Global Solution of the Cauchy Problem for a Class of Abstract Nonlinear Hyperbolic Equations (*).

PIERO D'ANCONA

Summary. – *This paper is concerned with the global solvability of the Cauchy problem for the abstract nonlinear equation*

$$u'' + \varphi_1(u)A_1u + \varphi_2(u)A_2u = 0$$

where A_1, A_2 are non-negative symmetric operators on an Hilbert space, while φ_1, φ_2 are locally Lipschitz continuous non-negative functions, in a Banach scale.

0. – Introduction.

This paper is concerned with the *global solvability* of the Cauchy problem

$$(1) \qquad \begin{cases} u'' + \varphi_1(u)A_1u + \varphi_2(u)A_2u = 0 \\ u(0) = u_0\,, \quad u'(0) = u_1 \end{cases}$$

where A_1, A_2 are nonnegative symmetric operators on an Hilbert space while φ_1, φ_2 are nonnegative functions.

A special attention will be paid to the case

$$(2) \qquad \varphi_i(u) = f_i(\langle P_i u, u\rangle) \qquad (i = 1, 2)$$

(with P_i not necessarly equal to A_i) in view of the applications to the PDE's. When $P_i = A_i$, problem (1) is of « variational type ».

A special case of (1) is the problem

$$(3) \qquad \begin{cases} u'' + f(\langle Au, u\rangle)Au = 0 \\ u(0) = u_0\,, \quad u(0) = u_1 \end{cases}$$

which was investigated by many authors (see e.g. [B], [M], [D], [Po], [R], [AS1]); here $A: V \to V'$ is a symmetric positive defined operator from a Banach space to

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its dual, i.e.

$$\begin{cases} \langle Au, v\rangle = \overline{\langle Av, u\rangle} \\ \langle Au, u\rangle \geqslant c\|u\|_V^2 \quad (c>0) \end{cases} \tag{4}$$

while $f(r)$ is a nonnegative function on $\boldsymbol{R}^+$.

Now POHOŽAEV ([P]) proved the global existence for (3) under the assumptions

$$\begin{cases} f \text{ is locally Lipschitz continuous} \quad \textit{and} \\ f(r) \geqslant \nu > 0 \quad \text{on } \boldsymbol{R}_+ \end{cases} \tag{5}$$

provided that the initial data u_0, u_1 are $A^{\frac{1}{2}}$-analytic vectors.

We recall that a vector $v \in V$ is called *$A^{\frac{1}{2}}$-analytic* if $\exists K, \Lambda \geqslant 0$ such that

$$\forall j \in \boldsymbol{N}, \quad A^j v \in V \quad \text{and} \quad |\langle A^j v, v\rangle|^{\frac{1}{2}} \leqslant K\Lambda^j j!\,. \tag{6}$$

A similar result was proved, under weaker hypotheses, in [AS1], where the global existence for problem (3) is obtained provided that

$$\begin{cases} f \text{ is continuous and } \geqslant 0 \quad \text{on } \boldsymbol{R}^+ \\ \int_0^{+\infty} f(r)\,dr = +\infty \quad \textit{or} \quad \sup_{\boldsymbol{R}^+} f < +\infty\,. \end{cases} \tag{7}$$

The most natural step towards a generalization of problem (3) is probably problem (1). Here (see Theorem 1 below) we prove a global existence result for (1) in a Banach scale generated by an n-tuple $\boldsymbol{B} = (B_1, \ldots, B_n)$ of operators.

More precisely, given a *Hilbert triplet* (V, H, V') (i.e. a reflexive Banach space V together with a bounded symmetric embedding I of V into its dual space V', H being the Hilbert space obtained by completing V in the inner product $(v, w)_H :=$ $:= \langle Iv, w\rangle$; $\langle\, ,\rangle$ is the duality map) and n operators $B_1, \ldots, B_n$ in $\mathcal{L}(V, H)$, such that

$$[B_h, B_k] = 0\,, \quad h, k = 1, \ldots, n \tag{8}$$

($[\, ,]$ denoting the commutator), we introduce the Banach spaces

$$X_r(\boldsymbol{B}) := \{v \in V\colon \|v\|_r < +\infty\}\,, \quad r > 0\,, \tag{9}$$

with the norms

$$\|v\|_r := \sup_{j \in \boldsymbol{N}} \|\boldsymbol{B}^j v\|_H \cdot \frac{r^j}{j!}\,. \tag{10}$$

We have set for brevity

$$(11)\qquad \begin{cases} \boldsymbol{B}^{\alpha} = B_1^{\alpha_1}\circ\ldots\circ B_n^{\alpha_n} \qquad (\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n) \\ \|\boldsymbol{B}^j v\|_H := \left(\sum\limits_{\substack{\alpha\in \boldsymbol{N}^n\\ |\alpha|=j}} \dfrac{\|\boldsymbol{B}^{\alpha}v\|_H^2}{\alpha!^2}\right)\cdot j! \qquad (j \in \boldsymbol{N})\,. \end{cases}$$

The family $\{X_r(\boldsymbol{B})\}_{r>0}$ is called the *Banach scale generated* by the n-tuple $\boldsymbol{B}$.
The elements of the Fréchet space

$$(12)\qquad X_{0^+}(\boldsymbol{B}) := \bigcap_{r>0} X_r(\boldsymbol{B})$$

are called the *$\boldsymbol{B}$-analytic vectors.*

Our main result can then be stated as follows (see Th. 5.1):

THEOREM 1. – *Let us consider Pb.* (1), *assuming that A_1, A_2 are bounded symmetric nonnegative linear operators from V into V', which satisfy the conditions*

$$(13)\qquad \begin{cases} \|A_i v\|_H \leqslant M(\|v\|_H + \|\boldsymbol{B}^1 v\|_H + \|\boldsymbol{B}^2 v\|_H) \\ \left(\sum\limits_{|\alpha|=j} \dfrac{\|[A, \boldsymbol{B}^{\alpha}]v\|_H^2}{\alpha!^2}\right)^{\frac12} \leqslant C(j+2)\left(\sum\limits_{|\alpha|=j} \dfrac{(A\boldsymbol{B}^{\alpha}v, \boldsymbol{B}^{\alpha}v)}{\alpha!^2}\right)^{\frac12} + \\ \qquad\qquad + C(j+1)(j+2)\sum\limits_{h=0}^{j} \|\boldsymbol{B}^h v\|_H / h! \end{cases}$$

$\forall j \in \boldsymbol{N}$ ($\alpha \in \boldsymbol{N}^n$), *for M, $C \geqslant 0$, while $\varphi_1(u)$, $\varphi_2(u)$ are locally Lipschitz continuous, bounded functions $\geqslant 0$ on V.*

Then Pb. (1) *is globally well-posed in $X_{0^+}(\boldsymbol{B})$, in the sense that for each u_0, u_1 in $X_{0^+}(\boldsymbol{B})$ there is an unique solution*

$$u \in C^2([0, +\infty[; X_{0^+}(\boldsymbol{B}))\,.$$

In the «variational» case in which

$$(14)\qquad \varphi_i(u) = f_i(\langle A_i u, u\rangle)\,, \qquad i = 1, 2$$

the functions f_i: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ are nonnegative, locally Lipschitz continuous, bounded on $\boldsymbol{R}^+$.

