Recovering a Centred Convex Body from the Areas of Its Shadows: a Stability Estimate (*).

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Summary. – *The main result of this paper is an estimate of the Hausdorff distance between two centrally symmetric bodies* T_1 *and* T_2 *of* $\boldsymbol{R}^3$ *by the* L^2*-norm of* $A(T_1; z) - A(T_2; z)$. *Here* $A(T_i; z)$, $i = 1, 2$, *is the area of the orthogonal projection of* T_i *in the direction* z.

1. – Let T be a convex body in $\boldsymbol{R}^3$ (i.e. a bounded closed convex subset of $\boldsymbol{R}^3$ with nonempty interior) and assume that T has a centre of symmetry at the origin of $\boldsymbol{R}^3$. We shall say that T is a *centred* convex body.

For any direction $z \in S^2$, $S^2 = \{z \in \boldsymbol{R}^3 \colon |z| = 1\}$, let $A(T; z)$ denote the area of the orthogonal projection of T onto a plane perpendicular to z. Clearly $A(T; z)$ is an even function on S^2, i.e. $A(T; z) = A(T; -z)$, $\forall z \in S^2$. The problem we deal with in the present paper consists in recovering a centred convex body T by the knowledge of $A(T; z)$, for every $z \in S^2$. Notice that this problem may be linked to problems in geometry (see [19]) and in other areas: for instance, inverse diffraction problems (see [17], p. 223) or object recognition from extended gaussian images (see [13]).

Uniqueness for the present problem can be proved by the following arguments. Denote by σ_T the *surface function* of ∂T, the boundary of T: for any subset E of S^2, $\sigma_T(E)$ is the area of such a part of ∂T whose spherical image is E (see, for instance, [15]). It is well known that σ_T is a measure, that is a completely additive function on the Borel subsets of S^2. Moreover, since T is centred, σ_T is even, i.e. $\sigma_T(E) = \sigma_T(-E)$.

The function $A(T; z)$ can be expressed in terms of σ_T as follows (see, for example, [19])

$$A(T; z) = \frac{1}{2}\int_{S^2} |\langle z, v\rangle|\, d\sigma_T(v)\,, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product.

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If T is a *regular* convex body, namely a strictly convex body whose boundary ∂T is a surface of class C^2 (see [5]; see also [9], [16]), then we can rewrite (1.1) as

$$A(T; z) = \frac{1}{2}\int_{S^2} |\langle z, v\rangle| [\gamma(v)]^{-1} d\sigma_v \,, \tag{1.2}$$

where $\gamma(v)$ is the Gauss curvature of ∂T at the point where v is the exterior normal and $d\sigma_v$ is the area element on S^2.

Let T_1 and T_2 be two centred convex bodies such that $A(T_1; z) = A(T_2; z)$ for all $z \in S^2$. Then from (1.1) it follows that

$$\int_{S^2} |\langle z, v\rangle|\, d\tau(v) = 0 \,, \quad \forall z \in S^2 \,, \tag{1.3}$$

where $\tau = \sigma_{T_1} - \sigma_{T_2}$. Since τ is even, identity (1.3) implies $\tau \equiv 0$ (see [19], p. 298). By virtue of Alexandrov uniqueness theorem for the generalized Minkowski problem (see [1 I, II], [5], [11], [15], [19]), it follows that $T_1 = T_2$. Thus uniqueness is established.

Now, the main question we are interested in is the following: if—in some sense—$A(T_1; z)$ is close to $A(T_2; z)$, can we infer that T_1 is close to T_2? A qualitative answer to this question can be found in a recent paper of GOODEY [12]. In particular he shows the following result: if T_n ($n = 1, 2, \ldots$), T are centred convex bodies such that $A(T_n; z) \to A(T; z)$ uniformly on S^2, as $n \to +\infty$, then the Hausdorff distance $\delta(T_n, T)$ between T_n and T tends to zero. He shows also that, in general, the continuous map $\pi: A(T; z) \to T$ is *not* uniformly continuous.

