Shape Sensitivity Analysis Via a Penalization Method (*).

M. C. Delfour - J.-P. Zolésio

Summary. – *The object of this paper is the development of a penalization technique to compute the shape derivative of cost functionals where the state is the solution of a non-linear equation and/or a linear variational inequality. This type of problem is frequently encountered in Shape Sensitivity Analysis.*

Résumé. – *Cet article présente le calcul des dérivées de forme de fonctionnelles définies sur un domaine géométrique par une méthode de pénalisation. On suppose que l'état est la solution d'une équation non-linéaire ou d'une inéquation linéaire. Ce type de problème est fréquemment rencontré en analyse de sensitivité des formes.*

1. – Introduction.

The object of this paper is the development of a penalization technique to compute the shape derivative of cost functionals where the state is the solution of a non-linear equation and/or a linear variational inequality. This type of problem is frequently encountered in Shape Sensitivity Analysis.

For partial differential equations where the state is the minimizing element of a quadratic energy functional over a linear subspace of a Hilbert space, the shape derivative can be computed by differentiating a Min Max problem with respect to an appropriate vector field (cf. Delfour and Zolésio [1, 2, 3]). This approach readily lends itself to some class of non-differentiable cost functions, but difficulties are encountered when the energy functional is non-linear or when the state is given as the minimizing element over a closed convex set which is not linear.

The reader should not be afraid by the list of some of the hypotheses. In fact, most of them are minimal and are verified under mild continuity hypotheses. What is important to notice is that we never ask any form of differentiability of the state

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(*) Entrata in Redazione il 15 giugno 1987.

This research was supported in part by the National Sciences and Engineering Council of Canada Operating Grant A-8730 and a FCAR Grant from the « Ministère de l'Education du Québec ».

Indirizzo degli AA.: M. C. Delfour: Centre de recherches mathématiques and Département de mathématiques et statistique, Université de Montréal, C.P. 6128, Succ. A, Montréal, Québec, Canada, H3C 3J7, U.S.A.; J.-P. Zolésio: Laboratoire de Physique Mathématique, Université des Sciences et Techniques du Languedoc, Pl. Eugène Bataillon, 34060 - Montpellier Cédex, France.

variable. In addition this paper constructively introduces a natural adjoint function and its corresponding variational inequality. All this is done in a non-standard way without an a priori Lagrangian formulation.

For illustration, we apply the theory to a numerical problem which is studied in DELFOUR-PAYRE and ZOLÉSIO [1]. The solution of the state equation does not have enough smoothness to justify the final expression by variational techniques. Other techniques using implicit functions theorem fail because the underlying function spaces are different. At best we could show by direct variational techniques that the state $y^t = y_t \circ T_t$ is differentiable in H^1-weak.

This paper also attempts to provide justifications for results which are usually obtained formally in the literature. One good example of such computations can be found in J. CÉA [1] who provides a quick and efficient tool to obtain the final expressions. It is important to notice that such expressions are usually not available for variational inequalities except in some special cases. The type of techniques we have used are related to the ones found in M. FORTIN and GLOWINSKI's [1] book, on augmented Lagrangian methods. Some of our results might also have some potential in dynamical problems such as the ones studied by G. DA PRATO [1]. Finally some of our results have been announced in DELFOUR and ZOLÉSIO [1].

2. – Statement of the problem and orientation.

Let $E\colon \mathbf{R}^+ \times K \to \mathbf{R}$ be an *energy functional* defined over a closed convex subset K of a Banach space B. Assume that for each t in $\mathbf{R}^+$, the map

$$\varphi \mapsto E(t, \varphi) \tag{1}$$

is convex and continuous on K and that there exists a unique solution $y = y(t) \in K$ to the minimization problem

$$E(t, y) = \operatorname*{Inf}_{\varphi \in K} E(t, \varphi) \stackrel{\Delta}{=} e(t) . \tag{2}$$

In particular y is completely characterized by the variational inequality

$$y \in K , \quad dE(t, y; 0, \varphi - y) \geqslant 0 , \quad \forall \varphi \in K , \tag{3}$$

where for each ψ in B

$$dE(t, y; 0, \psi) = \lim_{s \searrow 0} \frac{E(t, y + s\psi) - E(t, y)}{s} . \tag{4}$$

Associate with the above problem a cost function

$$J(t) = F\big(t, y(t)\big) \tag{5}$$

for some functional

$$F\colon \mathbf{R}^{+}\times K \to \mathbf{R}\,. \tag{6}$$

Assume that for all t in a neighborhood of 0 the map

$$\varphi \mapsto F(t, \varphi)$$

is convex and continuous on K for some topology $\mathcal{T}_B$ weaker than the norm topology of B.

Our objective is to investigate the existence the Gateaux semiderivative of J at 0

$$dJ(0) = \lim_{s\searrow 0} \frac{J(s) - J(0)}{s} \tag{8}$$

and to characterize it.

2.1. *Construction of a* Min Sup *problem*: *the Lagrangian approach.*

In many cases the above problem can be reformulated with the help of a Lagrangian of the form

$$L(t, \varphi; \psi) = F(t, \varphi) + dE(t, \varphi; 0, \psi)\,. \tag{9}$$

When $K = B$

$$J(t) = \operatorname*{Inf}_{\varphi\in B}\ \operatorname*{Sup}_{\psi\in B} L(t, \varphi; \psi)\,. \tag{10}$$

If in addition L is convex and lower semi continuous in φ and concave and upper semi continuous in ψ the Lagrangian has saddle points (φ_t, ψ_t) which are completely characterized by the following system of equations (we assume F and E are sufficiently differentiable in φ)

$$dF(t, \varphi_t; 0, \varphi) + d^2E(t, \varphi_t; 0, \psi_t; 0, \varphi) = 0\,, \quad \forall \varphi \in B \tag{11}$$

$$dE(t, \varphi_t; 0, \psi) = 0\,, \quad \forall \psi \in B\,. \tag{12}$$

For non-linear energy functionals $E(t, \varphi)$ the convexity of the Lagrangian with respect to φ is usually lost as can be seen from the thermal radiator problem (cf. DELFOUR, PAYRE and ZOLÉSIO [1]) where

$$E(\varphi) = \frac{1}{2}\int_{\Omega} |\nabla\varphi|^2\, dx + \int_{\Sigma_3} \left(\frac{1}{5}|\varphi|^5 - q_s\varphi\right) d\sigma - \int_{\Sigma_1} q_{in}\varphi\, d\delta \tag{13}$$

where $q_s > 0$, $q_{in} > 0$, Ω is a volume of revolution with boundary $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, Σ_1 is the interface between the radiator and the heat source, Σ_3 is the radiating surface and Σ_2 is the lateral adiabatic surface

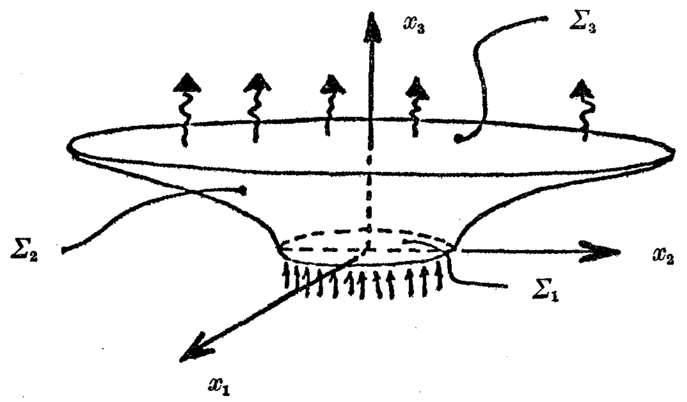

Figure 1. – Volume Ω and its boundary $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

It is readily seen that the underlying space B is

$$(14) \qquad B = \{\varphi \in H^1(\Omega) : \varphi|_{\Sigma_3} \in L^5(\Sigma_3)\}$$

which is a reflexive Banach space. However

$$(15) \qquad dE(\varphi; \psi) = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx + \int_{\Sigma_3} [|\varphi|^3 \varphi\psi - q_s \psi] \, d\sigma - \int_{\Sigma_1} q_{in} \varphi \, d\sigma$$

and when ψ is negative on a subset of non-zero measure of Σ_3, $dE(\varphi; \psi)$ is no longer convex with respect to φ.

Another interesting and difficult case is the one where K is no longer a linear subspace of B, but a closed convex set of B. There the Inf Sup formulation (10) could be modified as follows

$$(16) \qquad J(t) = \operatorname*{Inf}_{\varphi \in K} \operatorname*{Sup}_{\mu \geqslant 0} \operatorname*{Sup}_{\psi \in K} \{F(t, \varphi) - \mu \, dE(t, \varphi; 0, \psi - \varphi)\}.$$

but in general, we again loose the convexity with respect to φ. For the characterization of optimal controls in this context the reader is referred to SHI SHUZHONG [1]. It is also interesting to note that the right-hand-side of (16) can also be written in the following form

$$(16a) \qquad J(t) = \operatorname*{Inf}_{\varphi \in K} \operatorname*{Sup}_{\psi \in T_K(\varphi)} \{F(t, \varphi) - dE(t, \varphi; 0, \psi\}$$

where $T_K(\varphi)$ is the closure of the cone $\boldsymbol{R}^+(K - \varphi)$ in B.

Notice also that the following two variational inequalities are equivalent for a closed conyex set K

$$\exists y \in K\,, \quad \forall \psi \in K\,, \quad dE(y;\psi - y) \geqslant 0$$

and

$$\exists y \in K\,, \quad \forall \psi \in T_K(y)\,, \quad dE(y;\psi) \geqslant 0\,.$$

2.2. *Construction of a non-Lagrangian formulation.*

To get around the above difficulties we propose to replace the Lagrangian by the following functional

$$G(t,\varphi,\mu) = F(t,\varphi) + \mu[E(t,\varphi) - e(t)] \tag{17}$$

where $\mu \in \boldsymbol{R}^+$ and

$$e(t) = \operatorname*{Inf}_{\varphi\in K} E(t,\varphi) = E(t, y_t)\,. \tag{18}$$

It is readily seen that

$$J(t) = \operatorname*{Inf}_{\varphi\in K}\operatorname*{Sup}_{\mu\geqslant 0} G(t,\varphi,\mu)\,. \tag{19}$$

When $F(t,\varphi)$ and $E(t,\varphi)$ are convex with respect to φ, the functional G is convex in φ for all $\mu\geqslant 0$ and linear (hence concave) in μ for all φ.

