An Identification Problem for a Semilinear Parabolic Equation (*).

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Sunto. – *Si considera un problema sovradeterminato per l'operatore parabolico semilineare $\mathcal{D}(u) = D_t u - D_x^2 u - a(u)$ contenente un termine incognito $a(u)$ e si prova l'esistenza di almeno una soluzione (u, a).*

0. – Introduction.

We consider a semilinear parabolic equation of the form

$$(0.1) \qquad D_t u - D_x^2 u - a(u) = f \quad \text{in } Q_T = (0, l) \times (0, T) \qquad (l, T > 0)$$

subject to the following boundary and initial conditions

$$(0.2) \qquad D_x u(0, t) = \beta_1 \qquad 0 \leqslant t \leqslant T$$

$$(0.3) \qquad D_x u(l, t) = \beta_2 \qquad 0 \leqslant t \leqslant T$$

$$(0.4) \qquad u(x, 0) = g_3(x) \qquad 0 \leqslant x \leqslant l$$

β_1 and β_2 being *negative* constants.

For a *prescribed* function a it is well known that *problem* (0.1), (0.2), (0.3), (0.4), admits a unique solution u (e.g. from the anisotropic Hölder space $C^{2+\alpha,(2+\alpha)/2}(\overline{Q}_T)$), provided that a and the data f and g_3 belong to suitable Hölder spaces.

On the contrary, in our case the function a is assumed to be *unknown*. In order to determine the pair (u, a) it is evident that we need further information in addition to (0.2), ..., (0.4).

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The additional boundary conditions we are going to consider are the following

$$u(0, t) = g_1(t) \qquad 0 \leqslant t \leqslant T \tag{0.5}$$

$$u(l, t) = g_2(t) \qquad 0 \leqslant t \leqslant T\,. \tag{0.6}$$

However such conditions prove to be not sufficient to determine a: in fact using (0.5) and (0.6) we can determine a only in the ranges of g_1 and g_2, but not (in general) in the range of g_3.

Hence, in addition to (0.5), (0.6), we shall suppose that the function a (essentially an unknown source term depending on the «temperature» u) is *known* over some interval of temperatures coinciding with the range of g_3.

Thus we get the further information

$$a(\tau) = a_0(\tau) \qquad \textit{for any } \tau \textit{ in the range of } g_3. \tag{0.7}$$

Our identification problem consists therefore in determining a outside the range of g_3.

REMARK 0.1. – We observe that boundary conditions (0.2) and (0.3) are quite particular. Their introduction is intended only to simplify our exposition. The general case, where the constants β_1 and β_2 are replaced by a pair of non negative functions g_4 and g_5, is treated in the internal report [8].

Finally we stress that a problem similar to ours involving nonlinear parabolic equations in non-divergence form in the multidimensional case was first studied by ISKENDEROV [4]. He obtained mainly uniqueness and stability results. As far as existence is concerned, he outlined an iterative procedure strictly depending on the knowledge of the temperature-flux on the lateral boundary of the cylinder under consideration.

Taking advantage substantially of the same procedure BEZNOŠČENKO [1] proved later on some existence theorems (in the large) for solutions to inverse problems related to quasilinear parabolic equations.

Unfortunately such techniques seem not to apply when either the equation is in divergence form or the unknown function a does not appear in the boundary conditions.

1. – Statement of the main result.

Before stating our result, we have to specify the Hölder spaces which are, from our point of view, the functional framework appropriate for investigating the inverse problem (0.1), ..., (0.7). More exactly the unknown pair (u, a) is looked for respectively in the space $\mathcal{U} \times C^{1+\gamma}(\mathcal{R}(u))$, where $\mathcal{R}(u)$ denotes the range of u, $\alpha \in (0, \frac{1}{2})$,

$\gamma \in (\alpha, \frac{1}{2})$ and

$$\mathcal{U} = \{u \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T) : D_x u, D_t u \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)\} \tag{1.1}$$ (1)

We observe that, in order not to overburden our notations, throughout the paper we shall use the following notations

$$\|u\|_{n+\beta,(n+\beta)/2} = \|u\|_{C^{n+\beta,(n+\beta)/2}(\bar{Q}_T)} \tag{1.2}$$

$$\|\varphi\|_{n+\beta} = \|\varphi\|_{C^{n+\beta}(\bar{w})}, \tag{1.3}$$

$$\|\varphi\|_0 = \|\varphi\|_{L^\infty(\omega)} \tag{1.4}$$

where $n = 0, 1, 2, \ldots, \beta \in (0, 1)$ and $w \subset R$ denotes the domain of φ.

In different situations we shall use explicit notations.

Our result requires that data satisfy the following basic bounds

$$D_x f(x, t) \leqslant -m \qquad \forall (x, t) \in \bar{Q}_T \tag{1.5}$$

$$D_t g_1(t) \geqslant m, \quad D_t g_2(t) \leqslant -m \qquad \forall t \in [0, T] \tag{1.6}$$

$$D_x g_3(x) \leqslant -m \qquad \forall x \in [0, l] \tag{1.7}$$

for some prescribed positive constant m such that

$$m \leqslant \min(|\beta_1|, |\beta_2|). \tag{1.8}$$

REMARK 1.1. – According to bounds (1.6), (1.7) the chain of functions $\{g_1, g_3, g_2\}$ turns out to be monotonic non increasing on the parabolic boundary of Q_T.

As far as the smoothness of data is concerned, we shall assume that

$$f \in C^{4+\alpha,(4+\alpha)/2}(\bar{Q}_T) \tag{1.9}$$

$$g_1, g_2 \in C^{3+\gamma}([0, T]) \qquad (0 < \alpha < \gamma < \tfrac{1}{2}) \tag{1.10}$$

$$g_3 \in C^{6+\alpha}([0, l]) \tag{1.11}$$

$$a_0 \in C^{4+\alpha}([g_3(l), g_3(0)]). \tag{1.12}$$

Moreover the data f, g_1, g_2, g_3, a_0 have to satisfy suitable consistency conditions at $(0, 0)$ and $(l, 0)$. For the sake of brevity we do not list them, but we limit ourselves to asserting that they can be easily derived by equations (0.1), ..., (0.7) using the

(1) For the precise definition of Hölder spaces see [5, chpt. 1], where they are denoted by $H^{2+\alpha,(2+\alpha)}(Q_T)$.

following formulas

$$\lim_{x\to 0+} D_x^h D_t^j u(x, 0) = \lim_{t\to 0+} D_t^j D_x^h(0, t) \tag{1.13}$$

$$\lim_{x\to l-} D_x^h D_t^j u(x, 0) = \lim_{t\to 0+} D_t^j D_x^h u(l, t) . \tag{1.14}$$

We can now state our existence result that we shall prove in section 5.

THEOREM 1.1. – *Suppose that the data f, g_1, g_2, g_3, a_0 enjoy properties* (1.5), ..., (1.12) *for some $\alpha \in (0, \frac{1}{2})$ and $\gamma \in (\alpha, \frac{1}{2})$ and assume that they satisfy also the consistency conditions implied by* (0.1), ..., (0.7), (1.13), (1.14). *Then there exists a positive constant T for which the inverse problem* (0.1), ..., (0.7) *admits a solution $(u, a) \in \mathcal{U} \times C^{1+\gamma}(\mathcal{R}(u))$.*

2. – Some basic estimates for the solution to problem (0.1), ..., (0.4).

We observe that the unknown function a appearing in equation (0.1) may be determined at most on the range $\mathcal{R}(u)$ of u. Therefore it is basic to find out $\mathcal{R}(u)$ in terms of the functions g_1, g_2, g_3 only. Consequently our first task in this section consists in showing that, if $(u, a) \in \mathcal{U} \times C^{1+\gamma}(\mathcal{R}(u))$ is a solution to problem (0.1), ..., (0.4), then u attains its minimum and maximum values on the parabolic boundary of Q_T. Actually we are going to prove much more. Owing to the techniques developed in the sequel we are forced to show that $D_x u$ is bounded away from zero in $\bar{Q}_T$.

To that purpose we need to introduce the Banach space $C_b^{1+\gamma}(R)$ consisting of functions $a \in C(R)$ such that

$$\|a\|_{1+\gamma} = \operatorname*{Sup}_{\tau \in R} |a(\tau)| + \operatorname*{Sup}_{\tau,\, \sigma \in R;\, \tau \neq \sigma} |\tau - \sigma|^{-\gamma} |D_\tau a(\tau) - D_\tau a(\sigma)| < +\infty . \tag{2.1}$$

In the sequel we shall use also the following metric subspace of $C_b^{1+\gamma}(R)$

$$C_{a_0}^{1+\gamma}(R) = \{a \in C_b^{1+\gamma}(R) \colon a(\tau) = a_0(\tau),\ g_3(l) \leqslant \tau \leqslant g_3(0)\} , \tag{2.2}$$

where a_0 is the function in $C^{4+\alpha}([g_3(l), g_3(0)])$ appearing in (0.7).