An explicit stability estimate for regular bodies has been obtained by ANIKONOV and STEPANOV in [2]. Such an estimate—showing a not uniform Hölder type stability—involves not only the distance between $A(T_1; z)$ and $A(T_2; z)$ but also the distance between their derivatives of some order. In terms of such a distance it is possible to estimate, by equation (1.2), the difference $1/\gamma_1 - 1/\gamma_2$, γ_1, γ_2 being the Gauss curvatures of $\partial T_1, \partial T_2$ (see also [18]). Hence, by exploiting a stability result of [20] for Minkowski's problem, an estimate for the original problem follows. But such an estimate holds under strong a-priori conditions on the bodies and on their projection areas.

The purpose of the present paper is to obtain an a-priori estimate of the Hausdorff distance $\delta(T_1, T_2)$ between the bodies T_1 and T_2—not necessarily regular—in terms of an usual distance between $A(T_1; z)$ and $A(T_2; z)$ only, without any assumption on derivatives or similar restrictions. In other terms we will show that the stability is an intrinsic quality of the problem connected to the convexity of our bodies.

In a previous paper [7] the author obtained already a stability estimate for the considered problem in the special case of bodies enclosed by surfaces of revolution.

Here we are able to prove a general result:

THEOREM. – *Let* T_1 and T_2 *be centred convex bodies.*
Let

$$\text{(1.4)} \qquad \begin{cases} M_i = \max\limits_{z\in S^2} A(T_i; z)\,, \\ m_i = \min\limits_{z\in S^2} A(T_i; z)\,, \qquad i = 1, 2\,, \end{cases}$$

and $M = \max(M_1, M_2)$, $m = \min(m_1, m_2)$. *For any number* $p \in (0, \frac{1}{9})$, *there exists a constant* C, *which depends on* m, M *and* p *only* (*and is a continuous function of these arguments*) *such that*

$$\text{(1.5)} \qquad \delta(T_1, T_2) \leqslant C\|A(T_1; z) - A(T_2; z)\|^p_{L^2(S^2)}\,.$$

Recall that

$$\delta(T_1, T_2) = \inf\{t > 0\colon T_1 \subset T_{2,t} \text{ and } T_2 \subset T_{1,t}\}$$

with

$$T_{i,t} = \{x \in \boldsymbol{R}^3\colon \operatorname{dist}(x, T_i) < t\}\,, \qquad i = 1, 2\,.$$

The proof of this Theorem will be given in Sect. 4.

We will use some results of VOLKOV [20] and DISKANT [8] (see Sect. 2) which allow us to estimate $\delta(T_1, T_2)$ in terms of mixed volumes of T_1 and T_2. Moreover our proof is based upon an inequality—showed in Sect. 3—involving a sort of α-derivative, $1 \leqslant \alpha < \frac{3}{2}$, of the support function of a regular convex body T and the circumradius of T.

2. – Let T be a centred convex body. Let us denote by R the circumradius of T (the radius of the smallest sphere containing T) and by r the inradius (the largest radius of a sphere contained in T).

The numbers R and r can be estimated by the function $A(T; z)$. More precisely:

LEMMA 1 (Volkov). – *Let* $M = \max\limits_{z\in S^2} A(T; z)$, $m = \min\limits_{z\in S^2} A(T; z)$. *Setting*

$$\text{(2.1)} \qquad \bar{R} = \sqrt{\frac{\pi}{m}}\, M\,,$$

then

$$\text{(2.2)} \qquad R \leqslant \tfrac{1}{2}\bar{R}\,,$$

$$\text{(2.3)} \qquad r \geqslant \frac{m}{2\bar{R}}\,.$$

A more general version of this result is in [20]. It is enough to adapt that proof to our special situation with T centred (see also [15] p. 499).