In this case, the Inf Sup problem (19) is equivalent to the Inf Sup problem (14)

$$\operatorname*{Inf}_{\varphi\in K}\operatorname*{Sup}_{\mu\geqslant 0,\psi\in K}\{F(t,\varphi) - \mu\, dE(t,\varphi;\,0,\psi-\varphi)\}\,.$$

Indeed if

$$\exists\varphi\in K\,,\quad E(\varphi) = e$$

then φ is completely characterized by

$$dE(\varphi,\psi-\varphi)\geqslant 0\,,\quad \forall\psi\in K \Leftrightarrow \operatorname*{Sup}_{\psi\in K} - dE(\varphi,\psi-\varphi) = 0\,.$$

Conversely if

$$\exists\varphi\in K\,,\quad \operatorname*{Sup}_{\psi\in K} - dE(\varphi;\psi-\varphi) = 0\,,$$

then

$$E(\varphi) - \operatorname*{Inf}_{\psi\in K} E(\psi) = \operatorname*{Sup}_{\psi\in K}\{E(\varphi) - E(\psi)\} \leqslant \operatorname*{Sup}_{\psi\in K} - dE(\varphi;\psi-\varphi) = 0$$

which implies

$$\exists \varphi \in K, \quad E(\varphi) \leqslant \operatorname*{Inf}_{\psi \in K} E(\psi) \Rightarrow \exists \varphi \in K, \quad E(\varphi) = e .$$

The inequalities characterizing a saddle point $(\psi_t, \mu_t) \in K \times \mathbf{R}^+$ (if it exists) of (19) would be

$$dF(t, \varphi_t; 0, \varphi - \varphi_t) + \mu_t \, dE(t, \varphi_t; 0, \varphi - \varphi_t) \geqslant 0 , \quad \forall \varphi \in K \tag{20}$$

$$(\mu - \mu_t)[E(t, \varphi_t) - e(t)] \leqslant 0 , \quad \forall \mu \geqslant 0 . \tag{21}$$

The last inequality (21) is equivalent to

$$\begin{cases} \mu_t[E(t, \varphi_t) - e(t)] = 0 \\ \mu_t \geqslant 0 , \quad [E(t, \varphi_t) - e(t)] \leqslant 0 . \end{cases} \tag{22}$$

So if there exists a solution $(\varphi_t, \mu_t) \in K \times \mathbf{R}^+$ solution of (20)-(22)

$$E(t, \varphi_t) - e(t) = 0 \Rightarrow \varphi_t = y_t \quad \text{and } \mu_t \geqslant 0 \text{ arbitrary}$$

or

$$\mu_t = 0 \Rightarrow E(t, \varphi_t) - e(t) \leqslant 0 \Rightarrow \varphi_t = y_t .$$

If $\mu_t \geqslant 0$ is finite, equation (20) reduces to

$$dF(t, y_t; 0, \varphi - y_t) \geqslant 0 , \quad \forall \varphi \in K$$

which is equivalent to say that

$$F(t, y_t) = \operatorname*{Inf}_{\varphi \in K} F(t, \varphi) . \tag{23}$$

This implies that the solution y_t of (18) also minimizes $F(t, \varphi)$ over all φ in K. This is a special case. In all other cases $\mu_t = +\infty$ which makes it difficult to extract any information from (20).

At this stage the existence of saddle points is questionable and we have seemingly lost the adjoint state which quite naturally comes out of a Lagrangian formulation. To get around this difficulty we study the following family of problems indexed by $\varepsilon > 0$

$$J_\varepsilon(t) = \operatorname*{Inf}_{\varphi \in K} G_\varepsilon(t, \varphi) \tag{24}$$

where

$$G_\varepsilon(t, \varphi) = G\left(t, \varphi, \frac{1}{\varepsilon}\right) = F(t, \varphi) + \frac{1}{\varepsilon}[E(t, \varphi) - e(t)] . \tag{25}$$

Under appropriate hypotheses the minimizing elements φ_ε^t would be characterized by

$$dF(t, \varphi_\varepsilon^t;\, 0, \varphi - \varphi_\varepsilon^t) + \frac{1}{\varepsilon}\, dE(t, \varphi_\varepsilon^t;\, 0, \varphi - \varphi_\varepsilon^t) \geqslant 0\,, \qquad \forall \varphi \in K\,. \tag{16}$$

So the steps are now clear. We must introduce appropriate hypotheses so that

$$\lim_{s \searrow 0} J_\varepsilon(t) = J(t)\,.$$

In the process we shall construct the variable

$$p_\varepsilon^t = (\varphi_\varepsilon^t - \varphi_0^t)/\varepsilon$$

which will converge in an appropriate sense to a natural *adjoint state* variable p which is typical of a Lagrangian approach. Thus we shall recover everything without the afore mentioned limitation of a Lagrangian method.

3. – The family of problems indexed by t.

In this section a more precise problem formulation is given and specific hypotheses are introduced in order to make sense of the constructions outlined in the previous section.

3.1. *Problem formulation and hypotheses.*

Let $E\colon \boldsymbol{R}^+ \times K \to \boldsymbol{R}$ be an energy functional defined over a closed convex subset K of a Banach space B. Assume that the following hypothesis is verified.

H1 For each t in $0, T]$ the map

$$\varphi \mapsto E(t, \varphi) \tag{1}$$

is convex and continuous on K and there exists a unique solution $y = y(t) \in K$ to the minimization problem

$$E(t, y) = \operatorname{Inf}\{E(t, \varphi)\colon \varphi \in K\} \overset{\text{def}}{=} e(t)\,. \qquad \square \tag{2}$$

In particular y is completely characterized by the variational inequality

$$y \in K\,, \quad dE(t, y;\, 0, \varphi - y) \geqslant 0\,, \quad \forall \varphi \in K \tag{3}$$

where for each ψ in B

$$dE(t, y;\, 0, \psi) = \lim_{s \geqslant 0} \big(E(t, y + s\psi) - E(t, y)\big)/s\,. \tag{4}$$

Associate with the above problem a cost function:

$$J(t) = F(t, y(t)) \tag{5}$$

for some functional

$$F: \mathbf{R}^+ \times K \to \mathbf{R}. \tag{6}$$

For the moment, assume that the map $\varphi \mapsto F(t, \varphi)$ is convex and lower semi continuous on B.

Our main objective is to show that, under appropriate hypotheses, the cost function $J(t)$ can be expressed in the form

$$J(t) = J(0) + \int_0^t f(s)\, ds \tag{7}$$

for some function f in $L^\infty(0, T)$ which will be characterized in terms of the state $y(t)$ and the solution $p(t)$ to an appropriate adjoint unilateral problem for each t. Under an additional hypothesis we shall also show that f belongs to $C^0(0, T)$, that is J belongs to $C^1(0, T)$ and $dJ(0) = f(0)$.

3.2. *Penalized problems.*

Instead of tackling the problem directly we introduce a family of penalized problems indexed by $\varepsilon > 0$:

$$J_\varepsilon(t) = \operatorname*{Inf}_{\varphi \in K} \left\{ F(t, \varphi) + \frac{1}{\varepsilon}[E(t, \varphi) - e(t)] \right\}. \tag{8}$$

H2 (i) There exist $T > 0$ and $\bar{\varepsilon} > 0$ such that for all t in $[0, T]$ and ε in $[0, \bar{\varepsilon}]$ there exists a unique minimizing element y^t_ε in K of the functional

$$G_\varepsilon(t, \varphi) = F(t, \varphi) + \frac{1}{\varepsilon}[E(t, \varphi) - e(t)] \tag{9}$$

over all φ in K.

(ii) For all t in $[0, T]$

$$y^t_\varepsilon \to y^t_0 \quad \text{in } B. \quad \blacksquare \tag{10}$$

Hypothesis H2 contains hypothesis H1 and $y(t) = y^0_\varepsilon$.

Existence and uniqueness of solution y^t_ε in a neighborhood of $(t, \varepsilon) = (0, 0)$ may result from a positivity hypothesis on $F(t, \cdot)$ on K or from a growth property of $F(t, \varepsilon)$ as $\|\varphi\|$ goes to infinity. In the sequel we shall denote by y the solution $y(0) = y^0_0$.

To make sense of the adjoint state we need the following additional hypotheses in a neighborhood N of y in B.

H3 The map $\varphi \mapsto E(t, \varphi)$ is twice Gateaux differentiable in N: that is for all φ in N and ψ and ξ in B the following limit exist

$$dE(t, \varphi; 0, \psi) = \lim_{s \searrow 0} [E(t, \varphi + s\psi) - E(t, \varphi)]/s$$

$$d^2E(t, \varphi; 0, \psi; 0, \xi) = \lim_{s \searrow 0} [dE(t, \varphi + s\xi; 0, \psi) - dE(t, \varphi; 0, \psi)]/s . \quad \blacksquare$$

H4 There exists a Hilbert space V, $B \subset V$, with continuous embedding such that the map

$$\psi \mapsto F(t, \psi)$$

is convex and V-continuous. Moreover for all φ in $N \cap K$ the maps

$$\psi \mapsto dE(t, \varphi; 0, \psi) , \qquad (\psi, \xi) \mapsto d^2E(t, \varphi; 0, \psi; 0, \xi)$$

extend continuously to V and $V \times V$, respectively and

$$\exists \alpha > 0 \quad \text{such that} \quad \forall \psi \in V , \quad d^2E(t, 0; 0, \psi; 0, \psi) \geqslant \alpha \|\psi\|_V^2 . \quad \blacksquare$$

H5 Given convergent sequences $\varphi_n \to y_0^t$ in B, $\psi_n \to \psi$ in V (strong) and $\xi_n \to \xi$ in V (weak), there exists a subsequence $\{\varphi_{n_k}\}$ such that

$$d^2E(t, \varphi_{n_k}; 0, \psi_{n_k}; 0, \xi_{n_k}) \to d^2E(t, y_0^t; 0, \psi; 0, \xi) . \quad \blacksquare$$

As mentionned in section 2 we shall introduce the approximate adjoint state

$$p_\varepsilon^t = (y_\varepsilon^t - y_0^t)/\varepsilon \in B$$

and study its behaviour as ε goes to zero. This will require the following additional hypotheses.