THEOREM 2.1. – *Suppose that the data f and g_3 satisfy bounds* (1.5) *and* (1.7). *Then, if $(u, a) \in \mathcal{U} \times C_b^{1+\gamma}(R)$ is any solution to problem* (0.1), ..., (0.4), *we have*

$$D_x u(x, t) \leqslant -\min (mM^{-1}, m) \qquad \forall (x, t) \in \bar{Q}_T , \tag{2.3}$$

where M is a positive bound of $\|a\|_{1+\gamma}$.

PROOF. – Observe first that according to the results in [5, chapt. 5] the solution $u = U(a)$ to problem (0.1), ..., (0.4) belongs to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ together with the derivative $D_x u$.

Introduce now the function $w \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ so defined

$$w = D_x u + \nu\,, \tag{2.4}$$

where

$$\nu = \min\,(mM^{-1}, m)\,. \tag{2.5}$$

It is easy to check that w is a solution to the following Cauchy-Dirichlet problem

$$D_t w - D_x^2 w - w D_\tau a(u) = -\nu D_\tau a(u) + D_x f \quad \textit{in } Q_T \tag{2.6}$$

$$w(0, t) = \beta_1 + \nu \qquad 0 \leqslant t \leqslant T \tag{2.7}$$

$$w(x, 0) = D_x g_3 + \nu \qquad 0 \leqslant x \leqslant l \tag{2.8}$$

$$w(l, t) = \beta_2 + \nu \qquad 0 \leqslant t \leqslant T\,. \tag{2.9}$$

From bounds (1.5), (1.7) it is immediate to realize that our choice (2.5) of ν assures the nonpositivity of the right members in equations (2.7), ..., (2.9). The same is true also for the right member in (2.6). In fact $\forall (x, t) \in \bar{Q}_T$, $\forall \tau \in \mathcal{R}(u)$ we have

$$-\nu D_\tau a(\tau) + D_x f(x, t) \leqslant \nu M - m \leqslant 0\,. \tag{2.10}$$

From the maximum principle we infer that $w \leqslant 0$ in $\bar{Q}_T$. ■

COROLLARY 2.1. – *Under hypotheses* (1.5), (1.6), (1.7) *the range of the solution* u *to problem* (0.1), ..., (0.6) *is the interval* $[g_2(T), g_1(T)]$. *Moreover* u *satisfies the following bounds*

$$g_2(\tau) \leqslant u(x, t) \leqslant g_1(\tau) \qquad \forall (x, t) \in \bar{Q}_\tau\,, \quad \forall \tau \in (0, T]\,. \tag{2.11}$$

PROOF OF COROLLARY 2.1. – From bound (2.3) we immediately infer that

$$g_2(t) + \nu(l - x) \leqslant u(x, t) \leqslant g_1(t) - \nu x \qquad \forall (x, t) \in \bar{Q}_T\,. \tag{2.12}$$

Using the strict monotonicity of g_1 and g_2 (2), from (2.12) we easily derive (2.11). Since g_1 and g_2 are the boundary values of u, from (2.12) we conclude that the range of u is just the interval $[g_2(T), g_1(T)]$. ■

(2) See bounds (1.6).

Now we study the dependence of the solution $U(a)$ to problem (0.1), ..., (0.4) upon a, as a varies in the metric space $C_{a_0}^{1+\gamma}(R)$ defined by (2.2). More exactly we are interested in deriving several estimates assuring the boundedness and the continuity of the operator U. They will prove to be the basic tools to solve the transformed inverse problem of section 3.

To this purpose we state the following theorems:

THEOREM 2.2. – *Assume that $a \in C_{a_0}^{1+\gamma}(R)$ and that the data f, g_1, g_2, g_3, a_0 possess properties* (1.5), (1.7), (1.9), (1.11) *and satisfy consistency conditions implied by* (0.1), ..., (0.7), (1.13), (1.14). *Then problem* (0.1), ..., (0.4) *admits a unique solution $u = U(a)$ satisfying the following estimate*:

$$\|U(a)\|_{2+\alpha,(2+\alpha)/2} + \|DU(a)\|_{2+\alpha,(2+\alpha)/2} \leqslant C_1(T)\big[\|f\|_{2+\alpha,(2+\alpha)/2} + \|g_3\|_{4+\alpha} + 1\big] \;(^3) \quad \forall a \in C_{a_0}^{1+\gamma}(R)\,. \tag{2.13}$$

The « constant » $C_1(T)$ depends obviously on T, but it is also an increasing function of $\|a\|_{1+\gamma}$ and of the norms of data in the prescribed spaces. Moreover $C_1(T)$ remains bounded as $T \to 0^+$.

THEOREM 2.3. – *Let hypotheses listed in theorem 2.2 be satisfied. Then the map $a \to U(a)$ is continuous from $C_{a_0}^{1+\gamma}(R)$ into $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ and satisfies the estimate*

$$\|U(a_2) - U(a_1)\|_{2+\alpha,(2+\alpha)/2} \leqslant C_2(T)\|a_2 - a_1\|_\alpha \quad \forall a_1, a_2 \in C_{a_0}^{1+\gamma}(R)\,, \tag{2.14}$$

where the « constant » $C_2(T)$ enjoys properties similar to the ones of $C_1(T)$ in theorem 2.1.

Moreover the maps $a \to DU(a)$ and $a \to D_\tau a(U(a))$ are bounded and continuous from $C_{a_0}^{1+\gamma}(R)$ $(0 < \alpha < \gamma < \frac{1}{2})$ respectively to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ and $C^{\alpha,\alpha/2}(\bar{Q}_T)$.

We observe that estimate (2.13) is of the Schauder type, and can be inferred as in [5]. Therefore we omit the proof of theorem 2.2 and refer the reader interested in a detailed proof to the internal report [8].

On the contrary we shall give a proof of the less usual theorem 2.3. However, owing to its length, we postpone it to section 5.

3. – The inverse problem trasformed.

This section is devoted to transforming our inverse problem (0.1), ..., (0.7) into a new one, characterized by the appearence of the unknown function a in the boundary

(3) $Dv = (D_x v, D_t v)$ denotes the gradient of v.

conditions. The new problem involves a and the function v so defined

$$v = L\big(U(a), D\big)\, D_t U(a)\,, \tag{3.1}$$

where $U(a)$ is the solution to problem (0.1), ..., (0.4) and

$$L(u, D) = D_x u D_t - D_t u D_x\,. \tag{3.2}$$

Assuming for the moment that $v \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$, we can show that (v, a) is a solution to the following inverse problem

$$\begin{aligned} D_t v - D_x^2 v - 2[D_x^2 U(a)/D_x U(a)]D_x v - D_\tau a\big(U(a)\big) v = \\ = Q\big(U(a)\big) + D_t^2 f D_x U(a) - D_x D_t f D_t U(a) \quad \textit{in } Q_T \end{aligned} \tag{3.3}$$

$$v(0, t) = \tilde{g}_1(t) \qquad 0 \leqslant t \leqslant T \tag{3.4}$$

$$D_x v(0, t) = D_t g_1(t) D_t\big[a\big(g_1(t)\big)\big] - a\big(g_1(t)\big) D_t^2 g_1(t) + \tilde{g}_2(t) \qquad 0 \leqslant t \leqslant T \tag{3.5}$$

$$v(x, 0) = \tilde{g}_3(x) \qquad 0 \leqslant x \leqslant l \tag{3.6}$$

$$v(l, t) = \tilde{g}_4(t) \qquad 0 \leqslant t \leqslant T \tag{3.7}$$

$$D_x v(l, t) = D_t g_2(t) D_t\big[a\big(g_2(t)\big)\big] - a\big(g_2(t)\big) D_t^2(t) + \tilde{g}_5(t) \qquad 0 \leqslant t \leqslant T \tag{3.8}$$

$$a(\tau) = a_0(\tau) \qquad g_3(l) \leqslant \tau \leqslant g_3(0)\,. \tag{3.9}$$