Another result we need is an estimate concerning mixed volumes of two convex bodies T_1 and T_2. For the reader's convenience let us recall how the mixed volumes can be introduced (see, for example, [4], [5]). The volume of any linear combination $\lambda T_1 + \mu T_2$ (λ and μ being positive numbers) is given by

$$V(\lambda T_1 + \mu T_2) = \sum_{k=0}^{3} \binom{3}{k} \lambda^{3-k} \mu^k V_k(T_1, T_2)\,,$$

where the quantities $V_k(T_1, T_2)$ are just the *mixed volumes* of T_1 and T_2. Clearly

$$V_0(T_1, T_2) = V(T_1)\,, \qquad V_3(T_1, T_2) = V(T_2)\,,$$

$V(T_i)$ being the volume of T_i, $i = 1, 2$; moreover

$$V_1(T_1, T_2) = V_2(T_2, T_1)\,.$$

LEMMA 2. – *Let T_1 and T_2 be centred convex bodies. Let $M_i = \max\limits_{z \in S^2} A(T_i; z)$, $m_i = \min\limits_{z \in S^2} A(T_i; z)$, $i = 1, 2$, and $M = \max(M_1, M_2)$, $m = \min(m_1, m_2)$. Then there exists a constant K, which depends on M/m only (and is a continuous function of this argument) such that*

$$\delta(T_1, T_2) \leqslant K\big(|V(T_1) - V_2(T_1, T_2)|^{1/3} + |V(T_2) - V_1(T_1, T_2)|^{1/3}\big)\,. \tag{2.4}$$

PROOF. – The proof is based upon the results contained in the paper of DISKANT [8].

Precisely, let us introduce the deficiency coefficients

$$\lambda_1 = \sup\{\lambda \colon \lambda T_2 \subset T_1\}\,, \qquad \lambda_2 = \sup\{\lambda \colon \lambda T_1 \subset T_2\}$$

and let

$$\bar{\lambda} = \min(\lambda_1, \lambda_2),$$

$$\mu_1 = \frac{V^{1/2}(T_2) - V_1^{1/2}(T_1, T_2) + [V_1^{3/2}(T_1, T_2) - V(T_1)\,V^{1/2}(T_2)]^{1/3}}{V^{1/2}(T_2)}\,,$$

$\mu_2 =$ as above by interchanging T_1 and T_2.

From Lemma 4 and Theorem 1 of [8] it follows that

$$\delta(T_1, T_2) \leqslant \frac{1}{2} R \left(1 + \frac{1}{\bar{\lambda}}\right)(|\mu_1| + |\mu_2|)\,, \tag{2.5}$$

where R denotes the radius of the smallest sphere containing both T_1 and T_2.

Denote by r the radius of the largest sphere contained in both T_1 and T_2. By using Minkowski's inequality for the mixed volumes (see [20] p. 38) and their monotoneity property (see [5]), one can verify that

$$(2.6) \qquad |\mu_1| \leqslant \frac{\sqrt[3]{2}\, V_R^{1/6}}{V_r^{1/2}} \left[2|V_1(T_1, T_2) - V(T_2)|^{1/3} + \left(\frac{R}{r}\right)^{1/9} |V_2(T_1, T_2) - V(T_1)|^{1/3} \right],$$

where V_R and V_r are the volumes of the spheres of radius R and r respectively.

Obviously μ_2 satisfies an inequality quite analogous to (2.6).

Moreover, by virtue of Theorem 1 of [8] again, one has

$$(2.7) \qquad \bar{\lambda} \geqslant \tfrac{1}{3} (r/R)^6 .$$

Now, by (2.2) and (2.3) of Lemma 1,

$$(2.8) \qquad R/r \leqslant \pi (M/m)^2 .$$

Therefore, by using inequalities (2.6) (and the corresponding for μ_2), (2.7), (2.8), from (2.5) we deduce (2.4).

Notice that one can choose

$$K = \left(\frac{16}{3}\pi\right)^{-1/3} (1 + 3D^{12})(2 + D^{2/9}) D^3 ,$$

with $D = \sqrt{\pi}\,(M/m)$.

3. – This section is devoted to an inequality concerning the *support function* of a regular convex body T (not necessarily centred).

Recall that the support function h of a convex body T is defined by

$$(3.1) \qquad h(z) = \max_{v \in S^2} \langle z, v \rangle , \qquad z \in S^2 .$$

If T contains in its interior the origin of the coordinate system, then $h(z)$ is positive. If T is centred then $h(z) = h(-z)$, for every z in S^2.