However, in this special case the assumption of boundedness can be replaced by the following one:

$$(15)\qquad \int\limits_0^{+\infty} f_i(r)\, dr = +\infty$$

for one or both functions f_i.

Thus in particular we have the following result for PDE's (see Theorem 6.1):

COROLLARY 2. – *Consider the Cauchy problem, for real* $p, q \geq 1$:

$$(16)\qquad \begin{cases} u_{tt} - \left(\int_\Omega u_x^2\,dx\,dy\right)^p u_{xx} - \left(\int_\Omega u_y^2\,dx\,dy\right)^q u_{yy} = 0 \\ u(x, y, 0) = u_0(x, y) \\ u_t(x, y, 0) = u_1(x, y) \end{cases}$$

where Ω *is a given rectangle in* $\mathbf{R}^2$. *Then for every* u_0, u_1 *analytic* Ω*-periodic functions there exists an unique solution* u *in* $C^2(\mathbf{R}^2 \times [0, +\infty[)$. *Moreover,* $u(\cdot,\cdot,t)$ *and* $u_t(\cdot,\cdot,t)$ *are analytic* Ω*-periodic, for every* $t \geq 0$.

1. – Notations.

We begin with some preliminary definitions and results. For a more thorough treatment, see [AS2], [C].

DEFINITION 1.1. – A *Banach scale* is a family $\{X_r\}_{r\in I}$ of Banach spaces, I interval of $\mathbf{R}$, with norms $\|\cdot\|_r$, such that each X_r is continuously embedded in each $X_{r-\delta}$, $\delta > 0$ (it is usually supposed that $\|\cdot\|_{r-\delta} \leq \|\cdot\|_r$).

The spaces $X_{r^+} := \bigcup_{\delta>0} X_{r-\delta}$ and $X_\infty := \bigcap_I X_r$ are endowed with the locally convex inductive limit topology.

The space $X_{r^-} := \bigcap_{\delta>0} X_{r-\delta}$ will have the inverse limit topology with respect to the embeddings of the scale; it is a Fréchet space.

Finally, the *analyticity radius* of a vector $v \in X_s$ is defined as the number

$$r_v := \sup\{r \in I : v \in X_r\}.$$

DEFINITION 1.2. – A Banach scale is said to be *dense in itself* if, for every r, $r + \delta \in I$, $\delta > 0$, $X_{r+\delta}$ is dense in X_{r^-}.

DEFINITION 1.3. – A linear operator A on $\bigcup_I X_r$ is said to be of *order* m in the scale if, when $r, r-\delta \in I$, $\delta > 0$, $AX_r \subseteq X_{r-\delta}$ and there exists a constant K such that

$$(17)\qquad \|Av\|_{r-\delta} \leq \frac{K}{\delta^m}\|v\|_r .$$

In the introduction we defined the concept of a Banach scale generated by an n-tuple of operators. We repeat it here in a more general setting.

DEFINITION 1.4. – Let $\boldsymbol{B} = (B_1, \dots, B_n)$ an n-tuple of operators on the Hilbert space $\boldsymbol{H}$. With $D_\infty(\boldsymbol{B})$ we denote the intersection of all of the $D(B_{i_1} \circ \dots \circ B_{i_h})$, $i_1, \dots, i_h$ varying in $\{1, \dots, n\}$ and h in $\boldsymbol{N}$. $\boldsymbol{B}$ is said to be *closed* if, given any sequence $\{v_k\}_{\boldsymbol{N}}$ contained in all of the $D(B_i)$, converging to a $v \in \boldsymbol{H}$, and such that $B_i v_k \to w_i$, $i = 1, \dots, n$, then $v \in \bigcap_{i=1}^{n} D(B_i)$ and $w_i = B_i v$, $i = 1, \dots, n$. $\boldsymbol{B}$ is said to be *commuting* if $[B_h, B_k] = 0$ on $D_\infty(\boldsymbol{B})$.

DEFINITION 1.5. – Let $\boldsymbol{B}$ be a closed commuting n-tuple of linear operators on $\boldsymbol{H}$. The family of Banach spaces

$$X_r(\boldsymbol{B}) := \{v \in D_\infty(\boldsymbol{B}) \colon \|v\|_r < +\infty\} \qquad (r > 0)$$

with norms

$$\|v\|_r := \sup_{j \geqslant 0} \|\boldsymbol{B}^j v\|_H \frac{r^j}{j!} \tag{18}$$

(employing notation (11)) is called the *Banach scale generated by* $\boldsymbol{B}$.

In this frame it is possible to define a new kind of order for linear operators, stronger than the one in def. 1.3.

DEFINITION 1.6. – Given a pair (m, Λ) of positive numbers, a linear operator A on $D_\infty(\boldsymbol{B})$ is said to be of *ω-order* (m, Λ) with respect to $\boldsymbol{B}$ if there exists a constant K s.t.

$$\|\boldsymbol{B}^j A v\|_H \leqslant K \cdot (j + m)! \sum_{h=0}^{j+m} \|\boldsymbol{B}^h v\|_H \frac{\Lambda^{j+m-h}}{h!} \tag{19}$$

for every $j \in \boldsymbol{N}$ and $v \in D_\infty(\boldsymbol{B})$.

REMARK 1.7. – If an operator A has ω-order (m, Λ) with respect to $\boldsymbol{B}$, then it has order m in the scale $\{X_r(\boldsymbol{B})\}_{r<1/\Lambda}$ (def. 3). For a proof, see [AS2].