The functions $Q\big(U(a)\big)$ and $\tilde{g}_j$ $(j = 1, ..., 5)$ are defined by the following equations

$$\begin{aligned} Q\big(U(a)\big) = \{D_x^3 U(a) - 2[D_x^2 U(a)]^2/D_x U(a)\} D_t U(a) + \{-3 D_x D_t U(a) + \\ + 2D_x^2 U(a) D_t U(a)/D_x U(a)\} D_x^2 D_t U(a) + 2D_x^2 U(a)[D_x D_t U(a)]^2/D_x U(a)\,. \end{aligned} \tag{3.10}$$

$$\tilde{g}_1(t) = \beta_1 D_t^2 g_1(t) \qquad 0 \leqslant t \leqslant T \tag{3.11}$$

$$\tilde{g}_2(t) = D_t g_1(t) D_t f(0, t) - f(0, t) D_t^2 g_1(t) \qquad 0 \leqslant t \leqslant T \tag{3.12}$$

$$\begin{aligned} \tilde{g}_3(x) = D_x g_3(x)\{D_x^4 g_3(x) + D_x^2\big(a_0\big(g_3(x)\big)\big) + D_x^2 f(x, 0) - D_\tau a_0\big(g_3(x)\big)\cdot \\ \cdot\big[D_x^2 g_3(x) + a_0\big(g_3(x)\big) + f(x, 0)\big] + D_x f(x, 0)\big\} - \big[D_x^2 g_3(x) + a_0\big(g_3(x)\big) + f(x, 0)\big]\cdot \\ \cdot\big\{D_x^2 g_3(x) + D_x\big(a_0\big(g_3(x)\big)\big) + D_x f(x, 0)\big\} \qquad 0 \leqslant x \leqslant l \end{aligned} \tag{3.13}$$

$$\tilde{g}_4(t) = \beta_2 D_t^2 g_2(t) \qquad 0 \leqslant t \leqslant T \tag{3.14}$$

$$\tilde{g}_5(t) = D_t g_2(t) D_t f(l, t) - f(l, t) D_t^2 g_2(t) \qquad 0 \leqslant t \leqslant T\,. \tag{3.15}$$

In order to derive equations (3.3), ..., (3.9) it suffices to use estimate (2.3), the following identities and to perform standard computations

$$D_t V(a) - L(U(a), D) D_t^2 U(a) = 0 \tag{3.16}$$

$$D_x V(a) - D_x U(a) D_x D_t^2 U(a) = \\ = D_x^2 U(a) D_t^2 U(a) - D_t U(a) D_x^2 D_t U(a) - [D_x D_t U(a)]^2 \tag{3.17}$$

$$D_x^2 V(a) - L(U(a), D) D_x^2 D_t U(a) = \\ = Q(U(a)) + 2[D_x^2 U(a)/D_x U(a)] D_x L(U(a), D) D_t U(a), \tag{3.18}$$

where $Q(U(a))$ is defined by (3.10).

REMARK 3.1. – According to theorems 2.1 and 2.2, from definitions (3.10), ..., (3.15) we infer the following relations

$$b_1(a) \underset{\text{def}}{=} D_x^2 U(a)/D_x U(a); \quad b_1(a), D_x b_1(a), D_t b_1(a) \in C^{\alpha,\alpha/2}(\bar{Q}_T) \tag{3.19}$$

$$b_2(a) \underset{\text{def}}{=} D_\tau a(U(a)) \in C^{\alpha,\alpha/2}(\bar{Q}_T) \tag{3.20}$$

$$Q(U(a)) \in C^{\alpha,\alpha/2}(\bar{Q}_T) \tag{3.21}$$

$$\tilde{g}_1, \tilde{g}_4 \in C^{1+\gamma}([0, T]) \tag{3.22}$$

$$\tilde{g}_2, \tilde{g}_5 \in C^{(1+\alpha)/2}([0, T]) \tag{3.23}$$

$$\tilde{g}_3 \in C^{2+\alpha}([0, l]) . \tag{3.24}$$

We notice now that in our original case the function v belongs to $C^{\alpha,\alpha/2}(\bar{Q}_T)$, but it does not (in general) to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$. However we can prove, using an approximating and regularizing procedure concerning the function a and involving the classical problem (3.3), ..., (3.9), that (v, a) solves equation

$$\int_{Q_T} \varphi Q(U(a))\, dx\, dt = \int_{Q_T} v\{-D_t\varphi - D_x^2\varphi + D_x(\varphi b_1(a)) - b_2(a)\varphi\}\, dx\, dt + \\ - \int_0^l \varphi(\cdot, 0)\tilde{g}_3\, dx + \sum_{j=0}^{1}(-1)^j \int_0^T \varphi(jl, \cdot)\{D_t g_{1+j} D_t(a(g_{1+j})) - a(g_{1+j}) D_t^2 g_{1+j} + \\ + \tilde{g}_{2+3j} + b_1(a)(jl, \cdot)\tilde{g}_{1+3j}\}\, dt - \sum_{j=0}^{1}(-1)^j \int_0^T \tilde{g}_{1+3j} D_x\varphi(jl, \cdot)\, dt \qquad \forall \varphi \in \Phi \tag{3.25}$$

and satisfies conditions (3.4), (3.6), (3.7), (3.9).

The functional space Φ, consisting of test functions φ, is so defined:

$$\Phi = \{\varphi \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T) : \varphi(x, T) = 0,\ 0 \leqslant x \leqslant l\} . \tag{3.26}$$

We observe that equation (3.25) can be formally deduced from (3.3), ..., (3.9) by multiplying both members of equation (3.3) by a test function $\varphi \in \Phi$ and integrating by parts. Moreover every solution $(v, a) \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T) \times C^{1+\gamma}_{a_0}(R)$ to problem (3.25), (2.4), (3.6), (3.7), (3.9) is necessarily a solution to problem (3.3), ..., (3.9).

For a rigorous proof of equation (3.25) the reader may refer to the appendix in [8].

We want now to transform the classical problem (3.3), ..., (3.9) into a more suitable one by eliminating the unknown function a. This will be performed in two steps.

STEP 1. – We proceed to determining a in terms of $D_x v(0, \cdot)$, $D_x v(l, \cdot)$ and a_0 by using boundary conditions (3.10), ..., (3.13).

To this purpose we recall that g_1 and g_2 are strictly monotonic according to conditions (1.6). Then, using equations (3.5) and (3.8) and the following consistency conditions (implied by equations (0.1), ..., (0.7), (1.9), (1.10))

$$D_t g_1(0) - D_x^2 g_3(0) - f(0, 0) = a_0(g_3(0)) = a(g_3(0)) \tag{3.27}$$

$$D_t g_2(0) - D_x^2 g_3(l) - f(l, 0) = a_0(g_3(l)) = a(g_3(l)) \,, \tag{3.28}$$

we can express a in the closed interval $[g_2(T), g_1(T)]$ in terms of $D_x v(0, \cdot)$, $D_x v(l, \cdot)$ and a_0.

To this purpose we have to integrate equations (3.5), (3.8) which can be viewed as two first-order differential equations and to perform the changes of variables defined respectively by $t = g_1^{-1}(\tau)$ and $t = g_2^{-1}(\tau)$ [4]. We obtain the following formulas:

$$a(\tau) = D_t g_2(g_2^{-1}(\tau)) \Big\{ a_0(g_3(l))[D_t g_2(0)]^{-1} + \int_0^{g_2^{-1}(\tau)} [D_s g_2(s)]^{-2}[D_x v(l, s) - \tilde{g}_5(s)]\, ds \Big\} \qquad g_2(l) \leqslant \tau \leqslant g_2(0) \tag{3.29}$$

$$a(\tau) = a_0(\tau) \qquad g_3(l) \leqslant \tau \leqslant g_3(0) \tag{3.30}$$

$$a(\tau) = D_t g_1(g_1^{-1}(\tau)) \Big\{ a_0(g_3(0))[D_t g_1(0)]^{-1} + \int_0^{g_1^{-1}(\tau)} [D_s g_1(s)]^{-2}[D_x v(0, s) - \tilde{g}_2(s)]\, ds \Big\} \qquad g_1(0) \leqslant \tau \leqslant g_1(T) \,. \tag{3.31}$$

Using consistency conditions involving the data f, g_j $(j = 1, 2, 3)$ (for the details see [8] formulas (2.43), ..., (2.46)) we can prove that the function a defined by for-

[4] g^{-1} denotes the function inverse to g.

mulas (3.29), ..., (3.31) actually belongs to $C^1_{a_0}([g_2(T), g_1(T)])$. Moreover it is not difficult to realize that a verifies the following estimate

$$\|a\|_{C^{1+\gamma}([g_2(T),g_1(T)])} \leqslant C\{\|D_x v(0,\cdot)\|_\gamma + \|D_x v(l,\cdot)\|_\gamma + 1\} \qquad \gamma \in (\alpha, \tfrac{1}{2}) . \tag{3.32}$$

The positive constant C depends on m, β_1, β_2 and on suitable norms of data relevant to this context.

STEP 2. – Using step 1 we can represent a in the interval $[g_2(T), g_1(T)]$ in terms of v as follows

$$a = A(D_x v) , \tag{3.33}$$

where $A: C^{2\gamma,\gamma}(\bar{Q}_T) \to C^{1+\gamma}_{a_0}([g_2(T), g_1(T)])$ is the linear affine operator defined by formulas (3.29), ..., (3.31).