Let us introduce on S^2 the usual system of geographical coordinates θ, φ (θ = colatitude, φ = longitude).

For simplicity we continue to denote by $h(\theta, \varphi)$ the function $h(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

Let us expand $h(\theta, \varphi)$ into a series of spherical harmonics:

$$(3.2) \qquad h(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{n=-l}^{l} h_l^n \, Y_l^n(\theta, \varphi) ,$$

where

$$Y_l^n(\theta, \varphi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \left[\frac{(l-n)!}{(l+n)!}\right]^{1/2} P_l^n(\cos\theta)\, e^{in\varphi},$$

P_l^n being the associated Legendre function of the first kind of degree l and order n.

Let T be regular. It is well known (see [3]) that

$$\Delta_s h(\theta, \varphi) = -\sum_{l=1}^{\infty} l(l+1) \sum_{n=-l}^{l} h_l^n\, Y_l^n(\theta, \varphi),$$

where Δ_s denotes the Beltrami-Laplace operator:

$$\Delta_s h(\theta, \varphi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial h}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 h}{\partial\varphi^2}. \tag{3.3}$$

For any real number α, we shall define (see, for instance, [14])

$$(-\Delta_s)^{\alpha} h(\theta, \varphi) = \sum_{l=1}^{\infty} [l(l+1)]^{\alpha} \sum_{n=-l}^{l} h_l^n\, Y_l^n(\theta, \varphi), \tag{3.4}$$

provided the right-hand side of (3.4) converges.

Parseval's identity implies

$$\|(-\Delta_s)^{\alpha} h\|^2_{L^2(S^2)} = \sum_{l=1}^{\infty} [l(l+1)]^{2\alpha} \sum_{n=-l}^{l} |h_l^n|^2. \tag{3.5}$$

We want to prove the following

LEMMA 3. – *Let T be a regular convex body of $\boldsymbol{R}^3$ and $h(\theta, \varphi)$ be its support function. Let R denote the radius of the smallest centred sphere containing T. Then, for any $\varepsilon \in [0, \frac{1}{2})$, there exists a constant $\mathcal{C}(\varepsilon)$, depending only on ε, such that*

$$\|(-\Delta_s)^{(1+\varepsilon)/2} h\|_{L^2(S^2)} \leqslant \mathcal{C}(\varepsilon) R. \tag{3.6}$$

PROOF. – If $\varepsilon = 0$ the estimate has been already proved in [7]. A good evaluation of the constant $\mathcal{C}(0)$ is $(8\pi)^{1/2}$.

Let $0 < \varepsilon < \frac{1}{2}$. Let us first take the origin of the coordinate system to be an interior point of T. The proof is splitted into several steps.

i) Let us denote by $G(x)$, $x \in \boldsymbol{R}^3$, the rotation of angle $|x|$ about x, $G(0)$ being the identity. Let $z' = G(x) z$ be the image of the point z under the transformation $G(x)$.

Set

$$h(z') - h(z) = H(z, x), \quad z \in S^2, \quad x \in \boldsymbol{R}^3. \tag{3.7}$$

Moreover let us denote by ∇_s the surface gradient on S^2. Recall that

$$|\nabla_s h|^2 = h_\theta^2 + \frac{1}{\sin^2\theta} h_\varphi^2 . \tag{3.8}$$

From a geometrical point of view, $|\nabla_s h|$ has a precise meaning. Let us consider the supporting plane (that is, by our regularity condition, the tangent plane) of T whose exterior normal is $z = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and the contact point P of such a plane with T. If d denotes the distance between P and the origin, then (see, for instance, [4])

$$d^2 = |\nabla_s h(\theta, \varphi)|^2 + h^2(\theta, \varphi) .$$

Thus $|\nabla_s h|$ is the distance between P and the foot of the perpendicular from the origin to the supporting plane. Therefore

$$|\nabla_s h| \leqslant R . \tag{3.9}$$

We claim that

$$\|(-\Delta_s)^{(1+\varepsilon)/2} h\|^2_{L^2(S^2)} \leqslant K_1(\varepsilon) \int_{\mathbf{R}^3} |x|^{-(2\varepsilon+3)} dx \int_{S^2} |\nabla_s H(z, x)|^2 d\sigma_z , \tag{3.10}$$

where ∇_s acts on H with respect to z. As far as the constant $K_1(\varepsilon)$ is concerned one may take

$$K_1(\varepsilon) = \frac{3(2\varepsilon + 1)2^\varepsilon}{16\pi \cos(\pi\varepsilon)|\Gamma(-2\varepsilon)|} ,$$

where Γ is the Euler function.

