On Measures of Weak Noncompactness (*).

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Summary. – *In this paper an axiomatic approach to the notion of a measure of weak noncompactness is presented. Several properties of the defined measures are given. Moreover, we provide a few concrete realizations of the accepeted axiomatic system in some Banach spaces.*

1. – Introduction.

The notion of a measure of weak noncompactness was defined by DE BLASI in 1977 [6] (see also below). In contrast to the notion of a measure of noncompactness in strong sense (cf. [1, 4, 9, 10, 14]) it was rather seldom applied (see [3, 8, 12, 13]). This situation is caused by the fact that convenient criteria of weak compactness are rather unknown except for some few cases (compare [7, 11]). Therefore it is very difficult to construct some formulas allowing us to express De Blasi measure in a convenient form for applications.

In this paper we propose an axiomatic approach to the notion of measures of weak noncompactness which seems to solve the above mentioned problem in a positive sense. Roughly speaking, a measure of weak noncompactness (in our sense) is some function defined on the family of all nonempty and bounded subsets of a Banach space which vanishes on a family of some relatively weakly compact sets (not necessarily on all). This permits us to construct measures of weak noncompactness in several Banach spaces. Moreover, some nontrivial realizations of our axiomatics in reflexive spaces may be also given, while the classical measure due to De Blasi vanishes identically in this case. Actually, our definition will be illustrated by some examples.

Finally, let us mention that our approach to the notion of measures of weak noncompactness is very similar to an approach associated with the notion of measures of noncompactness in strong sense (cf. [1, 14]). This caused that many properties of these measures are similar. But, on the other hand, the theory of measures of

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noncompactness seem to be much more difficult and at the same time, much more interesting than that concerning measures of strong noncompactness.

2. – Notation and preliminaries.

Let $(E, \|\cdot\|)$ be a given Banach space with the zero element θ. Throughout this paper we will use the standard notation close to that from [1, 8]. For example, the open (closed) ball centered at x with radius r will be denoted by $K(x, r)$ $(\overline{K}(x, r))$. The symbol B will stand for the ball $K(\theta, 1)$. For a set $X \subset E$ we denote by $\overline{X}$, $\operatorname{diam} X$, $\operatorname{conv} X$, $\overline{\operatorname{conv}}\, X$ the closure, the diameter, the convex hull and the closed convex hull of X, respectively. The norm of a bounded nonempty subset X of E is the number $\|X\| = \sup\,[\|x\|: x \in X]$. The symbol $\overline{X}^w$ stands for the weak closure of a set X. For an arbitrary set X we denote by $K(X, r)$ the ball centered at X and of radius r

$$K(X, r) = \bigcup_{x \in X} K(x, r)\,.$$

In what follows denote by $\mathcal{M}_E$ (shortly $\mathcal{M}$) the family of all bounded subsets of E. For $X, Y \in \mathcal{M}_E$ put

$$d(X, Y) = \inf\,[r: X \subset K(Y, r)]\,,$$

$$D(X, Y) = \max\,[d(X, Y), d(Y, X)]\,.$$

The number $D(X, Y)$ is called Hausdorff distance between X and Y.

Analogously, denote by $\mathcal{N}_E$ the family of all nonempty and relatively compact subsets of E and by $\mathfrak{W}_E$ the family of all nonempty and relatively weakly compact subsets of E (shortly: $\mathcal{N}$, $\mathfrak{W}$). Obviously $\mathcal{N} \subset \mathfrak{W} \subset \mathcal{M}$ and $\mathfrak{W} = \mathcal{M}$ if and only if E is a reflexive space.

If $\mathfrak{Z}$ is a nonempty subfamily of $\mathcal{M}$ then by $\mathfrak{Z}^c$, $\mathfrak{Z}^{wc}$ we will denote its subfamilies consisting of all closed and weakly closed subsets of $\mathfrak{Z}$, respectively.

Let us mention that $\mathcal{M}^c$ forms a complete metric space with respect to the Hausdorff distance D, while $\mathcal{N}^c$, $\mathfrak{W}^{wc}$ are closed subspaces of $\mathcal{M}^c$ with respect to the topology generated by the Hausdorff distance.