Now we would eliminate a from problem (3.3), ..., (3.9). But, under our present hypotheses a, as a member of $C^{1+\gamma}_{a_0}(R)$, is defined on the whole of R. Such a difficulty can be overcome in the following way: according to bounds (2.11) in corollary 2.1 the domain of a in our inverse problem (0.1), ..., (0.7) turns out to coincide with the interval $[g_2(T), g_1(T)]$. This allows us to restrict the class $C^{1+\gamma}_{a_0}(R)$ to any class $\mathcal{A}$ more suitable for our problem without any danger of arbitrariness. In particular we can choose as our class $\mathcal{A}$ the image of $C^{1+\gamma}_{a_0}(R)$ under a (fixed) linear *extension* operator $\mathcal{E}$ acting from $C^{1+\gamma}([g_2(T), g_1(T)])$ to $C^{1+\gamma}_b(R)$ (5) such that

$$\|\mathcal{E}a\|_{C_b(\boldsymbol{R})} \leqslant C_1 \|a\|_{C([g_2(T),g_1(T)])} \qquad \forall a \in C([g_2(T), g_1(T)]) \tag{3.34}$$

$$\|D_\tau \mathcal{E}a\|_{C^\gamma_b(\boldsymbol{R})} \leqslant C_2 \|D_\tau a\|_{C^\gamma([g_2(T),g_1(T)])} \qquad \forall a \in C^{1+\gamma}_b([g_2(T), g_1(T)]) . \tag{3.35}$$

The positive constants C_1 and C_2 depends only on $g_1(T)$ and $g_2(T)$.

In order not to overburden our notations from now on we shall denote $\mathcal{E}a$ simply by a.

For our admissible functions $a \in \mathcal{A}$ from (3.32), (3.34), (3.35) we easily derive the following estimate

$$\|a\|_{C^{1+\gamma}_b(\boldsymbol{R})} \leqslant C(\|D_x v(0,\cdot)\|_\gamma + \|D_x v(l,\cdot)\|_\gamma + 1) \qquad \forall a \in \mathcal{A} . \tag{3.36}$$

The positive constant C depends on m, β_1, β_2 and suitable norms of data relevant to this context.

---

(5) For the definition of $C^{1+\gamma}_b(\boldsymbol{R})$ see formula (2.1).

We can now eliminate a from problem (3.3), ..., (3.9): we get the following semilinear Cauchy-Dirichlet problem

$$v \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T) \tag{3.37}$$

$$D_x v \in C^{2\gamma,\gamma}(\bar{Q}_T) \tag{3.38}$$

$$D_t v - D_x^2 v = B_1(D_x v) D_x v + B_2(D_x v) v + B_3(D_x v) \quad \text{in } Q_T \tag{3.39}$$

$$v(0, t) = \tilde{g}_1(t) \quad 0 \leqslant t \leqslant T \tag{3.40}$$

$$v(x, 0) = \tilde{g}_3(x) \quad 0 \leqslant x \leqslant l \tag{3.41}$$

$$v(l, t) = \tilde{g}_4(t) \quad 0 \leqslant t \leqslant T . \tag{3.42}$$

The nonlinear operators B_j $(j = 1, 2, 3)$ are defined by the equations

$$B_1(z) = 2D_x^2 U(A(z))/D_x U(A(z)) \tag{3.43}$$

$$B_2(z) = D_\tau A\big(U(A(z))\big) \tag{3.44}$$

$$B_3(z) = Q\big(U(A(z))\big) + D_t^2 f D_x U(A(z)) - D_x D_t f D_t U(A(z)) . \tag{3.45}$$

Problem (3.37), ..., (3.42) will be solved in the next section, while the remaining part of the present section is devoted to the study of some basic properties of operators B_j $(j = 1, 2, 3)$.

From theorems 2.1, 2.2, 2.3, definitions (3.10), (3.43), (3.44), (3.45) and estimate (3.32) we easily infer the following

LEMMA 3.1. – *The mappings B_j $(j = 1, 2, 3)$ are bounded and continuous from $C^{2\gamma,\gamma}(\bar{Q}_T)$ into $C^{\alpha,\alpha/2}(\bar{Q}_T)$ $(0 < \alpha < \gamma < \frac{1}{2})$. Moreover they satisfy the estimates*

$$\|B_j(z)\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)} \leqslant C(T, M) \quad \forall z \in K(M), \quad j = 1, 2, 3, \tag{3.46}$$

where C is a positive function bounded as $T \to 0^+$ for any (fixed) positive M and $K(M)$ is so defined

$$K(M) = \{z \in C^{2\gamma,\gamma}(\bar{Q}_T) : \|z\|_{C^{2\gamma,\gamma}(\bar{Q}_T)} \leqslant M\} . \tag{3.47}$$

Finally according to definitions (3.43), (3.44), (3.45), (3.29) it is not difficult to prove the following

LEMMA 3.2. – *Let v be a solution to problem (3.37), ..., (3.42). Then the values at $(0, 0)$ and $(l, 0)$ of $B_j(D_x v)$ $(j = 1, 2, 3)$, depend only upon the values at special points of data a_0, f, g_j $(j = 1, 2, 3)$ and of their derivatives. Such points and derivatives are* *listed in the following table*

function	*number of derivatives*	*points corresponding to*	
		(0, 0)	(l, 0)
a_0	1	$g_3(0)$	$g_3(l)$
f	4 w.r. to x, 2 w.r. to t	(0, 0)	(l, 0)
g_1	2	0	0
g_2	2	0	0
g_3	4	0	l

4. – An existence theorem for a semilinear parabolic problem.

In this section we solve problem (3.37), ..., (3.42), that we rewrite in a compact form as follows:

$$v \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)\,, \qquad \alpha \in (0, 1/2) \tag{4.1}$$

$$D_t v - D_x^2 v = B(v, D_x v) \qquad \text{in } Q_T \tag{4.2}$$

$$v(0, t) = \tilde{g}_1(t) \qquad 0 \leqslant t \leqslant T \tag{4.3}$$

$$v(x, 0) = \tilde{g}_3(x) \qquad 0 \leqslant x \leqslant l \tag{4.4}$$

$$v(l, t) = \tilde{g}_4(t) \qquad 0 \leqslant t \leqslant T\,. \tag{4.5}$$

The (nonlinear) operator B maps $C^{2\gamma,\gamma}(\bar{Q}_T) \times C^{2\gamma,\gamma}(\bar{Q}_T)$ $(0 < \alpha < \gamma < \frac{1}{2})$ into $C^{\alpha,\alpha/2}(\bar{Q}_T)$ and satisfies the following equations

$$B(v, D_x v)(0, 0) = D_t \tilde{g}_1(0) - D_x^2 \tilde{g}_3(0) \tag{4.6}$$

$$B(v, D_x v)(l, 0) = D_t \tilde{g}_4(0) - D_x^2 \tilde{g}_3(l)\,. \tag{4.7}$$

Such properties are easy consequences of lemmas 3.1 and 3.2.

REMARK 4.1. – From the definition of $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ it is immediate (see e.g. [5, chapt. 1, p. 7]) to deduce that such a space is continuously embedded into $C^{2\gamma,\gamma}(\bar{Q}_T)$ $\forall \gamma \in (0, \frac{1}{2}]$.

We observe now that problem (4.1), ..., (4.5) is equivalent to the following integro-differential problem: *to look for a function* $v \in C^{2\gamma,\gamma}(\bar{Q}_T)$ *with* $D_x v \in C^{2\gamma,\gamma}(\bar{Q}_T)$ *such that*

$$v(x, t) = \int_{Q_T} H(x, t, y, s) B(v, D_x v)(y, s)\, dy\, ds + F(x, t) \qquad \forall (x, t) \in \bar{Q}_T\,, \tag{4.8}$$

where the functions H and F are so defined

$$(4.9)\qquad H(x,t,y,s)=\frac{1}{l}\left[\theta\left(\frac{x-y}{l},\frac{t-s}{l^2}\right)-\theta\left(\frac{x+y}{l},\frac{t-s}{l^2}\right)\right]$$

$$(4.10)\qquad F(x,t)=\int_0^l H(x,t,y,0)\tilde{g}_3(y)\,dy-\frac{2}{l^2}\int_0^t D_x\theta\left(\frac{x}{l},\frac{t-s}{l^2}\right)\tilde{g}_1(s)\,ds+ \\ +\frac{2}{l^2}\int_0^t D_x\theta\left(\frac{x-l}{l},\frac{t-s}{l^2}\right)\tilde{g}_4(s)\,ds$$

in terms of the function θ defined by the equations

$$(4.11)\qquad \theta(x,t)=\sum_{n=-\infty}^{+\infty}E(x+2n,t)$$

where

$$(4.12)\qquad E(x,t)=(4\pi t)^{-1/2}\exp\left(-\tfrac{1}{4}x^2t^{-1}\right).$$

We recall that E is the well-known fundamental solution for the heat operator, while θ is the Green function (over the rectangle) for such a operator when Cauchy-Dirichlet conditions are prescribed. Finally F is the solution to problem (4.2), ..., (4.5) with $B=0$. We observe that F does not belong to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$, since data $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ do not verify (in general) condition (4.6), (4.7) with $B=0$. However, using properties (3.22), (3.24) and consistency conditions

$$(4.13)\qquad \tilde{g}_1(0)=\tilde{g}_3(0);\qquad \tilde{g}_4(0)=\tilde{g}_3(l),$$

it is not difficult to derive that $F\in C^{2,1}(Q_T)$ and D_tF, $D_x^2F\in L^\infty(Q_T)$.