Inequality (3.10) can be proved—owing to (3.8)—by the same procedure used in [6] (Lemma p. 249).

ii) Let us rewrite (3.10) as

$$\|(-\Delta_s)^{(1+\varepsilon)/2} h\|^2_{L^2(S^2)} \leqslant K_1(\varepsilon) \int_0^{+\infty} t^{-2\varepsilon-1} F(t)\, dt , \tag{3.11}$$

where

$$F(t) = \int_{S^2} d\sigma_w \int_{S^2} |\nabla_s H(z, tw)|^2 d\sigma_z . \tag{3.12}$$

Notice that, by (3.7), $F(0) = 0$.

One can show (see [6]) that if F and F' are bounded functions and $F(0)=0$ then

$$\int_0^{+\infty} t^{-(2\varepsilon+1)} F(t)\, dt \leqslant K_2(\varepsilon) \|F\|_{L^\infty}^{1-2\varepsilon} \|F'\|_{L^\infty}^{2\varepsilon}, \tag{3.13}$$

where

$$K_2(\varepsilon) = \frac{(2\varepsilon)^{-1-2\varepsilon}}{1-2\varepsilon}.$$

iii) Let us estimate $\|F\|_{L^\infty}$.

Setting

$$h(G(tw)z) = \tilde{h}(z, t, w),$$

by (3.7) one has

$$\nabla_s H(z, tw) = \nabla_s \tilde{h}(z, t, w) - \nabla_s h(z). \tag{3.14}$$

It is easy to check that

$$|\nabla_s \tilde{h}(z, t, w)|^2 \leqslant |\nabla_s h(z')|^2 \left(\left|\frac{\partial z'}{\partial \theta}\right|^2 + \frac{1}{\sin^2\theta} \left|\frac{\partial z'}{\partial \varphi}\right|^2 \right), \tag{3.15}$$

where $z = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and $z' = G(tw)z$; hence, since G is a rotation, it follows that

$$|\nabla_s \tilde{h}(z, t, w)|^2 \leqslant 2 |\nabla_s h(z')|^2. \tag{3.16}$$

Therefore, from (3.12), (3.14), (3.16) and (3.9) it follows that

$$F(t) \leqslant 6(4\pi R)^2. \tag{3.17}$$

iv) To compute $F'(t)$, let us notice that by Gauss-Green theorem (see [3]) one has

$$\int_{S^2} |\nabla_s H(z, x)|^2 d\sigma_z = -\int_{S^2} H(z, x) \Delta_S H(z, x)\, d\sigma_z, \tag{3.18}$$

where Δ_S too acts on H with respect to z. Writing down the explicit form for H, as in (3.7), it follows that

$$\int_{S^2} |\nabla_s H(z, x)|^2 d\sigma_z = -2\int_{S^2} h(z) \Delta_S h(z)\, d\sigma_z + \int_{S^2} \Delta_S h(z)[h(z') + h(z'')]\, d\sigma_z, \tag{3.19}$$

where $z' = G(x)z$ and $z'' = [G(x)]^{-1} z$. Thus, by (3.12) and (3.19),

$$F'(t) = \int_{S^2} d\sigma_w \int_{S^2} \Delta_S h(z) \frac{\partial}{\partial t} [\tilde{h}(z, t, w) + \tilde{h}(z, t, -w)] \, d\sigma_z . \tag{3.20}$$

From (3.9) it follows that

$$\left| \frac{\partial}{\partial t} \tilde{h}(z, t, w) \right| \leqslant R . \tag{3.21}$$