Finally, for $\mathfrak{Z} \subset \mathcal{M}$ let us denote

$$D(X, \mathfrak{Z}) = \inf\,[D(X, Y): Y \in \mathfrak{Z}]\,.$$

In the sequel we accept the following definition

DEFINITION 1. – *A function $\gamma: \mathcal{M} \to \langle 0, +\infty)$ is said to be a measure of weak noncompactness if it is subject to the following conditions:*

1) *The family* $\ker \gamma = [X \in \mathcal{M}: \gamma(X) = 0]$ *is nonempty and* $\ker \gamma \subset \mathfrak{W}$,

2) $X \subset Y \Rightarrow \gamma(X) \leqslant \gamma(Y)$,

3) $\gamma(\overline{\text{conv}}\, X) = \gamma(X)$,

4) $\gamma(\lambda X + (1-\lambda) Y) \leqslant \lambda\gamma(X) + (1-\lambda)\gamma(Y)$, *for* $\lambda \in \langle 0, 1 \rangle$,

5) *if* $X_n \in \mathcal{M}^{wc}$, $X_{n+1} \subset X_n$ *for* $n = 1, 2, \ldots$ *and if* $\lim_{n\to\infty} \gamma(X_n) = 0$, *then* $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker \gamma$ described in 1) will be called the kernel of the measure γ.

Notice that the measure γ has the following property

6) $\gamma(\overline{X}^w) = \gamma(X)$.

Indeed, the relation

$$X \subset \overline{X}^w \subset \overline{\text{conv}}\, X$$

and 2), 3) imply 6).

Moreover, let us note that the kernel $\ker \gamma$ forms a subfamily of $\mathcal{M}$ being closed with respect to taking closure and weak closure of sets. Further, the axiom 4) implies that this family is convex. Summing up, it may be shown that $(\ker \gamma)^{wc}$ forms a closed subspace of $\mathcal{M}^{wc}$ with respect to the Hausdorff distance.

Let us mention yet that the set X_∞ described in 5) must belong to $\ker \gamma$ what can be easily infer from the relation $X_\infty \subset X_n$ for $n = 1, 2, \ldots$.

Now we indicate some important properties of a measure of weak noncompactness.

THEOREM 1. – *Each measure of noncompactness is locally Lipschitzian (hence continuous) with respect to the Hausdorff distance.*

THEOREM 2. – *Let* $t_1, t_2, \ldots, t_n$ *be nonnegative reals such that* $\sum_{i=1}^{n} t_i \leqslant 1$ *and let* $\{x_0\} \in \ker \gamma$. *Then*

$$\gamma\Big(x_0 + \sum_{i=1}^{n} t_i X_i\Big) \leqslant \sum_{i=1}^{n} t_i \gamma(x_0 + X_i)\,.$$

The proofs of these theorems are exactly the same as the proofs of analogous properties for a measure of strong noncompactness and is therefore omitted (cf. [1, 2]).

Also the proof of the below given theorem may be patterned on the proof of suitable theorem from [2].

THEOREM 3. – *If* $\|X\| \leqslant 1$, *then*

$$\gamma(X + Y) \leqslant \gamma(Y) + \|X\|\gamma(K(Y, 1))\,.$$

In what follows we define a class of measures of weak noncompactness having additional, good properties.

A measure γ will be referred to as a measure with the maximum property provided

7) $\gamma(X \cup Y) = \max[\gamma(X), \gamma(Y)]$.

The measure γ such that for any $X \in \mathcal{M}$ and $\lambda \in R$

8) $\gamma(\lambda X) = |\lambda| \gamma(X)$

is said to be homogeneous, and if it satisfies

9) $\gamma(X + Y) \leqslant \gamma(X) + \gamma(Y)$

it is called subadditive. It is called sublinear if 8) and 9) hold.

DEFINITION 2. – *The measure γ will be called regular if it is sublinear, has maximum property and* $\ker \gamma = \mathfrak{W}$.

Now let us note that the measure of weak noncompactness defined by DE BLASI [6] in the following way:

$$\omega(X) = \inf[t > 0 \colon \text{there exists } C \in \mathfrak{W} \text{ such that } X \subset C + tB],$$

is an example of regular measure.

Actually, this measure may be expressed in the following concise form

$$\omega(X) = D(X, \mathfrak{W}) . \tag{2.1}$$

Furthermore, it can be shown that $\omega(B) = 1$ $\big($generally: $\omega(K(x_0, r)) = r\big)$ in the case when E is nonreflexive [6] and $\omega(B) = 0$ (even $\omega(X) = 0$ for every $X \in \mathcal{M}$) in the case when E is reflexive. This last assertion follows from (2.1), for instance.