H6 Given any two sequences $\{\varphi_n\}$ in $N \cap K$ and $\{\psi_n\}$ in V such that $\varphi_n \to y_0^t$ in B and $\psi_n \to \psi$ weakly in V for some ψ in V, there exist subsequences (still denoted $\{\varphi_n, \psi_n\}$) such that

$$\liminf_{n \to \infty} d^2E(t, \varphi_n; 0, \psi_n; 0, \psi_n) \geqslant d^2E(t, y_0^t; 0, \psi; 0, \psi) . \quad \blacksquare$$

3.3. *A priori estimates for the penalized problems.*

LEMMA 1. – Assume that hypotheses H2 to H4 are verified. There exist a constant $c(t) > 0$ such that

$$|E(t, y_\varepsilon^t) - E(t, y_0^t)| < \varepsilon c(t) \|y_\varepsilon^t - y_0^t\|_V \tag{11}$$

(12) $$\|y_\varepsilon^t - y_0^t\|_V \leqslant \varepsilon c(t)/\alpha$$

(13) $$\|p_\varepsilon^t\|_V \leqslant c(t)/\alpha .$$

PROOF. – By definition of the minimizing element y_ε^t we have

(14) $$F(t, y_\varepsilon^t) + \frac{1}{\varepsilon}[E(t, y_\varepsilon^t) - E(t, y_0^t)] \leqslant F(t, y_0^t) .$$

By V-continuity and convexity of $\varphi \mapsto F(t, \varphi)$, there exists a support functional to $F(t, \varphi)$ at $\varphi = y_0^t$, that is

$$\exists x_0^* \in V' , \quad \forall \varphi \in V , \quad F(t, \varphi) \geqslant F(t, y_0^t) + \langle x_0^*, \varphi - y_0^t \rangle .$$

Hence

(15) $$F(t, \varphi) \geqslant F(t, y_0^t) - c(t)\|\varphi - y_0^t\|_V , \quad \forall \varphi \in V ,$$

with $c(t) = \|x_0^*\|_{V'}$. From (14) we have

$$|E(t, y_\varepsilon^t) - E(t, y_0^t)| \leqslant \varepsilon |F(t, y_\varepsilon^t) - F(t, y_0^t)| .$$

But from (15)

$$0 \leqslant F(t, y_0^t) - F(t, y_\varepsilon^t) \leqslant c(t)\|y_0^t - y_\varepsilon^t\| .$$

and hence (11). By hypothesis H3, there exists $\theta \in]0, 1[$ such that (use the variational inequality (3))

$$E(t, y_\varepsilon^t) - E(t, y_0^t) \geqslant d^2 E(t, y_0^t + \theta(y_\varepsilon^t - y_0^t); 0, y_\varepsilon^t - y_0^t; 0, y_\varepsilon^t - y_0^t)$$

and by hypothesis H4

$$E(t, y_\varepsilon^t) - E(t, y_0^t) \geqslant \alpha \|y_\varepsilon^t - y_0^t\|_V^2 .$$

Combining the last inequality with (11) we obtain (12) and (13). ■

LEMMA 2. – Under hypothesis H2 to H4, for all t in $[0, T]$

(16) $$J_\varepsilon(t) \to J_0(t) \quad \text{as } \varepsilon \to 0$$

and for each $\varepsilon > 0$ there exists $\theta \in]0, 1[$ such that

(17) $$0 \leqslant \frac{1}{\varepsilon} dE(t, y_0^t; 0, p_\varepsilon^t) + d^2 E(t, y_0^t + \theta(y_\varepsilon^t - y_0^t); 0, p_\varepsilon^t; 0, p_\varepsilon^t) \leqslant -dF(t, y_0^t; 0, p_\varepsilon^t) .$$

But

$$-dF(t, y_0^t; 0, p_\varepsilon^t) \leqslant c(t)^2/\alpha \tag{18}$$

and

$$0 \leqslant \frac{1}{\varepsilon} dE(t, y_0^t; 0, p_\varepsilon^t) \leqslant c(t)^2/\alpha \tag{19}$$

$$\begin{aligned} 0 \leqslant \limsup \frac{1}{\varepsilon} dE(t, y_0^t; 0, p_\varepsilon^t) &\leqslant \\ \leqslant -\liminf \{dF(t, y_0^t; 0, p_\varepsilon^t) &+ d^2E(t, y_0^t + \theta(y_\varepsilon^t - y_0^t); 0, p_\varepsilon^t; 0, p_\varepsilon^t)\} . \end{aligned} \tag{20}$$

PROOF. - By definition $J_\varepsilon(t) \leqslant J_0(t)$ from (14). But

$$E(t, y_\varepsilon^t) \geqslant E(t, y_0^t) \Rightarrow F(t, y_\varepsilon^t) \leqslant J_\varepsilon(t)$$

and necessarily

$$F(t, y_\varepsilon^t) \leqslant J_\varepsilon(t) \leqslant J_0(t) = F(t, y_0^t) .$$

By hypothesis H4 and estimate (12) in Lemma 1, we obtain (16). We know by hypothesis H2 to H4 that

$$E(t, y_\varepsilon^t) - E(t, y_0^t) \geqslant dE(t, y_0^t; 0, y_\varepsilon^t - y_0^t) \geqslant 0 .$$

Combining this with (11) and (12) we obtain (19). Now by hypothesis H3, there exists $\theta \in]0, 1[$ such that

$$E(t, y_\varepsilon^t) - E(t, y_0^t) = dE(t, y_0^t; 0, y_\varepsilon^t - y_0^t) + d^2E(t, y_0^t + \theta(y_\varepsilon^t - y_0^t); 0, y_\varepsilon^t - y_0^t) . \tag{21}$$

But y_ε^t verifies the variational inequality

$$dF(t, y_\varepsilon^t; 0, \varphi - y_\varepsilon^t) + \frac{1}{\varepsilon} dE(t, y_\varepsilon^t; 0, \varphi - y_\varepsilon^t) \geqslant 0 , \qquad \forall \varphi \in K \tag{22}$$

and

$$dF(t, y_\varepsilon^t; 0, \varphi - y_\varepsilon^t) + dF(t, \varphi; 0, y_\varepsilon^t - \varphi) \leqslant 0 . \tag{23}$$

By setting $\varphi = y_0^t$ in the above inequalities we obtain (17) and (20). ∎

REMARK 1. - For $0 \leqslant \varepsilon_1 \leqslant \varepsilon_2$

$$E(y_0) \leqslant E(y_{\varepsilon_1}) \leqslant E(y_{\varepsilon_2}) , \qquad F(y_{\varepsilon_2}) \leqslant F(y_{\varepsilon_1}) \leqslant F(y_0)$$

$$J_0 = F(y_0) \geqslant J_{\varepsilon_1} \geqslant J_{\varepsilon_2} , \qquad 0 \geqslant \frac{J_{\varepsilon_1} - J_0}{\varepsilon_1} \geqslant \frac{J_{\varepsilon_2} - J_0}{\varepsilon_2}$$

$$0 \geqslant \frac{F(y_{\varepsilon_1}) - F(y_0)}{\varepsilon_1} \geqslant \frac{F(y_{\varepsilon_2}) - F(y_0)}{\varepsilon_2} .$$

But

$$0 \leqslant dE(y_0; p_\varepsilon) \leqslant \frac{1}{\varepsilon}[E(y_\varepsilon) - E(y_0)] \leqslant F(y_0) - F(y_\varepsilon) \leqslant - dF(y_0; y_\varepsilon - y_0)$$

and

$$\lim_{\varepsilon\searrow 0} dE(y_0; p_\varepsilon) = 0 , \quad \limsup_{\varepsilon\searrow 0} dF(y_0; p_\varepsilon) \leqslant 0 .$$

Hence

$$0 \geqslant dF_0 = \lim_{\varepsilon\searrow 0} \frac{F(y_\varepsilon) - F(y_0)}{\varepsilon} \geqslant \frac{F(y_{\varepsilon_1}) - F(y_0)}{\varepsilon_1} \geqslant \frac{F(y_{\varepsilon_2}) - F(y_0)}{\varepsilon_2}$$

$$0 \geqslant dJ_0 = \lim_{\varepsilon\searrow 0} \frac{J\varepsilon - J_0}{\varepsilon} \geqslant \frac{J_{\varepsilon_1} - J_0}{\varepsilon_1} \geqslant \frac{J_{\varepsilon_2} - J_0}{\varepsilon_2}$$

and

$$\lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon^2}[E(y_\varepsilon) - E(y_0)] = dJ_0 - dF_0 \geqslant 0 .$$

Also

$$0 \geqslant dF_0 \geqslant \limsup_{\varepsilon\searrow 0} dF(y_0; p_\varepsilon)$$

$$0 \geqslant dJ_0 \geqslant \limsup_{\varepsilon\searrow 0} dF(y_0; p_\varepsilon) + \limsup_{\varepsilon\searrow 0} \frac{1}{\varepsilon} dE(y_0; p_\varepsilon)$$

and

$$0 \leqslant \limsup_{\varepsilon\searrow 0} \frac{1}{\varepsilon} dE(y_0; p_\varepsilon) \leqslant dJ_0 - dF_0 . \quad \blacksquare$$

3.4. *Limiting behaviour of p_ε^t as ε goes to zero.*

In lemma 1 we have seen that the elements p_ε^t are bounded in V. So by construction they have weak limit points in the tangent convex cone

(24) $$T_K(y_0^t) = V\text{-closure } \{\lambda(\varphi - y_0^t): \varphi \in K, \lambda \geqslant 0\} .$$

LEMMA 3. – Assume that hypotheses H1 to H4 and H6 are verified and that p in V is a weak limit point of $\{p_\varepsilon^t: \varepsilon > 0\}$. Then

(25) $$dE(t, y_0^t; 0, p) = 0 , \quad p \in T_K(y_0^t)$$

(26) $$0 \leqslant \liminf \frac{1}{\varepsilon} dE(t, y_0^t; 0, p_\varepsilon^t)$$

(27) $$0 \leqslant \limsup \frac{1}{\varepsilon} dE(t, y_0^t; 0, p_\varepsilon^t) \leqslant - [dF(t, y_0^t; 0, p) + d^2E(t, y_0^t; 0, p; 0, p)] .$$

PROOF. – Identity (25) is a direct consequence of inequalities (19). As for (26) it follows from (3) by setting $\varphi = y_0^t$ and dividing by $\varepsilon > 0$. Finally (27) follows from (20) and is a consequence of the weak lower semicontinuity of $\psi \mapsto dF(t, y_0^t; 0, \psi)$ and hypothesis H6. $\blacksquare$

So the weak limit points of $\{p_\varepsilon^t;\ \varepsilon > 0\}$ belong to the closed convex cone

$$S(t) = T_K(y_0^t) \cap \nabla E(t, y_0^t)^\perp , \tag{28}$$

where

$$\nabla E(t, y_0^t)^\perp = V\text{-closure } \{\psi \in B\colon dE(t, y_0^t;\ 0, \psi) = 0\} . \tag{29}$$

In fact they belong to a smaller set for which the condition

$$0 \leqslant \limsup \frac{1}{\varepsilon} dE(t, y_0^t;\ 0, p_\varepsilon^t) \leqslant c(t)^2/\alpha ,$$

holds, but that set is hard to characterize.

3.5. *Variational inequality for the limit points.*

We now construct a cone $A(t)$ and a variational inequality for the limit points of $\{p_\varepsilon^t\} = \{p_\varepsilon^t\colon \varepsilon > 0\}$. Let

$$A(t) = \left\{ \psi \in V \;\middle|\; \begin{array}{l} \exists \{\varphi_\varepsilon\colon \varepsilon > 0\} \subset K , \quad \psi_\varepsilon = (\varphi_\varepsilon - y_0^t)/\varepsilon \text{ such that} \\ \psi_\varepsilon \to \psi \quad \text{in } V \text{ (weak) as } \varepsilon > 0 \to 0 \quad \text{and} \\ \lim\limits_{\varepsilon \searrow 0} \dfrac{1}{\varepsilon} dE(t, y_0^t;\ 0, \psi_\varepsilon) = 0 . \end{array} \right. \tag{30}$$

LEMMA 4. – (i) The set $A(t)$ is a cone with vertex at 0 in V. Moreover

$$\mathbf{R}^+(K - y_0^t) \cap \nabla E(t, y_0^t)^\perp \subset A(t) \subset T_K(y_0^t) \cap \nabla E(t, y_0^t)^\perp . \tag{31}$$

(ii) If

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} dE(t, y_0^t;\ 0, p_\varepsilon^t) = 0 , \tag{32}$$

then all weak points of $\{p_\varepsilon^t\}$ are in $\overline{\text{co}}\, A(t)$.