REMARK 1.8. – Two Banach scales $\{X_r\}_I$, $\{Y_r\}_I$ are said to be *equivalent* if

$$Y_{r-\delta} \subsetneq X_r \subsetneq Y_{r+\delta}$$

with continuous embeddings, whenever $\delta > 0$, $r - \delta, r, r + \delta \in I$.

In this case, the first Banach scale is dense in itself (def. 1.2) iff the second is.

In particular, the scale generated by $\boldsymbol{B}$ is equivalent to the scale

$$\tilde{X}_r(\boldsymbol{B}) := \{v \in D_\infty(\boldsymbol{B}) \colon |||v|||_r < +\infty\} \qquad (r > 0) \tag{20}$$

with norms

$$|||v|||_r := \left(\sum_{j \in \boldsymbol{N}} \|\boldsymbol{B}^j v\|_H^2 \frac{r^{2j}}{j!^2}\right)^{\frac{1}{2}}. \tag{21}$$

2. – The global existence in the linear case.

The theorem of this paragraph, due to AROSIO and SPAGNOLO ([AS2], [S]), will be crucial in order to get our global existence result for Pb. (1); thus we reproduce it here for sake of completeness.

We begin by establishing our basic setting, in which both linear and non-linear global existence results (see § 5) are obtained.

Suppose that:

(22)

i) a Hilbert triplet (V, H, V') is given;

ii) an n-tuple $\boldsymbol{B} = (B_1, \ldots, B_n)$ of commuting operators of $\mathcal{L}(V, H)$ is given;

iii) the norms $\|v\|_V$ and $\|v\|_H + \sum_{i=1}^{n} \|B_i v\|_H$ are equivalent on V. For sake of simplicity, we will assume that

$$\|v\|_V = \Big(\|v\|_H^2 + \sum_{i=1}^{n} \|B_i v\|_H^2\Big)^{\frac{1}{2}};$$

iv) a positive constant Λ is given, and the Banach scale

$$\{X_r(\boldsymbol{B})\}_{0<r<1/\Lambda}$$

is dense in itself (def. 1.2).

In this same setting, we will say that a linear operator A on $D_\infty(\boldsymbol{B})$ satisfies a *quasi-commutativity condition* with $\boldsymbol{B}$ if

$$\Big(\sum_{\substack{|\alpha|=j\\ \alpha\in N^n}} \frac{\|[A, \boldsymbol{B}^\alpha]v\|_H^2}{\alpha!^2}\Big)^{\frac{1}{2}} \leqslant c(j+2)\Lambda\Big(\sum_{\substack{|\alpha|=j\\ \alpha\in N}} \frac{(A\boldsymbol{B}^\alpha v, \boldsymbol{B}^\alpha v)}{\alpha!^2}\Big)^{\frac{1}{2}} + \\ + c(j+1)(j+2)\sum_{h=0}^{j} \|\boldsymbol{B}^h v\|_H \frac{\Lambda^{j+2-h}}{h!} \tag{23}$$

for some nonnegative constant c, for every $j \in \boldsymbol{N}$, $v \in X_{0^+}(\boldsymbol{B})$. $[\ ,\ ]$ denotes the commutator, and we have employed notations (11).

We recall that a linear operator $A\colon V \to V'$ is said to be *symmetric* if

$$\langle Av, w\rangle = \overline{\langle Aw, v\rangle}\,, \tag{24}$$

and *nonnegative* if

$$\langle Av, v\rangle \geqslant 0\,. \tag{25}$$

We can now state the global existence theorem for the linear problem

$$\begin{cases} v'' + A(t)v = 0 \\ v(0) = v_0\,, \quad v'(0) = v_1\,. \end{cases} \tag{26}$$

THEOREM 2.1. – *Let the setting be as in* (22). *Let*

$$A \in L^1_{\text{loc}}(0, +\infty; \mathcal{L}(V, V')) \tag{27}$$

be a family of symmetric nonnegative linear operators $A(t)\colon V \to V'$, *operating on* $X_{0^+}(\boldsymbol{B})$.

Assume that, for every $j \in \boldsymbol{N}$, $\alpha \in \boldsymbol{N}^n$ *and* $v \in X_{0^+}(\boldsymbol{B})$

$$\boldsymbol{B}^\alpha A(\cdot)v \text{ is } H\text{-measurable}\,, \tag{28}$$

$A(t)$ *satisfies, for each* $t \geqslant 0$, *the quasi-commutativity condition* (23) *with* $\boldsymbol{B}$, *and precisely*

$$\Bigl(\sum_{\substack{|\alpha|=j\\ \alpha\in \boldsymbol{N}^n}} \frac{\|[A, \boldsymbol{B}^\alpha]v\|_H^2}{\alpha!^2}\Bigr)^{\frac12} \leqslant \sqrt{\alpha(t)}\,(j+2)\Lambda\Bigl(\sum_{\substack{|\alpha|=j\\ \alpha\in \boldsymbol{N}^n}} \frac{(A\boldsymbol{B}^\alpha v, \boldsymbol{B}^\alpha v)}{\alpha!^2}\Bigr)^{\frac12} + \\ + \alpha(t)\cdot(j+1)(j+2)\sum_{h=0}^{j}\|\boldsymbol{B}^h v\|_H \Lambda^{j+2-h}/h! \tag{29}$$

$$\|A(t)v\|_H \leqslant \beta(t)\bigl(\|v\|_H + \|\boldsymbol{B}^1 v\|_H + \|\boldsymbol{B}^2 v\|_H\bigr) \tag{30}$$

for some nonnegative locally integrable functions $\alpha(t)$ *and* $\beta(t)$.

Then Pb. (2) *is globally solvable in* $X_{0^+}(\boldsymbol{B})$.

More precisely, if the initial data v_0, v_1 *belong to* $X_{r_0}(\boldsymbol{B})$ *for some* $r_0 < 1/\Lambda$, *and we set*

$$r(t) := r_0 \exp\Bigl[-\Lambda\Bigl(1 + \frac{2}{\sqrt{1 - r_0\Lambda}}\Bigr)\int_0^t \sqrt{\alpha(s)}\, ds\Bigr] \tag{31}$$

then there exists an unique solution

$$u \in \bigcap_{T>0}\ \bigcap_{r<r(T)} H^1(0, T; X_r(\boldsymbol{B}))\,.$$

REMARK 2.2. – Actually, we won't need this theorem in its full generality; in fact, we are interested in the special case of the problem

$$\begin{cases} v'' + \bigl(a_1(t)A_1 + a_2(t)A_2\bigr)v = 0 \\ v(0) = v_0\,, \qquad v'(0) = v_1\,. \end{cases} \tag{32}$$

It is easy to see that, in order to satisfy hypotheses (27)-(30), it is sufficient to require that, for $k = 1, 2$:

(33) $\quad a_k \in L^1_{\text{loc}}(0, +\infty)$;

(34) $\quad a_k \geqslant 0$;

(35) $A_k \in \mathcal{L}(V, V')$ is symmetric and nonnegative;

(36) A_k satisfies the quasi-commutativity condition (23) with $\boldsymbol{B}$;

(37) A_k is of ω-order $(2, \Lambda)$ with respect to $\boldsymbol{B}$ (def. 1.6) (and therefore operates on $X_{0^+}(\boldsymbol{B})$: remark 1.7).

