A Class of Strongly Nonlinear Functional Differential Equations (*).

ENZO MITIDIERI - IOAN I. VRABIE (**)

Summary. – *Let H be a real Hilbert space, $\varphi: H \to [0, +\infty]$ a proper l.s.c., convex function with $L_k := \{u \in H; \|u\|^2 + \varphi(u) \leq k\}$ compact for every $k > 0$, let $\tau > 0$ be a given constant and $C_{\partial\varphi}([-\tau, 0]; H) := \{v \in C([-\tau, 0]; H);\ v(t) \in D(\partial\varphi)$ a.e. for $t \in (-\tau, 0)\}$. We prove an existence result for strong solutions to a class of functional differential equations of the form*

$$u'(t) + \partial\varphi(u(t)) \in F(t, u(t), u_t)\,, \qquad 0 < t < T$$
$$u(s) = v(s)\,, \qquad -\tau \leq s \leq 0\,,$$

where $F: [0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H) \to H$ satisfies a certain demiclosedness condition, while $v \in C_{\partial\varphi}([-\tau, 0]; H)$, $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2 ds < +\infty$.

1. – Introduction.

Throughout this paper H is a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, $\varphi: H \to [0, +\infty]$ is a proper l.s.c. convex function and $\partial\varphi$ is the subdifferential of φ.

If $[a, b]$ is an interval in $\mathbb{R}$, then $C_{\partial\varphi}([a, b]; H)$ denotes the subset of all functions v belonging to $C([a, b]; H)$ and satisfying $v(t) \in D(\partial\varphi)$ a.e. for $t \in (a, b)$. We emphasize that in all that follows $C_{\partial\varphi}([a, b]; H)$ is endowed with the sup. norm topology of $C([a, b]; H)$. Let $\tau > 0$ and $T > 0$ be two given constants.

We recall that if $u \in C_{\partial\varphi}([-\tau, T]; H)$ and $t \in [0, T]$, then $u_t: [-\tau, 0] \to H$ is defined by $u_t(s) := u(t + s)$, for every $s \in [-\tau, 0]$. Obviously, whenever $u \in C_{\partial\varphi}([-\tau, T]; H)$ and $t \in [0, T]$, u_t belongs to $C_{\partial\varphi}([-\tau, 0]; H)$.

Let $F: [0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H) \to H$ be a given function, and let us

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Indirizzo degli AA.: E. MITIDIERI: Dipartimento di Scienze Matematiche, Università degli Studi di Trieste, Piazzale Europa 1, 34100 Trieste (Italy); I. I. VRABIE: Department of Mathematics, Polytechnic, Institute of Iași R. 6600 and P. O. Box 180, Ro, Is 1 6600, Iași 6600, R.S. Romania.

consider the following functional differential equation

$$(1.1)\qquad \begin{cases} u'(t) + \partial\varphi(u(t)) \ni F(t, u(t), u_t)\,, & 0 < t < T\,, \\ u(s) = v(s)\,, & -\tau \leqq s \leqq 0\,, \end{cases}$$

where $v \in C_{\partial\varphi}([-\tau, 0]; H)$, $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2\, ds < +\infty$.

The main goal of this paper is to prove an existence result for strong solutions to (1.1). Basic sources of references for this kind of problems are the monograph of HALE [10] and the survey paper of WEBB [23].

Although functional differential equations have been intensively studied over the past several years by many authors (see for instance [7-14, 16-20, 23] and the references therein) as far as we know, this is the first attempt to overcome the difficulties encountered when F is defined merely on $[0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H)$ and $-F$ lacks monotonicity with respect to its second and third arguments. The possibility of considering such functions F which, in general, are discontinuous with respect to the induced strong topology on both domain and range, allows us to obtain as particular cases of our main existence theorem new results refering to strongly nonlinear partial differential equations of functional type.

The interesting feature of this class of equations consists in that the state of the system depends not only on its history but also on the history of its « diffusion ». See the examples in Section 6.

We emphasize that our results seem to be new even in the semilinear case, i.e., when $\partial\varphi$ is linear.

The method of proof which is partially inspired from [15] is mainly based on a fixed point theorem due to ARINO, GAUTIER and PENOT [1] and rests heavily on a deep regularity result due to BREZIS [3].

The paper is divided into six Sections, the second one being mainly devoted to some notations, definitions and results widely used in all that follows. In Section 3 we state our main result, while Section 4 is merely concerned with its complete proof. Section 5 contains two results concerning the continuation of the solutions, while in the last Section 6 we analyze some examples in order to emphasize the effectiveness of the abstract results.

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2. – Preliminaries.

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ and let $\varphi\colon H \to [0, +\infty]$ be a proper, l.s.c., convex function. Set

$$D(\varphi) := \{u \in H;\ \varphi(u) < +\infty\}\,,$$

$$\partial\varphi(u) := \{v \in H;\ \varphi(w) - \varphi(u) \geqq (v, w - u),\ w \in H\}\,,$$

and

$$D(\partial\varphi) := \{u \in H;\ \partial\varphi(u) \neq \Phi\}\ .$$

We recall that $D(\varphi)$ is called the *effective domain* of φ, $\partial\varphi(u)$ the *subdifferential of φ calculated at u*, and the operator $\partial\varphi$: $D(\partial\varphi) \subset H \to 2^H$ which assigns to each $u \in D(\partial\varphi)$ the set $\partial\varphi(u)$—the *subdifferential* of φ. We have

$$D(\partial\varphi) \subset D(\varphi) \quad \text{and} \quad \overline{D(\partial\varphi)} = \overline{D(\varphi)}\ .$$

It is well known [4] that $\partial\varphi$ is a maximal monotone operator.

Let us consider the following quasi-autonomous evolution equation

$$\tag{2.1} \begin{cases} u'(t) + \partial\varphi(u(t)) \ni f(t)\,, & 0 < t < T\,, \\ u(0) = u_0\,, \end{cases}$$

where φ is as above, $f \in L^2([0, T]; H)$ and $u_0 \in \overline{D(\varphi)}$.

By a strong solution to (2.1) we mean a function $u \in W^{1,2}([0, T]; H)$ with $u(0) = u_0$, $u(t) \in D(\partial\varphi)$ a.e. for $t \in (0, T)$ and satisfying

$$u'(t) + \big(\partial\varphi(u(t)) - f(t)\big)^0 = 0 \qquad \text{a.e. for } t \in (0, T)\,,$$

where, if C is a nonempty, closed convex set in H, C^0 denotes the unique element of C having minimal norm.

We recall for easy reference the following result due to Brézis (see [3]).

THEOREM 2.1. – *Let φ: $H \to [0, +\infty]$ be a proper, l.s.c., convex function. Then, for each $u_0 \in D(\varphi)$ and $f \in L^2([0, T]; H)$ there exists a unique strong solution to* (2.1) *such that $t \to \varphi(u(t))$ is absolutely continuous from $[0, T]$ into $\mathbb{R}^+$ and which, in addition satisfies*

$$\tag{2.2} \|u'(t)\|^2 + \frac{d}{dt}\varphi(u(t)) = (f(t), u'(t)) \qquad \text{a.e. for } t \in (0, T)\,,$$

and

$$\tag{2.3} \left(\int_0^{T_0} \|u'(t)\|^2\, dt\right)^{\frac{1}{2}} \leq \left(\int_0^{T_0} \|f(t)\|^2\, dt\right)^{\frac{1}{2}} + \sqrt{\varphi(u_0)}\ .$$

DEFINITION 2.1. – A proper, l.s.c., convex function φ: $H \to [0, +\infty]$ is called of *compact type* if for each $k > 0$ the set

$$K_k := \{u \in H;\ \|u\|^2 + \varphi(u) \leq k\}$$

is compact in H.

REMARK 2.1. – Obviously, φ is of compact type if and only if for each $k > 0$ the set

$$M_k := \{u \in H;\ \|u\| + \varphi(u) \leqq k\}$$

is compact in H.

