On Infinite-Horizon Lower Closure Results for Optimal Control (*).

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Summary. – *Recently, Carlson gave a new infinite-horizon lower closure result* [12, 14]. *Here an infinite-dimensional generalization of this result is derived by combining a new extension of Chacon's biting lemma with a known infinite-dimensional lower semicontinuity result for problems with a finite time horizon.*

0. – Introduction.

In this paper we present a new approach to lower closure problems for optimal control problems with infinite time horizon. Thus far, the most general results obtained for such problems were not nearly as strong as their counterparts for finite time horizon problems. Here, however, we shall demonstrate that the approach to lower closure problems suggested in [2, p. 588] can be brought to bear on infinite-horizon lower closure problems as well. This gives results which are of exactly the same degree of generality.

An important tool developed here is an infinite-dimensional extension of Chacon's biting lemma for a σ-finite underlying measure space. This result is obtained in section 1 of this paper (Lemma 1.7). In section 2 this result is then combined with an infinite-dimensional lower semicontinuity result for a finite underlying measure space (Theorem 2.2) so as to yield the desired infinite-dimensional lower closure result for a σ-finite underlying measure space (Theorem 2.5).

Previous infinite-horizon lower closure results are due to BAUM [9], BATES [8], the present author [1], and CARLSON [12, 14]. Of these, Carlson's result [12, Thm. 2.4.3], [14, Thm. 3.3] ([13, Thm. 2.3]) is the most general one. As explained in section 2, our Theorem 2.5 generalizes this result and also extends it to infinite dimensions and an abstract underlying σ-finite measure space.

1. – An extension of Chacon's biting lemma.

First, we shall introduce some notation. Let $(\Omega, \mathcal{A}, \nu)$ be a finite measure space and E a separable Banach space. The set of all integrable functions from Ω into E is denoted by $\mathcal{L}^1_E(\Omega) := \mathcal{L}^1_E(\Omega, \mathcal{A}, \nu)$. (Note that by separability of E strong and

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scalar measurability coincide [16]). Also, the set of all *scalarly* measurable essentially bounded functions from Ω into E' (the continuous dual of E) is denoted by $\mathfrak{L}^{\infty}_{E'}(\Omega)$.

DEFINITION 1.1. – A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathfrak{L}^1_E(\Omega)$ is said to *w^2-converge* to f_0 in $\mathfrak{L}^1_E(\Omega)$ if there exists a nonincreasing sequence $\{B_p\}_{p=1}^{\infty}$ of sets in $\mathcal{A}$ with $\lim_p \nu(B_p) = 0$, such that for every $p \in \boldsymbol{N}$

$$\lim_k \int_{\Omega \setminus B_p} \langle f_k, h \rangle \, d\nu = \int_{\Omega \setminus B_p} \langle f_0, h \rangle \, d\nu \qquad \text{for all } h \in \mathfrak{L}^{\infty}_{E'}(\Omega) .$$

Here $\langle \, , \rangle$ denotes the duality between E and E'.

DEFINITION 1.2. – A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathfrak{L}^1_E(\Omega)$ is said to *w-converge* to f_0 in $\mathfrak{L}^1_E(\Omega)$ if

$$\lim_k \int_{\Omega} \langle f_k, h \rangle \, d\nu = \int_{\Omega} \langle f_0, h \rangle \, d\nu \qquad \text{for all } h \in \mathfrak{L}^{\infty}_{E'}(\Omega) .$$

Evidently, w-convergence is the well-known convergence in the weak topology $\sigma(\mathfrak{L}^1_E(\Omega), \mathfrak{L}^{\infty}_{E'}(\Omega))$; it is stronger than w^2-convergence.

REMARK 1.3. – If $\{f_k\}_{k=1}^{\infty}$ w^2-converges (or w-converges) to f_0 in $\mathfrak{L}^1_E$ the following holds:

$$f_0(t) \in \bigcap_{n=1}^{\infty} \text{cl co} \{f_k(\omega) \colon k \geqslant n\} \text{ a.e.}$$

Here cl co stands for closed convex hull.

This remark already indicates a certain connection between w^2-convergence on the one hand and Tonelli's notion of seminormality and Cesari's notion of property (Q) on the other. For more information regarding this we refer to [5]. For a another connection, with relaxation theory and its associated weak limit concepts, the reader should consult [2, 4, 6]. (Actually, the main result of this paper can also be proven by using the main results of [4], without use of Chacon's biting lemma.)

