, $n$ odd and $(n, l) > 1$. There exists an integer $k$ such that $i \equiv i + (n, l) + lk \pmod n$; therefore we have $a_i = a_{i+(n,l)+lk} = a_{i+(n,l)}$ for each $i \in N_{(n,l)}$.

By induction on $r \in Z$, it follows that $a_i = a_{i+r(n,l)}$. Then the relation

$$a_0 a_1 \dots a_{n-1} = 1$$

is equivalent to

$$(a_1 a_2 \dots a_{(n,l)})^{n/(n,l)} = 1\,.$$

Since $i \not\equiv j \pmod n$ if $1 \leqslant i < j \leqslant (n, l)$ and by setting $z = a_1 a_2 \dots a_{(n,l)}$, we obtain

$$H_1(M(n, l)) = \langle a_1, a_2, \dots, a_{(n,l)-1}, z : z^{n/(n,l)} = 1,\ a_i a_j = a_j a_i,\ z a_i = a_i z\rangle \simeq$$
$$\simeq \underbrace{Z\times Z\dots\times Z}_{(n,l)-1\ \text{times}}\times Z_{n/(n,l)}\,.$$

II.3) $l$ even, $n$ even and $(n, l) = 2$. There exists an integer $k$ such that $i \equiv i + 2 + lk \pmod n$. Then we have $a_i = a_{i+2+lk} = a_{i+2}$ for each $i \in \Delta_{n-1}$. The relation $a_0 a_1 \dots a_{n-1} = 1$ becomes $(a_1)^{n/2}(a_2)^{n/2} = 1$. From the other relations of $H_1(M(n, l))$, we get $(a_1)^{-l/2}(a_2)^{l/2} = 1$. By setting $x = a_1 a_2$, $y = a_1$, it follows that $x^{n/2} = 1$ and $y^l = x^{l/2}$. Further we have $y^{nl/2} = (y^l)^{n/2} = (x^{n/2})^{l/2} = 1$ and

$$x = x^{(p(l/2)+q(n/2))} = x^{p(l/2)}x^{q(n/2)} = x^{p(l/2)} = y^{pl}$$

for some integers $p$ and $q$. Thus $H_1(M(n, l)) = \langle y / y^{nl/2} = 1\rangle \simeq Z_{nl/2}$.

II.4) $l$ even, $n$ even and $(n, l) > 2$. There exists an integer $k$ such that $i + lk \equiv i + (n, l) \pmod n$. Thus we have $a_i = a_{i+lk} = a_{i+(n,l)}$. The relation $a_0 a_1 \dots a_{n-1} = 1$ becomes $(a_1 a_2 \dots a_{(n,l)})^{n/(n,l)} = 1$. From each relation

$$a_i^{-1} a_{i-2}^{-1} \dots a_{i-1} a_{i+1} = 1 ,$$

we get

$$(a_2 a_4 \dots a_{(n,l)})^{l/(n,l)} = (a_1 a_3 \dots a_{(n,l)-1})^{l/(n,l)} .$$

By setting $x = a_1 a_2 \dots a_{(n,l)}$ and $y = a_1 a_3 \dots a_{(n,l)-1}$, it follows that $x^{n/(n,l)} = 1$ and $x^{l/(n,l)} = y^{2l/(n,l)}$, i.e.

$$H_1(M(n, l)) = \langle a_1, a_2, \dots, a_{(n,l)-2}, x, y : x^{n/(n,l)} = 1, x^{l/(n,l)} = y^{2l/(n,l)}, \\ a_i a_j = a_j a_i, a_i x = x a_i, a_i y = y a_i \rangle .$$

Further we have

$$y^{2ln/(n,l)^2} = (y^{2l/(n,l)})^{n/(n,l)} = x^{ln/(n,l)^2} = 1$$

and

$$x = x^{pn/(n,l)} y^{ql/(n,l)} = x^{ql/(n,l)} = y^{2ql/(n,l)}$$

for some integers $p$, $q$. Finally

$$H_1(M(n, l)) = \langle a_1, a_2, \dots, a_{(n,l)-2}, y : y^{2ln/(n,l)^2} = 1, a_i a_j = a_j a_i, y a_i = a_i y \rangle \simeq \\ \simeq \underbrace{Z \times \dots \times Z}_{(n,l)-2 \text{ times}} \times Z_{2nl/(n,l)^2} . \quad ■$$