Now let us note that each regular measure of weak noncompactness is comparable with De Blasi measure ω. Namely, we have

THEOREM 4. – *If γ is a regular measure, then*

$$\gamma(X) \leqslant \gamma(B)\omega(X) .$$

PROOF. – The case of reflexive space is obvious so let us assume that E is nonreflexive.

Denote $r = \omega(X)$. Let us take an arbitrary $\varepsilon > 0$. Then, in view of (2.1) there exists a set $Y \in \mathfrak{W}$ such that $X \subset K(Y, r + \varepsilon)$. Hence and in virtue of the obvious relation

$$K(Y, r + \varepsilon) = Y + (r + \varepsilon)B ,$$

we obtain

$$\gamma(X) = \gamma(Y + (r + \varepsilon)B) \leqslant (r + \varepsilon)\gamma(B) .$$

The arbitrariness of ε completes the proof.

The following simple theorem will be useful in the sequel.

THEOREM 5. – *Let* $\mu\colon \mathcal{M} \to \langle 0, +\infty)$ *be a function satisfying the axioms* 1), 2), 7) *and such that* $\mu(\{x\}) = 0$ *for any* $x \in E$. *Then* μ *satisfies the property* 5).

PROOF. – Let $(X_n)_{n\in N}$ be a sequence of sets from $\mathcal{M}$ such that $\bar{X}_n^w = X_n$, $X_n \supset X_{n+1}$, $n = 1, 2, \dots$, and $\lim_{n\to\infty} \mu(X_n) = 0$.

Further, take an arbitrary sequence of points $(x_n)_{n\in N}$ such that $x_n \in X_n$ for $n = 1, 2, \dots$. Then we have

$$\mu(\{x_1, x_2, x_3, \dots\}) = \mu(\{x_1, x_2, \dots, x_{n-1}\} \cup \{x_n, x_{n+1}, \dots\}) =$$
$$= \max[\mu(\{x_1, x_2, \dots, x_{n-1}\}), \mu(\{x_n, x_{n+1}, \dots\})] = \mu(\{x_n, x_{n+1}, \dots\}) \leqslant \mu(X_n) ,$$

what in view of made assumptions implies that

$$\mu(\{x_1, x_2, \dots\}) = 0 .$$

Hence, by virtue of 1) the set $\{x_1, x_2, \dots\}$ is relatively weakly compact so that it has at least one weak cluster point x. Because $\{x_n, x_{n+1}, \dots\} \subset X_n$ and $\bar{X}_n^w = X_n$, thus $x \in X_n$ for any $n = 1, 2, \dots$: Hence $x \in X_\infty = \bigcap_{n=1}^{\infty} X_n$ and the proof is complete.

3. – Measures of weak noncompactness in L^1 space.

Consider the space $L^1 = L^1(a, b)$ consisting of all functions $x\colon (a, b) \to R$ which are measurable and Lebesgue integrable on the interval (a, b). The space L^1 will be equipped with the usual norm

$$\|x\| = \int_a^b |x(t)|\, dt .$$

It is well known that L^1 is nonreflexive [7]. But on the other hand in this space the following convenient criterion of weak compactness is known [7]:

THEOREM 6. – *A set* $X \in \mathcal{M}_{L^1}$ *is relatively weakly compact if and only if*

$$\lim_{\substack{m(E)\to 0\\ E\subset(a,b)}} \int_E x(t)\, dt = 0$$

uniformly with respect to $x \in X$.

Actually the symbol $m(E)$ stands for the Lebesgue measure of E in R.

Let us notice that the above theorem may be rewritten in the following equivalent form:

THEOREM 7. – *A set $X \in \mathcal{M}_{L^1}$ is relatively weakly compact iff*

$$\lim_{\varepsilon \to 0} \left\{ \sup \left[\int_c^d |x(t)|\, dt \colon a \leqslant c \leqslant d \leqslant b,\ d - c \leqslant \varepsilon \right] \right\} = 0$$

uniformly with respect to $x \in X$.

Now, for an arbitrary $X \in \mathcal{M}_{L^1}$ let us define

$$\gamma(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_c^d |x(t)| \colon a \leqslant c \leqslant d \leqslant b,\ d - c \leqslant \varepsilon \right] \right\} \right\}. \tag{3.1}$$

Note that, according to Theorem 7 we have

$$\gamma(X) = 0 \Leftrightarrow \bar{X}^w \quad \text{is weakly compact in } L^1 .$$

Hence $\gamma(\{x\}) = 0$ for any $x \in L^1$.