This implies that $F\in C^{2\gamma,\gamma}(\bar{Q}_T)$ $(0<\alpha<\gamma<\frac{1}{2})$. Moreover by a straightforward inspection we get that also $D_xF\in C^{2\gamma,\gamma}(\bar{Q}_T)$.

Taking advantage now of the representation formula for the solution to a Cauchy-Dirichlet problem related to the heat equation (6), it is an easy task to check that every solution $v\in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ to problem (4.2), ..., (4.5) is a solution to the integro-differential equation (4.8).

Conversely every solution $v\in C^{2\gamma,\gamma}(\bar{Q}_T)$ (with $D_xv\in C^{2\gamma,\gamma}(\bar{Q}_T)$) to the integro-differential equation (4.8) is easily seen (by differentiation) to be a solution to problem (4.2), ..., (4.5). Since $B(v,D_xv)\in C^{\alpha,\alpha/2}(\bar{Q}_T)$ and satisfies (4.6), (4.7), we infer that v really belongs to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$.

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(6) See e.g. [2, theorem 19.3.4].

Then we observe that $(v, D_x v)$ is a solution in $C^{2\gamma,\gamma}(\overline{Q}_T)\times C^{2\gamma,\gamma}(\overline{Q}_T)$ to the following system of Volterra integral equations

$$(4.14)\qquad v(x,t)=\int_{Q_T} H(x,t,y,s)B(v,w)(y,s)\,dy\,ds+F(x,t)\qquad \forall(x,t)\in\overline{Q}_T$$

$$(4.15)\qquad w(x,t)=\int_{Q_T} D_x H(x,t,y,s)B(v,w)(y,s)\,dy\,ds+D_x F(x,t)\qquad \forall(x,t)\in\overline{Q}_T\,,$$

when v is a solution to the integro-differential equation (4.8).

Conversely, if $(v, w)\in C^{2\gamma,\gamma}(\overline{Q}_T)\times C^{2\gamma,\gamma}(\overline{Q}_T)$ is a solution to system (4.14), (4.15), it is immediate to derive that $w = D_x v$.

Now we can state the following

THEOREM 4.1. – *Problem* (4.14), (4.15) *admits* (*at least*) *a solution* $(v, w)\in C^{2\gamma,\gamma}(\overline{Q}_T)\times C^{2\gamma,\gamma}(\overline{Q}_T)$ $(0<\alpha<\gamma<\frac{1}{2})$ *for* T *small enough.*

By virtue of theorem 4.1 we infer

COROLLARY 4.1. – *Problem* (4.1), ..., (4.5) *admits at least a solution* $v\in C^{2+\alpha,(2+\alpha)/2}(\overline{Q}_T)$ *for some small* T.

PROOF OF THEOREM 4.1 (Sketch). – We are going to solve the Volterra integral system (4.14), (4.15) by using Shauder's fixed-point theorem.

To this purpose we introduce the nonlinear operator $\mathcal{B}=(\mathcal{B}_0,\mathcal{B}_1)$ where the component operators $\mathcal{B}_0$ and $\mathcal{B}_1$ are defined respectively by the right-hand sides in equations (4.14), (4.15).

We observe that $\mathcal{B}$ maps $C^{2\gamma,\gamma}(\overline{Q}_T)\times C^{2\gamma,\gamma}(\overline{Q}_T)$ $(0<\alpha<\gamma<\frac{1}{2})$ into $C^{\alpha,\alpha/2}(\overline{Q}_T)\times C^{\alpha,\alpha/2}(\overline{Q}_T)$. Consequently we shall look for the fixed point of $\mathcal{B}$ in the closed ball $\mathcal{K}(M_1)$ so defined.

$$(4.16)\qquad \mathcal{K}(M_1)=\{(v,w)\in C^{2\gamma,\gamma}(\overline{Q}_T)\times C^{2\gamma,\gamma}(\overline{Q}_T):\ \|v\|_{2\gamma,\gamma}\le M_1,\ \|w\|_{2\gamma,\gamma}\le M_1\}\,,$$

M_1 being a (large enough) positive constant.

To apply Schauder's theorem we have to show that $\mathcal{B}$ maps $\mathcal{K}(M_1)$ into itself and is compact. The first property is implied by lemma 3.1 choosing T small enough, while the latter is a consequence of the following

LEMMA 4.1. – *If* $\beta\in(\delta,1]$, $C^{\beta,\beta/2}(\overline{Q}_T)$ *is compactly embedded into* $C^{\delta,\delta/2}(\overline{Q}_T)$.

And of the following estimates

$$(4.17)\qquad |D_x^j H(x,t,y,s)|\le C(T)(t-s)^{-(1+j)/2}\exp[-c(t-s)^{-1}(x-y)^2]$$
$$0<x<l,\ 0<y<l,\ 0<s<t<T,\ j=0,1\,.$$

where c is a constant in $(0, \frac{1}{4})$ and $C(T)$ is a positive constant which remains bounded as $T \to 0^+$.

In fact we can show that $\mathcal{B}$ maps $\mathcal{K}(M_1)$ into $\mathcal{K}(M_1) \cap \mathcal{K}_\beta(M_1)$, where $\mathcal{K}_\beta(M_1)$ is a bounded set in $C^{2\beta,\beta}(\bar{Q}_T)$ for any $\beta \in (\gamma, \frac{1}{2})$. ∎

5. – Proofs of Theorems 1.1 and 2.3.

PROOF OF THEOREM 1.1. – According to corollary 4.1 the transformed inverse problem (3.3), ..., (3.9) admits a solution $(\bar{v}, \bar{a}) \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T) \times \mathcal{A}$ [7] for some T small enough.

In order to show that our inverse problem (0.1), ..., (0.7) is, in turn, solvable, we shall show that the pair $(\bar{u}, \bar{a})$, where

$$\bar{u} = U(\bar{a}) , \tag{5.1}$$

is actually a solution.

To this purpose we begin by recalling that $U(\bar{a})$ is a solution to the Cauchy-Neumann problem (0.1), ..., (0.4) (with $a = \bar{a}$) and that $\bar{a}$ satisfies equation (0.7).

Hence the pair $(\bar{u}, \bar{a})$ will turn to be a solution to our inverse problem if, and only if, we show that $U(\bar{a})$ verifies also equations (0.5) and (0.6). This, in turn, is equivalent to proving that the functions $\bar{g}_1$ and $\bar{g}_2$ so defined

$$\bar{g}_1(t) = U(\bar{a})(0, t) \qquad 0 \leqslant t \leqslant T \tag{5.2}$$

$$\bar{g}_2(t) = U(\bar{a})(l, t) \qquad 0 \leqslant t \leqslant T \tag{5.3}$$

coincide respectively with g_1 and g_2.

We observe also that $\bar{g}_1$ and $\bar{g}_2$ belong to $C^{(4+\alpha)/2}([0, T])$, since $U(\bar{a})$ belongs to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ together with its gradient $DU(\bar{a})$.

In order to prove the equations $\bar{g}_j = g_j$ $(j = 1, 2)$ we need to introduce the function $V(\bar{a}) \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ so defined

$$V(\bar{a}) = L\big(U(\bar{a}), D\big) D_t U(\bar{a}) . \tag{5.4}$$

Taking advantage of the same argument used in section 3, we can prove that $\big(V(\bar{a}), \bar{a}\big)$ is a solution to the problem obtained from (3.25), (3.4), (3.6), (3.7), (3.9) by substituting the pair $(\bar{g}_1, \bar{g}_2)$ for (g_1, g_2) in the definitions of the functions $\tilde{g}_j$ $(j = 1, 2, 4, 5)$.