We emphasize that we prefer to use L_k instead of M_k because, in applications, whenever $\partial\varphi$ is a partial differential operator, it is much easier to check the compactness of L_k than that of M_K.

The next result is an obvious instance of [21, Corollary 2.3.2].

PROPOSITION 2.1. – *Let φ: $H \to [0, +\infty]$ be a proper, l.s.c., convex function of compact type and let $u_0 \in D(\varphi)$ be fixed.*

Then the solution mapping $f \to u$—which assigns to each f belonging to $L^2([0, T]; H)$ the unique strong solution to (2.1) corresponding to u_0 and to f—is sequentially continuous from $L^2([0, T]; H)$ endowed with its weak topology into $C([0, T]; H)$ endowed with its strong topology.

Finally, we recall the following fixed point theorem due to ARINO, GAUTIER and PENOT [1].

THEOREM 2.2. – *Let C be a nonempty, convex and weakly compact subset in a separated locally convex vector space. If $\mathbb{P}$: $C \to C$ is a function which is weakly weakly sequentially continuous, then $\mathbb{P}$ has at least one fixed point.*

2. – The main result.

We begin by explaining what we mean by a strong solution to (1.1). Namely, we introduce

DEFINITION 3.1. – A function u: $[-\tau, T] \to H$ is called a *strong solution* to the problem (1.1) on $[0, T]$ if

(S_1) $u(s) = v(s)$ for each $s \in [-\tau, 0]$.

(S_2) $u_t \in C_{\partial\varphi}([-\tau, 0]; H)$ for each $t \in [0, T]$.

(S_3) The function f: $[0, T] \to H$, defined by $f(t) := F(t, u(t), u_t)$ a.e. for $t \in (0, T)$, belongs to $L^2([0, T]; H)$.

(S_4) u is a strong solution of (2.1) with f as above and $u_0 = v(0)$ in the sense indicated in Section 2.

Let $v \in C_{\partial\varphi}([-\tau, 0]; H)$ be fixed and let $u \in C([0, T_0]; H)$ be a given function with $u(0) = v(0)$. In all that follows we denote by $\hat{u}$ the function $\hat{u}$: $[-\tau, T_0] \to H$

defined by

$$\hat{u}(t) := \begin{cases} v(t) & \text{for } t \in [-\tau, 0]\,, \\ u(t) & \text{for } t \in (0, T_0]\,. \end{cases} \tag{3.1}$$

DEFINITION 3.2. – A function F: $[0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H)$ is called *$\partial\varphi$-demiclosed* if for each $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2\, ds < +\infty$, and for each $T_0 \in (0, T]$, the following conditions are satisfied

(i) For each $u \in C_{\partial\varphi}([0, T_0]; H) \cap W^{1,2}([0, T_0]; H)$ with $u(0) = v(0)$ and for which there exists $g \in L^2([0, T_0]; H)$ with $g(t) \in \partial\varphi(u(t))$ a.e. for $t \in (0, T_0)$, the function $t \to F(t, u(t), u_t)$ is strongly measurable from $[0, T_0]$ into H.

(ii) If (u_n) is a sequence in $C_{\partial\varphi}([0, T_0]; H) \cap W^{1,2}([0, T_0]; H)$ with $u_n(0) = v(0)$ for each $n \in \mathbb{N}$ and for which there exists (g_n) in $L^2([0, T]; H)$ with $g_n(t) \in \partial\varphi(u_n(t))$ for each $n \in \mathbb{N}$ and a.e. for $t \in (0, T_0)$ and if, in addition $\lim u_n = u$ in $C([0, T_0]; H)$, $u \in C_{\partial\varphi}([0, T_0]; H)$,

$$w\text{-}\lim u'_u = u' \quad \text{in } L^2([0, T_0]; H)\,,$$

$$w\text{-}\lim g_n = g \quad \text{in } L^2([0, T_0]; H)\,,$$

$$w\text{-}\lim p_n = p \quad \text{in } L^2([0, T_0]; H)\,,$$

where, for each $n \in \mathbb{N}$, p_n: $[0, T_0] \to H$ is defined by

$$p_n(t) := F(t, u_n(t), \hat{u}_{n_t}) \quad \text{a.e. for } t \in (0, T_0)\,,$$

then

$$p(t) = F(t, u(t), \hat{u}_t) \quad \text{a.e. for } t \in (0, T_0)\,.$$

DEFINITION 3.3. – A function F: $[0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H) \to H$ is called *$\partial\varphi$-dominated* if there exists two non decreasing functions $l(\cdot)$, $l_0(\cdot)$ from $\mathbb{R}^+$ into $\mathbb{R}^+$, $c_1, c_2 \in L^1([0, T]; \mathbb{R}^+)$ and a constant $k \in (0, 1)$ such that

$$\|F(t, u, v)\|^2 \leq k\|\partial\varphi^0(u)\|^2 + l(\|u\| + \varphi(u)) \cdot \left[\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2\, ds + l_0(\varphi(u))\, c_1(t) + c_2(t)\right]$$

a.e. for $t \in (0, T)$, for each $u \in D(\partial\varphi)$ and $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2\, ds < +\infty$.

Now, we are able to proceed to the statement of our main result.

THEOREM 3.1. – *Let φ: $H \to [0, +\infty]$ be a proper, l.s.c., convex function of compact type and let F: $[0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H) \to H$ be a function which is both $\partial\varphi$-dominated and $\partial\varphi$-demiclosed. Then, for each $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2 \, ds < +\infty$, there exists $T_0 = T_0(v) \in (0, T)$ such that the problem* (1.1) *has at least one strong solution defined on* $[0, T_0]$.

Concerning the semilinear case, i.e., when $\partial\varphi$ is linear, we have

COROLLARY 3.1. – *Let A: $D(A) \subset H \to H$ be a densely defined linear, self-adjoint, m-accretive operator and let φ: $H \to \mathbb{R}^+$ be defined by*

$$\varphi(u) := \begin{cases} \frac{1}{2}\|A^{\frac{1}{2}}u\|^2, & \text{if } u \in D(A^{\frac{1}{2}}), \\ +\infty & \text{otherwise}. \end{cases}$$

Let F: $[0, T] \times D(\partial\varphi) \times C_{\partial\varphi}([-\tau, 0]; H) \to H$ be a function which is both $\partial\varphi$-demiclosed and $\partial\varphi$-dominated. If $(I + A)^{-1}$ is compact, then for each $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(A^{\frac{1}{2}})$ and $\int_{-\tau}^{0} \|Av(s)\|^2 \, ds < +\infty$, there exists $T_0 = T_0(v) \in (0, T]$ such that the problem

$$\begin{aligned} u'(t) + Au(t) &= F(t, u(t), u_t), & 0 < t < T, \\ u(s) &= v(s), & -\tau \leqq s \leqq 0, \end{aligned}$$

has at least one strong solution defined on $[0, T_0]$.

4. – The proof of the main result.

We shall use a fixed point argument as follows. Let $r > 0$ and $T_0 \in (0, T]$ be fixed and let us denote by $\mathbb{K}^r_{T_0}$ the closed ball with centre 0 and radius r in $L^2([0, T_0]; H)$, namely

$$\mathbb{K}^r_{T_0} := \left\{ h \in L^2([0, T_0]; H); \int_0^{T_0} \|h(s)\|^2 \, ds \leqq r^2 \right\}.$$

Let v be an arbitrary, but fixed, element in $C_{\partial\varphi}([-\tau, 0]; H)$ satisfying $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\varphi(v(s))\|^2 \, ds < +\infty$, and let us consider the problem

$$\begin{cases} u_h'(t) = \partial\varphi(u_h(t)) \ni h(t), & 0 < t < T_0, \\ u_h(0) = v(0), \end{cases} \tag{4.1}$$

where $h \in \mathbb{K}^r_{T_0}$.