PROOF. – (i) To show that $0 \in A(t)$, choose $\varphi_\varepsilon = y_0^t$, $\forall \varepsilon > 0$. Given $\lambda > 0$ and $\psi \in A(t)$

$$\exists \{\varphi_\varepsilon\} \subset K , \quad \psi_\varepsilon = (\varphi_\varepsilon - y_0^t)/\varepsilon \to \psi \quad \text{in } V \text{ (weak) as } \varepsilon \to 0 .$$

and

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} dE(t, y_0^t;\ 0, \psi_\varepsilon) = 0 .$$

Then choose

$$\bar{\varphi}_\varepsilon = \varphi_{\varepsilon\lambda} , \quad \bar{\varphi}_\varepsilon = (\bar{\varphi}_\varepsilon - y_0^t)/\varepsilon$$

and notice that

$$\bar{\psi}_\varepsilon = \lambda \frac{\varphi_{\varepsilon\lambda} - y_0^t}{\varepsilon\lambda} = \lambda \psi_{\varepsilon\lambda} \to \lambda\psi \quad \text{in } V(\text{weak}) \text{ as } \varepsilon \to 0 .$$

Moreover

$$\frac{1}{\varepsilon} dE(t, y_0^t; \bar{\psi}_\varepsilon) = \lambda \frac{1}{\varepsilon\lambda} dE(t, y_0^t; 0, \psi_{\varepsilon\lambda}) \to \lambda \cdot 0 = 0 .$$

So we have shown that $A(t)$ is a cone with vertex at 0.

The next step is to show that any element

$$\psi \in \boldsymbol{R}^+(K - y_0^t) \cap \nabla E(t, y_0^t)^\perp$$

belongs to $A(t)$. This is equivalent to show that $\forall\lambda \geqslant 0$ and $\varphi \in K$ such that

$$dE(t, y_0^t; 0, \varphi - y_0^t) = 0 .$$

Then

$$\psi = \lambda(\varphi - y_0^t) \in A(t) .$$

To see that choose for ε a λ such that $\varepsilon\lambda \leqslant 1$

$$\varphi_\varepsilon = (1 - \varepsilon\lambda) y_0^t + \varepsilon\lambda\varphi \in K .$$

Then

$$\psi_\varepsilon = (\varphi_\varepsilon - y_0^t)/\varepsilon = \lambda(\varphi - y_0^t) , \qquad \frac{1}{\varepsilon} dE(t, y^t{}_0; 0, \psi_\varepsilon) = 0$$

and $\psi \in A(t)$. This proves the first part of (31). For the second one, it is clear that

$$\psi_\varepsilon \in \boldsymbol{R}_+(K - y_0^t) \Rightarrow \psi \in T_K(y_0^t) .$$

Moreover there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leqslant \varepsilon_0$

$$0 \leqslant dE(t, y_0^t; 0, \psi_\varepsilon) \leqslant \varepsilon .$$

But ε goes to zero and necessarily

$$0 \leqslant dE(t, y_0^t; 0, \psi) \leqslant 0 .$$

(ii) By the weak closure of $\overline{\text{co}}\, A(t)$. ■

REMARK 2. – If

$$dE(t, y_0^t; 0, \varphi) = 0\,, \qquad \forall \varphi \in B \tag{33}$$

then

$$\bar{A}(t) = T_K(y_0^t) = \overline{\text{co}}\, A(t) \tag{34}$$

and all limit points of $\{p_\varepsilon^t\}$ belong to $\bar{A}(t)$. ■

REMARK 3. – If (32) is true and y_0^t minimizes $F(t, \varphi)$ over K, then $p_\varepsilon^t \to 0$ in V (strong). To see this use (20) and hypothesis H6. ■

THEOREM 1. – (i) Under hypothesis H1 to H6 any limit point p of $\{p_\varepsilon^t\}$ in V (weak) belongs to

$$S(t) = T_K(y_0^t) \cap \nabla E(t, y_0^t)^\perp \tag{35}$$

and verifies the variational inequality

$$dF(t, y_0^t; 0, \psi) + d^2E(t, y_0^t; 0, \psi; 0, p) \geqslant 0\,, \qquad \forall \psi \in A(t) \tag{36}$$

and the inequality

$$dF(t, y_0^t; 0, p) + d^2E(t, y_0^t; 0, p; 0, p) \leqslant 0\,. \tag{37}$$

(ii) If

$$\overline{\text{co}}\, A(t) = S(t) \tag{H7}$$

and the map

$$\psi \mapsto dF(t, y_0^t; 0, \psi) \tag{38}$$

is linear, then $p_\varepsilon^t \to p$ in V (weak), where p is the unique solution in $S(t)$ of the variational inequality

$$\begin{cases} p \in S(t)\,, \qquad \forall \psi \in S(t) \\ dF(t, y_0^t; 0, \psi - p) + d^2E(t, y_0^t; 0, \psi - p; 0, p) \geqslant 0\,. \end{cases} \tag{39}$$ ■

REMARK 4. – Hypothesis H7 seems to be weaker than the classical hypothesis

$$V\text{-closure}\ \{\mathbf{R}^+(K - y_0^t) \cap \nabla E(t, y_0^t)^\perp\} = S(t) \tag{40}$$

(cf. F. MIGNOT [1], J. SOKOLOWSKI [1]). ■

PROOF OF THEOREM 1. – Since the parameter t is fixed, we shall drop it everywhere in the proof. (i) We already know that the weak limit points of $\{p_\varepsilon^t\}$ belongs to $S(t)$ and that (37) is verified. To established (36) we fix a weak limit point p of $\{p_\varepsilon\}$ and the associated sequence $\{\varepsilon_k > 0\}$, $\varepsilon_k \to 0$ such that

$$p_k = p_{\varepsilon_k} \to p \quad \text{in } V \text{ (weak)}.$$

Consider an arbitrary element ψ in $A(t)$ and its associated $\{\varphi_\varepsilon : \varepsilon > 0\} \subset K$ such that

$$\psi = \text{weak} \lim_{\varepsilon \searrow 0} \psi_\varepsilon, \quad \psi_\varepsilon = (\varphi_\varepsilon - \varphi_0)/\varepsilon \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} dE(y_0; \psi_\varepsilon) = 0.$$

The above properties remain true with ε_k in place of ε.

We now turn to the variational equation for $y_k = y_{\varepsilon_k}$

$$dF(y_k; \varphi - y_k) + \frac{1}{\varepsilon_k} dE(y_k; \varphi - y_k) \geqslant 0, \quad \forall \varphi \in K. \tag{41}$$

Let $\varphi = \varphi_k = \varphi_{\varepsilon_k}$ in (41). By hypothesis H4, there exists $\theta_k \in]0, 1[$ such that

$$dE(y_k; \varphi_k - y_k) = dE(y_0; \varphi_k - y_k) + d^2E(y_0 + \theta_k(y_k - y_0); \varphi_k - y_k; y_k - y_0).$$

So (41) yields

$$\begin{aligned} 0 \leqslant dF\left(y_k; \frac{\varphi_k - y_0}{\varepsilon_k}\right) + \frac{1}{\varepsilon_k} dE\left(y_0; \frac{\varphi_k - y_0}{\varepsilon_k}\right) + d^2E\left(y_0 + \theta_k(y_k - y_0); \frac{\varphi_k - y_0}{\varepsilon_k}; p_k\right) - \\ - dF(y_0; p_k) - \frac{1}{\varepsilon_k} dE(y_0; p_k) - d^2E(y_0 + \theta_k(y_k - y_0); p_k; p_k) \end{aligned} \tag{42}$$

where we have used the fact that

$$dF(y_0; y_k - y_0) + dF(y_k; y_0 - y_k) \leqslant 0.$$

Multiply (42) by λ_k^n and sum over k from n to N_n:

$$\begin{aligned} 0 \leqslant -\sum \lambda_k^n \frac{1}{\varepsilon_k} dE(y_0; p_k) - \sum \lambda_k^n [dF(y_0; p_k) + d^2E(y_0 + \theta_k(y_k - y_0); p_k; p_k)] \\ + \sum \lambda_k^n \frac{1}{\varepsilon_k} dE(y_0; \psi_k) + \sum \lambda_k^n dF(y_k; \psi_k) + \sum \lambda_k^n d^2E(y_0 + \theta_k(y_k - y_0); \psi_k; p_k); \end{aligned} \tag{43}$$

where

$$\sum_{k=n}^{N_n} \lambda_k^n = 1, \quad \lambda_k^n \geqslant 0, \quad \psi_k = (\varphi_k - y_0)/\varepsilon_k. \tag{44}$$

The first term on the first line of (43) is negative. Take the lim sup of the remaining terms on the Right-Hand-Side of (43) and use the following result: given a sequence $\{f_k\}$ of real numbers such that $f_k \to f$ in $\mathbf{R}$, then

$$\bar{f}_n = \sum_{k=n}^{N_n} \lambda_k^n f_k \to f\,, \qquad \sum_{k=n}^{N_n} \lambda_k^n = 1\,, \qquad \lambda_k^n \geqslant 0\,. \tag{45}$$

By lemma 3

$$\liminf_{\varepsilon \searrow 0} \left[dF(y_0; p_\varepsilon) + d^2E(y_0 + \theta_\varepsilon(y_\varepsilon - y_0); p_\varepsilon; p_\varepsilon)\right] = a \leqslant 0 \tag{46}$$

exists and is negative (cf. (20) in lemma 2).