LEMMA 1.4 (Chacon). – Suppose that $\mathcal{F}$ is a subset of $\mathfrak{L}^1_{\mathfrak{R}}(\Omega)$ such that

$$\sup_{f \in \mathcal{F}} \int_{\Omega} |f| \, d\mu < +\infty .$$

Then $\mathcal{F}$ is relatively sequentially w^2-compact, i.e. for every sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{F}$ there exists a subsequence $\{\ell\}$ of $\{k\}$ and a function $f_* \in \mathfrak{L}^1_{\mathfrak{R}}(\Omega)$ such that

$$\{f_\ell\} \ w^2\text{-converges to } f_* \text{ in } \mathfrak{L}^1_{\mathfrak{R}}(\Omega) .$$

A relatively involved proof of this result has been given in [11]. A much simpler proof, depending only on the Yosida-Hewitt decomposition theorem, Tychonov's theorem and standard measure theory, was found by THOMSEN and PLACHKY. It can be found in [18, pp. 186-188] and [7, Appendix].

THEOREM 1.5 (Dunford-Pettis theorem). – Suppose that the separable Banach space E is reflexive. For every set $\mathcal{F}$ of functions in $\mathfrak{L}^1_E(\Omega)$ the following are equivalent:
$\mathcal{F}$ is relatively sequentially w-compact in $\mathfrak{L}^1_E(\Omega)$,

$$(\|f\| : f \in \mathcal{F}\} \text{ is relatively sequentially } w\text{-compact in } \mathfrak{L}^1_{\mathfrak{R}}(\Omega)\,,$$

This result follows immediately from [16, Thm. IV.2.1] and the Eberlein-Smulian theorem. Observe (cf. [17, IV.2.3]) that the latter statement is also equivalent to

$$\{\|f\| : f \in \mathcal{F}\} \text{ is uniformly } \nu\text{-integrable}\,.$$

Let μ be a σ-finite measure on $(\Omega, \mathcal{A})$. Let $\{\Omega_j\}_{j=1}^\infty$ be a fixed nondecreasing sequence in $\mathcal{A}$ such that $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$ and $\mu(\Omega_j) < +\infty$ for all $j \in \boldsymbol{N}$. By $\mathfrak{L}^{1,\mathrm{loc}}_E(\{\Omega_j\})$ we shall denote the set of all measurable functions f from Ω into E such that

$$f|\Omega_j \text{ belongs to } \mathfrak{L}^1_E(\Omega_j) \text{ for all } j \in \boldsymbol{N}\,.$$

Here $f|\Omega_j$ denotes the restriction of f to the finite measure space $(\Omega_j, \mathcal{A} \cap \Omega_j, \mu)$.

DEFINITION 1.6. – A sequence $\{f_k\}_{k=1}^\infty$ in $\mathfrak{L}^{1,\mathrm{loc}}_E(\{\Omega_j\})$ is said to *w^2-converge* to f_0 in $\mathfrak{L}^{1,\mathrm{loc}}_E(\{\Omega_j\})$ if

$$\{f_k|\Omega_j\}_{k=1}^\infty \; w^2\text{-converges to } f_0|\Omega_j \text{ for every } j \in \boldsymbol{N}\,.$$

It is easy to see that Remark 1.3 also applies to w^2-convergence in $\mathfrak{L}^{1,\mathrm{loc}}_E(\{\Omega_j\})$. Our next result extends Lemma 1.4 to infinite dimensions:

LEMMA 1.7. – Suppose that the separable Banach space E is reflexive and that $\mathcal{F}$ is a subset of $\mathfrak{L}^{1,\mathrm{loc}}_E(\{\Omega_j\})$ such that

$$\sup_{f \in \mathcal{F}} \int_{\Omega_j} \|f\|\, d\mu < +\infty \qquad \text{for every } j \in \boldsymbol{N}\,.$$

Then $\mathcal{F}$ is relatively sequentially w^2-compact in $\mathfrak{L}^{1,\mathrm{loc}}_E(\{\Omega_j\})$.