In view of Theorem 2.1, for each $h \in \mathbb{K}^r_{T_0}$, the problem (4.1) has a unique strong solution u_h defined on $[0, T_0]$.

Now, let us define the operator $\mathbb{P}$: $D(\mathbb{P})\subset \mathbb{K}^r_{T_0}\to L^2([0, T_0]; H)$ by

$$(\mathbb{P}h)(t) := F(t, u_h(t), \hat{u}_{h_t}) \quad \text{a.e. for } t\in(0, T_0]\,, \tag{4.2}$$

and for each

$$h\in D(\mathbb{P}) = \{h\in \mathbb{K}_{T_0};\ t\to F(t, u_h(t), \hat{u}_{h_t}) \text{ belongs to } L^2([0, T_0]; H)\}\,.$$

We emphasize that in the definition of $\mathbb{P}$, u_h is the unique strong solution to (4.1) corresponding to $v(0)$—which is fixed—and to $h\in D(\mathbb{P})$, while $\hat{u}_h$ is given by (3.1).

At this point it is quite transparent that (1.1) has at least one strong solution defined on $[0, T_0]$ if and only if $\mathbb{P}$ has at least one fixed point. Indeed, h is a fixed point of $\mathbb{P}$ if and only if u_h is a strong solution of (1.1) defined on $[0, T_0]$.

In order to show that $\mathbb{P}$ has at least one fixed point we resort to Theorem 2.2. To this aim, we need the following two lemmas.

LEMMA 4.1. – *For each $v\in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0)\in D(\varphi)$ and $\int_{-\tau}^{0}\|\partial\varphi(v(s))\|^2 ds < +\infty$ there exists $r>0$ and $T_0\in(0, T]$ such that the operator $\mathbb{P}$ given by (4.2) is defined on all $\mathbb{K}^r_{T_0}$ and maps the latter into itself, i.e., $D(\mathbb{P}) = \mathbb{K}^r_{T_0}$ and $\mathbb{P}(\mathbb{K}^r_{T_0})\subset\mathbb{K}^r_{T_0}$.*

PROOF. – First, let us choose $r>0$, such that

$$r^2 \geqslant 2(1+3k)(1-k)^{-1}\varphi(v(0)) \tag{4.3}$$

where $k\in(0, 1)$ is the constant in Definition 3.3.

Fix $T_0\in(0, \min\{T, \tau\}]$—which will be defined more precisely latter—and let us observe that, for r and T_0 as above, in view of Theorem 2.1, we have

$$u_h\in C_{\partial\varphi}([0, T_0]; H)\cap W^{1,2}([0, T_0]; H) \qquad \text{for each } h\in\mathbb{K}^r_{T_0}\,.$$

Therefore, by (i) in Definition 3.1, we easily conclude that the function $t\to F(t, u_h(t), \hat{u}_{h_t})$ is strongly measurable from $[0, T_0]$ into H, for each $h\in\mathbb{K}^r_{T_0}$. Consequently, since F is $\partial\varphi$-dominated, we have

$$\begin{aligned}\int_0^{T_0}\|F(t, u_{h_t}(t), \hat{u}_h)\|^2 dt \leq k\int_0^{T_0}\|\partial\varphi^0(u_h(t))\|^2 dt + \\ +\int_0^{T_0} l\big(\|u_h(t)\| + \varphi(u_h(t))\big)\int_{-\tau}^{0}\|\partial\varphi^0(\hat{u}_1(t+s))\|^2 ds\, dt + \\ +\int_0^{T_0} l\big(\|u_1(t)\| + \varphi(u_1(t))\big)\, l_0\big((u_h(t))\big)\, c_1(t)\, dt + \int_0^{T_0} l\big(\|u_h(t)\| + \varphi(u_1(t))\big)\, dt\,,\end{aligned} \tag{4.4}$$

for each $h \in \mathbb{K}^r_{T_0}$. At this stage we do not know whether or not the right hand side of (4.4) is finite. Thus, our aim now is to show that for $T_0 \in (0, \min\{T, \tau\}]$—sufficiently small—this is indeed the case. To this end, for each $h \in \mathbb{K}^r_{T_0}$, let us define $g_h\colon [0, T_0] \to H$ by

$$g_h(t) := -u_h'(t) + h(t) \qquad \text{a.e. for } t \in (0, T_0) \tag{4.5}$$

and let us observe that from (2.2) we easily obtain

$$\frac{d}{ds}\varphi(u_h(s)) + \|g_h(s)\|^2 = (g_h(s), h(s)) \qquad \text{a.e. for } s \in (0, T_0)\,.$$

Integrating this equality both sides over $[0, t] \subset [0, T_0]$ and using Cauchy's inequality, after some obvious rearrangements, we get

$$\varphi(u_h(t)) + \tfrac{1}{2}\int_0^{T_0}\|g_h(s)\|^2\,ds \leqq \varphi(v(0)) + \tfrac{1}{2}\int_0^{T_0}\|h(s)\|^2\,ds\,,$$

for each $h \in \mathbb{K}^r_{T_0}$ and $t \in [0, T_0]$. This inequality implies both

$$\varphi(u_h(t)) \leqq \varphi(v(0)) + \tfrac{1}{2}\int_0^{T_0}\|h(s)\|^2\,ds \leqq \varphi(v(0)) + \tfrac{1}{2}r^2 \tag{4.6}$$

and

$$\int_0^{T_0}\|g_h(s)\|^2\,ds \leqq 2\varphi(v(0)) + \int_0^{T_0}\|h(s)\|^2\,ds \leqq 2\varphi(v(0)) + r^2\,, \tag{4.7}$$

for each $h \in \mathbb{K}^r_{T_0}$.

From (4.7) it readily follows that

$$(1 - 2\varepsilon)\int_0^{T_0}\|g_h(s)\|^2\,ds \leqq 2\varphi(v(0)) + r^2$$

for each $h \in \mathbb{K}^r_{T_0}$ and $\varepsilon \in [0, \frac{1}{2}]$. Taking $\varepsilon = (1 - k)[2(1 + 3k)]^{-1}$—which is possible because $k \in (0, 1)$ implies $(1 - k)[2(1 + 3k)]^{-1} \in (0, \frac{1}{2})$—after some standard calculations involving (4.3) we obtain

$$k\int_0^{T_0}\|g_h(s)\|^2\,ds \leqq \frac{1 + k}{2}\,r^2 \tag{4.8}$$

for each $h \in \mathbb{K}^r_{T_0}$.

Now, let us observe that, in view of (4.6), and since $\mathbb{K}^r_{T_0}$ is bounded in $L^2([0, T_0]; H)$, there exists $m > 0$—which does *not* depend on $T_0 \in (0, \min\{T, \tau\}]$—such that

$$\|u_h(t)\| \leqq m \qquad \text{and } \varphi(u_h(t)) \leqq m \tag{4.9}$$

for each $h \in \mathbb{K}^r_{T_0}$ and $t \in [0, T_0]$.

Using (4.4), (4.8), (4.9), the fact that $\|\partial\varphi^0(u_h(t))\| \leqq \|g_h(t)\|$ for each $h \in \mathbb{K}^r_{T_0}$ and a.e. for $t \in (0, T_0)$—see (4.5) and Theorem 2.1—and recalling that $l(\cdot)$ and $l_0(\cdot)$ are nondecreasing, we conclude that

$$\int_0^{T_0} \|F(t, u_h(t), \hat{u}_{h_t})\|^2\, dt \leqq \frac{1+k}{2} r^2 + l(2m) \int_0^{T_0}\int_{-\tau}^0 \|\partial\varphi^0(\hat{u}_h(t+s))\|^2\, ds\, dt + \\ + l(2m)\, l_0(m) \int_0^{T_0} c_1(t)\, dt + l(2m) \int_0^{T_0} c_2(t)\, dt$$

for each $h \in \mathbb{K}^r_{T_0}$.