So using (45) and (46), the second term in the first line of (43) is less than $-a$ as k goes to ∞. By definition of ψ we know that

$$\lim_{k \to \infty} \frac{1}{\varepsilon_k} dE(y_0; \psi_k) = 0$$

and by using (45), the first term in the second line of (43) goes to 0 as k goes to ∞. By hypothesis H5 there exists a subsequence of $\{\varepsilon_k\}$, still denoted $\{\varepsilon_k\}$ such that

$$d^2E(y_0 + \theta_k(y_k - y_0); \psi_k; p_k) \to d^2E(y_0; \psi; p)$$

and by using (45) again the term in the last line of (43) goes to $d^2E(y_0; \psi; p)$. The only term left is

$$g_n = \sum_{k=n}^{N_n} \lambda_k^n \, dF(y_k; \psi_k)\,. \tag{47}$$

Recall that for a convex continuous function F, the map

$$\psi \mapsto dF(\varphi; \psi): \ V \text{ (strong)} \to \mathbf{R}$$

is convex and locally Lipschitz continuous and that

$$(\varphi, \psi) \mapsto dF(\varphi; \psi): \ V \text{ (strong)} \times V \text{ (strong)} \to \mathbf{R}$$

is upper semicontinuous. As a result

$$\begin{aligned} g_n - dF(y_0; \psi) &= \sum \lambda_k^n [dF(y_k; \psi_k) - dF(y_0; \psi)] = \\ &= \sum \lambda_k^n [dF(y_k; \psi_k) - dF(y_k; \psi)] + \sum \lambda_k^n [dF(y_k; \psi) - dF(y_0; \psi)]\,. \end{aligned} \tag{48}$$

By local Lipschitz continuity, there exists a neighborhood N of y_0 and a constant $c > 0$ such that

$$\forall y \in N\,, \qquad \forall \psi_1, \psi_2 \in V\,, \qquad |dF(y; \psi_2) - dF(y; \psi_1)| \leqslant c\|\psi_2 - \psi_1\|_V\,. \tag{49}$$

As a result the first term on the Right-Hand-Side of (48) is bounded by

$$\sum \lambda_k^n c\|\psi_k - \psi\|_V = c\| \sum \lambda_k^n \psi_k - \psi\|_V \to 0 . \tag{50}$$

As for the second term denote by

$$l = \limsup_{k\to\infty} dF(y_k; \psi) \leqslant dF(y_0; \psi) . \tag{51}$$

Then always by (45)

$$\limsup_{n\to\infty} \sum \lambda_k^n dF(y_k; \psi) = \limsup_{k\to\infty} dF(y_k; \psi) \tag{52}$$

and the second term is negative.

In conclusion we have shown the following inequality for all ψ in $A(t)$

$$0 \leqslant -a + dF(y_0; \psi) + d^2E(y_0; \psi; p) .$$

But in view of lemma 3, we know that

$$a = dF(y_0; p) + d^2E(y_0; p; p) \leqslant 0 .$$

Recall that the set $A(t)$ in a cone; So for any ψ in $A(t)$ and $\lambda > 0$

$$dF(y_0; \lambda\psi) + d^2E(y_0; \lambda\psi; p) \geqslant a$$

and

$$dF(y_0; \psi) + d^2E(y_0; \psi; p) \geqslant \operatorname{Inf}\{a/\lambda \colon \lambda > p\} = 0 .$$

(ii) When (38) is linear, inequality (36) holds for all ψ in $\overline{\text{co}}\, A(t)$ and by combining it with (37)

$$\begin{cases} p \in S(t) , \quad \forall \psi \in \overline{\text{co}}\, A(t) \\ dF(y_0; \psi - p) + d^2E(y_0; \psi - p; p) \geqslant 0 . \end{cases} \tag{53}$$

So when hypothesis H7 is true, (39) has a unique solution which necessarily coincides with all weak limit points of $\{p_\varepsilon\}$.

This yields the uniqueness of the weak limit point and its complete characterization. ■

REMARK 4. – Another interesting cone with vertex at 0 for which inequality (36) holds is

$$B(t) = \left\{ \psi \in \nabla E(t, y_0^t)^\perp \;\middle|\; \begin{array}{l} \exists\{\lambda_\varepsilon > 0\},\ \exists\{\varphi_\varepsilon\} \subset K,\ \psi_\varepsilon = \lambda_\varepsilon(\varphi_\varepsilon - y_\varepsilon^t)/\varepsilon \\ \text{such that } \psi_\varepsilon \to \psi \text{ in } V(\text{strong}) \text{ as } \varepsilon \to 0 \text{ and} \\ \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} dE(t, y_0^t; 0, \psi_\varepsilon) \leqslant 0 . \end{array} \right\} \tag{54}$$

By definition, it is easy to check that

$$\overline{\text{co}}\, A(t) - p \subset \overline{\text{co}}\, B(t)$$

for all limit points p of $\{p_\varepsilon^t\}$ in V (weak). It is easy to show that

$$\boldsymbol{R}^+(K - y_0^t) \cap \nabla E(t, y_0^t)^\perp \subset B(t) \subset T_k(y_0^t) \cap \nabla E(t, y_0^t)^\perp . \tag{55}$$

So condition **H7** could be further weakened to

H7 $$\overline{\text{co}}\, \{C(t), B(t)\} = S(t) . \quad \blacksquare$$

REMARK 5. – If inequality (36) is to be verified only on $\boldsymbol{R}^+(K - y_0^t) \cap \nabla E(t, y_0^t)^\perp$, then hypothesis H5 can be weakened to

H5′ There exists a dense subspace D of V such that

$$\forall \varphi \in N \cap K , \quad \forall \psi \in D , \quad (\varphi, \xi) \mapsto d^2E(t, \varphi; 0, \xi; 0, \psi)$$

is continuous from $B \times V$ (weak) into $\boldsymbol{R}$.

4. – Limiting behaviour of $J_\varepsilon(t)$ as a function of t and derivative of $J_0(t)$.

The object of this section is to determine conditions under which $J_0 \in W^{1,1}(0, T)$ and study the limit of $dJ_0(t)$ as t goes to zero.

4.1. *Differentiability of $J_\varepsilon(t)$ with respect to t.*

We first compute the derivative of $J_\varepsilon(t)$, $t \in [0, T]$ from the right

$$dJ_\varepsilon(t) = \lim_{\varepsilon \searrow 0} [J_\varepsilon(t + s) - J_\varepsilon(t)]/s \tag{1}$$

where J_ε is defined by (8) as

$$J_\varepsilon(t) = \text{Min}\, \{G_\varepsilon(t, \varphi) \colon \varphi \in K\} \tag{2}$$

with

$$G_\varepsilon(t, \varphi) = F(t, \varphi) + \frac{1}{\varepsilon} [E(t, \varphi) - e(t)] . \tag{3}$$

Introduce the sets

$$A_\varepsilon(t) = \{\psi \in K \colon G(t, \psi) = J_\varepsilon(t)\} . \tag{4}$$

We first need an intermediate result from J. P. ZOLÉSIO [3] which will be applied to $e(t)$ and $J_\varepsilon(t)$.

THEOREM 1. – Let $G: \boldsymbol{R}\times B \to \boldsymbol{R}$ be a functional defined on a reflexive Banach space B and be $\mathcal{A}$ a subset of B. Let

$$J(t) = \operatorname{Inf}\{G(t, \varphi): \varphi \in \mathcal{A}\}, \quad A(t) = \{\psi \in \mathcal{A}: J(t) = G(t, \psi)\} \tag{5}$$

with the following hypothesis: there exists $T > 0$ such that:

HH1 $A(t) \neq \emptyset$, $0 \leqslant t \leqslant T$;

HH2 $\forall y^0 \in A(0)$, $\forall y^t \in A(t)$, the functions $s \mapsto G(s, y^0)$ and $s \mapsto G(s, y^t)$ are differentiable in a neighborhood of zero;

HH3 $\forall y^0 \in A(0)$, $s \mapsto \partial_s G(s, y^0)$ is upper semi-continuous;

HH4 $\exists$ a topology $\mathcal{T}$ on B such that;

(i) $\forall \{t_n\}$, $0 \leqslant t_n \leqslant T$, such that $t_n \to 0$, $\exists y^0 \in A(0)$, $\exists$ a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that for all k, $\exists y_k \in A(t_{n_k})$ and $y_k \to y^0$ in the $\mathcal{T}$ topology.

(ii) The map $(s, \varphi) \mapsto \partial_s G(s, \varphi)$ is lower semi-continuous on $\{0\}\times A(0)$ for the topology $\mathcal{T}$.

Then the Right-Hand-Side derivative of J is given by

$$dJ(t) = \operatorname{Inf}\{\partial_t G(0, \varphi): \varphi \in A(0)\}. \quad \blacksquare \tag{6}$$

We now proceed in two steps. First we use Theorem 1 to show that under appropriate hypotheses, $e(t)$ is continuously differentiable on $[0, T[$. Then using that result and Theorem 1 once more, we obtain the differentiability of J_ε in $[0, T]$.

LEMMA 1. – Assume that hypothesis H1 is verified and that

H8 $$\forall \varphi \in N, \quad t \mapsto E(t, \varphi): [0, T] \to \boldsymbol{R}$$

is of class C^1 and the map

$$t, \varphi \mapsto E(t, \varphi) \quad \text{and} \quad (t, \varphi) \mapsto dE(t, \varphi; 1, 0)$$

are weakly lower semi-continuous on $[0, T]\times B$.

Then the function $e(t)$ is of class C^1 on $[0, T]$ and

$$e'(t) = de(t; 1) = dE(t, y_0^t; 1, 0), \quad 0 \leqslant t \leqslant T. \tag{7}$$

PROOF. – By direct application of Theorem 1, we obtain the R.H.S. derivative $de(t;1)$ given by (7). But since the set $A(0)$ is reduced to the single element y_0^t, then

$$de(t;1) = -\,de(t;-1) = e'(t)$$

is the usual derivative at t. ■

LEMMA 2. – Assume that hypothesis H1, H2 and H8 are verified and that

H9 for each $\varepsilon \geqslant 0$, the function $t \mapsto y_\varepsilon^t\colon [0,T] \to B$ is continuous;

H10 $\forall \varphi \in N$ the functions $t \mapsto F(t,\varphi)\colon [0,T] \to \boldsymbol{R}$ is of class C^1 and the maps

$$(t,\varphi) \mapsto F(t,\varphi)\,, \qquad (t,\varphi) \mapsto dF(t,\varphi;1,0)$$

are weakly lower semi-continuous on $[0,T] \times B$.

Then for each $\varepsilon > 0$ and $0 \leqslant t \leqslant T$,

$$dJ_\varepsilon(t) = dF(t,y_\varepsilon^t;1,0) + \frac{1}{\varepsilon}[dE(t,y_\varepsilon^t;1,0) - dE(t,y_0^t;1,0)]\,. \tag{8}$$

PROOF. – Direct application of Theorem 1. ■

4.2. *Absolute continuity of* J_0.

We first construct the pointwise limit $f(t)$ of $dJ_\varepsilon(t)$ as ε goes to zero. Then we use a boundedness hypothesis to get the absolute continuity of the limit function $J_0(t)$ on $[0,T]$.

H11 The map

$$\varphi \mapsto dF(t,\varphi;1,0)\colon V \to \boldsymbol{R}$$

is continuous in N.

H12 For all ψ in B and t in $[0,T]$, the limit

$$d^2E(t,\varphi;1,0;0,\psi) = \lim_{s \searrow 0}\,[dE(t+s,\varphi;1,0;0,\psi) - dE(t,\varphi;1,0;0,\psi)]/s$$

exists for all φ in N.

H13 For all t in $[0,T]$, the map

$$\varphi,\psi \mapsto d^2E(t,\varphi;1,0;0,\psi)$$

is continuous on $N \times V$ (weak).