In fact

$$\left| \frac{\partial}{\partial t} \tilde{h}(z, t, w) \right| \leqslant |\nabla_s h(z')| ,$$

since

$$\left| \frac{\partial z'}{\partial t} \right| = 1 \qquad (\text{see [6]}) .$$

Analogously

$$\left| \frac{\partial}{\partial t} \tilde{h}(z, t, -w) \right| \leqslant R . \tag{3.22}$$

Therefore (3.20), (3.21), (3.22) imply

$$|F'(t)| \leqslant 8\pi R \int_{S^2} |\Delta_S h(z)| \, d\sigma_z . \tag{3.23}$$

In order to estimate the integral in the right-hand side of (3.23), let us recall that (see, for example, [4] p. 66)

$$2h(z) + \Delta_S h(z) = \mathfrak{R}_1 + \mathfrak{R}_2 , \tag{3.24}$$

where $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are the principal radii of curvature of ∂T at the point where the exterior normal is z.

Therefore

$$2h(z) + \Delta_S h(z) > 0 \tag{3.25}$$

for every z.

As $h(z)$ is positive, (3.25) implies

$$|\Delta_S h(z)| \leqslant \Delta_S h(z) + 4h(z) . \tag{3.26}$$

Coupling then (3.23) and (3.26) yields

$$|F'(t)| \leqslant 2(8\pi R)^2 , \tag{3.27}$$

since $h(z) \leqslant R$ and

$$\int_{S^2} \Delta_s h(z)\, d\sigma_z = 0 .$$

Assembling now (3.11), (3.13), (3.17) and (3.27) proves Lemma 3 in the case T has the origin as an interior point. Notice that in such a case the constant $\mathcal{C}(\varepsilon)$ in (3.6) can be chosen as follows:

$$\mathcal{C}(\varepsilon) = 4\pi(\tfrac{4}{3})^{\varepsilon}[6K_1(\varepsilon)K_2(\varepsilon)]^{1/2} .$$

If T does not contain the origin in its interior, let us consider the parallel translation of T by a vector u such that $T' = T + u$ has the origin as an interior point. Clearly $|u| < R$ and T' is contained in the centred sphere of radius $R - |u|$. If h_u denotes the support function of T', then

$$h_u(z) = h(z) + U(z) , \tag{3.28}$$

where $U(z) = \langle u, z\rangle$.

Notice that

$$\begin{cases} U_l^n = 0 , \quad \text{for every } l \neq 1 , \\ \sum_{n=-1}^{1} |U_1^n|^2 = \dfrac{4\pi}{3} |u|^2 , \end{cases} \tag{3.29}$$

U_l^n being the coefficients in the expansion of U into spherical harmonics. Therefore, by using (3.6) for h_u and taking into account (3.29), from (3.28) it is easy to deduce that Lemma 3 holds also in the case T does not contain the origin.

4. – PROOF OF THE THEOREM. – Firstly we prove the Theorem assuming T_1 and T_2 are regular bodies. Let h_1 and h_2 be the corresponding support functions. Let us start from inequality (2.4) of Lemma 2 and estimate $|V(T_1) - V_2(T_1, T_2)|$. To this end recall that (see, for instance, [5])

$$V(T_1) - V_2(T_1, T_2) = \tfrac{1}{3}\int_{S^2} h_1(z)[\varrho_1(z) - \varrho_2(z)]\, d\sigma_z , \tag{4.1}$$

where $\varrho_i(z) = [\gamma_i(z)]^{-1}$ $(i = 1, 2)$, and $\gamma_i(z)$ is the Gauss curvature of ∂T_i at the point P with the exterior normal z. Now remember that $\varrho_i(z)$ is related to $A(T_i; z)$ by equation (1.2).

Setting then $A(T_i; z) = A_i(z)$ let us expand A_i and ϱ_i into spherical harmonics Y_l^n and let $A_{i,l}^n$ and $\varrho_{i,l}^n$ denote relevant coefficients. Since A_i and ϱ_i are even functions $A_{i,l}^n = \varrho_{i,l}^n = 0$, when l is an odd number.