Next, let us denote by

$$\Sigma^- := \{(t, s) \in [0, T_0] \times [-\tau, 0];\ t + s \leqq 0\}$$

and by

$$\Sigma^+ := \{(t, s) \in [0, T_0] \times [-\tau, 0];\ t + s > 0\}\,.$$

Recalling the definition of $\hat{u}_h$—see (3.1)—the last inequality can be equivalently rewritten as

$$\int_0^{T_0} \|F(t, u_h(t), \hat{u}_{h_t}\|^2\, dt \leqq \frac{1+k}{2} r^2 + l(2m) \iint_{\Sigma^-} \|\partial\varphi^0(v(t+s))\|^2\, ds\, dt + \\ + l(2m) \iint_{\Sigma^+} \|\partial\varphi^0(u_h(t+s))\|^2\, ds\, dt + l(2m)\, l_0(m) \int_0^{T_0} c_1(t)\, dt + \\ + l(2m) \int_0^{T_0} c_2(t)\, dt\,, \quad \text{for each } h \in \mathbb{K}^r_{T_0}\,.$$

Using once again the fact that $\|\partial\varphi^0(u_h(t))\| \leqq \|g_h(t)\|$ a.e. for $t \in (0, T_0)$ and for each $h \in \mathbb{K}^r_{T_0}$, we obtain

$$\int_0^{T_0} \|F(t, u_h(t), \hat{u}_{h_t})\|^2\, dt \leqq \frac{1+k}{2} r^2 + l(2m) \int_0^{T_0}\int_{-\tau}^{-t} \|\partial\varphi^0(v(t+s))\|^2\, ds\, dt + \\ + l(2m) \int_0^{T_0}\int_{-t}^0 \|g_h(t+s)\|^2\, ds\, dt + l(2m)\, l_0(m) \int_0^{T_0} c_1(t)\, dt + l(2m) \int_0^{T_0} c_2(t)\, dt$$

for each $h \in \mathbb{K}^r_{T_0}$.

After the change of variable $\theta = t + s$, we conclude that

$$\int_0^{T_0} \|F(t, u_h(t), \hat{u}_{h_t})\|^2\, dt \leqq \frac{1+k}{2} r^2 + l(2m) \int_0^{T_0}\int_{t-\tau}^0 \|\partial\varphi^0(v(\theta)(\|^2\, d\theta\, dt + \\ + l(2m) \int_0^{T_0}\int_0^t \|g_h(\theta)\|^2\, d\theta\, dt + l(2m) \cdot l_0(m) \int_0^{T_0} c_1(t)\, dt + l(2m) \int_0^{T_0} c_2(t)\, dt\,,$$

for each $h \in \mathbb{K}^r_{T_0}$. Since $t \in [0, T_0] \subset [0, \tau]$, the last inequality in conjunction with (4.7) yields

$$\int_0^{T_0} \| F(t, u_h(t), \hat{u}_{h_t}) \|^2 \, dt \leqq \frac{1+k}{2} r^2 + l(2m) T_0 \int_{-\tau}^{0} \| \partial\varphi^0(v(\theta)) \|^2 \, d\theta + \\ + l(2m) T_0 [2\varphi(v(0)) + r^2] + l(2m) l_0(m) \int_0^{T_0} c_1(t) \, dt + l(2m) \int_0^{T_0} c_2(t) \, dt$$

for each $h \in \mathbb{K}^r_{T_0}$.

Recalling that

$$v(0) \in D(\varphi) , \quad \int_{-\tau}^{0} \| \partial\varphi^0(v(s)) \|^2 \, ds < +\infty , \quad k \in (0, 1) , \quad c_1, c_2 \in L^1([0, T]; \mathbb{R}_+) ,$$

and taking into account that $m > 0$ does *not* depend on T_0, we conclude that for T_0 small enough

$$l(2m) T_0 \int_{-\tau}^{0} \| \partial\varphi^0(v(\theta)) \|^2 \, d\theta + l(2m) T_0 [2\varphi(v(0)) + r^2] + \\ + l(2m) l_0(m) \int_0^{T_0} c_1(t) \, dt + l(2m) \int_0^{T_0} c_2(t) \, dt \leqq \frac{1-k}{2} r^2$$

for each $h \in \mathbb{K}^r_{T_0}$.

From the last two inequalities we easily deduce that, for $r > 0$ defined by (4.3) and T_0 as above,

$$\int_0^{T_0} \| F(t, u_h(t), \hat{u}_{h_t}) \|^2 \, dt \leqq r^2$$

for each $h \in \mathbb{K}^r_{T_0}$. Since this inequality shows that the operator $\mathbb{P}$ given by (4.2) is defined on all $\mathbb{K}^r_{T_0}$ and maps the latter into itself, the proof of Lemma 4.1 is complete.

LEMMA 4.2. – *Let* $v \in C_{\partial\varphi}([-\tau, 0]; H)$ *be arbitrary with*

$$v(0) \in D(\varphi) \quad \textit{and} \quad \int_{-\tau}^{0} \| \partial\varphi^0(v(s)) \|^2 \, ds < +\infty ,$$

and let $r > 0$ *and* $T_0 \in (0, T]$ *be such that the operator* $\mathbb{P}$ *given by* (4.2) *is defined on all of* $\mathbb{K}^r_{T_0}$ *and maps the latter into itself.*

Then $\mathbb{P}$ *is weakly-weakly sequentially continuous from* $\mathbb{K}^r_{T_0}$ *into itself.*

PROOF. – First of all, let us remark that $\mathbb{K}^r_{T_0}$ is nonempty and weakly compact in $L^2([0, T_0]; H)$. Therefore, to complete the proof, it suffices to show that the graph of $\mathbb{P}$ is weakly sequentially closed in $\mathbb{K}^r_{T_0} \times \mathbb{K}^r_{T_0}$.

To this aim, let $((h_n, p_n))$ be an arbitrary sequence in the graph of $\mathbb{P}$ such that

$$\text{w-lim}\, h_n = h \quad \text{in } L^2([0, T_0]; H)$$

and

$$\text{w-lim}\, p_n = p \quad \text{in } L^2([0, T_0]; H)\,.$$

To simplify the notations, let us denote by (u_n) the sequence (u_{h_n}) of strong solutions to (4.1) corresponding to (h_n), and by (g_n) the sequence (g_{h_n}) as defined by (4.5), respectively. From Proposition 2.1 combined with (2.3) in Theorem 2.1 it follows that we may assume with no loss of generality—by extracting a subsequence if necessary—that

$$\text{(4.10)} \qquad \begin{cases} \lim u_n \;\;= u & \text{in } C([0, T_0]; H)\,,\ u \in C_{\partial\varphi}(0, T_0]; H)\,, \\ w\text{-lim}\, u: = w & \text{in } L^2([0, T_0]; H) \end{cases}$$

and

$$\text{(4.11)} \qquad w\text{-lim}\, g_n = g \quad \text{in } L^2([0, T_0]; H)$$

where u is the unique strong solution to (4.1) corresponding to h.

In order to use the $\partial\varphi$-demiclosedness condition on F to conclude that $p = \mathbb{P}h$ we have merely to show that $w = u'$, and $g(t) = -u'(t) + h(t)$ a.e. for $t \in (0, T_0)$. See (ii) in Definition 3.2. For this purpose, let us define the operator

$$A\colon D(A) \subset L^2([0, T_0]; H) \to 2^{L^2([0, T_0]; H)}$$

by

$$A\tilde{f} := \{f \in L^2([0, T_0]; H)\,;\ \tilde{f}(t) \in \partial\varphi(f(t)) \text{ a.e. for } t \in (0, T_0)\}$$

for each $f \in D(A)$, where

$$D(A) = \{f \in L^2([0, T_0]; H)\,;\ f(t) \in D(\partial\varphi) \text{ a.e. for } t \in (0, T_0)$$
$$\text{and there exists } \tilde{f} \in L^2([0, T_0]; H),\ \tilde{f}(t) \in \partial\varphi(f(t)) \text{ a.e. for } t \in (0, T_0)\}\,.$$

Clearly A is maximal monotone in $L^2([0, T_0]; H)$ and, in addition $g_n \in Au_n$ for each $n \in \mathbb{N}$. Since $L^2([0, T_0]; H)$ is obviously uniformly convex, the graph of A is strongly weakly sequentially closed in $L^2([0, T_0]; H) \times L^2([0, T_0]; H)$. See [2, Proposition 3.5, p. 75]. Thus, inasmuch as $C([0, T_0]; H)$ is continuously imbedded in $L^2([0, T_0]; H)$, the last remark in conjunction with (4.10) and (4.11) shows that $g \in Au$, or equivalently that $g(t) \in \partial\varphi(u(t))$ a.e. for $t \in (0, T_0)$.