Next, let us notice that γ satisfies the properties 2), 4), 7), 8), 9) listed in the previous section. Thus, keeping in mind Theorem 5 we infer that γ satisfies also the axiom 5).

Further, let us mention the following simple relation

$$\gamma(\operatorname{conv} X) = \gamma(X)\,, \tag{3.2}$$

for each $X \in \mathcal{M}_{L^1}$: Moreover, the axiom 2) follows

$$\gamma(X) \leqslant \gamma(\bar{X}) \tag{3.3}$$

for any $X \in \mathcal{M}_{L^1}$. In order to prove the reverse inequality let us take $x \in \bar{X}$. Then there exists a sequence $(x_n)_{n \in N} \subset X$ such that $\lim_{n \to \infty} x_n = x$. Fixing an arbitrary $\varepsilon > 0$ and $c, d \in (a, b)$ such that $c \leqslant d$ and $d - c \leqslant \varepsilon$, we have

$$\int_c^d |x(t)|\, dt \leqslant \int_c^d |x(t) - x_n(t)|\, dt + \int_c^d |x_n(t)|\, dt \leqslant$$
$$\leqslant \int_a^b |x_n(t) - x(t)|\, dt + \int_c^d |x_n(t)|\, dt \leqslant \|x_n - x\| + \int_c^d |x_n(t)|\, dt\,,$$

and consequently

$$\sup\left[\int_c^d |x(t)|\,dt\colon a\leqslant c\leqslant d\leqslant b,\ d-c\leqslant\varepsilon\right]\leqslant\|x-x_n\|+$$
$$+\sup\left[\int_c^d |x_n(t)|\,dt\colon a\leqslant c\leqslant d\leqslant b,\ d-c\leqslant\varepsilon\right].$$

Thus, taking into account that the number $\|x_n-x\|$ is arbitrarily small, we obtain

$$\gamma(\bar{X})\leqslant\gamma(X)\,. \tag{3.4}$$

Now, combining (3.2), (3.3) and (3.4) we infer that the function γ possesses also the property 3).

Finally we can formulate the following theorem

THEOREM 8. – *The function $\gamma(X)$, defined by the formula* (3.1) *is a regular measure of noncompactness in the space $L^1(a, b)$ such that $\gamma(X)\leqslant\omega(X)$ for any X.*

The last assertion from the above theorem follows easily form Theorem 4 and from the relation

$$\gamma(B)=1\,,$$

which can be easily verified.

It will be interesting to seek if the converse inequality, i.e. the inequality $\omega(X)\leqslant\leqslant\gamma(X)$ is true. Unfortunately we are not able to recognize its validity.

Now, we are going to provide another example of a measure of weak noncompactness in the space L^1. At the beginning let us assume that the function $\beta\colon\langle 0,+\infty)\to\langle 0,+\infty)$ is such that $\beta(0)=\lim_{\varepsilon\to 0}\beta(\varepsilon)=0$, is given.

For arbitrary $x\in L^1$ let us define

$$\delta(X,\beta,\varepsilon)=\sup\left[\int_c^d |x(t)|\,dt-\beta(d-c)\colon a\leqslant c\leqslant d\leqslant b\leqslant,\ d-c\leqslant\varepsilon\right].$$

If $X\in\mathcal{M}_{L^1}$, then we define

$$\delta(X,\beta,\varepsilon)=\sup\left[\delta(x,\beta,\varepsilon)\colon x\in X\right]. \tag{3.5}$$

The function $\delta(x,\beta,\varepsilon)$ will be called the integral modulus of continuity of the function x with respect to β.

It is easy to check that the function $\delta(X,\beta,\varepsilon)\colon\mathcal{M}\to\langle 0,+\infty)$ is a measure of weak noncompactness in the space L^1, which has the maximum property and is nonsublinear. Moreover, we have the following equality

$$\lim_{\varepsilon\to 0} \delta(X, \beta, \varepsilon) = \gamma(X)\,, \tag{3.6}$$

where γ is defined by the formula (3.1). In order to prove it let us notice that the inequality

$$\lim_{\varepsilon\to 0} \delta(X, \beta, \varepsilon) \leqslant \gamma(X) \tag{3.7}$$

is trivial and follows inmediately from (3.1) and (3.5). In order to prove the reverse inequality let us denote $\sigma = \lim_{\varepsilon\to 0} \delta(X, \beta, \varepsilon)$. Then, for an arbitrary $\eta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\delta(X, \beta, \varepsilon) \leqslant \sigma + \eta$$