3. – An extension of a theorem by Nishida.

In order to prove our global existence result for Pb. (1) we shall firstly prove the local existence. To this end, we will use a nonlinear version of Ovčiannikov's Theorem; such a result was obtained by KANO and NISHIDA for a first order equation (see [KN] for the original proof). It's not too difficult to generalize it to the case of an m-th order equation, for $m \geqslant 2$.

THEOREM 3.1. – *Let $\{X_\varrho\}_I$ be a Banach scale, $I = [\bar{\varrho}, \varrho_0]$. Consider the Problem*

$$(38)\qquad \begin{cases} \dfrac{d^m u}{dt^m} = F(t, u(t)), & 0 \leqslant t \leqslant T \\ u^{(j)}(0) = u_j, & j = 1, \dots, m-1, \; u_j \in X_{\varrho_0}. \end{cases}$$

Let $R_0 = \sum_{j=0}^{m-1} T^j \|u_j\|_{\varrho_0}$, and $R > R_0$.

We will assume that:

i) *the mapping $(t, u) \mapsto F(t, u)$ is continuous on $[0, T] \times \{u \in X_\varrho : \|u\|_\varrho < R\}$ with values in $X_{\varrho-\delta}$ (for every ϱ, $\varrho - \delta \in [\bar{\varrho}, \varrho_0[$, $\delta > 0$).*
Moreover, there exists a $k \geqslant 0$ s.t.

$$\|F(t, 0)\|_\varrho \leqslant \frac{k}{(\varrho_0 - \varrho)^m} \qquad \text{for every } \varrho \in [\bar{\varrho}, \varrho_0[\,.$$

ii) *For every $\varrho, \varrho' \in [\bar{\varrho}, \varrho_0[$, with $\varrho' > \varrho$, and u, v in $X_{\varrho'}$ with $\|u\|_{\varrho'} < R$, $\|v\|_{\varrho'} < R$,*

$$(39)\qquad \|F(t, u) - F(t, v)\|_\varrho \leqslant C \frac{\|u - v\|_{\varrho'}}{(\varrho' - \varrho)^m}.$$

Then there exists a positive constant α such that (38) has an unique solution u in $C^m([0, \alpha(\varrho_0 - \varrho)[; X_\varrho)$ for every $\varrho \in [\bar{\varrho}, \varrho_0[$. Moreover, $\|u(t)\|_\varrho < R$.

PROOF. – The proof closely follows the idea of Nishida's theorem. However, we exhibit it for sake of completeness. We divide it into several steps.

1) *We can suppose $\bar{\varrho} = 0$.* – Simply translate the scale, i.e. set $Y_\varrho := X_{\varrho + \bar{\varrho}}$ and reformulate the theorem in this frame.

2) *Integral version of the problem.* – (38) is equivalent to

$$u(t) = U_0(t) + \int_0^t dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_m} ds\, F(s, u(s))$$

where $U_0(t) := \sum_{j=0}^{m} t^j u_j$. Note that any solution of the integral equation is C^m as soon as it is continuous.

3) *We can suppose* $F(t, 0) = 0$. – It is sufficient to set

$$\overline{U}_0 = U_0 + \iiint F(s, 0)\, ds\,, \qquad \overline{F}(t, w) := F(t, w) - F(t, 0)\,.$$

4) *The basic space.* – Let α be a positive constant, whose value will be precised in the following. Let X be the Banach space of functions belonging to $C([0, \alpha(\varrho_0 - \varrho)[; X_\varrho)$ for every $\varrho \in [0, \varrho_0[$, with the norm

$$M[u] := \sup_{\varrho\in[0,\varrho_0[}\ \sup_{t\in[0,\alpha(\varrho_0-\varrho)[} \|u(t)\|_\varrho \left(1 - \frac{t}{\alpha(\varrho_0 - \varrho)}\right)$$

(finite on X). Define inductively the approximate solutions

$$\begin{aligned} u_0(t) &= U_0(t)\\ u_{k+1}(t) &= U_0(t) + \iiint F(s, u_k(s))\, ds \end{aligned}$$

and set $\alpha_0 = 2\alpha$, $\alpha_{k+1} = \alpha_k - \alpha_0 \cdot 2^{-k-2}$, so that $\alpha_k \downarrow \alpha$.

We have to show that the u_k's are well defined (namely, that their values are in the domain of F, for suitable values of t).

We will proceed by induction. Suppose that for $k = 0, \dots, n$

$$\|u_k(t)\|_\varrho < R \qquad \text{when } \varrho \in [0, \varrho_0[,\ t \in [0, \alpha_k(\varrho_0 - \varrho)[\,.$$

In this case we see immediately that u_{n+1} is well defined, continuous on $[0, \alpha_{n+1}(\varrho_0 - \varrho)[$ with values in X_ϱ. The same is true for $v_k := u_{k+1} - u_k$ (for $k = 0, \dots, n$).

5) *More norms.* – Put, for $k \geq 0$,

$$M_k[u] := \sup_{\varrho\in[0,\varrho_0[}\ \sup_{t\in[0,\alpha_k(\varrho_0-\varrho)[} \|u(t)\|_\varrho \left(1 - \frac{t}{\alpha_k(\varrho_0 - \varrho)}\right)$$

and note that, for $k = 1, \dots, n$

$$\lambda_k \equiv M_k[v_k] < +\infty\,.$$

In fact, if $\|v_{k-1}(t)\|_\varrho \leqslant C_0$ for $\varrho \in [0, \varrho_0[$, $t \in [0, \alpha_{k-1}(\varrho_0 - \varrho)[$, then $\|v_k(t)\|_\varrho \leqslant \leqslant \int_0^t \ldots \int_0^{t_m} ds\, C \cdot \|v_{k-1}(s)\|_{\varrho'} (\varrho' - \varrho)^{-m} \leqslant C \cdot C_0 \cdot \alpha_k^m$ for $t \in [0, \alpha_{k-1}(\varrho_0 - \varrho)[$ (and any $\varrho' \in]\varrho, \varrho_0[$) (apply (39)).