Next, since the operator $\mathbb{B}$: $W^{1,2}([0, T_0]; H) \to L^2([0, T_0]; H)$ defined by

$$\mathbb{B}f := f'$$

for each $f \in W^{1,2}([0, T_0]; H)$ is linear continuous, it readily follows that it is also weakly-weakly continuous. Hence

$$w\text{-}\lim u_n' = u' \quad \text{in } L^2([0, T_0]; H)\,,$$

and $g(t) = -u'(t) + h(t)$ a.e. for $t \in (0, T_0)$.

Finally, an appeal to (ii) in Definition 3.2 shows that

$$p(t) = (\mathbb{P}h)(t) \qquad \text{a.e. for } t \in (0, T_0)\,.$$

Consequently the graph of $\mathbb{P}$ is weakly-weakly closed in $\mathbb{K}^r_{T_0} \times \mathbb{K}^r_{T_0}$, and this complete the proof of Lemma 4.2.

PROOF OF THEOREM 3.1. – From Lemma 4.1 it follows that for each $v \in C_{\partial\varphi}\cdot$ $\cdot([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2\, ds < +\infty$ there exist $r > 0$ and $T_0 \in (0, T]$ such that the operator $\mathbb{P}$ given by (4.2) is defined on all of $\mathbb{K}^r_{T_0}$ and maps the latter into itself. From Lemma 4.2 we conclude that, for $r > 0$ and $T_0 \in (0, T]$ as above, the operator $\mathbb{P}$ is weakly-weakly sequentially continuous from $\mathbb{K}^r_{T_0}$ into itself. Since $\mathbb{K}^r_{T_0}$ is nonempty, convex and weakly compact in $L^2([0, T_0]; H)$, by Theorem 2.2, $\mathbb{P}$ has at least one fixed point $h \in \mathbb{K}_{T_0}$.

Thus u_h is a strong solution to the problem (1.1) and this completes the proof of Theorem 2.1.

REMARK 4.1. – A glance at the proof of Theorem 3.1 shows that a similar result holds true if $\tau = -\infty$, i.e. for the case of an infinite delay. We note that in this case we have to consider instead of $C_{\partial\varphi}([-\tau, 0]; H)$ the space $C^{ub}_{\partial\varphi}((-\infty, 0); H)$ of all uniformly continuous and bounded functions v from $(-\infty, 0]$ into H with $v(s) \in D(\partial\varphi)$ a.e. for $s \in (-\infty, 0)$, endowed with the usual sup-norm.

REMARK 4.2. – It is also evident—see (4.8)—that in Theorem 3.1 we may assume that F is merely defined, either on $[0, T] \times D(\partial\varphi) \times \tilde{C}_{\partial\varphi}([-\tau, 0]; H)$, or on $[0, T] \times$ $\times D(\partial\varphi) \times \tilde{C}^{ub}_{\partial\varphi}((-\infty, 0]; H)$ where

$$\tilde{C}^{ub}_{\partial\varphi}((-\tau, 0]; H) := \{v \in C_{\partial\varphi}([-\tau, 0]; H)\,;\, \partial\varphi^0(v) \in L^2([-\tau, 0]; H)$$

and

$$\tilde{C}^{ub}_{\partial\varphi}((-\infty, 0]; H) := \{v \in C^{ub}_{\partial\varphi}((-\infty, 0]; H)\,;\, \partial\varphi^0(v) \in L^2((-\infty, 0]; H)\,.$$

5. – Continuation of the solutions.

The proof of the next result follows, except some minor modifications, the same lines as those in the proof of [21, Theorem 3.2.2] and therefore we do not give details

THEOREM 5.1. – *Let $\varphi\colon H \to [0, +\infty]$ be a proper, l.s.c., convex function of compact type and let*

$$F\colon [0, T]\times D(\partial\varphi)\times C_{\partial\varphi}([-\tau, 0]; H) \to H$$

be a function which is both $\partial\varphi$-demiclosed and $\partial\varphi$-dominated. Assume also that for each bounded subset B_1 in $D(\partial\varphi)$ and each subset B_2 in $C_{\partial\varphi}([-\tau, 0]; H)$ with $\{t \to \partial\varphi^0(v(t)); v \in B_2\}$ bounded fn $L^2([-\tau, 0]; H)$, there exists $f_{B_1\times B_2} \in L^2([0, T]; \mathbb{R})$ such that

$$\|F)t, u, v)\| \leqq f_{B_1\times B_2}(t)$$

a.e. for $t \in (0, T)$ and for each $(u, v) \in B_1\times B_2$.

Let $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0}\|\partial\varphi^0(v(s))\|^2\,ds < +\infty$ and let u be a noncontinuable strong solution to (1.1).

Then, either u is defined on $[0, T]$, or u is defined on $[0, T_m)$ with $0 < T_m \leqq T$ and in this case we have

$$\lim_{t\uparrow T_m} \|u(t)\| = +\infty .$$

Concerning the existence of global strong solutions to (1.1) we prove

THEOREM 5.2. – *Let $\varphi\colon H \to [0, +\infty]$ be a proper, l.s.c., convex function of compact type and let*

$$F\colon [0, T]\times D(\partial\varphi)\times C_{\partial\varphi}([-\tau, 0]; H) \to H$$

be a $\partial\varphi$-demiclosed function for which there exists $k \in (0,1)$, $k_1 > 0$, and $c \in L^1([0, T]; \mathbb{R}^+)$ such that

$$\|F(t, u, v)\|^2 \leqq k\|\partial\varphi(u)\|^2 + k_1\int_{-\tau}^{0}\|\partial\varphi^0(v(s))\|^2\,ds + c(t)$$

a.e. for $t \in (0, T)$, for each $u \in D(\partial\varphi)$ and $v \in C_{\partial\varphi}([-\tau, o]; H)$. Let $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0}\|\partial\varphi^0(v(s))\|^2\,ds < +\infty$, and let u be a noncontinuable strong solution to (1.1).

Then u is defined on $[0, T]$.

PROOF. – Clearly F is a $\partial\varphi$-dominated and thus Theorem 3.1 applies. Let $v \in C_{\partial\varphi}([-\tau, 0]; H)$ with $v(0) \in D(\varphi)$ and $\int_{-\tau}^{0} \|\partial\varphi^0(v(s))\|^2\, ds < +\infty$, and let u be a noncontinuable strong solution to (1.1) whose existence is ensured by Theorem 3.1 and Zorn's lemma.

Let us assume by contradiction that u is defined on $[0, T_0)$ with $0 < T_0 \leqq T$ To complete the proof it suffices to show that the limit

$$u(T_0 - 0) := \lim_{t \uparrow T_0} u(t) \tag{5.1}$$

exists

$$u(T_0 - 0) \in D(\varphi)\,, \tag{5.2}$$

and

$$\int_{-\tau}^{0} \|\partial\varphi^0(u(T_0 + s))\|^2\, ds < +\infty\,. \tag{5.3}$$

Indeed, once (5.1), (5.2) and (5.3) are proved, by a simple translation argument combined with Theorem 3.1, we conclude that, either u can be continued to the right of T_0 as a strong solution of (1.1) if $T_0 < T$, or u can be extended to $[0, T]$ if $T_0 = T$, thereby contradicting the initial supposition that u is noncontinuable.