PROOF. – Let $\{f_k\}_{k=1}^\infty$ be an arbitrary sequence in $\mathcal{F}$. By Lemma 1.4 the following is true *a fortiori*: for arbitrary $j \in \boldsymbol{N}$ there exists a nonincreasing sequence $\{B_{pj}\}_{p=1}^\infty$ in $\mathcal{A} \cap \Omega_j$, $\lim_{p\to\infty} \mu(B_{pj}) = 0$, such that for every $p \in \boldsymbol{N}$

$$(\|f_k|\Omega_j \setminus B_{pj}\|\}_{k=1}^\infty \text{ is relatively sequentially } w\text{-compact in } \mathfrak{L}^1_{\mathfrak{R}}(\Omega_j \setminus B_{pj})\,.$$

Hence, by Theorem 1.5 for every $p \in \boldsymbol{N}$

$$\{f_k|\Omega_j \setminus B_{pj}\}_{k=1}^{\infty} \text{ is relatively sequentially } w\text{-compact in } \mathfrak{L}_E^1(\Omega_j \setminus B_{pj}) .$$

By a standard diagonal extraction argument this implies that there exist a subsequence $\{\ell\}$ of $\{k\}$ and a sequence $\{f_{*pj}\}_{p=1}^{\infty}$, $f_{*pj} \in \mathfrak{L}_E^1(\Omega_j \setminus B_{pj})$, such that for every $p \in \boldsymbol{N}$

$$\{f_\ell|\Omega_j \setminus B_{pj}\}_\ell \ w\text{-converges to } f_{*pj} \text{ in } \mathfrak{L}_E^1(\Omega_j \setminus B_{pj}) . \tag{1.1}$$

Note that, as a consequence, for every $p \in \boldsymbol{N}$

$$\sup_\ell \int_{\Omega_j \setminus B_{pj}} \|f_\ell\| \, d\mu \leqslant \sup_{f \in \mathcal{F}} \int_{\Omega_j} \|f\| \, d\mu < +\infty . \tag{1.2}$$

Since $\Omega_j \setminus B_{pj}$ is contained in $\Omega_j \setminus B_{p+1,j}$ for every p, it follows elementarily from (1.1) that $f_{*pj} = f_{*p+1,j}|\Omega_j \setminus B_{pj}$ μ-a.e. on $\Omega_j \setminus B_{pj}$ for every p. Thus, setting $f_j^* := f_{*pj}$ on $\Omega_j \setminus B_{pj}$, $p \in \boldsymbol{N}$, makes sense and defines an element of $\mathfrak{L}_E^1(\Omega_j)$, since it follows from applying the monotone convergence theorem to (1.2) that

$$\int_{\Omega_j} \|f_j^*\| \, d\mu \leqslant \sup_{f \in \mathcal{F}} \int_{\Omega_j} \|f\| \, d\mu . \tag{1.3}$$

Observe that (1.1) can now be reformulated as

$$\{f_\ell|\Omega_j\} \ w^2\text{-converges to } f_j^* .$$

In all of this j was an arbitrary element of $\boldsymbol{N}$. Since Ω_{j+1} contains Ω_j for every $j \in \boldsymbol{N}$, it follows simply from the definition of w^2-convergence that $f_j^* = f_{j+1}^*|\Omega_j$ μ-a.e. on Ω_j for all j. Hence, setting $f_* := f_j^*$ on Ω_j, $j \in \boldsymbol{N}$, unambiguously defines an element f_* of $\mathfrak{L}_E^{1,\mathrm{loc}}(\{\Omega_j\})$, in view of (1.3). ∎

A similar proof is well-known for w-convergence in $\mathfrak{L}_E^{1,\mathrm{loc}}(\{\Omega_j\})$; see for instance [1, Appendix A].

2. – Main result.

In this section we obtain a very general inite-dimensional infinite-horizon lower closure result by combining Lemma 1.7, the infinite-dimensional infinite-horizon extension of Chacon's biting lemma, with a well-known infinite-dimensionalfinite-horizon lower semicontinuity result, which will now be stated. Let $(\Omega, \mathcal{A}, \nu)$ be a finite measure space and X a metric space (metric d). The set of all Borel measurable functions from Ω into X is denoted by $\mathfrak{L}_X^0(\Omega)$. (We note already that this definition can also be given for a σ-finite measure space, with no alterations needed in the formulation.)