6) *Estimate of* λ_k. – Set $\varrho(s) = \frac{1}{2}(\varrho_0 - s/\alpha_0 + \varrho)$; from (39) and the assumption in step 3 we have

$$\|F(t, u_0(t))\|_\varrho \leqslant C \cdot \left(\frac{1}{2}\left(\varrho_0 - \frac{s}{\alpha_0}\right)\right)^{-m} \cdot \|u_0(t)\|_{\varrho(s)}$$

(note that, if $s \in [0, \alpha_0(\varrho_0 - \varrho)[$, then $\varrho(s) \in]\varrho, \varrho_0[$) whence

$$\lambda_0 \leqslant \sup_{\varrho \in [0, \varrho_0[} \sup_{t \in [0, \alpha_0(\varrho_0 - \varrho)[} \left\{ CR_0 \left(\iiint \frac{2^m \cdot \alpha_0^m}{(\alpha_0(\varrho_0 - \varrho) - s)^m} ds \right) \cdot \left(1 - \frac{t}{\alpha_0(\varrho_0 - \varrho)}\right)\right\}.$$

The integral here equals (writing $\sigma = \alpha_0(\varrho_0 - \varrho)$)

$$\frac{2^m \alpha_0^m}{(m-1)!} \ln \frac{\sigma}{\sigma - t} \leqslant \frac{2^m \alpha_0^m}{(m-1)!} \frac{t}{\sigma - t}$$

so that

$$\lambda_0 \leqslant \frac{2^m C R_0}{(m-1)!}\, \alpha_0^m . \tag{40}$$

Now λ_k: in this case we set $\varrho(s) = \frac{1}{2}(\varrho_0 - s/\alpha_{k+1} + \varrho)$; we have

$$\|v_{k+1}(t)\|_\varrho \leqslant C \int_0^t \ldots \int_0^{t_m} ds \frac{\|v_k(s)\|_{\varrho(s)}}{(\varrho(s) - \varrho)^m} \leqslant C \iiint \frac{\lambda_k}{\left(1 - s/\alpha_k(\varrho_0 - \varrho(s))\right)} \frac{ds}{(\varrho(s) - \varrho_0)^m}$$

by the definition of λ_k. If in the last formula we replace α_k with α_{k+1} (note that $\alpha_{k+1} < \alpha_k$), after a few passages we have

$$\|v_{k+1}(t)\|_\varrho \leqslant C \lambda_k 2^m \alpha_{k+1}^m \iiint \frac{\alpha_{k+1}(\varrho_0 - \varrho) + s}{\{\alpha_{k+1}(\varrho_0 - \varrho) - s\}^{m+1}} ds .$$

Applying the estimate of the Appendix, with $\sigma = \alpha_{k+1}(\varrho_0 - \varrho)$, we finally obtain

$$\|v_{k+1}(t_\varrho)\| \leqslant 2^{m+1} \frac{m-1}{m!} c \alpha_{k+1}^m \lambda_k \cdot \left(1 - \frac{t}{\alpha_{k+1}(\varrho_0 - \varrho)}\right)^{-1}$$

for $\varrho \in [0, \varrho_0[$, $t \in [0, \alpha_{k+1}(\varrho_0 - \varrho)[$. Therefore

$$\lambda_{k+1} \leqslant 2^{m+1} \frac{m-1}{m!} c \alpha_{k+1}^m \lambda_k \leqslant 2^{m+1} \frac{m-1}{m!} c \alpha_0^m \cdot \lambda_k .$$

If α_0 is sufficiently small, we have then, for $k = 0, \ldots, n-1$

$$\lambda_{k+1} \leqslant \tfrac{1}{3}\lambda_k . \tag{41}$$

7) *Well-definition of u_k's.* – First of all note that, for $k = 0, \ldots, n$

$$\|v_k(t)\|_\varrho \leqslant \lambda_k \cdot (1 - \alpha_k/\alpha_{k+1})^{-1}$$

when $\varrho \in [0, \varrho_0[$, $t \in [0, \alpha_{k+1}(\varrho_0 - \varrho)[$ (by def. of λ_k) so that

$$\|u_{k+1}(t)\|_\varrho \leqslant \lambda_k(1 - \alpha_{k+1}/\alpha_k)^{-1} + \|u_k(t)\|_\varrho \leqslant \sum_{j=0}^{k} \lambda_j(1 - \alpha_{j+1}/\alpha_j)^{-1} + \|u_0(t)\|_\varrho$$

for $k = 0, \ldots, n$ (by summing up for $j = 0$ to k). From (41) it follows

$$\sum_{j=0}^{k} \lambda_j(1 - \alpha_{j+1}/\alpha_j)^{-1} \leqslant 12\,\lambda_0$$

(as $(1 - \alpha_{j+1}/\alpha_j)^{-1} = 2^{j+2}\alpha_j/\alpha_j \leqslant 2^{j+2}$). But from (40), if we choose a (possibly) smaller α_0, we can obtain $\lambda_0 < (R - R_0)/24$ and finally, for $k = 0, \ldots, n$, $\varrho \in [0, \varrho_0[$, $t \in [0, \alpha_{k+1}(\varrho_0 - \varrho)[$

$$\|u_{k+1}(t)\|_\varrho \leqslant \frac{R - R_0}{2} + R_0 < R$$

whence the desired well-definition condition on u_{n+1}. But note that the estimates (40), (41) are independent of n, so that with our choice of α_0 (and consequently of $\alpha = \alpha_0/2$) the approximated solutions u_k will be well defined and continuous for $k \geqslant 0$.

8) *Convergence of $\{u_k\}$.* – On $[0, \alpha(\varrho_0 - \varrho)[$ $(\varrho \in [0, \varrho_0[)$

$$\|u_{k+1}(t) - u_k(t)\|_\varrho = \|v_k(t)\|_\varrho \leqslant \lambda_k\left(1 - \frac{t}{\alpha_k(\varrho_0 - \varrho)}\right)^{-1} < \lambda_k\left(1 - \frac{t}{\alpha(\varrho_0 - \varrho)}\right)^{-1}$$

therefore

$$M[u_{k+1} - u_k] \leqslant \lambda_k .$$

The convergence of $\sum \lambda_k$ implies the convergence of $\{u_k\}$ in X to an $u(t)$, such that

$$\|u(t)\|_\varrho \leqslant R_0 + \frac{R - R_0}{2}$$

for t in $[0, \alpha(\varrho_0 - \varrho)[$.