In order to prove (5.1), (5.2) and (5.3), let us rewrite the equation in (1.1) as

$$u'(s) + h(s) = F(s, u(s), u_s) \qquad \text{a.e. for } d \in (0, T_0)\,,$$

where $h\colon [0, T_0] \to H$ satisfies $h(s) \in \partial\varphi(u(s))$ a.e. for $s \in (0, T_0)$. From the above equation we obviously have

$$\|u'(s)\|^2 + 2(u'(s), h(s)) + \|h(s)\|^2 = \|F(s, u(s), u_s)\|^2$$

a.e. for $s \in (0, T_0)$. Since $h(s) \in \partial\varphi(u(s))$ a.e. for $s \in (0, T_0)$, we then conclude that

$$\|u'(s)\|^2 + 2\,\frac{d}{ds}\varphi(u(s)) + \|h(s)\|^2 = \|F(s, u(s), u_s)\|^2$$

a.e. for $s \in (0, T_0)$. Integrating this equality both sides over $[0, t] \subset [0, T_0)$, neglecting the first integral on the left hand side and taking into account that

$$\|\partial\varphi^0(u(s))\| \leqq \|h(s)\| \qquad \text{a.e. for } s \in (0, T_0)\,,$$

we get

$$2\varphi(u(t)) + \int_0^t \|\partial\varphi^0(u(s))\|^2\, ds \leqq 2\varphi(v(0)) + \int_0^t \|F(s, u(s), u_s)\|^2\, ds$$

for each $t \in [0, T_0)$. Now, using the growth condition that F satisfies, after some obvious rearrangement, we obtain

$$2\varphi(u(t)) + (1-k)\int_0^t \|\partial\varphi^0(u(s))\|^2 \, ds \leqq 2\varphi(v(0)) + k_1\int_0^t \int_{-\tau}^0 \|\partial\varphi^0(u(s+\theta))\|^2 \, d\theta \, ds + \int_0^t c(s)\, ds$$

for each $t \in [0, T_0)$.

After the change of variable $s + \theta = \xi$, we arrive at

$$2\varphi(u(t)) + (1-k)\int_0^t \|\partial\varphi^0(u(s))\|^2 \, ds \leqq 2\varphi(v(0)) + K_1\int_0^t \int_{s-\tau}^s \|\partial\varphi^0(u(\xi))\|^2 \, d\xi \, ds + \int_0^t c(s)\, ds$$

and hence

$$2\varphi(u(t)) + (1-k)\int_0^t \|\partial\varphi^0(u(s))\|^2 \, ds \leqq 2\varphi(v(0)) + K_1\int_0^t \int_{-\tau}^s \|\partial\varphi^0(u(\xi))\|^2 \, d\xi \, ds + \int_0^t c(s)\, ds$$

for each $t \in [0, T_0)$.

From this inequality, we easily obtain

$$2\varphi(u(t)) + (1-k)\int_0^t \|\partial\varphi^0(u(s))\|^2 \leqq m + k_1\int_0^t \int_0^s \|\partial\varphi^0(u(\xi))\|^2 \, d\xi \, ds$$

for each $t \in [0, T_0)$, where

$$m = 2\varphi(v(0)) + k_1 T\int_{-\tau}^0 \|\partial\varphi^0(v(\xi))\|^2 \, d\xi \, .$$

Inasmuch as φ is nonnegative and $k \in (0, 1)$, from Gronwall's inequality we conclude both

$$\int_0^{T_0} \|\partial\varphi^0(u(s))\|^2 \, ds < +\infty \tag{5.4}$$

and

$$\sup_{t\in[0,T_0)} \varphi(u(t)) < +\infty \, . \tag{5.5}$$

Therefore

$$\int_{T_0-\tau}^{T_0} \|\partial\varphi^0(u(s))\|^2 \, ds \leqq \int_{-\tau}^{T_0} \|\partial\varphi^0(u(s))\|^2 \, ds = \int_{-\tau}^0 \|\partial\varphi^0(v(s))\|^2 \, ds + \int_0^{T_0} \|\partial\varphi^0(u(s))\|^2 \, ds < +\infty \, ,$$

and thus (5.3) holds.

Next, recalling the growth condition that F satisfies and (5.4), we deduce that $t \to F(t, u(t), u_t)$ belongs to $L^2([0, T_0]; H)$. As a consequence, the problem

$$w'(t) + \partial\varphi(w(t)) \ni F(t, u(t), u_t)\,, \qquad 0 < t < T_0\,,$$
$$w(0) = v(0)\,,$$

has a unique strong solution w defined on $[0, T_0]$ which obviously must coincide with u on $[0, T_0)$. Since w is continuous on $[0, T_0]$ this implies (5.1). Finally, from (5.1), (5.5) and the fact that φ is l.s.c.; we obtain (5.2), thereby completing the proof.

REMARK 5.1. – The reader may easily verify that both Theorems 5.1 and 5.2 can be suitably restated to handle as well as the case when the delay is infinite. Of course, as pointed out in *Remark* 4.1, in order to do that one has to replace $C_{\partial\varphi}([-\tau, 0]; H)$ by $C^{ub}_{\partial\varphi}((-\infty, 0]; H)$—the latter being endowed with its usual sup-norm.

6. – Examples.

As a first example we consider the nonlinear diffusion equation with infinite delay

$$\text{(6.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta\alpha(u) = \displaystyle\int_{-\infty}^{t} k(t-s)\,\Delta\beta(u)\,ds\,, & \text{a.e. for } (t, x) \in (0, T) \times \Omega\,, \\ \alpha(u(t, x)) = 0\,, \quad \text{a.e. for } (t, x) \in (0, T) \times \partial\Omega\,, \\ u(s, x) = v(s, x)\,, \quad \text{for each } s \in (-\infty, 0] \text{ and a.e. for } x \in \Omega\,. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geqq 1$, with smooth boundary $\partial\Omega$, $\alpha, \beta\colon \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing with $\alpha(0) = \beta(0) = 0$, and $k \in L^\infty([0, +\infty); \mathbb{R})$.

For the specific choice $\alpha(r) = \beta(r) = r|r|^{m-1}$ for each $r \in \mathbb{R}$, where $m > 0$ is a given constant, this equation describes the gas flow through a porous medium in which the concentration speed depends not only on the instantaneous diffusion, but also on the cummulative history of the diffusion.

Before proceeding to the exact statement of the existence result concerning (6.1) we introduce some notations.

First, let us define $J\colon \mathbb{R} \to \mathbb{R}$ by

$$J(r) := \int_0^r \alpha(\theta)\,d\theta \qquad \text{for each } r \in \mathbb{R}\,,$$

and let us recall that $C^{ub}((-\infty, 0]; H)$ is the space of all uniformly continuous and bounded functions from $(-\infty, 0]$ into H. In all that follows, we denote by $\|\cdot\|_{0,1}$ the norm of $H_0^1(\Omega)$ defined by

$$\|w\|_{0,1} := \left(\int_\Omega |\nabla w|^2\, dx\right)^{\frac{1}{2}} \qquad \text{for each } w \in H_0^1(\Omega)\,,$$

where $|\cdot|$ stands for the euclidean norm of $\mathbb{R}^N$, and by $\|\cdot\|_{-1}$ the norm of $H^{-1}(\Omega)$ —the dual of $H_0^1(\Omega)$.

We have

$$\|\Delta w\|_{-1} = \|w\|_{0,1} \qquad \text{for each } w \in H_0^1(\Omega)\,,$$

where $-\Delta\colon H_1^0(\Omega) \to H_0^{-1}(\Omega)$ is the natural isomorphism between $H_0^1(\Omega)$ and $H_0^{-1}(\Omega)$, i.e.,

$$(-\Delta w)(v) := \int_\Omega \nabla w \cdot \nabla v\, dx \qquad \text{for each } v, w \in H_0^3(\Omega)\,.$$

Now, we are preparated to formulate the main existence result referring to (6.1).