It is then an easy task to check that the function

$$v = \bar{v} - V(\bar{a}) \tag{5.5}$$

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(7) The space $\mathcal{A}$ is defined in step 2 in section 3.

solves the problem

$$(5.6)\qquad \int_{Q_T} v\{-D_s\varphi - D_y^2\varphi + D_y(\varphi b_1(\bar a)) - \varphi b_2(\bar a)\}\,dy\,ds = -\sum_{j=0}^{1}(-1)^j\int_0^T \varphi(jl,s)\cdot$$

$$\cdot\{d_{1+j}(s) + h_{3+j}(s) - b_1(\bar a)(jl,s)h_{1+j}(s)\}\,ds + \sum_{j=0}^{1}(-1)^j\int_0^T h_{1+j}(s)D_y\varphi(jl,s)\,ds \qquad \forall\varphi\in\Phi$$

$$(5.7)\qquad v(0,t) = h_1(t) \qquad 0\leqslant t\leqslant T$$

$$(5.8)\qquad v(x,0) = 0 \qquad 0\leqslant x\leqslant l$$

$$(5.9)\qquad v(l,t) = h_2(t) \qquad 0\leqslant t\leqslant T\,.$$

We observe that the function space Φ is defined by (3.26), while the functions $b_1(\bar a), b_2(\bar a), h_1, h_2, h_3, h_4, d_1, d_2$ are defined respectively by formulas (3.19), (3.20) and the following ones

$$(5.10)\qquad h_{1+j}(t) = D_t^2 g_{1+j}(t) - D_t^2\bar g_{1+j}(t) \qquad 0\leqslant t\leqslant T\,,\quad j=0,1$$

$$(5.11)\qquad h_{3+j}(t) = -\{f(jl,t) + \bar a(\bar g_{1+j}(t))\}[D_t^2 g_{1+j}(t) - D_t^2\bar g_{1+j}(t)] +$$
$$+\{D_t f(jl,t) + D_t[\bar a(\bar g_{1+j}(t))]\}[D_t g_{1+j}(t) - D_t\bar g_{1+j}(t)] +$$
$$+ D_t^2 g_{1+j}(t)[\bar a(g_{1+j}(t)) - \bar a(\bar g_{1+j}(t))] \qquad 0\leqslant t\leqslant T\,,\quad j=0,1$$

$$(5.12)\qquad d_{1+j}(t) = D_t g_{1+j}(t)D_t[\bar a(g_{1+j}(t)) - \bar a(\bar g_{1+j}(t))] \qquad 0\leqslant t\leqslant T\,,\quad j=0,1\,.$$

Our aim consists in showing that the function

$$(5.13)\qquad \zeta(t) = |D_t g_2(t) - D_t\bar g_2(t)| + |D_t g_1(t) - D_t\bar g_1(t)| \qquad 0\leqslant t\leqslant T$$

satisfies the following integral inequality

$$(5.14)\qquad \zeta(\tau)\leqslant C\int_0^\tau(\tau-t)^{-1/2}\zeta(t)\,dt \qquad 0<\tau\leqslant T\,,$$

where the positive constant C depends upon admissible norms of data (see assumptions (1.9), ..., (1.12)).

From (5.14) and lemma 1.1 in [7] we infer that

$$(5.15)\qquad \zeta(\tau) = 0\,,\qquad 0<\tau\leqslant T\,.$$

From (5.13) and (5.15) we immediately deduce that

$$(5.16)\qquad D_t g_1(t) = D_t\bar g_1(t)\,;\qquad D_t g_2(t) = D_t\bar g_2(t) \qquad \forall t\in[0,T]\,.$$

Our assertion $g_1 = \bar{g}_1$ and $g_2 = \bar{g}_2$ is immediately implied by the equations

$$g_1(0) - \bar{g}_1(0) = g_2(0) - \bar{g}_2(0) = 0 . \tag{5.17}$$

We postpone for the moment the proof of (5.14). We show instead that equations (5.17) hold true. This depends on the fact that $(U(a), a)$ and $(U(\bar{a}), \bar{a})$ are solutions respectively to problems (0.1), ..., (0.7) and (0.1), ..., (0.4), (5.2), (5.3), (0.7). Hence $\bar{g}_1$ and $\bar{g}_2$ satisfy the same compatibility conditions as g_1 and g_2. This implies that

$$g_1(0) - \bar{g}_1(0) = g_2(0) - \bar{g}_2(0) = D_t g_1(0) - D_t \bar{g}_1(0) = D_t g_2(0) - D_t \bar{g}_2(0) = 0 . \tag{5.18}$$

We proceed now to proving estimate (5.14) by taking advantage of a representation formula for the function v solution to equation (5.6).

To this purpose we introduce the pair of linear differential operators L and L^* so defined in $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$:

$$L = D_t - D_x^2 - b_1(\bar{a}) D_x - b_2(\bar{a}) \tag{5.19}$$

$$L^* = - D_t - D_x^2 + D_x[b_1(\bar{a}) \cdot] - b_2(\bar{a}) . \tag{5.20}$$

Consider then the solution φ to the following Cauchy problem, where $\psi \in C_0^\infty(Q_T)$:

$$\begin{cases} L^*\varphi = \psi & in\ Q_T \\ \varphi(x, T) = 0 & 0 \leqslant x \leqslant l \\ D_x\varphi(0, t) = 0 & 0 \leqslant t \leqslant T \\ D_x\varphi(l, t) = 0 & 0 \leqslant t \leqslant T . \end{cases} \tag{5.21}$$

As is well-known, φ belongs to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ (see e.g. [5, chapt. 4]) and can be so represented

$$\varphi(y, s) = \int_0^l dx \int_s^T G^*(y, s, x, t)\psi(x, t)\, dt \qquad \forall (y, s) \in \bar{Q}_T , \tag{5.22}$$

where G^* is the Green function related to problem (5.21). It can be shown that G^* satisfies the following estimates

$$|D_t^h G^*(y, s, x, t)| + |D_s^h G^*(y, s, x, t)| \leqslant C(t - s)^{-(1+2h)/2} \exp\left(-c(t-s)^{-1}(x-y)^2\right) \tag{5.23}$$

$$|(D_t + D_s) G^*(y, s, x, t)| \leqslant C(t - s)^{-1+\alpha/2} \exp\left(-c(t-s)^{-1}(x-y)^2\right) \tag{5.24}$$

$$h = 0, 1 , \quad 0 < c < \tfrac{1}{4} .$$

Substituting function φ defined by (5.22) in (5.6) gives the identity

$$(5.25)\qquad \int_{Q_T} v(x,t)\psi(x,t)\,dx\,dt = -\sum_{j=0}^{1}(-1)^j\int_0^T [d_{1+j}(s) + h_{3+j}(s) - b_1(\bar{a})(jl,s)h_{1+j}(s)]\,ds\cdot \cdot\int_0^l dx\int_s^T G^*(jl,s,x,t)\psi(x,t)\,dt \qquad \forall\psi\in C_0^\infty(Q_T)\,.$$

From (5.22) we infer the following representation for v, which turns out to be the same as in the regular case:

$$(5.26)\qquad v(x,t) = -\sum_{j=0}^{1}(-1)^j\int_0^t G^*(jl,s,x,t)[d_{1+j}(s) + h_{3+j}(s) - b_1(\bar{a})(jl,s)h_{1+j}(s)]\,ds \qquad \forall(x,t)\in Q_T\,.$$

Arguing as in theorem 3.17 in [3] it is not difficult to prove the identity

$$(5.27)\qquad G^*(y,s,x,t) = G(x,t,y,s) \qquad x,y\in[0,l]\,,\quad 0\leqslant s<t\leqslant T\,,$$

G being the Green function related to the problem

$$(5.28)\qquad \begin{cases} L\varrho = F & in\ Q_T \\ \varrho(x,0)=0 & 0\leqslant x\leqslant l \\ D_x\varrho(0,t) - b_1(\bar{a})(0,t)\varrho(0,t) = 0 & 0\leqslant t\leqslant T \\ D_x\varrho(l,t) - b_1(\bar{a})(l,t)\varrho(l,t) = 0 & 0\leqslant t\leqslant T\,. \end{cases}$$

Compute now the traces of v along the segments $x=0$ and $x=l$. From (5.7), (5.9), (5.26), and (5.27) we derive the following equations

$$(5.28)\qquad h_1(t) = -2\sum_{j=0}^{1}(-1)^j\int_0^t G(0,t,jl,s)\{d_{1+j}(s) + h_{3+j}(s) - b_j(\bar{a})(jl,s)h_{1+j}(s)\}\,ds \qquad 0\leqslant t\leqslant T$$

$$(5.29)\qquad h_2(t) = -2\sum_{j=0}^{1}(-1)^j\int_0^t G(l,t,jl,s)\{d_{1+j}(s) + h_{3+j}(s) - b_1(\bar{a})(jl,s)h_{1+j}(s)\}\,ds \qquad 0\leqslant t\leqslant T\,.$$

Taking advantage of formulas (5.10), ..., (5.13) and integrating both members of equations (5.28) and (5.29) over $[0, \tau]$ $(\tau \in (0, T])$, we obtain the following equations:

$$(5.30)\qquad D_\tau g_{1+j}(\tau) - D_\tau \bar{g}_{1+j}(\tau) = -2\sum_{i=0}^{1}(-1)^i \beta_{1+i}^{-1}\int_0^\tau dt \int_0^t G(jl, t, il, s)\cdot$$

$$\cdot\{h_{8+i}(s)[D_t g_{1+i}(s) - D_t\bar{g}_{1+i}(s)] + h_{10+i}(s)[\bar{a}(g_{1+j}(s) - \bar{a}(\bar{g}_{1+i}(s))]\}\, ds +$$

$$-2\sum_{i=0}^{1}(-1)^i\beta_{1+i}^{-1}\int_0^\tau dt\int_0^t G(jl, t, il, s)\, h_{6+i}(s)\,[D_s^2 g_{1+i}(s) - D_s^2\bar{g}_{1+i}(s)] +$$

$$+2\sum_{i=0}^{1}(-1)^i\beta_{1+i}^{-1}\int_0^\tau dt\int_0^t G(jl, t, il, s)\, b_1(\bar{a})(il, s)(D_t g_{1+i}(s))^{-1}\cdot$$

$$\cdot D_x[\bar{a}(g_{1+i}(s)) - \bar{a}(\bar{g}_{1+i}(s))]\, ds \qquad j = 0, 1\,.$$

where

$$(5.31)\qquad h_{6+j}(s) = f(jl, s) + \beta_{1+j} b_1(\bar{a})(jl, s) + \bar{a}(\bar{g}_{1+j}(s)) \qquad j = 0, 1$$

$$(5.32)\qquad h_{8+j}(s) = D_s f(jl, s) + D_s[\bar{a}(\bar{g}_{1+j}(s))] \qquad j = 0, 1\,.$$