9) *u is a solution.* – Fix $\varrho' \in]\varrho, \varrho_0[$; for $t \in [0, \alpha(\varrho_0 - \varrho)[$

$$\left\| U_0(t) + \iiint F(s, u(s))\, ds - u(s) \right\|_\varrho \leqslant$$

$$\leqslant \iiint \|F(s, u(s)) - F(s, u_k(s))\|_\varrho\, ds + \|u_{k+1}(t) - u(t)\|_\varrho \leqslant$$

$$\leqslant \frac{c}{(\varrho' - \varrho)^m} \iiint \|u(s) - u_k(s)\|_{\varrho'}\, ds + \|u_{k+1}(t) - u(t)\|_\varrho$$

and observe that the convergence in X implies the uniform convergence of u_k with values in X_ϱ (for any $\varrho \in [0, \varrho_0[$).

10) *Uniqueness.* – Fix a $\varrho_1 \in]0, \varrho_0[$, and set on X

$$M^1[u] := \sup_{\varrho \in [0, \varrho_1]} \sup_{t \in [0, \alpha(\varrho_1 - \varrho)[} \|u(t)\|_\varrho \left(1 - \frac{t}{\alpha(\varrho_1 - \varrho)}\right).$$

If u, v are solutions, then $M^1[u]$, $M^1[v]$ are finite (as $M^1 \leqslant M$).
Setting $w = u - v$, $\varrho(s) = \frac{1}{2}(\varrho + \varrho_1 - s/\alpha)$,

$$\|w(t)\|_\varrho \leqslant C \iiint \|w(s)\|_{\varrho(s)} (\varrho(s) - \varrho)^{-m}\, ds$$

on $[0, \alpha(\varrho_1 - \varrho)[$, and therefore

$$\|w(t)\|_\varrho \leqslant 2^m C \alpha^{m+1} M^1[w] \iiint \frac{\alpha(\varrho_1 - \varrho) + s}{\{\alpha(\varrho_1 - \varrho) - s\}^{m+1}}\, ds\,.$$

Applying again the estimate given in the Appendix (with $\sigma = \alpha(\varrho_1 - \varrho)$) we have

$$\|w(t)\|_\varrho \leqslant 2^{m+1} C \cdot \frac{1}{m!} \cdot \alpha^m \cdot \left(1 - \frac{t}{\alpha(\varrho_1 - \varrho)}\right)^{-1}$$

so that

$$M^1[w] \leqslant \frac{2^{m+1}}{m!} C \alpha^m M^1[w]\,.$$

If α is small enough, this implies $M^1[w] = 0$, that is,

$$w(t) = 0 \quad \text{on } [0, \alpha(\varrho_1 - \varrho)[\,,$$

and this result holds for any $\varrho_1 \in]0, \varrho_0[$. ■

REMARK 3.2. – Actually, an analogous proof is valid for the integral equation

$$u(t) = u_0(t) + \int_0^t dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_m} ds\, F(t, s, u(s))$$

for any u_0 continuous on $[0, A_0(\varrho_0 - \varrho)[$ with values in X_ϱ ($\varrho \in [0, \varrho_0[$, $A_0 > 0$), and such that $\|u_0(t)\|_\varrho \leqslant R_0$.

4. – The local existence.

In this section, we first specialize Th. 3.1 to Pb. (1); secondly, to the particular case $\varphi_k(u) = f_k(\langle P_k u, u \rangle)$, where P_k can be different from A_k.

In the following section, it will be showed that the local solution thus obtained can be prolonged to a global one, under suitable assumptions.

Suppose we have a Banach space V, together with its dual space V', and a Banach scale $\{X_r\}_{r>0}$ in V. Actually, all that is needed is a scale $\{X_r\}_I$, with I containing a (left) neighbourhood of r_0, r_0 being the index of the space X_{r_0} where the initial data are chosen.

We consider problem (1) with the following assumptions ($k = 1, 2$):

(42) $A_k\colon X_{\bar r} \to X_{\bar r}$ is a linear operator of order 2 in the scale $\{X_r\}_{r \geqslant \bar r}$ (def. 1.3), with $0 < \bar r < r_0$,

and

(43) $$\varphi_k\colon X_{\bar r} \to \boldsymbol{C} \quad \text{is locally Lipschitz continuous}\,.$$

THEOREM 4.1. – *Problem* (1) *under assumptions* (42), (43) *is locally solvable in the scale* $\{X_r\}_{r \geqslant \bar r}$; *more precisely, for every initial data* u_0, u_1 *in* X_{r_0}, $r_0 > \bar r$, *there exists a positive constant* α *such that* (1) *has an unique solution in* $C^2([0, \alpha(r_0 - r)[; X_r)$ *for every* $r \in [\bar r, r_0[$.

PROOF. – Set, for $u \in X_{\bar r}$,

$$F(u) = -\varphi_1(u) A_1 u - \varphi_2(u) A_2 u\,.$$

Pb. (1) becomes a particular case of (38), with $m = 2$ and F independent of t. We need only prove that F satisfies assumptions i), ii) of Th. 3.1. Continuity is self-evident; we have to estimate the difference $F(u) - F(v)$.

Fix an $R > \|u_0\|_{r_0} + T\|u_1\|_{r_0}$ ($T > 0$ arbitrary). Denote with L_i the Lipschitz constant of φ_i on $\{u \in X_{\bar r}\colon \|u\|_{\bar r} < R\}$; it will be as well its Lipschitz constant on all of the sets $\{u \in X_r\colon \|u\|_r < R\}$ with $\bar r \leqslant r \leqslant r_0$. Denote with k_i the constants such that, whenever $r' > r \geqslant \bar r$

$$\|A_i v\|_{r'} \leqslant k_i \frac{\|v\|_r}{(r' - r)^2}$$

for every $v \in X_{r'}$. Then we have, if $u, v \in X_{r'}$, $\|u\|_{r'} < R$, $\|v\|_{r'} < R$

$$\|\varphi_i(u) A_i u - \varphi_i(v) A_i v\|_r \leqslant \|\varphi_i(u) A_i u - \varphi_i(v) A_i u\|_r + \|\varphi_i(v) A_i u - \varphi_i(v) A_i v\|_r \leqslant$$
$$\leqslant (L_i R + |\varphi_i(v)|) \cdot \frac{k_i}{(r' - r)^2} \cdot \|u - v\|_{r'}$$

and majorizing $|\varphi_i(v)|$ with

$$|\varphi_i(0)| + |\varphi_i(v) - \varphi_i(0)| \leqslant |\varphi_i(0)| + R \cdot L_i$$

we obtain

$$\|F(u) - F(v)\|_r \leqslant C \frac{\|u - v\|_{r'}}{(r' - r)^2}$$

with $C = \sum_{i=1,2} (2L_i R + |\varphi_i(0)| \cdot k_i)$. ∎

For the next result we have to make an additional hypothesis on the Banach scale: we will suppose that

(44) the embeddings $X_r \hookrightarrow V$ are continuous for $r \geqslant \bar{r}$.