COROLLARY 6. – *If $n$, $n'$ are even and $l$, $l'$ are odd, then $G(n, l)$ is isomorphic to $G(n', l')$ iff $\{n, l\} = \{n', l'\}$.*

*If $n$, $n'$, $l$, $l'$ are odd (resp. even), then $G(n, l)$ is not isomorphic to $G(n', l')$ whenever $(n, l) \neq (n', l')$.*

PROOF. – By noting that $G(n, l) \simeq G(l, n)$, the proof is a direct consequence of proposition 5. ■

4. – In this section we study some algebraic properties of the groups $G(4, l) \simeq G(l, 4)$ for $l = 4q + 3$, $q \geq 0$, and $G(3, l) \simeq G(l, 3)$ for $l = 6q + 4$, $q \geq 0$. We have the following

PROPOSITION 7. – *The group $G(4, 4q + 3)$, $q \geq 0$, has the finite presentation*

$$\langle X, Y : X^{4q+3} = Y^{4q+3} = (Y^{q+1} X^{q+1})^2 \rangle .$$

*In particular* $G(4, 3)$ *is the binary tetrahedral group* $\langle 2, 3, 3\rangle$. *Moreover* $G(4, 4q+3)$ *is isomorphic to* $G(4, 4q'+3)$ *iff* $q = q'$.

PROOF. – A finite presentation for $G(4, l)$ ($l = 4q + 3$, $q \geqslant 0$) is given by

$$G(4, l) = \langle a_0, a_1, a_2, a_3 \colon a_0^{-1}(a_2^{-1}a_0^{-1})^q(a_3a_1)^{q+1} = 1,\ a_3^{-1}(a_1^{-1}a_3^{-1})^q(a_2a_0)^{q+1} = 1\,,$$
$$a_2^{-1}(a_0^{-1}a_2^{-1})^q(a_1a_3)^{q+1} = 1,\ a_1^{-1}(a_3^{-1}a_1^{-1})^q(a_0a_2)^{q+1} = 1\rangle\,.$$

Since $(a_3a_1)^{q+1} = a_3(a_1a_3)^q a_1$, we have

$$a_0^{-1}(a_2^{-1}a_0^{-1})^q a_3(a_1a_3)^q a_1 = 1 \tag{7}$$

$$a_3^{-1}(a_1^{-1}a_3^{-1})^q a_2(a_0a_2)^q a_0 = 1 \tag{8}$$

$$a_2^{-1}(a_0^{-1}a_2^{-1})^q a_1(a_3a_1)^q a_3 = 1 \tag{9}$$

$$a_1^{-1}(a_3^{-1}a_1^{-1})^q a_0(a_2a_0)^q a_2 = 1\,. \tag{10}$$

The relation (10) is a consequence of the other relations. In fact, by successively multiplying (7) (8) (9), it follows that

$$a_2^{-1}(a_0^{-1}a_2^{-1})^q a_1(a_3a_1)^q a_3 a_3^{-1}(a_1^{-1}a_3^{-1})^q a_2(a_0a_2)^q a_0 a_0^{-1}(a_2^{-1}a_0^{-1})^q a_3(a_1a_3)^q a_1 = 1\,,$$

i.e. $a_2^{-1}(a_0^{-1}a_2^{-1})^q a_1a_2a_3(a_1a_3)^q a_1 = 1$. Since $a_1a_2a_3 = a_0^{-1}$, we obtain $a_2^{-1}(a_0^{-1}a_2^{-1})^q \cdot a_0^{-1}(a_1a_3)^q a_1 = 1$, which is the inverse of the relation (10). By making use of the formula $a_0^{-1} = a_1a_2a_3$ (or equivalently $a_0 = a_3^{-1}a_2^{-1}a_1^{-1}$), we can only consider the three following relations for $G(4, l)$:

$$a_1a_2a_3(a_2^{-1}\,a_1a_2a_3)^q(a_3a_1)^{q+1} = 1$$
$$a_3^{-1}(a_1^{-1}a_3^{-1})^q(a_2a_3^{-1}a_2^{-1}a_1^{-1})^{q+1} = 1$$
$$a_2^{-1}(a_1a_2a_3a_2^{-1})^q(a_1a_3)^{q+1} = 1$$

or equivalently

$$(a_2^{-1}a_1a_2a_3)^{q+1}(a_3a_1)^{q+1} = a_2^{-1}$$
$$(a_3a_1)^{-q-1}(a_1a_2a_3a_2^{-1})^{-q-1} = a_1^{-1}$$
$$a_2^{-1}(a_1a_2a_3a_2^{-1})^q(a_1a_3)^{q+1} = 1\,.$$

Since $(a_2^{-1}a_1a_2a_3)^q = a_2^{-1}(a_1a_2a_3a_2^{-1})^q a_2$ and $(a_1a_3)^q = a_1(a_3a_1)^q a_1^{-1}$, we have

$$(a_2^{-1}a_1a_2a_3)^{q+1}a_1^{-1}(a_1a_3)^{q+1}a_1 = a_2^{-1}$$
$$(a_1a_3)^{-q-1}a_1a_2(a_2^{-1}a_1a_2a_3)^{-q-1}a_2^{-1} = 1$$
$$(a_2^{-1}a_1a_2a_3)^q a_2^{-1}(a_1a_3)^{q+1} = 1\,.$$

By setting $X = a_1 a_3$ and $Z = a_2^{-1} a_1 a_2 a_3$, it follows that

$$Z^{q+1} a_1^{-1} X^{q+1} a_1 = a_2^{-1}$$
$$X^{-q-1} a_1 a_2 Z^{-q-1} a_2^{-1} = 1$$
$$Z^q a_2^{-1} X^{q+1} = 1 .$$

Then we obtain

$$Z^{q+1} X^{q+1} Z^{-q-1} X^{q+1} Z^{2q+1} = 1$$
$$Z = Z^{-q} X^{q+1} Z^{2q+1} X^{q+1} Z^{-q-1} X^{-2q-1} .$$

Finally the group $G(4, l)$ admits the following finite presentation

$$G(4, l) = \langle X, Z \colon X^l = (Z^{-q-1} X^{q+1})^2 = Z^{-l} \rangle ,$$

where $l = 4q + 3$ $(q \geqslant 0)$. By setting $Y = Z^{-1}$, we have

$$G(4, l) = \langle X, Y \colon X^l = Y^l = (Y^{q+1} X^{q+1})^2 \rangle .$$

If $q = 0$ (i.e. $l = 3$), then $G(4, 3) = \langle X, Y \colon X^3 = Y^3 = (YX)^2 \rangle$ is isomorphic to the binary tetrahedral group $\langle 2, 3, 3 \rangle$ (see [8], p. 69). Finally the statement of proposition 7 follows by using corollary 6. ■

PROPOSITION 8. – *The group* $G(3, 6q + 4)$, $q \geqslant 0$, *has the finite presentation*

$$\langle X, Y \colon X^{3q+2} = Y^3 = (Y^{-1} X^{2q+1})^3 \rangle .$$

*In particular* $G(3, 4)$ *is isomorphic to the binary tetrahedral group* $\langle 2, 3, 3 \rangle$. *Furthermore* $G(3, 6q + 4)$ *is isomorphic to* $G(3, 6q' + 4)$ *iff* $q = q'$.