LEMMA 3. – Assume that hypotheses H1 to H13 are verified and that the map (3.38) is linear, then

$$\forall t \in [0, T], \quad dJ_\varepsilon(t) \to f(t) \quad \text{as} \quad \varepsilon \to 0$$

where

$$f(t) = dF(t, y_0^t; 1, 0) + d^2E(t, y_0^t; 1, 0; 0, p_0^t). \tag{9}$$

PROOF. – From H12, there exists θ, $0 < \theta < 1$, such that

$$[dE(t, y_\varepsilon^t; 1, 0) - dE(t, y_0^t; 1, 0)]/\varepsilon = d^2E(t, y_0^t + \theta(y_\varepsilon^t - y_0^t); 1, 0; 0, p_\varepsilon^t).$$

By H13, the R.H.S. of the above expression goes to

$$d^2E(t, y_0^t; 1, 0; 0, p_0^t).$$

Similarily by H11

$$dF(t, y_\varepsilon^t; 1, 0) \to dF(t, y_0^t; 1, 0).$$

Then (9) is obtained by going to the limit in (8) as ε goes to zero. ∎

We now introduce the boundedness hypothesis to apply Lebesgue Dominated Convergence Theorem and

$$J_0(t) = \lim_{\varepsilon\searrow 0} J_\varepsilon(t) = J_0(0) + \lim_{\varepsilon\searrow 0} \int_0^t dJ_\varepsilon(s)\, ds = J_0(0) + \int_0^t f(s)\, ds.$$

Recall from Remark 3.1 that

$$J_\varepsilon(t) \nearrow J_0(t) \quad \text{as } \varepsilon \to 0.$$

The boundedness hypothesis is

H14 $\exists M > 0$ such that $\forall t \in [0, T]$, $\forall \varphi \in N$, $\forall \psi \in V$

$$|d^2E(t, \varphi; 1, 0; 0, \psi)| \leqslant M \|\psi\|_V$$

and the map

$$t, \varphi \mapsto dF(t, \varphi; 1, 0)$$

is bounded in $0, T] \times N$.

THEOREM 2. – Under hypotheses H1 to H14, the linearity of the map (3.28) and the density hypothesis H7 for all t in $[0, T]$, the function J_0 is absolutely continuous.

Its derivative coincides almost everywhere with the function f in $L^\infty(0, T)$ and hence J_0 belong to $W^{1,\infty}(0, T)$:

$$dJ_0(t) = dF(t, y_0^t; 1, 0) + d^2E(t, y_0^t; 1, 0; 0, p_0^t) \quad \text{a.s. in } [0, T]\,, \tag{10}$$

where p_0^t is the unique solution in S_t of the variational inequality: for all ψ in S_t

$$dF(t, y_0^t; 0, \psi - p_0^t) + d^2E(t, y_0^t; 0, \psi - p_0^t; 0, p_0^t) \geqslant 0\,. \quad \blacksquare \tag{11}$$

REMARK 1. – Hypothesis H9 requires the continuity of the function $t \mapsto y_\varepsilon^t$, $\varepsilon \geqslant 0$, in the B-norm. It is clear that the technique of lemma 3.1 would only give the continuity in V. Thus a stronger result is required which can be obtained in each case depending on the structure of E and F. ■

4.3. *Differentiability of $J_0(t)$ at $t = 0$.*

As this juncture Theorem 2 seems to be the most reasonable result when K is not a subspace of V. The delicate point is the continuity of p_0^t as a function of t at 0 in V (weak). It is crucially related to the limiting behaviour of the sets

$$S_t = T_K(y_0^t) \cap \nabla E(t, y_0^t)^\perp\,. \tag{12}$$

This point is readily explained in the following one-dimensional example.

EXAMPLE. – $K = \{\varphi \in \mathbf{R} \colon \varphi \geqslant 0\}$

$$E(u, \varphi) = \tfrac{1}{2}\varphi^2 + u\varphi\,, \qquad F(u, \varphi) = \tfrac{1}{2}(\varphi - 1)^2\,. \tag{13}$$

It is easy to verify that

$$y_u = \begin{cases} 0\,, & \text{if } u \geqslant 0 \\ -u\,, & \text{otherwise} \end{cases}$$

and that

$$J(u) = \tfrac{1}{2}(y_u - 1)^2 = \begin{cases} \tfrac{1}{2}\,, & u \geqslant 0 \\ (u+1)^2/2\,, & \text{otherwise}\,. \end{cases}$$

For $t = 0$ as a function of u the function $J(u)$ is represented in Figure 2.

The directional derivative at u in the direction v is

$$dJ(u; v) = \begin{cases} 0\,, & u \geqslant 0 \\ (u+1)v\,, & \text{for } u < 0\,. \end{cases} \tag{14}$$

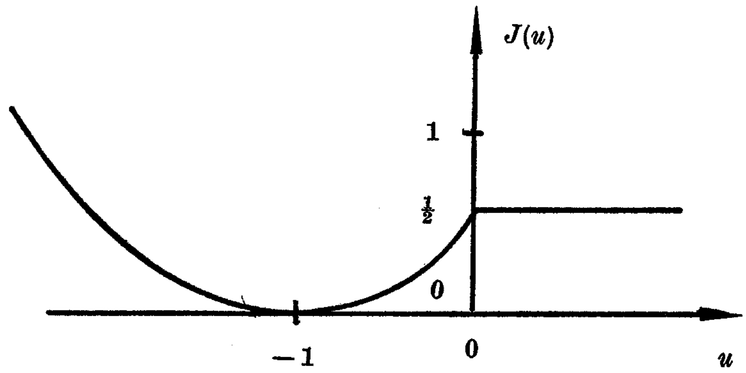

Figure 2.

So J is differentiable everywhere except at $u = 0$

$$dJ(0; v) = \min\{0, v\}.$$

Now fix u, v and $t \geqslant 0$

$$\tilde{E}(t, \varphi) = E(u + tv, \varphi), \quad \tilde{F}(t, \varphi) = F(u + tv, \varphi)$$
$$y_t = y_{u+tv}, \quad \tilde{J}(t) = J(u + tv).$$

Choose $u = 0$. Then for $t \geqslant 0$

$$y_t = \begin{cases} 0, & \text{if } v \geqslant 0 \\ -tv, & \text{if } v < 0 \end{cases}$$

for $v = 1$ and $t > 0$

$$y^t_\varepsilon = \left\{ \begin{array}{ll} (\varepsilon - t)/(\varepsilon + t), & 0 \leqslant t \leqslant \varepsilon \\ 0, & \varepsilon < t \end{array} \right\} y^t_0 = 0,$$

$$p^t_\varepsilon = \left\{ \begin{array}{ll} (\varepsilon - t)/\varepsilon(\varepsilon + 1), & 0 \leqslant t \leqslant \varepsilon \\ 0, & \varepsilon < t \end{array} \right\} p^t_0 = 0.$$

But for $t = 0$

$$y^0_\varepsilon = \varepsilon/(\varepsilon + 1), \quad y^0_0 = 0, \quad p^0_t = 1/(\varepsilon + 1), \quad p^0_0 = 1.$$

As a result

$$\lim_{t \searrow 0} p^t_0 = 0 \neq 1 = p^0_0.$$

For $v = -1$ and $t \geqslant 0$

$$y^t_\varepsilon = \frac{\varepsilon + t}{\varepsilon + 1}, \quad p^t_\varepsilon = \frac{1 - t}{\varepsilon + 1}, \quad p^t_0 = 1 - t \to p^0_0 = 1.$$

Finally

$$dJ_\varepsilon(t) = p_\varepsilon^t \tag{15}$$

and in each case we recover the results at the begining. ■

PROPOSITION 1. – (i) Assume that hypotheses H1 to H14 (H7 for all t in $[0, T]$) hold, that the map (3.38) is linear and that

H15 $t, \psi \mapsto d^2E(t, y_0^t; 1, 0; 0, \psi)$: $0, T] \times V$ (weak) is continuous.

H16 $t \mapsto dF(t, y_0^t; 1, 0)$ is continuous at $t = 0$.

H17 $p_0^t \to p$ (unique) in V (weak).

Then

$$dJ_0(0) = dF(0, y_0^0; 1, 0) + d^2E(0, y_0^0; 1, 0; 0, p) . \tag{16}$$

(ii) If, in addition, $p = p_0^0$, then p is completely characterized by (11) with $t = 0$. ■

When the cones $S(t)$ have an appropriate behaviour as t goes to 0, it is possible to obtain a variational equation for the limit point p of p_0^t as t goes to zero.

PROPOSITION 2. – Assume that the hypotheses of Proposition 1 (i) hold and that

H18
$$\lim_{t \searrow 0} dF(t, y_0^t; 0, \psi) = dF(0, y_0^0; 0, \psi) , \quad \forall \psi \in V ,$$
$$\liminf_{t \searrow 0} dF(t, y_0^t; 0, p_0^t) \geq dF(0, y_0^0; 0, p) .$$

H19
$$\lim_{t \searrow 0} d^2E(t, y_0^t; 0, \psi; 0, p_0^t) = d^2E(0, y_0^0; 0, \psi; 0, p) ,$$
$$\liminf_{s \searrow 0} d^2E(t, y_0^t; 0, p_0^t; 0, p_0^t) \geq d^2E(0, y_0^0; 0, p; 0, p) .$$

H20 $\exists T > 0$ such that

$$\forall 0 < t_1 \leq t_2 \leq T , \quad S(t_1) \subset S(t_2) . \tag{17}$$

Then p is the unique solution in the closed convex cone

$$S = \bigcap_{0 < t \leq T} S(t) \tag{18}$$

of the variational inequality

$$\begin{cases} p \in S , \quad \forall \psi \in S \\ dF(0, y_0; 0, \psi - p) + d^2E(0, y_0; 0, \psi - p; 0, p) \geq 0 . \end{cases} \tag{19}$$ ■

5. – Shape derivative for the radiator problem.

Let $V = V(t, x_1, x_2, z)$ be a velocity field, $V \in C^0([0, T], C^1(\boldsymbol{R}^3; \boldsymbol{R}^3))$ such that

$$V(t, x_1, x_2, 0) = 0 . \tag{1}$$

since Σ_1 is invariant in the deformation of the domain. If the field V is written as $V = (V_{x_1}, V_{x_2}, V_z)$, then the condition

$$V_z(t, x_1, x_2, L) = 0 \tag{2}$$

implies that Σ_3 remains linear in the deformation.