On the other hand from (1.2) one can deduce (see [2], [18], [10]) that

$$A^n_{i,l} = b_l \varrho^n_{i,l} \tag{4.2}$$

with

$$b_l = \frac{\sqrt{\pi}\,(-1)^{l/2+1}}{2(l/2+1)!}\,\Gamma\left(\frac{l-1}{2}\right), \quad l \text{ being an even number}. \tag{4.3}$$

Notice that b_l behaves asymptotically as $l^{-5/2}$. For simplicity let us set $\varrho_1(z) - \varrho_2(z) = \varrho(z)$, $A_1(z) - A_2(z) = A(z)$, $h_1(z) = h(z)$ and denote by $\varrho^n_l, A^n_l, h^n_l$ the relevant coefficients of the expansions into spherical harmonics.

Thus we can rewrite (4.1) as

$$V(T_1) - V_2(T_1, T_2) = \tfrac{1}{3}\left(h^0_0 \varrho^0_0 + \sum_{l>0}\sum_{n=-l}^{l} h^n_l \bar{\varrho}^n_l\right). \tag{4.4}$$

From (4.2), (4.3) it follows that

$$|h^0_0 \varrho^0_0| \leqslant \frac{2}{\sqrt{\pi}} R_1 |A^0_0|, \tag{4.5}$$

where R_1 is the circumradius of T_1.

Let us estimate now the other term in the right-hand side of (4.4).

For fixed $p \in (0, \frac{1}{3})$, set $q = (1-p)/p$ and choose $\varepsilon \in (0, \frac{1}{2})$, $\eta \in (0, \varepsilon)$ such that

$$q = \frac{\frac{3}{2} - \varepsilon}{\eta}.$$

Cauchy-Schwarz inequality implies that

$$\left|\sum_{l>0}\sum_{n=-l}^{l} h^n_l \bar{\varrho}^n_l\right| \leqslant \|(-\Delta_s)^{(1+\varepsilon)/2} h\|_{L^2(S^2)} \|(-\Delta_s)^{-(1+\varepsilon)/2} \varrho\|_{L^2(S^2)} \tag{4.6}$$

(see definition (3.4) and identity (3.5)), where the first term on the right-hand side can be estimated by (3.6). Let us estimate the other.

Let $Q_l = [l(l+1)]^{-1-\varepsilon} \sum_{n=-l}^{l} |\varrho^n_l|^2$. Jensen's inequality implies that

$$\left(\frac{\sum_{l>0} Q_l}{\sum_{l>0}[l(l+1)]^{\eta} Q_l}\right)^{1+q} \leqslant \frac{\sum_{l>0}[l(l+1)]^{-\eta q} Q_l}{\sum_{l>0}[l(l+1)]^{\eta} Q_l},$$

that is

$$\sum_{l>0}[l(l+1)]^{-(1+\varepsilon)} \sum_{n=-l}^{l} |\varrho^n_l|^2 \leqslant \left(\sum_{l>0}[l(l+1)]^{-5/2} \sum_{n=-l}^{l} |\varrho^n_l|^2\right)^{p} \times \left(\sum_{l>0}[l(l+1)]^{-1-\varepsilon+\eta} \sum_{n=-l}^{l} |\varrho^n_l|^2\right)^{1-p}. \tag{4.7}$$

By (4.2), (4.3) one obtains

$$\sum_{l>0}[l(l+1)]^{-5/2}\sum_{n=-l}^{l}|\varrho_l^n|^2\leqslant\frac{1}{8\pi}\sum_{l>0}\sum_{n=-l}^{l}|A_l^n|^2\,, \tag{4.8}$$

since

$$[l(l+1)]^{-5/2}<\frac{1}{8\pi}b_l^2\,,$$

for every even positive l.

On the other hand, since

$$\varrho_l^n=\int_{S^2}\varrho(z)\,\overline{Y}_l^n(z)\,d\sigma_z\,,$$

from the addition formula for spherical harmonics we deduce that

$$\sum_{n=-l}^{l}|\varrho_l^n|^2=(2l+1)\int_{S^2}d\sigma_z\int_{S^2}\varrho(z)\varrho(z')P_l(\cos\widehat{zz'})\,d\sigma_{z'}\,, \tag{4.9}$$

P_l being the Legendre polynomial of degree l and $\widehat{zz'}$ the angle between the vectors z and z'.