for any $\varepsilon \leqslant \varepsilon_0$. Hence

$$\delta(x, \beta, \varepsilon) \leqslant \sigma + \eta$$

for any $x \in X$ and $\varepsilon \leqslant \varepsilon_0$, and consequently

$$\int_c^d |x(t)|\, dt \leqslant \sigma + \eta + \beta(\varepsilon)$$

for $x \in X$, $\varepsilon \leqslant \varepsilon_0$ and for any $c, d \in (a, b)$ such that $c < d$ and $d - c \leqslant \varepsilon$. The last inequality and (3.7) gives the desired equality (3.6).

Recapitulating the above assertions we have the following

THEOREM 9. – *The function* $\delta(X, \beta, \varepsilon)\colon \mathcal{M}_{L^1} \to \langle 0, +\infty)$ *is a measure of weak noncompactness in the space* L^1, *having the maximum property and being nonsublinear. The kernel* $\ker \delta(X, \beta, \varepsilon)$ *of this measure consists of all* $X \in \mathcal{M}_{L^1}$ *such that*

$$\int_c^d |x(t)|\, dt \leqslant \beta(\varepsilon)$$

for any $x \in X$ *and for all* $c, d \in (a, b)$, $c \leqslant d$ *and such that* $d - c \leqslant \varepsilon$. *Moreover, this measure satisfies the relation* (3.6).

4. – Other examples.

In this section we give some scheme allowing to construct nontrivial measures of weak noncompactness in the case of a reflexive Banach space. Actually those measures have to be irregular because all regular measures in reflexive spaces vanish identically.

Then, let us assume that E is a given reflexive Banach space and let F be another Banach space. Further we assume that μ is a measure of noncompactness (in strong sense; cf. [1]) defined in the space F.

Finally assume that $T: E \to F$ is a linear continuous operator. Thus, of course, $T: \mathcal{N}_E \to \mathcal{N}_F$.

In order to construct some nontrivial examples we will additionally assume that there exist at least one set $A \in \mathcal{M}_E$ such that $T(A) \notin \ker \mu$ and at least one set $B \in \mathcal{M}_E - \mathcal{N}_E$ such that $T(B) \in \ker \mu$.

Then we have the following

THEOREM 10. – *The function* $\gamma: \mathcal{M}_E \to \langle 0, +\infty)$, *defined by the formula*

$$\gamma(X) = \mu(T(X))$$

is a measure of weak noncompactness in the space E such that $\ker \gamma \neq \mathcal{M}_E = \mathcal{W}_E$ *and* $\ker \gamma \neq \mathcal{N}_E$.

PROOF. – We provide only a sketch of the proof.

First notice that the axioms 1), 2), 4) from Definition 1 are obvious. In order to prove the axiom 3) let us observe that in view of the linearity of T and the properties of μ we get

$$\gamma(\operatorname{conv} X) = \gamma(X) . \tag{4.1}$$

Moreover, we obtain

$$\gamma(\bar{X}) = \mu(T(\bar{X})(\leqslant \mu(\overline{T(X)}) = \mu(T(X)) = \gamma(X)$$

so that by 2) we have

$$\gamma(\bar{X}) = \gamma(X) . \tag{4.2}$$

Combining (4.1) and (4.2) we see that the axiom 3) is satisfied.

Finally observe that the property 5) is a simple consequence of the reflexivity of the space E.

Thus, the proof is complete.

Now we give two examples of measures of weak noncompactness realized according to Theorem 10.

EXAMPLE 1. – Let us take the Hilbert space l^2 with a simple measure of strong noncompactness, $\operatorname{diam} X$. Let: $l^2 \to l^2$ denote the projection operator i.e.

$$T(x) = T(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) ,$$

where n is fixed.

For $X \subset l^2$ let us define

$$\gamma(X) = \operatorname{diam}\left(T(X)\right).$$

It is easy to check that all conditions associated with Theorem 10 are satisfied. Then γ is a measure of weak noncompactness in the space l^2. Its kernel consists of all sets $y + X \in \mathcal{M}_{l^2}$ such that the first n components of every $x \in X$ are equal to zero and $y \in l^2$ is arbitrarily taken.