In order to show that the function ζ satisfies the integral inequality (5.14) we need to estimate the right-hand side in (5.30). To this purpose we use the following lemma 5.1 ([8]) and equations (5.17):

LEMMA 5.1. – *Let $q, r \in C^1([0, T])$ and let $r(0) = 0$. Then the following bounds hold*

$$(5.33)\qquad \left|\int_0^t d\sigma\int_0^\sigma G(kl, \sigma, jl, s)\, q(s)\, D_s^i r(s)\, ds\right| \leqslant C\int_0^t (t-s)^{-\frac{1}{2}}|r(s)|\, ds \qquad i, j, k = 0, 1\,,$$

C being a positive constant depending upon $\|q\|_1$.

Thus we get the integral inequality

$$(5.34)\qquad |\zeta(\tau)| \leqslant C\int_0^\tau (\tau - t)^{-1/2}\Big[|\zeta(t)| + \big|\bar{a}(g_{1+j}(t)) - \bar{a}(\bar{g}_{1+j}(t))\big|\Big]\, dt \qquad \forall \tau \in (0, T]\,.$$

([8]) For a proof see e.g. [8].

Using the representation formulas

$$g_j(t) - \bar{g}_j(t) = \int_0^t [D_t g_j(s) - D_t \bar{g}_j(s)]\, ds \tag{5.35}$$

from (5.34) we immediately derive (5.14).

This concludes the proof of theorem 1.1. ■

PROOF OF THEOREM 2.3. – We proceed first to proving estimate (2.14). To this purpose, we recall that the solution $u = U(a)$ to problem (0.1), ..., (0.4) is also a solution to the following nonlinear Volterra integral equation

$$u(x, t) = \int_{Q_T} G(x, t, y, s)\, a(u(y, s))\, dy\, ds + F(x, t) \qquad \forall (x, t) \in \bar{Q}_T . \tag{5.36}$$

Here G denotes the Green function related to the heat operator $D_t - D_x^2$ and homogeneous Cauchy-Neumann conditions. It can be so represented (see e.g. [2, theorem 19.3.5])

$$G(x, t, y, s) = \frac{1}{l}\left[\theta\left(\frac{x-y}{l}, \frac{t-s}{l^2}\right) + \theta\left(\frac{x+y}{l}, \frac{t-s}{l^2}\right)\right], \tag{5.37}$$

where the functions θ and E are defined respectively by formulas (4.14) and (4.15).

From the quoted theorem 19.3.5 in [2] we infer also that F can be represented as follows

$$F(x, t) = \int_{Q_T} G(x, t, y, s) f(y, s)\, dy\, ds + \int_0^l G(x, t, y, 0) g_3(y)\, dy + \\ - \frac{2}{l}\beta_1 \int_0^t \theta\left(\frac{x}{l}, \frac{s}{l^2}\right) ds + \frac{2}{l}\beta_2 \int_0^t \theta\left(\frac{x-l}{l}, \frac{s}{l^2}\right) ds \qquad \forall (x, t) \in \bar{Q}_T . \tag{5.38}$$

Observe now that F is a solution to problem (0.1), ..., (0.4) with $a = 0$.

Since our data belong to the suitable Hölder spaces and satisfy the consistency conditions $\beta_1 = D_x g_3(0)$ and $\beta_2 = D_x g_3(l)$, we easily infer that $F \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ and the following estimate holds true

$$\|F\|_{2+\alpha,(2+\alpha)/2} \leqslant C(T)\{\|f\|_{\alpha,\alpha/2} + \|g_3\|_{2+\alpha} + 1\} , \tag{5.39}$$

where owing to theorem 19.3.5 in [2] the positive constant $C(T)$ remains bounded as $T \to 0^+$ (9).

(9) We agree that throughout this proof $C(T)$ will denote a positive function which remains bounded as $T \to 0+$.

Suppose now that a_j $(j = 1, 2)$ are two functions in $C^{1+\gamma}_{a_0}(R)$ and let $U(a_j)$ be the corresponding solutions to problem (0.1), ..., (0.4 with $a = a_j$.

Then $U(a_2) - U(a_1)$ satisfies the estimate

$$\|U(a_2) - U(a_1)\|_{1,1/2} \leqslant C(T)\|a_2 - a_1\|_0 \qquad \forall a_1, a_2 \in C^{1+\gamma}_{a_0}(R), \tag{5.40}$$

$C(T)$ being also an increasing function in $\|a_1\|_1$ and $\|a_2\|_1$.

To prove (5.40) we consider the identity

$$\begin{aligned}[U(a_2) - U(a_1)](x, t) = &\int_{Q_T} G(x, t, y, s)[a_2(U(a_2)(y, s)) - a_2(U(a_1)(y, s))]\, dy\, ds + \\ &+ \int_{Q_T} G(x, t, y, s)(a_2 - a_1)(U(a_1)(y, s))\, dy\, ds \qquad \forall (x, t) \in \bar{Q}_T.\end{aligned} \tag{5.41}$$

From definitions (5.37), (4.14) and (4.15) we infer that $G \in C^\infty(\Omega_T)$, where

$$\Omega_T = \{(x, t, y, s) \in R^4 \colon x, y \in (0, l), 0 < s < t < T\} \tag{5.42}$$

and satisfies the bound

$$\begin{aligned}|D^h_x D^j_t G(x, t, y, s)| \leqslant C(T)(t - s)^{-(1+h+2j)/2} \exp\left(-c(t - s)^{-1}(x - y)^2\right) \\ \forall (x, t, y, s) \in Q_T, \quad 0 < c < \tfrac{1}{4}, \quad 0 \leqslant h + 2j \leqslant 3.\end{aligned} \tag{5.43}$$

Taking the $C^{1,1/2}(\bar{Q}_T)$-norms of both members in (5.41) and applying lemma 5.2 reported below, we obtain the integral inequality

$$\begin{aligned}\|U(a_2) - U(a_1)\|_{C^{1,1/2}(\bar{Q}_T)} \leqslant C(T)\|a_2\|_1 \int_0^\tau (\tau - s)^{-1/2}\|U(a_2) - U(a_1)\|_{L^\infty(Q_s)}\, ds + \\ + C(T)\|a_2 - a_1\|_0 \leqslant C(T)\|a_2\|_1 \int_0^\tau (\tau - s)^{-1/2}\|U(a_2) - U(a_1)\|_{C^{1,1/2}(\bar{Q}_s)}\, ds + C(T)\|a_2 - a_1\|_0 \\ \forall a_1, a_2 \in C^{1+\gamma}_{a_0}(R).\end{aligned} \tag{5.44}$$

From lemma 1.1 in [7] we finally infer estimate (5.40).

LEMMA 5.2. – *Let $I \in C(\Omega_T)$ be a function satisfying the following estimate*

$$\begin{aligned}|D^h_x D^j_t I(x, t, y, s)| \leqslant C(T)(t - s)^{-(1+h+2j)/2} \exp\left[-c(t - s)^{-1}(x - y)^2\right] \\ \forall (x, t, y, s) \in \Omega_T, \quad 0 < c < \tfrac{1}{4}, \quad 0 \leqslant h \leqslant 1, \quad 0 \leqslant j \leqslant 1,\end{aligned} \tag{5.45}$$

where the positive constant $C(T)$ remains bounded as $T \to 0+$.

Then the linear operator $\mathfrak{I}$ *so defined*

$$\mathfrak{I}f(x,t)=\int_{Q_t} I(x,t,y,s)\,f(y,s)\,dy\,ds \qquad \forall (x,t)\in Q_T \tag{5.46}$$

maps $L^\infty(Q_T)$ *into* $C^{1,1/2}(\overline{Q}_T)$. *Moreover the following estimate holds*

$$\|\mathfrak{I}f\|_{C^{1,1/2}(Q_\tau)}\leqslant C(T)\int_0^\tau(\tau-s)^{-1/2}\|f\|_{L^\infty(Q_s)}ds \qquad \forall \tau\in(0,T]\,,\quad \forall f\in L^\infty(Q_T)\,, \tag{5.47}$$

where the positive constant $C(T)$ *remains bounded as* $T\to 0+$.