This is always the case for Banach scales generated by operators (as it is readily seen).

We will consider the problem

$$(45) \qquad \begin{cases} u'' + f_1(\langle P_1 u, u\rangle) A_1 u + f_2(\langle P_2 u, u\rangle) A_2 u = 0 \\ u(0) = u_0\,, \quad u'(0) = u_1 \end{cases}$$

where, for $k = 1, 2$,

(46) $f_k \colon \boldsymbol{C} \to \boldsymbol{C}$ is locally Lipschitz continuous.

COROLLARY 4.2. – *Consider Pb.* (45), *where A_k and f_k satisfy* (42) *and* (46) *respectively, while P_k $(k = 1, 2)$ satisfies one of the following assumptions*:

(47) $P_k \in \mathfrak{L}(V, V')$

or

(48) *$P_k \colon X_{\bar{r}} \to X_{\bar{r}}$ is of finite order in the scale (def.* 1.3), *and a continuous embedding $V \hookrightarrow V'$ is given.*

Finally, we suppose that the scale satisfies (44).

Then Pb. (45) *is locally solvable in the scale $\{X_r\}_{r > \bar{r}}$; more precisely, for every initial data u_0, u_1 in X_{r_0}, $r_0 > \bar{r}$, there exists a positive constant α such that* (45) *has an unique solution in $C^2([0, \alpha(r_0 - r)[; X_r)$ for every $r \in [\bar{r}, r_0[$.*

PROOF. – We just have to prove, under both hypotheses on P_i, that $\varphi_i(u) = f_i(\langle P_i u, u\rangle)$ is locally Lipschitz continuous on $X_{\bar{r}}$. Fix an $R > 0$, and $u, v \in X_{\bar{r}}$ with $\|u\|_{\bar{r}}, \|v\|_{\bar{r}} < R$.

In the first case (47)

$$|\langle P_i u, v\rangle| \leqslant \|P_i\|_{\mathcal{L}(V,V')}\cdot \sigma^2 \|u\|_{\bar r}\cdot \|v\|_{\bar r}$$

if σ is such that $\|\cdot\|_V \leqslant \sigma \|\cdot\|_{\bar r}$ (hyp. (44)).

In the second case (48)

$$|\langle P_i u, v\rangle| \leqslant \|P_i u\|_{V'}\cdot \|v\|_V ;$$

the continuity of $V \underset{\to}{\subsetneq} V'$ implies $\|\cdot\|_{V'} \leqslant \sigma_1 \|\cdot\|_V$, and fixed a $\delta > 0$, the continuity of $X_{\bar r+\delta} \underset{\to}{\subsetneq} V$ implies $\|\cdot\|_V \leqslant \sigma_2 \|\cdot\|_{\bar r+\delta}$, so that

$$|\langle P_i u, v\rangle| \leqslant \sigma\sigma_1\sigma_2 \|P_i u\|_{\bar r+\delta}\cdot \|v\|_{\bar r} \leqslant \frac{k_i}{\delta^{m_i}} \sigma\sigma_1\sigma_2 \|u\|_{\bar r} \|v\|_{\bar r}$$

(where m_i is the order of P_i in the scale and k_i its constant).

In both cases, we see that $|\langle P_i u, v\rangle|$ is bounded by a constant dependent only on R, if $\|u\|_{\bar r}$, $\|v\|_{\bar r} < R$; therefore f_i can be considered to be Lipschitz continuous, with constant L_i, in the following inequalities:

$$|\varphi_i(u) - \varphi_i(v)| \leqslant |f_i(\langle P_i u, u\rangle) - f_i(\langle P_i u, v\rangle)| + |f_i(\langle P_i u, v\rangle) - f_i(\langle P_i v, v\rangle)| \leqslant$$
$$\leqslant L_i(|\langle P_i u, u - v\rangle| + |\langle P_i(u-v), v\rangle|)$$

and, applying the preceding estimates, we obtain the thesis. ■

5. – The global existence in the nonlinear case.

We have now all the necessary tools to prove our global existence results for Pb. (1). The setting will be the one stated in (22): a Hilbert triplet (V, H, V'), a commuting n-tuple $\boldsymbol{B}$ in $\mathcal{L}(V, H)$ generating the norm of V, and the resulting Banach scale $\{X_r(\boldsymbol{B})\}_{r>0}$, dense in itself for $r \in]0, 1/\Lambda[$.

THEOREM 5.1. – *Let us consider problem* (1) *under the following assumptions,* $k = 1, 2$:

(49) $A_k \in \mathcal{L}(V, V')$ *is a symmetric nonnegative operator of* ω-*order* $(2, \Lambda)$ *with respect to* $\boldsymbol{B}$ (*def.* 1.6);

(50) A_k *satisfies the quasi-commutativity condition* (23) *with* $\boldsymbol{B}$; $\varphi_k\colon X_{0^+}(\boldsymbol{B}) \to \mathbf{R}^+$ *is a bounded nonnegative function, locally Lipschitz continuous on* X_r *for any* $r > 0$.

Then Pb. (1) *is globally solvable in* $X_{0^+}(\boldsymbol{B})$.