THEOREM 6.1. – *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqq 1$, whose boundary $\partial\Omega$ is an $(N-1)$-dimensional manifold of class C^2, let $\alpha, \beta \in C(\mathbb{R}; \mathbb{R}) \cap C^1(\mathbb{R}; \mathbb{R} - \{0\})$ be two given functions satisfying:*

(H$_1$) *There exist $c > 0$ and $p > (N-2)/N$ if $N > 2$, $p > 0$ if $1 \leqq N \leqq 2$, such that*

(6.2) $$\alpha'(r) \geqq c|r|^{p-1} \qquad \textit{for each } r \in \mathbb{R} - \{0\}\,;$$

(H$_2$) *There exists $c_0 > 0$ such that*

(6.3) $$0 \leqq \beta'(r) \leqq c_0 \alpha'(r) \qquad \textit{for each } r \in \mathbb{R} - \{0\}$$

and let $k \in L^\infty([0, +\infty); \mathbb{R})$. Then, for each $v \in C^{ub}((-\infty, 0]; H^{-1}(\Omega))$ with $v(s) \in L^1(\Omega) \cap H^{-1}(\Omega)$, $\alpha(v(s)) \in H_0^1(\Omega)$ a.e. for $s \in (-\infty, 0)$. $v(0) \in L^1(\Omega)$, $J(v(0)) \in L^1(\Omega)$, and $\int_{-\infty}^0 \|\Delta\alpha(v(s))\|_{-1}^2\, ds < +\infty$, there exists at least one global solution u to (6.1) in the following sense:

$$u \in C([0, T]; H^{-1}(\Omega))\,, \qquad u(s) = v(s) \qquad \textit{for \{ach } s \in (-\infty, 0]\,,$$

$$\alpha(u), \beta(u) \in L^2([0, T]; H_0^1(\Omega))\,, \qquad \frac{\partial u}{\partial t} \in L^2([0, T]; H^{-1}(\Omega))\,,$$

and u satisfies (6.1) *in the sense of distributions* ($-\Delta$ *is understood as the natural isomorphism between* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$).

PROOF. – In order to appear to Theorems 3.1 and 5.2—see also *Remarks* 4.1 and 5.1—set $H = H^{-1}(\Omega)$ and let us define $\varphi\colon H^{-1}(\Omega) \to \bar{\mathbb{R}}^+$ by

$$\varphi(u) := \begin{cases} \int_\Omega J(u(x))\,dx & \text{if } u \in L^1(\Omega) \text{ and } J(u) \in L^1(\Omega)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

It is well known that φ is proper, l.s.c. and convex and its subdifferential is given by

$$\partial\varphi(u) = -\Delta\alpha(u)$$

for each $u \in D(\partial\varphi)$, where $D(\partial\varphi) = \{u \in H^{-1}(\Omega) \cap L^1(\Omega);\ \alpha(u) \in H_0^1(\Omega)\}$. See for instance [5, or 2, Proposition 2.10, p. 67].

Since α satisfies (6.2), by Sobolev's embedding theorem combined with Schauder's theorem [6, Théorème VI.4, p. 90], we easily conclude that φ is of compact type.

Next, let us define—see Remark 4.2—

$$F\colon [0, T] \times D(\partial\varphi) \times C^{ub}_{\partial\varphi}((-\infty, 0]; H^{-1}(\Omega)) \to H^{-1}(\Omega)$$

by

$$F(t, u, v) := \int_{-\infty}^{0} k(-\theta)\, \Delta\beta(v(\theta))\, d\theta$$

for each $(t, u, v) \in [0, T] \times D(\partial\varphi) \times \tilde{C}^{ub}_{\partial\varphi}((-\infty, 0]; H^{-1}(\Omega))$.

Clearly, F does not depend on $(t, u) \in [0, T] \times D(\partial\varphi)$, and in addition, by (6.3), we have

$$\begin{aligned} \|F(t, u, v)\|_{-1}^2 &= \left\| \int_{-\infty}^{0} k(-\theta)\, \Delta\beta(v(\theta))\, d\theta \right\|_{-1}^2 = \\ &= \left\| \Delta \int_{-\infty}^{0} k(-\theta) \beta(v(\theta))\, d\theta \right\|_{-1}^2 = \left\| \int_{-\infty}^{0} k(-\theta) \beta(v(\theta))\, d\theta \right\|_{0,1}^2 = \\ &\leq \int_\Omega \int_{-\infty}^{0} |k(-\theta)|^2 \beta'(\theta)^2 |\nabla v(\theta)|^2\, d\theta\, dx \leq c_0 \int_\Omega \int_{-\infty}^{0} |k(-\theta)|^2 \alpha'(\theta)^2 |\nabla v(\theta)|^2\, d\theta\, dx\,. \end{aligned}$$

Since $k \in L^\infty([0, +\infty); \mathbb{R})$, this inequality shows that

$$\|F(t, u, v)\|_{-1}^2 \leq K \int_{-\infty}^{0} \|\Delta\alpha(v(\theta))\|_{-1}^2\, d\theta = K \int_{-\infty}^{0} \|\partial\varphi(v(\theta))\|_{-1}^2\, d\theta$$

for each $(t, u, v) \in [0, T] \times D(\partial\varphi) \times \tilde{C}^{ub}_{\partial\varphi}((-\infty, 0]; H^{-1}(\Omega))$, where $K > 0$ is a given constant. Thus, F is $\partial\varphi$-dominated, and more than this, it satisfies the growth condition in Theorem 5.2. Next, we prove that F is $\partial\varphi$-demiclosed. To this aim let us define $\mathbb{B}\colon D(\mathbb{B}) \subset H^{-1}(\Omega) \to H^{-1}(\Omega)$ by

$$\mathbb{B}u := -\Delta\beta(u)$$

for each $u \in D(\mathbb{B})$, where $D(\mathbb{B})$ is defined in a similar way as $D(\partial\varphi)$. Furthermore let us define

$$\Theta\colon D(\Theta) \subset L^2([0, T]; H^{-1}(\Omega)) \to 2^{L^2([0,T];H^{-1}(\Omega))}$$

by

$$\Theta f := \{\tilde{f} \in L^2([0, T]; H^{-1}(\Omega));\ f(t) \in \mathbb{B}\tilde{f}(t) \text{ a.e. for } t \in (0, T)\}$$

for each $f \in D(\Theta)$, where

$$D(\Theta) = \{f \in L^2([0, T]; H^{-1}(\Omega)),\ f(t) \in D(\mathbb{B}) \text{ a.e. for } t \in (0, T)$$
$$\text{and there exists } \tilde{f} \in L^2([0, T]; H^{-1}(\Omega)),\ \tilde{f}(t) \in \mathbb{B}f(t) \text{ a.e. for } t \in (0, T)\}\,.$$

Since Θ is m-accretive in $L^2([0, T]; H^{-1}(\Omega))$, a simple argument involving [2, Proposition 3.5, p. 75] shows that F is $\partial\varphi$-demiclosed.

Finally, let us observe that (6.1) may be rewritten as

$$\begin{aligned} &u'(t) + \partial\varphi(u(t)) = F(t, u(t), u_t)\,, && 0 < t < T\,,\\ &u(s) = v(s)\,, && -\infty < s \leqq 0\,, \end{aligned}$$

where φ and F satisfy all the hypotheses of Theorems 3.1 and 5.2—see also *Remarks* 4.1 and 5.1—and this completes the proof.