EXAMPLE 2. – Now, let l^2 be the same space as previously. Moreover, consider the space l^∞ of all bounded sequences, furnished with the norm

$$\|x\| = \|(x_1, x_2, \dots)\| = \sup\left[|x_n| : n = 1, 2, \dots\right].$$

Assume that a measure of strong noncompactness is given in l^∞

$$\mu(X) = \lim_{n\to\infty}\left\{\sup_{x\in X}\left\{\sup\left[|x_k| : k \geqslant n\right]\right\}\right\}$$

(cf. [1]). Finally, let us consider the operator $T: l^2 \to l^\infty$ defined in the following way

$$Tx = T(x_1, x_2, \dots) = (y_1, y_2, \dots),$$

where $y_1 = x_1$, $y_2 = x_2/\sqrt{2} + x_3/\sqrt{2}$, and generally,

$$y_n = x_n/\sqrt{2^{n-1}} + \sum_{k=1}^{n-1} x_{n+k}/\sqrt{2^k},$$

for $n = 2, 3, \dots$.

Obviously our operator maps l^2 into l^∞ and is linear. Moreover, it is easy to check that T is bounded and

$$\|T\| \leqslant \sqrt{2}/(\sqrt{2} - 1).$$

Thus T maps each relatively compact set in l^2 into a relatively compact set in l^∞. Moreover, it is easy to verify that $T(X) \in \ker \mu$ for any $X \in \mathcal{N}_{l^2}$.

On the other hand, $\ker \gamma \supsetneq \mathcal{N}_{l^2}$. Indeed, let us take the set X containing all vectors of orthonormal bases in l^2, i.e. $X = \{e_1, e_2, \dots\}$, where $e_k = (0, 0, \dots, 0, 1, 0, \dots)$. Then $T(X) = \{T(e_1), T(e_2), \dots\}$, where $T(e_k) = (0, 0, \dots, 1/\sqrt{2^{k-1}}, 0, \dots)$. Thus $T(X) \in \ker \mu$ what means that $\gamma(X) = 0$. But we have $X \notin \mathcal{N}_{l^2}$.

Moreover, we show that $\ker \gamma \subsetneq \mathcal{N}_{l^2}$ what means that γ is a nontrivial measure in the space l^2. In fact, let us take the set $X = \{x_1, x_2, \dots\}$, where

$$x_1 = (1, 0, 0, \dots)$$
$$x_2 = (0, 1, 1/\sqrt{2}, 0, 0, \dots)$$
$$x_3 = (0, 0, 1, 1/\sqrt{2}, 1/\sqrt{2^2}, 0, \dots)$$

....

and so on. It is easy to check that the greatest component of $T(x_n)$, placed on the n-th coordinate, is equal to $1/\sqrt{2^{n-1}} + 1/2 + 1/4 + \ldots + 1/2^{n-1}$. Thus $\gamma(X) = \mu(TX) \geqslant 1$.

This shows that our last claim is valid.

Finally let us mention that in this case it is rather difficult to give a full description of the kernel $\ker\gamma$ because T is defined via a complicated formula.

5. – Theorem of Darbo type.

This last section is devoted to indicate some very useful theorem, proved first by DARBO [5] in the case of the so-called Kuratowski measure of noncompactness (in strong sense). In the case of an arbitrary measure of noncompactness (strong) this theorem was proved in [1] (cf. also [14]). Moreover, an analog of this theorem with the use of De Blasi measure has been proved by DE BLASI [6] in the case of separable spaces and by EMMANUELE [8] in the case of an arbitrary Banach space.

Below we provide the version of this theorem associated with an arbitrary measure of weak noncompactness.

Let us assume that γ is a measure of weak noncompactness defined in the space E. Further, let C be a nonempty, convex, closed and bounded subset of E. Moreover, let $T\colon C \to C$ be a weakly continuous operator such that there exists $k \in \langle 0, 1)$ with the property

$$\gamma(T(X)) \leqslant\leqslant k\gamma(X)$$

for any $X \subset C$. Then we have

THEOREM 11. – *Under the above assumptions, the operator T has at least one fixed point in the set C. Moreover, the set of all fixed points of T,* $\operatorname{Fix} T = [x \in C\colon Tx = x]$ *belongs to* $\ker\gamma$.

We omit the simple proof of this theorem which can be carried over analogously as in [8].

Let us only mention that the information that $\operatorname{Fix} T \subset \ker\gamma$ is very important because it allows us to characterize the solutions of some equations where existence is proved with the help of Theorem 11. Some applications of Theorem 11, based on this idea, will appear elsewhere.

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