DEFINITION 2.1. – A sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathfrak{L}_X^0(\Omega)$ is said to *converge in measure* to x_0 if

$$\lim_k \nu\Big(\big\{\omega \in \Omega\colon d(x_k(\omega), x_0(\omega)) > \varepsilon\big\}\Big) = 0 \quad \text{for every } \varepsilon > 0 .$$

THEOREM 2.2. – Let X be a metric space, and V a separable reflexive Banach space. Suppose that the sequences $\{x_k\}_{k=0}^{\infty} \in \mathfrak{L}_X^0(\Omega)$ and $\{v_k\}_{k=0}^{\infty} \subset \mathfrak{L}_V^1(\Omega)$ are such that

$$\{x_k\}_{k=1}^{\infty} \text{ converges in measure to } x_0 \text{ in } \mathfrak{L}_X^0(\Omega) ,$$
$$\{v_k\}_{k=1}^{\infty} \; w^2\text{-converges to } v_0 \text{ in } \mathfrak{L}_V^1(\Omega) .$$

Suppose also that the function $l\colon \Omega\times X\times V \to (-\infty, +\infty]$ is such that for ν-a.e. ω in Ω:

(2.1) $\quad l(\omega, \cdot, \cdot)$ is sequentially l.s.c. at every point of $\{x_0(\omega)\}\times V$,

(2.2) $\quad l(\omega, x_0(\omega), \cdot)$ is convex on V ,

and is also such that for some uniformly ν-integrable sequence $\{\lambda_k\} \subset \mathfrak{L}_{\mathfrak{R}}^1(\Omega)$:

$$l(\cdot, x_k, v_k) \geqslant \lambda_k \quad \text{for all } k \in \boldsymbol{N} .$$

Then

$$\liminf_k \tilde{\int_0} l(\cdot, x_k, v_k)\, d\nu \geqslant \tilde{\int_0} l(\cdot, x_0, v_0)\, d\nu ,$$

where $\tilde{\int_0}$ indicates outer integration (cf. [2, 3]).

The above result is of a well-known type (at least if w^2-convergence is replaced by w-convergence); see for instance [15] for a large number of references. The infinite-dimensional version given here (again with w- instead of w^2-convergence) can be found in [10] for Ω locally compact with ordinary integration, and in [4, Thm. 3.1] and [5, Cor. 4.11] for general abstract Ω and outer integration (the latter results go further than Theorem 2.2). Moreover, it is evident from the definition of w^2-convergence that Theorem 2.2 continues to hold for that type of convergence. Actually, a completely similar reasoning lies behind the proofs of lower closure results in [5] (see also the remark made in [2, p. 588]). There it is already made evident that, from a technical point of view, the only difference between lower closure and lower semicontinuity results lies in the occurrence of relative w^2-compactness, in addition to w-convergence.

It is possible to rephrase Theorem 2.2 in terms of orientor fields: Let $Q\colon \Omega \times X \rightrightarrows V$ be a multifunction (the «orientor field»).

DEFINITION 2.3. – The orientor field Q is said to have *property* (K) in the variable x at the point $(\omega, x^0) \in \Omega \times X$ if

$$Q(\omega, x^0) = \bigcap_{\delta > 0} \text{seq-cl} \cup \{Q(\omega, x)\colon d(x, x^0) < \delta\}\,.$$

Here seq-cl stands for the weak sequential closure in V.

In contrast to what is usually done in the literature we take the domain of Q to be all of $\Omega \times X$. In [2] it was shown that by allowing Q to take empty values on $\Omega \times X$ this simplification can be made without any loss of generality.

COROLLARY 2.4. – Suppose that the sequences $\{x_k\}_{k=0}^{\infty} \subset \mathfrak{L}_X^0(\Omega)$ and $\{v_k\}_{k=0}^{\infty} \subset \mathfrak{L}_V^1(\Omega)$ are such that

$$\{x_k\}_{k=1}^{\infty} \text{ converges in measure to } x_0\,,$$
$$\{v_k\}_{k=1}^{\infty}\ w^2\text{-converges to } v_0 \text{ in } \mathfrak{L}_V^1(\Omega)\,.$$

Suppose also that $Q\colon \Omega \times X \rightrightarrows V$ is such that for ν-a.e. $\omega \in \Omega$

$$Q \text{ has property } (K) \text{ in the variable } x \text{ at every point } (\omega, x_0(\omega))\,,$$
$$Q(\omega, x_0(\omega)) \text{ is a convex set}\,,$$
$$v_k(\omega) \in Q(\omega, x_k(\omega)) \text{ for all } k \in \boldsymbol{N}\,.$$

Then $v_0(\omega) \in Q(\omega, x_0(\omega))$ for ν-a.e. $\omega \in \Omega$.