The second example we analyze refers to the Navier-Stokes equations with delay on the viscosity. Namely, let us consider the problem (6.4) below, where Ω is a bounded domain in $\mathbb{R}^3$

$$(6.4)\qquad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + \displaystyle\sum_{i=1}^{3} u_i \frac{\partial \boldsymbol{u}}{\partial x_i} = \boldsymbol{f} - \nabla p_* + \int_{-\infty}^{t} k(t-s)\,\Delta\boldsymbol{u}\,ds & \text{a.e. for } (t, x) \in (,0, T) \times \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{a.e. for } (t, x) \in (0, T) \times \Omega\,,\\ \boldsymbol{u} = 0 & \text{a.e. for } (t, x) \in (0, T) \times \partial\Omega\,,\\ \boldsymbol{u}(s, x) = \boldsymbol{v}(s, x) & \text{for each } s \in (-\infty, 0] \text{ and a.e. for } x \in \Omega\,. \end{cases}$$

In order to rewrite this problem in the form (1.1) let us introduce the function spaces

$$C_\sigma^\infty(\Omega) := \{\boldsymbol{u} = (u_1, u_2, u_3);\, u_i \in C_0^\infty(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}\ ;$$

$$H(\Omega) := [L^2(\Omega)]^3\ ;$$

$$H_\sigma(\Omega) := \text{the completion of } C_\sigma^\infty(\Omega) \text{ in the } H(\Omega) \text{ norm}.$$

Let us denote by P_Ω: $H(\Omega) \to H_\sigma(\Omega)$ the orthogonal projection from $H(\Omega)$ onto $H_\sigma(\Omega)$, and let us define also

$$H_\sigma^1(\Omega) := [H_0^1(\Omega)]^3 \cap H_\sigma(\Omega)\,.$$

Now, let us consider the function φ: $H_\sigma(\Omega) \to \mathbb{R}^+$, given by

$$\varphi(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \sum_{i,j}^{3} \left|\frac{\partial u_j}{\partial x_i}\right|^2 dx & \text{if } \boldsymbol{u} \in H_\sigma^1(\Omega)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

It is known that φ is proper, l.s.c. convex and of compact type, and also that

$$\partial\varphi(\boldsymbol{u}) = -P_\Omega \Delta \boldsymbol{u}$$

for each

$$\boldsymbol{u} \in D(\partial\varphi) = [H^2(\Omega)]^3 \cap [H_0^1(\Omega)]^3 \cap H_\sigma(\Omega)\,.$$

Thus, (6.4) may be rewritten as an equation in $H_\sigma(\Omega)$ of the form

$$(6.5)\qquad \begin{cases} \boldsymbol{u}'(t) + \partial\varphi(\boldsymbol{u}(t)) = F(t, \boldsymbol{u}(t), \boldsymbol{u}_t)\,, & 0 < t < T\,, \\ \boldsymbol{u}(s) = \boldsymbol{v}(s)\,, & -\infty < s \leqq 0\,, \end{cases}$$

where φ is as above while

$$F\colon [0, T] \times D(\partial\varphi) \times \tilde{C}_{\partial\varphi}^{ub}((-\infty, 0];\, H(\Omega)) \to H_\sigma(\Omega)$$

is defined by

$$F(t, \boldsymbol{u}, \boldsymbol{v}) := -P_\Omega\left(\sum_{i=1}^{3} u_i \frac{\partial \boldsymbol{u}}{\partial x_i}\right) + P_\Omega \boldsymbol{f}(t) + \int_{-\infty}^{0} k(-\theta)\, \partial\varphi(\boldsymbol{v})\, d\theta$$

for each $(t, \boldsymbol{u}, \boldsymbol{v}) \in [0, T] \times D(\partial\varphi) \times \tilde{C}_{\partial\varphi}^{ub}((-\infty, 0];\, H_\sigma(\Omega))$. See Remark 4.2.

In all that follows we denote by $\|\cdot\|$ the usual norm of $H(\Omega)$.

The main existence result concerning (6.4), or (6.5) is the following.

THEOREM 6.2. – *Let Ω be a bounded domain in $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a 2-dimensional manifold of class C^3 and let $k \in L^\infty([0, +\infty); \mathbb{R})$. Then, for each $f \in L^2([0, T]; H(\Omega))$ and each $\boldsymbol{v} \in \tilde{C}^{\iota b}_{\partial\varphi}((-\infty, 0]; H(\Omega))$ with $\boldsymbol{v}(0) \in H^1_\sigma(\Omega)$ and $\int_{-\infty}^{0} \|P_\Omega \Delta \boldsymbol{v}(s)\|^2\, ds < +\infty$ there exists $T_0 \in (0, T]$ such that the problem* (6.5) *has at least one strong solution defined on $[0, T_0]$.*

PROOF. – Since, as we already pointed out φ is proper, l.s.c., convex and of compact type, in order to apply Theorem 3.1, we have merely to check that F is $\partial\varphi$-demiclosed and $\partial\varphi$-dominated. The proof of the fact that F is $\partial\varphi$-demiclosed follows exactly the same lines in the proof of [21, Lemma 4.10.5] and therefore we omit it.

To show that F is $\partial\varphi$-dominated we resort to the following lemma whose proof may be found in [22, p. 119].

LEMMA 6.1. – *There exists a constant $C > 0$ depending only on Ω such that*

$$\left\| P_\Omega \left(\sum_{i=1}^{3} u_i \frac{\partial \boldsymbol{u}}{\partial x_i} \right) \right\| \leqq C(\varphi(\boldsymbol{u}))^{\frac{3}{4}} \|P_\Omega \Delta \boldsymbol{u}\|^{\frac{1}{2}}$$

for each $u \in D(\partial\varphi)$.

From Lemma 6.1 we easily deduce, via Cauchy's inequality that

$$\left\| P_\Omega \left(\sum_{i=1}^{3} u_i \frac{\partial \boldsymbol{u}}{\partial x_i} \right) \right\| \leqq \frac{\varepsilon}{2} \|P_\Omega \Delta \boldsymbol{u}\| + \frac{C}{2\varepsilon} (\varphi(\boldsymbol{u}))^{\frac{3}{2}}$$

for each $\boldsymbol{u} \in D(\partial\varphi)$, and therefore, after some standard calculations, we conclude that F is $\partial\varphi$-dominated.

Thus the conclusion is a direct consequence of Theorem 3.1, and this complete the proof.

REMARK 6.1. – Using similar arguments we can also prove a local existence result for the following system of Lotka-Volterra type

$$\frac{\partial u}{\partial t} - \Delta u = u M_1(u, v) + \int_{-\infty}^{t} k_1(t-s) \operatorname{div}(g_1(\nabla u))\, ds \qquad \text{a.e. for } (t, x) \in (0, T) \times \Omega\,,$$

$$\frac{\partial v}{\partial t} - \Delta v = v M_2(u, v) + \int_{-\infty}^{t} k_2(t-s) \operatorname{div}(g_2(\nabla v))\, ds\,,$$

$$-\frac{\partial u}{\partial n} \in \beta_1(u) \quad \text{a.e. for } (t, x) \in (0, T) \times \partial\Omega ,$$

$$-\frac{\partial v}{\partial n} \in \beta_2(v) \quad \text{a.e. for } (t, x) \in (0, T) \times \partial\Omega ,$$

$$u(s, x) = u_1(s, x) \quad \text{for each } s \in (-\infty, 0] \text{ and a.e. for } x \in \Omega ,$$

$$v(s, x) = v_1(s, x) \quad \text{for each } s \in (-\infty, 0] \text{ and a.e. for } x \in \Omega .$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, whose boundary $\partial\Omega$ is an $(N-1)$-dimensional mannifold of class C^2, $M_1(\cdot, \cdot)$, $M_2(\cdot, \cdot)$ are continuous functions from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$, $k_1, k_2 \in L^\infty([0, +\infty); \mathbb{R})$, $g_1(\cdot)$, $g_2(\cdot)$ are C^1 and globally Lipschitz from $\mathbb{R}$ into $\mathbb{R}$, β_1, β are maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta_i(0)$, $i = 1, 2$, while $u_1, v_1 \in C^{ub}((-\infty, 0]; L^2(\Omega))$ satisfy $u_1(0), v_1(0) \in L^\infty(\Omega)$.

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