PROOF. – A finite presentation for $G(3, l)$ $(l = 6q + 4, q \geqslant 0)$ is given by

$$G(3, l) = \langle a_0, a_1, a_2 \colon$$

$$a_0^{-1} a_1^{-1} (a_2^{-1} a_0^{-1} a_1^{-1})^q (a_2 a_1 a_0)^q a_2 a_1 = 1 , \tag{11}$$

$$a_2^{-1} a_0^{-1} (a_1^{-1} a_2^{-1} a_0^{-1})^q (a_1 a_0 a_2)^q a_1 a_0 = 1 , \tag{12}$$

$$a_1^{-1} a_2^{-1} (a_0^{-1} a_1^{-1} a_2^{-1})^q (a_0 a_2 a_1)^q a_0 a_2 = 1 , \tag{13}$$

$$a_1 a_2 a_0 = 1 \rangle . \tag{14}$$

Since the relation (13) is a consequence of (11), (12) and the formula (14) gives $a_0^{-1} = a_1 a_2$ (or equivalently $a_0 = a_2^{-1} a_1^{-1}$), we just need to consider the two following relations for $G(3, l)$;

$$a_1 a_2 a_1^{-1}(a_2^{-1} a_1 a_2 a_1^{-1})^q (a_2 a_1 a_2^{-1} a_1^{-1})^q a_2 a_1 = 1 ,$$
$$a_2^{-1} a_1 a_2 (a_1^{-1} a_2^{-1} a_1 a_2)^q (a_1 a_2^{-1} a_1^{-1} a_2)^q a_1 a_2^{-1} a_1^{-1} = 1 .$$

Since

$$(a_1^{-1} a_2^{-1} a_1 a_2)^{q+1} = a_1^{-1}(a_2^{-1} a_1 a_2 a_1^{-1})^{q+1} a_1$$

and

$$(a_2 a_1 a_2^{-1} a_1^{-1})^{q+1} = a_2 (a_1 a_2^{-1} a_1^{-1} a_2)^{q+1} a_2^{-1},$$

we obtain

$$(a_2^{-1} a_1 a_2 a_1^{-1})^{q+1} a_2 (a_1 a_2^{-1} a_1^{-1} a_2)^{q+1} a_2^{-1} = a_2^{-2} a_1^{-1}$$
$$(a_2^{-1} a_1 a_2 a_1^{-1})^{q+1} a_1 (a_1 a_2^{-1} a_1^{-1} a_2)^{q+1} = a_2 .$$

By setting $X = a_2^{-1} a_1 a_2 a_1^{-1}$, it follows that

$$X^{q+1} a_2 X^{-q-1} a_2^{-1} = a_2^{-2} a_1^{-1}$$
$$X^{q+1} a_1 X^{-q-1} = a_2 .$$

Then we have

$$a_1 X^{q+1} a_1 X^{q+1} a_1 X^{-q-1} = X^q$$
$$X = X^{q+1} a_1^{-1} X^{-q-1} a_1 X^{q+1} a_1 X^{-q-1} a_1^{-1}$$

or equivalently

$$(a_1 X^{q+1})^3 = X^{l/2} ,$$
$$X = X^{q+1} a_1^{-1} X^{-q-1} a_1 X^{q+1} a_1 X^{-q-1} a_1^{-1} .$$

By setting $Y = a_1 X^{q+1}$, we obtain

$$Y^3 = X^{3q+2} ,$$
$$X = X^{2q+2} Y^{-1} X^{-q-1} Y^2 X^{-q-1} Y^{-1} .$$

From these relations, it follows that

$$Y^3 = X^{3q+2} ,$$
$$X^{3q+2} = (Y^{-1} X^{2q+1})^3 .$$

By setting $Z = Y^{-1}X^{2q+1}$, the group $G(3, l)$ ($l = 6q + 4$, $q \geq 0$) admits the following finite presentation

$$G(3, l) = \langle X, Y, Z\colon Y^3 = X^{3q+2} = Z^3,\ YZ = X^{2q+1}\rangle .$$

If $q = 0$ (i.e. $l = 4$), then

$$G(3, 4) = \langle X, Y, Z\colon Y^3 = X^2 = Z^3,\ YZ = X\rangle = \\ = \langle Y, Z\colon Y^3 = Z^3 = (YZ)^2\rangle \simeq \langle 2, 3, 3\rangle .$$

Finally the statement of proposition 8 follows by using corollary 6 and the isomorphism $G(3, 6q + 4) \simeq G(6q + 4, 3)$. ■

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