PROOF. – Define the function $l_Q\colon \Omega \times X \times V \to \{0, +\infty\}$ by $l_Q(\omega, x, v) := 0$ if $v \in Q(\omega, x)$, $l_Q(\omega, x, v) =: +\infty$ otherwise. It is well-known that the conditions of Theorem 2.2 are precisely met for l_Q by those of the present corollary; e.g., see [2]. The result then follows immediately from applying Theorem 2.2. ■

We are now ready to state and prove the main result of this note. As in section 1, let μ be a σ-finite measure on $(\Omega, \mathcal{A})$, and let $\{\Omega_j\}_{j=1}^{\infty}$ be a fixed nondecreasing sequence in $\mathcal{A}$ such that $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$ and $\mu(\Omega_j) < +\infty$ for all $j \in \boldsymbol{N}$.

THEOREM 2.5. – Let X be a metric space and V and W separable reflexive Banach spaces. Suppose that the sequences $\{x_k\}_{k=0}^{\infty} \subset \mathfrak{L}_X^0(\Omega)$, $\{v_k\}_{k=0}^{\infty} \subset \mathfrak{L}_V^{1,\mathrm{loc}}(\{\Omega_j\})$ and $\{w_k\} \subset \mathfrak{L}_W^{1,\mathrm{loc}}(\{\Omega_j\})$ are such that for every $j \in \boldsymbol{N}$

$$\{x_k|\Omega_j\}_{k=1}^{\infty} \text{ converges in measure to } x_0|\Omega_j \text{ in } \mathfrak{L}_X^0(\Omega_j)\,,$$

$$\{v_k|\Omega_j\}_{k=1}^{\infty}\ w\text{-converges to } v_0|\Omega_j\,, \tag{2.3}$$

$$\sup_k \int_{\Omega_j} \|w_k\|\, d\mu < +\infty\,. \tag{2.4}$$

Then there exist a subsequence $\{k'\}$ of $\{k\}$ and a function $w_* \in \mathcal{L}_W^{1,\mathrm{loc}}(\{\Omega_j\})$ such that for every function $l\colon \Omega \times X \times V \times W \to (-\infty, +\infty]$ the following inequality holds for every $j \in \boldsymbol{N}$:

$$\liminf_{k'} \tilde{\int_{\Omega_j}} l(\cdot, x_{k'}, v_{k'}, w_{k'})\, d\mu \geqslant \tilde{\int_{\Omega_j}} l(\cdot, x_0, v_0, w_*)\, d\mu ,$$

provided that l satisfies (2.1)-(2.2) and is also such that for every $j \in \boldsymbol{N}$

$$l(\cdot, x_k, v_k, w_k) \geqslant \lambda_{kj} \quad \text{on } \Omega_j \text{ for all } k \in \boldsymbol{N}.$$

for some uniformly μ-integrable sequence $\{\lambda_{kj}\}_{k=1}^{\infty} \subset \mathcal{L}_{\mathfrak{R}}^1(\Omega_j)$.

PROOF. – From (2.4) it follows by Lemma 2.4, the extension of Chacon's biting lemma, that there exist a subsequence $\{k'\}$ of $\{k\}$ and a function w_* in $\mathcal{L}_W^{1,\mathrm{loc}}(\{\Omega_j\})$ such that

$$\{w_{k'}\} \ w^2\text{-converges to } w_* \text{ in } \mathcal{L}_W^{1,\mathrm{loc}}(\{\Omega_j\}) .$$

By (2.3) it follows elementarily that

$$\{(v_{k'}, w_{k'})\} \ w^2\text{-converges to } (v_0, w_*) \text{ in } \mathcal{L}_{V \times W}^{1,\mathrm{loc}}(\{\Omega_j\}) ,$$

so the proof is finished by invoking Theorem 2.2 for each j separately. ■

With Corollary 2.4 in mind it is now easy to see that Theorem 2.5 generalizes the lower closure result of [12, Thm. 2.4.3], [14, Thm. 3.3] in a number of ways: There Ω is the positive real axis, equipped with the Lebesgue measure, X is a complete separable metric space, V is finite-dimensional, and W is 1-dimensional. Also, only one function l, viz. $l(\omega, x, v, w) = w$, is considered. The method of proof in [12, 14] rests on the approach presented in [1], and uses in addition Helly's selection theorem, so that any extensions of the kind reached here lie definitely outside the scope of [12, 14].

Note added in proof: Professor Michel Valadier has kindly pointed out to the author that the proof of Chacon's biting lemma as given by Thomsen and Plachky [11] is incomplete. It is still an open problem how their proof should be reconstructed.

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