Denote by $T_t = T_t(V)$, the transformation associated to V

$$\frac{dT}{dt}(V)X = V(t, T_t(V)X) , \quad T_0(V)X = X , \quad t \geqslant 0 .$$

Consider the matrix

$$A(t) = J(t)(DT_t)^{-1} \cdot {}^*(DT_t)^{-1}$$

where DT_t is the Jacobian matrix of T_t and $J(t) = \det(DT_t)$. On Σ_3 $J(t)$ is to be understood as $\det(D\tilde{T}_t)$ where $\tilde{T}_t$ is a mapping from $\boldsymbol{R}^2$ in $\boldsymbol{R}^2$, namely

$$\tilde{T}_t(x_1, x_2) = T_t(x_1, x_2, L)$$

and $D\tilde{T}_t$ is the 2×2 matrix. In fact $\tilde{T}_t$ is the transformation associated with the velocity field

$$\tilde{V}(t, x_{1_1}, x_2) = \big(V_x(t, x_1, x_2, L), V_{x_2}(t, x_1, x_2, L)\big) .$$

For $t \in [0, T]$ and any matrix norm we have

$$\left\{ \begin{array}{ll} \|A(t)(x)\| \leqslant C \|T_t\|_{W^{1,\infty}(\Omega)} , & \forall x \in \bar{\Omega} \\ |J(t)(x)| \leqslant C \|T_t\|_{W^{1,\infty}(\Omega)} , & \forall x \in \bar{\Omega} . \end{array} \right. \tag{3}$$

Now the norm $\|T_t\|_{W^{1,\infty}(\Omega)}$ is continuous in t and a fortiori bounded in $0, T]$. We also recall (from J. P. ZOLÉSIO [1]) that the following continuity properties

$$\left\{ \begin{array}{ll} \|A(t) - A(s)\|_{L^\infty(\Omega)} \to 0 & \text{when } s \to t \\ |J(t) - J(s)|_{L^\infty(\Omega)} \to 0 & \text{when } s \to t . \end{array} \right. \tag{4}$$

Denote by Ω_t the perturbed domain

$$\Omega_t = T_t(V)\,(\Omega)$$

with its boundory in three pieces:

$$\Sigma_i^t = T_t(V)(\Sigma_i)\,, \quad 1 \leqslant i \leqslant 3\,.$$

But from (1) $\Sigma_1^t = \Sigma_1$.

For each t in $[0, T]$ we consider the Banach space

$$B_t = B(\Omega_t) = \{\varphi \in H^1(\Omega_t):\ \varphi|_{\Sigma_3^t} \in L^5(\Sigma_3^t)\}$$

and the energy functional defined on this space:

$$E_t(\varphi) = \int_{\Omega_t} \tfrac{1}{2}|\nabla\varphi|^2\,dx + \int_{\Sigma_3^t} (\tfrac{1}{5}|\varphi|^5 - \varphi q_s)\,d\Sigma - \int_{\Sigma_1} q_i\varphi\,d\Sigma$$

E_t is convex, lower semi-continuous on B_t and there exists a unique element $y_t \in B_t$ which minimizes E_t on B_t (see M. C. DELFOUR, G. PAYRE, J. P. ZOLÉSIO [1, 2]).

The cost function associated to the radiator problem is $F_t\colon B_t \to \mathbf{R}^+$ defined by

$$F_t(\varphi) = \int_{\Omega_t} [(\varphi - T_1)^+]^2\,dx\,.$$

Then a unique element $y_{\varepsilon,t} \in B_t$ minimizes on B_t the penalized energy:

(5) $$\varepsilon > 0\,, \quad E_t(y_{\varepsilon,t}) + \varepsilon F_t(y_{\varepsilon,t}) \leqslant E_t(\varphi) + \varepsilon F_t(\varphi)\,, \quad \forall \varphi \in B_t\,.$$

Consider the function $\hat{y} = \max(y_{\varepsilon,t}, q_s^{\frac{1}{4}})$ and assume that

(6) $$T_1 > q_s^{\frac{1}{4}}$$

then

$$(\hat{y} - T_1)^+ = (y - T_1)^+$$

and $F_t(\hat{y}) = F_t(y_{\varepsilon,t})$; from M. C. DELFOUR, G. PAYRE, J. P. ZOLÉSIO [1, 2] we then know that

$$E_t(\hat{y}) + \varepsilon F_t(\hat{y}) \leqslant E_t(y_{\varepsilon,t}) + \varepsilon F_i(y_{\varepsilon,t})\,.$$

By uniqueness of the minimum in (5) we get $\hat{y} = y_{\varepsilon,t}$ that is:

(7) $$y_{\varepsilon,t} \geqslant q_s^{\frac{1}{4}} \quad \text{on } \Omega_t\,.$$

It is immediate that

(8) $$\Delta y_{\varepsilon,t} = \varepsilon(y_{\varepsilon,t} - T_1)^+ \quad \text{in } \Omega_t\,.$$

Then, $\Delta y_{\varepsilon,t}$ being in $L^2(\Omega_t)$, $(\partial/\partial n) y_{\varepsilon,t}$ is defined on $H^{-\frac{1}{2}}(\partial\Omega_t)$ and

$$\frac{\partial}{\partial n} y_{\varepsilon,t} = 0 \quad \text{on } \Sigma_2^t \tag{9}$$

$$\frac{\partial}{\partial n} y_{\varepsilon,t} = q_i \quad \text{on } \Sigma_1 . \tag{10}$$

And the radiating (non linear) condition

$$\frac{\partial}{\partial n} y_{\varepsilon,t} + (y_{\varepsilon,t})^4 = q_s \quad \text{on } \Sigma_3^t . \tag{11}$$

We suppose now that $y_{\varepsilon,t}|\Sigma_3^t$ has an upper bound (which is compatible with the fact that, from (11) and (7) $(\partial/\partial n) y_{\varepsilon,t} \leqslant 0$ on Σ_0^t).

It can be easily verified that $y_{\varepsilon,t}$ is continuous outside of $\overline{\Sigma_3^t}$ in $\bar{\Omega}_t$: for example by introducing, for any $\alpha > 0$, the function

$$g_\alpha(x_1, x_2, z) = y_{\varepsilon,t}(x_1, x_2\, z)\, \varrho_\alpha(z)$$

where $0 \leqslant \varrho_\alpha \leqslant 1$ is a C^∞ function on $[0, L]$ such that $\varrho(z) = 1$ for$0 \leqslant z \leqslant L - 2\alpha$, and $\varrho(z) = 0$ for $L - \alpha \leqslant z \leqslant L$. In particular $g_\alpha = y_{\varepsilon,t}$ in a neighbourhood of Σ_1; we have

$$g_\alpha = 0 \qquad \text{on } \{z = L - \alpha\} \cap \bar{\Omega}_t$$

$$\frac{\partial}{\partial n} g_\alpha = 0 \qquad \text{on } \Sigma_2^t \cap \{z \leqslant L - \alpha\}$$

$$\frac{\partial}{\partial n} g_\alpha = q_i \qquad \text{on } \Sigma_1$$

and

$$\Delta g_\alpha = \varrho''_\alpha(z) y_{\varepsilon,t} + 2\varrho'_\alpha \frac{\partial}{\partial z} y_{\varepsilon,t}$$

belongs to $L^2(\Omega_t)$.

The g_α is the solution of a linear well posed boundary problem on $\Omega_t \cap \{z < L - \alpha\}$ and we know that $g_\alpha \in C^0(\bar{\Omega}_t)$; then by the Masimum Principle (see PROTTER and WEINBERGER [1]) we know that the maximum for g_α on $\bar{\Omega}_t$ is achieved at a boundary point M at which $(\partial/\partial n) g(M) > 0$.

This point M can only be located on Σ_1. Then for each $\alpha > 0$, $y_{\varepsilon,t}\, \Omega_t \cap \{z < L - 2\alpha\}$ reaches its maximum on Σ_1. But since $y_{\varepsilon,t}$ is upper bounded on Σ_3^t we also have $y_{\varepsilon,t}$ reaching its maximum on Σ_1. Now it would be possible to obtain the continuity with respect to (t, ε) of $\max\{y_{\varepsilon,t}(x) : x \in \Sigma_1\} = \max\{g_\alpha : x \in \Sigma_1\}$. Thus this maximum is bounded for $(t, \varepsilon) \in [0, T] \times [0, \bar{\varepsilon}]$:

$$\begin{cases} \exists M, \quad \forall \varepsilon \in [0, \bar{\varepsilon}], \quad \forall t \in [0, T], \quad \forall x \in \bar{\Omega}_t \\ q_s^{\frac{1}{4}} \leqslant y_{\varepsilon,t}(x) \leqslant M . \end{cases} \tag{12}$$

Consider now

$$y_\varepsilon^t = y_{\varepsilon,t} \circ T_t \,. \tag{13}$$

It is the unique element of $B(\Omega) = B_0$ which minimizes on B_0 the functional

$$E(t, \varphi) + \varepsilon F(t, \varphi)$$

where

$$E(t, \varphi) = E_t(\varphi \circ T_t^{-1}) = \\ = \tfrac{1}{2}\int_\Omega \langle A(t)\cdot \nabla\varphi, \nabla\varphi\rangle\, dx + \int_{\Sigma_3} \left(\tfrac{1}{5}|\varphi|^5 - q_s\varphi\right) J(t)\, d\Sigma - \int_{\Sigma_1} q_i \varphi\, d\Sigma \tag{14}$$

and

$$F(t, \varphi) = F_t(\varphi \circ T^{-1}) = \int_\Omega [(\varphi - T_1)^+]^2 J(t)\, dx \,. \tag{15}$$

Obviously from (45) and (50), we have

$$u_e = q_i^{\frac{1}{4}} \leqslant y_\varepsilon^t(x) \leqslant M\,, \qquad \forall x \in \bar{\Omega},\ \forall t \in [0, T],\ \forall \varepsilon \in [0, \bar{\varepsilon}] \,. \tag{16}$$

To obtain the coercivity of the second derivative

$$\varphi \mapsto d^2E(t, y_0^t;\, 0, \varphi;\, 0, \varphi)$$

we need now to introduce the closed convex subset of $B(\Omega)$:

$$K = \left\{\varphi \in B(\Omega) : \frac{u_e}{2} \leqslant \varphi \leqslant M \text{ a.e. on } \bar{\Omega}_t\right\}. \tag{17}$$

From (16) we get $y_\varepsilon^t \in K$ for any ε and t.

We now turn to the verification of hypothesis H8, the continuity of $t \mapsto y_\varepsilon^t$ in $B(\Omega)$; $\varepsilon > 0$.

LEMMA 1. – $\exists C > 0$, s.t. $\forall t \in [0, T]$, $\forall \varepsilon \in [0, \bar{\varepsilon}]$,

$$\|y^t\|_{B(\Omega)} \leqslant C \,. \tag{18}$$

PROOF. – We have

$$\|y^t\|_B \leqslant \|y_{\varepsilon,t}\|_{B_t} \|T_t\|_{W^{1,\infty}(\Omega)} \,.$$

But,

$$E_t(y_{\varepsilon,t}) \leqslant E_t(0) = 0\,,$$

that is

$$\int_{\Omega_t} \tfrac{1}{2}|\nabla y_{\varepsilon,t}|^2\,dx + \tfrac{1}{5}\int_{\Sigma_3^t} |y_{\varepsilon,t}|^5\,d\Sigma \leqslant \int_{\Sigma_1} \{|y_{\varepsilon,t}|q_i + \varepsilon[(y_{\varepsilon,t} - T_1)^+]^2\}\,d\Sigma\,.$$

By (16) we get

$$\leqslant [C + \varepsilon(C - T_1)^2] \quad \text{measure} \quad (\Sigma_1) = a\,.$$

Then it is immediate that: $\|y_\varepsilon^t\|_{B_t} \leqslant \sqrt{a} + a^{\frac{1}{5}}$. ∎

LEMMA 2. – $\forall \varepsilon \geqslant 0$, $\|y_\varepsilon^s - y_\varepsilon^t\|_{B(\Omega)} \to 0$ as $s \to t$.