Since

$$|P_l(x)|\leqslant 1\,,\qquad \text{for every } x\in[-1,1]\,,$$

from (4.9) it follows that

$$\sum_{n=-l}^{l}|\varrho_l^n|^2\leqslant(2l+1)\left(\int_{S^2}|\varrho(z)|\,d\sigma_z\right)^2. \tag{4.10}$$

Notice that

$$\int_{S^2}|\varrho(z)|\,d\sigma_z\leqslant\int_{S^2}[\varrho_1(z)+\varrho_2(z)]\,d\sigma_z=S_1+S_2\,, \tag{4.11}$$

S_i being the area of ∂T_i, $i=1,2$.

Therefore (4.10) and (4.11) yield

$$\sum_{l>0}[l(l+1)]^{-1-\varepsilon+\eta}\sum_{n=-l}^{l}|\varrho_l^n|^2\leqslant S_{\varepsilon,\eta}(S_1+S_2)\,, \tag{4.12}$$

where

$$S_{\varepsilon,\eta}=\sum_{l\text{ even}}\frac{2l+1}{[l(l+1)]^{1+\varepsilon-\eta}}\,.$$

By using (4.7), (4.8), (4.12) and the inequality $S_i \leq 4\pi R_i^2$ ($R_i =$ circumradius of T_i, $i = 1, 2$), we obtain

$$\|(-\Delta_s)^{-(1+\varepsilon)/2}\varrho\|_{L^2(S^2)} \leq C_1 \|\tilde{A}\|^p_{L^2(S^2)}, \tag{4.13}$$

where $\tilde{A}(z) = A(z) - A_0^0/(2\sqrt{\pi})$ and

$$C_1 = \frac{[32\pi^2 S_{\varepsilon,\eta}(R_1^2 + R_2^2)]^{(1-p)/2}}{\sqrt{8\pi}}.$$

Finally, assembling (4.4), (4.5), (3.6) and (4.13) produces

$$|V(T_1) - V_2(T_1, T_2)| \leq \frac{R_1}{3}\left(\frac{2}{\sqrt{\pi}}|A_0^0| + C_1 \mathrm{C}(\varepsilon)\|\tilde{A}\|^p_{L^2(S^2)}\right). \tag{4.14}$$

Quite analogously we can obtain that

$$|V(T_2) - V_1(T_1, T_2)| \leq \frac{R_2}{3}\left(\frac{2}{\sqrt{\pi}}|A_0^0| + C_1 \mathrm{C}(\varepsilon)\|\tilde{A}\|^p_{L^2(S^2)}\right). \tag{4.15}$$

Now, to get the estimate (1.5), it suffices to insert (4.14) and (4.15) in (2.4) and to use the inequality (2.2) of Lemma 1. Notice that

$$|A_0^0| + \|\tilde{A}\|^p_{L^2(S^2)}| \leq 2^{1-p/2}[1 + (4\sqrt{\pi}M)^{1-p}]\|A\|^p_{L^2(S^2)}.$$

The Theorem is so proven in the case of regular bodies. We want to emphasize that from the above proof it would be possible to deduce an explicit (though rather complicate) expression for the constant C, appearing in (1.5), in terms of M, m and p. Here we omit it for simplicity. In the general case (T_1 and T_2 not necessarily regular bodies), we can find two sequences $\{T_{1,n}\}$, $\{T_{2,n}\}$ of regular bodies such that

$$\delta(T_i, T_{i,n}) \to 0, \quad \text{as } n \to +\infty, \quad i = 1, 2$$

(see [4], p. 36).

We can apply the Theorem to $T_{1,n}$ and $T_{2,n}$ and write down (1.5) for them. By letting $n \to +\infty$ and using the fact that

$$A(T_{i,n}; z) \to A(T_i; z), \quad i = 1, 2,$$

uniformly with respect to z (see [12], Theorem 1), one concludes that the Theorem is valid in the general case too.

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