To derive an analogous estimate for $U(a_2)-U(a_1)$ in $C^{2+\alpha,(2+\alpha)/2}(\overline{Q}_T)$ we observe that $v=U(a_2)-U(a_1)$ is a solution to the problem

$$\begin{cases} D_t v - D_x^2 v = a_2(U(a_2)) - a_1(U(a_1)) & \text{in } Q_T\\ D_x v(0,t) = 0 & 0\leqslant t\leqslant T\\ v(x,0)=0 & 0\leqslant x\leqslant l\\ D_x v(l,t)=0 & 0\leqslant t\leqslant T\,. \end{cases} \tag{5.48}$$

Since $a_1=a_2=a_0$ in $[g_3(l), g_3(0)]$ (10) and $U(a_1)$ and $U(a_2)$ agree at $(0,0)$ and $(l,0)$, the function $a_2(U(a_2))-a_1(U(a_1))$ vanishes at $(0,0)$ and $(l,0)$. From classical results we infer the estimate

$$\|U(a_2)-U(a_1)\|_{2+\alpha,(2+\alpha)/2}\leqslant \leqslant C(T)\{\|a_1(U(a_2))-a_1(U(a_1))\|_{\alpha,\alpha/2}+\|(a_2-a_1)(U(a_2))\|_{\alpha,\alpha/2}\}\,. \tag{5.49}$$

In order to estimate the first norm in the right-hand side of (5.49), we take now advantage of lemma 4.2 (with $\gamma=1$ and $\varepsilon=\alpha$) in [6], which we report here as lemma 5.3 for the convenience of the reader:

LEMMA 5.3. – *Let* $u_1, u_2\in C^{1,1/2}$ *and let* $a\in C_b^{1+\alpha}(R)$. *Then the function* $a(u_2)-a(u_1)$ *belongs to* $C^{\alpha,\alpha/2}(\overline{Q}_T)$ *and satisfies the estimate*

$$\|a(u_2)-a(u_1)\|_{\alpha,\alpha/2}\leqslant C\Big\{\|D_\tau a\|_0|u_2-u_1|_{\alpha,\alpha/2}+|D_\tau a|_\alpha\|u_2-u_1\|_0\sum_{j=1}^{2}|u_j|_{1,1/2}^{\alpha}\Big\} \tag{5.50}$$ (11)

where C *is a positive constant depending only on* α.

(10) See definition (2.2).

(11) $|u|_{1,1/2}=\operatorname{Sup}\{(|x_2-x_1|^2+|t_2-t_1|)^{-1/2}|u(x_2,t_2)-u(x_1,t_1)|:(x_1,t_1),(x_2,t_2)\in\overline{Q}_T,$ $(x_1,t_1)\neq(x_2,t_2)\}$.

From (5.49), (5.50), (5.40) we easily derive the estimate

$$(5.51)\qquad \|U(a_2)-U(a_1)\|_{2+\alpha,(2+\alpha)/2}\leqslant C(T)[1+\|a_1\|_{1+\gamma}]\Big[1+\sum_{j=1}^{2}\|U(a_j)\|^{\alpha}_{1,1/2}\Big]\times$$
$$\times[\|U(a_2)-U(a_1)\|_{\alpha,\alpha/2}+\|a_2-a_1\|_{\alpha}]\qquad \forall a_1,a_2\in C^{1+\gamma}_{a_0}(R)\,.$$

From (5.51), (2.13) and the first inequality in (5.44) we immediately infer estimate (2.14).

Then we observe that the boundedness of the maps $a\to D(U(a))$ and $a\to\to D_\tau a(U(a))$ from $C^{1+\gamma}_{a_0}(R)$ respectively to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ and $C^{\alpha,\alpha/2}(\bar{Q}_T)$ can be inferred by a classical regularizing procedure (12) taking into account the fact that our data f, g_1, g_2, g_3 and a_0 satisfy the appropriate consistency conditions. In order to prove the continuity of such mappings we consider the following estimate, which can be shown by applying the quoted technique to problem (5.48):

$$(5.52)\qquad \|DU(a_2)-DU(a_1)\|_{2+\alpha,(2+\alpha)/2}\leqslant C(T)\|D[a_2(U(a_2))-a_1(U(a_1))]\|_{\alpha,\alpha/2}$$
$$\forall a_1,a_2\in C^{1+\gamma}_{a_0}(R)\,.$$

Consequently it suffices to prove that the mapping $a\to D[a(U(a))]$ is uniformly continuous from $C^{1+\gamma}_{a_0}(R)$, endowed with the metric of $C^{1+\gamma}_{b}(R)$, to $C^{\alpha,\alpha/2}(\bar{Q}_T)$. To this purpose assume that $a_1, a_2\in C^{1+\gamma}_{a_0}(R)$ and take advantage of estimates (2.14), (5.50), (5.52). After boring computations we deduce the inequality

$$(5.53)\qquad \|D[a_2(U(a_2))]-D[a_1(U(a_1))]\|_{\alpha,\alpha/2}\leqslant$$
$$\leqslant C(T)\{\|a_2-a_1\|_{1+\alpha}+|D_\tau a_1(U(a_2))-D_\tau a_1(U(a_1))|_{\alpha,\alpha/2}\}\qquad \forall a_1,a_2\in C^{1+\gamma}_{a_0}(R)\,.$$

We notice that the positive constant $C(T)$ depends also on $\|a_1\|_{1+\gamma}$ and $\|a_2\|_{1+\gamma}$ (as an increasing function) and on admissible norms of data (see assumptions (1.10), ..., (1.13)).

In order to estimate the seminorm appearing in the last hand-side of (5.53), we introduce the function $A\colon R^2\to R$ so defined

$$(5.54)\qquad A(u_1,u_2)=\begin{cases}|u_2-u_1|^{-\alpha}[D_\tau a_1(u_2)-D_\tau a_1(u_1)] & u_1\neq u_2\\ 0 & u_1=u_2\,.\end{cases}$$

Since $D_\tau a_1\in C^{\gamma}_{b}(R)$ $(\alpha<\gamma<\frac{1}{2})$, $A\in C(R^2)$ and satisfies the bound

$$(5.55)\qquad |A(u_1,u_2)|\leqslant|D_\tau a_1|_{C^{\gamma}_{b}(R)}|u_2-u_1|^{\gamma-\alpha}\leqslant\|a_1\|_{1+\gamma}|u_2-u_1|^{\gamma-\alpha}\qquad \forall u_1,u_2\in R\,.$$

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(12) For the details see e.g. [8, theorem 1.2].

Setting then $u_j = U(a_j)$ $(j = 1, 2)$ and performing long and tedious computations we obtain the following inequality, where Sup denotes the Supremum as (x_1, t_1), (x_2, t_2) run over $\bar{Q}_T$, $(x_1, t_1) \neq (x_2, t_2)$,

$$(5.56)\quad |D_\tau a_1(U(a_2)) - D_\tau a_1(U(a_1))|_{\alpha,\alpha/2} \leqslant \\ \leqslant \{|u_2|^{\alpha}_{1,1/2} \operatorname{Sup} |A(u_2(x_2, t_2), u_2(x_1, t_1)) - A(u_1(x_2, t_2), u_1(x_1, t_1))| + \\ + |u_2 - u_1|^{\alpha}_{1,1/2} \|a_1\|_{1+\gamma} |u_1|^{\alpha-\gamma}_{\alpha,\alpha/2} (l^2 + T)^{(\alpha-\delta)/2}\}\,.$$

From theorem 2.2 we infer that u_1 and u_2 satisfy the estimates

$$(5.57)\qquad \|u_j\|_{1,1/2} \leqslant M_1 \quad j = 1, 2$$

for some positive constant M_1 depending only on T, l, $\|f\|_{\alpha,\alpha/2}$, $\|g_3\|_{2+\alpha}$ and M, the latter being a positive bound for $\|a_1\|_{1+\gamma}$ and $\|a_2\|_{1+\gamma}$.

Taking advantage of the uniform continuity of the function A over $[-M_1, M_1] \times \times [-M_1, M_1]$ and using estimate (5.44), from (5.53) we easily derive that with each $\varepsilon > 0$ we can associate a $\delta > 0$ depending on ε, M, T, l and the admissible norms of data such that

$$(5.58)\quad \|a_2 - a_1\|_{1+\gamma} \leqslant \delta \text{ and } \|a_j\|_{1+\gamma} \leqslant M \ (j = 1, 2) \Rightarrow \\ \Rightarrow \|D_\tau a_1(U(a_2)) - D_\tau a_1(U(a_1))\|_{\alpha,\alpha/2} \leqslant \varepsilon\,.$$

From (5.50), (5.53) and (5.58) we easily infer the continuity of the mappings $a \to DU(a)$ and $a \to D_\tau a(U(a))$ from $C^{1+\gamma}_{a_0}(R)$ respectively to $C^{2+\alpha,(2+\alpha)/2}(\bar{Q}_T)$ and $C^{\alpha,\alpha/2}(\bar{Q}_T)$. ■

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