PROOF. – y_ε^s and y_ε^t are the two elements of B characterized by the variational equations:

$$\forall \varphi \in B\,, \quad dE(s, y_\varepsilon^r; 0, \varphi) + \varepsilon\, dF(s, y_\varepsilon^s; 0, \varphi) = 0$$

$$dE(t, y_\varepsilon^t; 0, \varphi) + \varepsilon\, dF(t, y_\varepsilon^t; 0, \varphi) = 0\,.$$

By substracting these equations, taking $z = y_\varepsilon^t - y_\varepsilon^s$ and $\varphi = z$ we get:

$$\text{(19)} \quad \int_\Omega \langle A(t)\cdot\nabla z, \nabla z\rangle\,dx + \int_{\Sigma_3} J(t)[(y_\varepsilon^t)^4 - (y_\varepsilon^s)^4]z\,d\Sigma + \\ + \varepsilon\int_\Omega [(y_\varepsilon^t - M)^+ - (y_\varepsilon^s - M)^+]J(t)\,dx = -\int_\Omega \langle (A(t) - A(s))\cdot\nabla y_\varepsilon^s, \nabla z\rangle\,dx - \\ - \int_{\Sigma_3} (J(t) - J(s))(y_\varepsilon^s)^4 z\,d\Sigma - \varepsilon\int_\Omega (J(t) - J(s))(y_\varepsilon^s - M)^+ z\,dx\,.$$

From (18) and (3), (4) it can easily be verified that the Right-Hand-Side of (19) goes to zero as s goes to t.

On the other side we have the monotony inequalities

$$(a^4 - b^4)(a - b) \geqslant \tfrac{1}{8}|a - b|^5 \quad \text{and} \quad [(a - T_1)^+ - (b - T_1)^+](a - b) \geqslant 0\,,$$

combining these two inequalities with the fact that $J(t) \geqslant 0$ on $\bar{\Omega}$ (for $J(t) \to 1$ in $C^0(\bar{\Omega})$ when $t \to 0$) we get in (19):

$$\int_\Omega \langle A(t)\cdot\nabla z, \nabla z\rangle\,dx \to 0\,, \quad s \to t$$

and

$$\int_{\Sigma_3} J(t)|z|^5\,d_{\Sigma_3} \to 0\,; \quad s \to t\,.$$

Now going back to the moving domain Ω_t we get $\|z\circ T_t^{-1}\|_{B(\Omega_t)}\to 0$ but

$$\|z\circ T_t^{-1}\|_{B(\Omega_t)}\geqslant \|T_t\|^{-1}_{W^{1,\infty}(\Omega)}\|z\|_{B(\Omega)}\,.$$

Then we get $\|z\|_{B(\Omega)}\to 0$ as $s\to T$. ■

5.1. *Derivatives of E and F.*

We recall (from J. P. ZOLÉSIO [1], [2]) that $t\mapsto A(t)$ and $t\mapsto J(t)$ are differentiable from $[0, T]$ in $L^\infty(\Omega)$ and that the derivatives are given by

$$A'(t)=\operatorname{div} V(t)I_d-\left(DV(t)+{}^*DV(t)\right($$

$$J'(t)=\operatorname{div} V(t)\,.$$

Then for all φ in B we get the existence of

$$dE(t,\varphi;1,0)=\int_\Omega \tfrac{1}{2}\langle A'(t)\cdot\nabla\varphi,\nabla\varphi\rangle\,dx+\int_{\Sigma_3}\left(\tfrac{1}{5}|\varphi|^5-q_s\varphi\right)J'(t)\,d\Sigma$$

also we have, for $\varphi,\psi\in B(\Omega)$:

$$dE(t,\varphi;0,\psi)=\int_\Omega\langle A(t)\cdot\nabla\varphi,\nabla\psi\rangle\,dx+\int_{\Sigma_3}\left(|\varphi|^3\varphi-q_s\right)\psi J(t)\,d\Sigma$$

and for $\varphi\in K$, ξ, $\psi\in B$

$$d^2E(t;\varphi;0,\psi;0,\xi)=\int_\Omega\langle A(t)\cdot\nabla\xi,\nabla\psi\rangle\,dx+4\int_{\Sigma_3}|\varphi|^3\psi\xi J(t)\,d\Sigma\,.$$

Moreover:

$$d^2E(t,\varphi;0,\psi;0,\psi)\geqslant\int_{\Omega_t}|\nabla(\psi\circ T_t^{-1})|^2\,dx+\frac{u_e^3}{2}\int_{\Sigma_3^t}(\psi\circ T_t^{-1})^2\,d\Sigma\geqslant$$

$$\geqslant\operatorname{Min}\left(1,\frac{u_e^3}{2}\right)\|\psi\circ T_t^{-1}\|_{H^1(\Omega_t)}\geqslant\operatorname{Min}\left(1,\frac{u_e^3}{2}\right)\|T_t\|^{-1}_{W^{1,\infty}(\Omega)}\|\psi\|^2_{H^1(\Omega)}\,.$$

5.2. *Characterization of the convex set S_t.*

The gradient of $E(t,\cdot)$ at y_0^t is zero for y_0^t minimizes $E(t,0)$ on all the Banach space B, that is $dE(t,y_0^t;0,\varphi)=0$, $\forall\varphi\in B$. Then:

$$\{\varphi\in V \text{ s.t. } dE(t,y_0^t;0,\varphi)=0\}=H^1(\Omega)\,.$$

Then to characterize S_t we just have to consider the tangent cone: for this we have the

LEMMA 3.

$$T_{y_0^t}(K) = H^1(\Omega) .$$

PROOF. – We first obtain

$$\{\lambda(\varphi - y_0^t) \text{ s.t. } \lambda \geqslant 0,\ \varphi \in K\} = L^\infty(\Omega) \cap H^1(\Omega)$$

for $K \subset L^\infty(\Omega)$ and y_0^t an interior point (in $L^\infty(\Omega)$ topology to K). Then we conclude by density of $L^\infty(\Omega) \cap H^1(\Omega)$ in $H^1(\Omega)$. ■

We turn now to the verification of hypotheses H5, H6 and H20.

Let $p_n = p_{\varepsilon_n}^{t_n}$ converge weakly in $H^1(\Omega)$ to q (since p_ε^t is bounded in $H^1(\Omega)$, from Lemma 1, independently on $\varepsilon \geqslant 0$ and t).

Then this convergence is true in $H^s(\Omega)$, strongly for any $s < 1$ and the traces on Σ_3 converge in $H^{s-\frac{1}{2}}(\Sigma_3)$ then in $L^\alpha(\Sigma_3)$ for any $\alpha < 4$. In particular $(p_n)^2$ converges to q^2 strongly in $L^{\frac{3}{2}}(\Sigma_3)$. To verify H5, H6 and H20 it is now a direct application of the following.

LEMMA 4. – $\forall \varepsilon \geqslant 0$, for any sequence $t_n \to s$ there exists a subsequence t_m such that

$$y_\varepsilon^{t_m}|_{\Sigma_3} \to y_\varepsilon^t|_{\Sigma_3} \quad \text{in } L^p(\Sigma_3),\ m \to \infty ,$$

for any p, $1 \leqslant p < \infty$.

(This subsequence converges in all the $L^p(\Sigma_3)'s$).

PROOF. – We have established that $y_\varepsilon^{t_n}$ converges to y_ε^0 in $B(\Omega)$; then the traces on Σ_3 converge in $L^5(\Sigma_3)$. So there exists a subsequence which converges almost every where on Σ_3. But

$$|y_\varepsilon^{t_n}| \leqslant M \quad \text{on } \Sigma_3 ;$$

so this subsequence, written y^m for simplicity, verifies

$$|y^m|^p \to |y_\varepsilon^t|^p \quad \text{a.e. on } \Sigma_3$$

$$|y^m|^p \leqslant M^p \quad \text{a.e. on } \Sigma_3 .$$

By the Lebesgue convergence theorem we get the convergence of $|y^m|^p$ to $|y_\varepsilon^t|^p$ in $L^1(\Sigma_3)$ that is that y^m converges to y_s^t in $L^p(\Sigma_3)$. ■

Now Proposition 7 (in DELFOUR-PAYRE-ZOLÉSIO 1]) can be directly applied to the radiator problem and we get the

THEOREM 4. – The domain Ω being described in the first section, let $y(\Omega) \in B(\Omega)$ be the solution of

$$\operatorname*{Min}_{\varphi \in B(\Omega)} \int_\Omega \tfrac{1}{2}|\nabla\varphi|^2\,dx + \int_{\Sigma_3} \left(\tfrac{1}{5}|\varphi|^5 - \varphi q_s\right) d\Sigma - \int_{\Sigma_3} q_i \varphi\, d\Sigma\,.$$

For any admissible velocity field V (such that (39), (40)) let $y(\Omega_t)$ be the associated solution on Ω_t and

$$J(\Omega_t) = \int_{\Omega_t} [(y(\Omega_t) - T_1)^+]^2\, d\Sigma$$

with $T_1 > q_s^{\frac{1}{4}}$.

Then the Eulerian derivative of J at Ω in the direction $V \in C^0([0, T], C^1(\mathbf{R}^3; \mathbf{R}^3))$ exists and is given by

$$dJ(\Omega; V) \stackrel{\text{def}}{=} \lim_{\substack{\geqslant \\ t \to 0}} \big(J(\Omega_t) - J(\Omega)\big)/t = \\ = \int_\Omega (y - T_1)^+ p\, dx + \int_\Omega \langle A'(0)\cdot\nabla y, \nabla p\rangle\, dx + \int_{\Sigma_3} J'(0)(y^4 - q_s)p\, d\Sigma + \\ \tfrac{1}{2}\int_\Omega J_0(0)[(y - T_1)^+]^2\, dx$$

where $y = y(\Omega)$ and $p = p(\Omega)$ are respectively the element of $B(\Omega)$ and $H^1(\Omega)$ characterized by the problems

$$\int_\Omega \langle \nabla y, \nabla\varphi\rangle\, dx + \int_{\Omega_3} (|y|^3 y - q_s)\varphi\, d\sigma = \int_{\Sigma_1} q_i \varphi\, d\sigma\,, \qquad \forall \varphi \in B$$

$$\int_\Omega \langle \nabla p, \nabla\psi\rangle\, dx + 4\int_{\Sigma_3} y^3 p\psi\, d\Sigma = \int_\Omega (y - T_1)^+ \psi\, d\Sigma\,, \qquad \forall \psi \in H^1(\Omega)$$

and

$$A'(0) = \operatorname{div} V(0) I_d - \big(DV(0) + {}^*DV(0)\big)$$

$$J'(0) = \partial_{x_1} V_{x_1}(0, x_1, x_2, L) + \partial_{x_2} V_{x_2}(0, x_1, x_2, L) \qquad \text{on } \Sigma_3\,.$$

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