More precisely, if φ_1, φ_2 *are bounded by the constant* M, *and we set*

$$r(t) := r_0 \exp\left[-\Lambda\left(1 + \frac{2}{\sqrt{1 - r_0\Lambda}}\right)Mt\right],$$

then for every choice of the initial data u_0, u_1 *in* X_{r_0}, *with* $0 < r_0 < 1/\Lambda$, *there exists an unique solution*

$$u \in \bigcap_{T>0} \bigcap_{r<r(T)} C^2([0, T]; X_r(\boldsymbol{B})) .$$

PROOF. – 1) LEMMA. – *If* $\bar{u} \in C^2([0, \bar{T}[; X_{\bar{r}}(\boldsymbol{B}))$, *with* $0 < \bar{r} < r_0$, *is a solution of Pb.* (1), *then* $\bar{u} \in C^2([0, T]; X_r(\boldsymbol{B}))$ *for every* $r < r(T)$, $T < \bar{T}$ (*where* $r(t)$ *is the function defined above*); *moreover, when* $t \to \bar{T}^-$, $\bar{u}(t)$ *and* $\bar{u}'(t)$ *converge in* X_r *for* $r < r(T)$.

To prove this, put ($k = 1, 2$)

$$a_k(t) := \begin{cases} \varphi_k(u(t)) & \text{for } t \in [0, \bar{T}[\,, \\ 0 & \text{for } t \geqslant \bar{T}\,. \end{cases}$$

Each a_k is nonnegative, bounded by M and continuous on $[0, \bar{T}[$ (as φ_k is continuous on X_r).

Then the problem

$$\begin{cases} v'' + a_1(t)A_1 v + a_2(t)A_2 v = 0 \\ v(0) = u_0\,, \quad v'(0) = u_1 \end{cases} \tag{51}$$

satisfies conditions (32)-(37) of Remark 2.2, therefore (as the solution of (51) is unique)

$$\bar{u} = v \in \bigcap_{\bar{T} > T > 0} \bigcap_{r<r(T)} H^1(0, T; X_r(\boldsymbol{B})) .$$

Note that, if $v \in C([0, T[; X_r(\boldsymbol{B}))$, ($T < \bar{T}$), then $A_k v$ is in $C([0, T[; X_{r'}(\boldsymbol{B}))$ for every $r' < r$ (as A_k is of finite order in the scale), whence $v \in C^2([0, T[; X_{r'}(\boldsymbol{B}))$; but v is in $C([0, T[; X_r(\boldsymbol{B}))$ when $0 < T < \bar{T}$, $0 < r < r(T)$, so that

$$\bar{u} = v \in C^2([0, T]; X_r(\boldsymbol{B}))$$

(from the continuity of $r(t)$).

Finally, remarking that

$$v \in C^1([0, T[; X_r(\boldsymbol{B}))$$

when $T > 0$, $0 < r < r(T)$, (in fact, v'' has its only discontinuity point in $\overline{T}$, but has finite right and left limits there) we complete the proof of the Lemma.

2) It is easy to see that all of the assumptions of Theorem 4.1 are satisfied (see in particular Remark 1.7). This guarantees the existence of a local solution.

Let now T^* be the supremum of the $T > 0$ such that $\exists r_T \in]0, r_0[$, $\exists u \in C^2([0, T[; X_{r_T}(\boldsymbol{B}))$ solution of (1). We will show that $T^* = +\infty$.

From the uniqueness part of Th. 4.1 it follows that two any local solutions coincide on the intersection of their domains.

This allows us to define a «maximal» solution $\overline{u}$ on $[0, T^*[$. For every $T < T^*$, $\overline{u}$ is in $C^2([0, T[; X_{r_T}(\boldsymbol{B}))$ and therefore in

$$\bigcap_{T^* > T > 0} \ \bigcap_{r < r(T)} C^2([0, T]; X_r(\boldsymbol{B}))$$

(Lemma). Suppose now that $T^* < +\infty$; $\overline{u}$ and $\overline{u}'$ have a left limit in T^* (in X_r with $r < r(T^*)$), so that we can apply Th. 4.1 again, starting in $t = T^*$. This produces a solution $\overline{\overline{u}}$ on $[T^*, T^* + \varepsilon[$, and $\overline{u}$ prolonged with $\overline{\overline{u}}$ is at least C^1. Proceeding as in the proof of the Lemma, i.e., linearizing the equation, we obtain that the prolonged solution is in

$$\bigcap_{T^* + \varepsilon > T > 0} \ \bigcap_{r < r(T)} C^2([0, T]; X_r(\boldsymbol{B}))$$

thus contradicting the maximality of T^*.

A last application of the Lemma gives us the final result; uniqueness is an obvious consequence of the uniqueness of the local solution. ■

In Corollary 4.2 we gave a local existence result for a particular form of Pb. (1), namely

$$\begin{cases} u'' + f_1(\langle P_1 u, u \rangle) A_1 u + f_2(\langle P_2 u, u \rangle) A_2 u = 0 \\ u(0) = u_0 , \quad u'(0) = u_1 . \end{cases} \tag{45}$$

We will suppose now in addition that, for $k = 1, 2$

(52) $f_k \colon \boldsymbol{C} \to \boldsymbol{R}^+$ is a nonnegative, bounded, locally Lipschitz continuous function.

COROLLARY 5.2. – *Suppose that, for $k = 1, 2$, A_k verifies* (49), (50), *while f_k verifies* (52). *If each P_k satisfies one of the following conditions*

$$P_k \in \mathcal{L}(V, V')$$

or

P_k is a linear operator on $X_{0^+}(\boldsymbol{B})$, of finite order in the scale $\{X_r(\boldsymbol{B})\}_{r>0}$ (def. 1.3) *then Pb.* (45) *is globally solvable in $X_{0^+}(\boldsymbol{B})$.*

More precisely, the conclusion of Th. 5.1 *holds true (if f_1, f_2 are bounded by the constant M).*

PROOF. – Cor. 4.2 holds under assumption (44), that is always verified by Banach scales generated by operators; moreover, V is continuously embedded in V' in an Hilbert triplet. Proceed now as in the proof of Th. 5.1, using Cor. 4.2 instead of Th. 4.1. ∎

We finally focus our attention on a very special case of Pbs. (1), (45), the « variational » problem

$$\begin{cases} u'' + f_1(\langle A_1 u, u\rangle) A_1 u + f_2(\langle A_2 u, u\rangle) A_2 u = 0 \\ u(0) = u_0\,, \quad u'(0) = u_1\,, \end{cases} \tag{53}$$

which is particularly interesting, as in this case we can replace the hypothesis

$$f_k \text{ bounded} \tag{54}$$

with the more useful one

$$\int_0^{+\infty} f(r)\, dr = +\infty\,. \tag{55}$$

THEOREM 5.3. – *Suppose that, for $k = 1, 2$, A_k verifies* (49), (50), *while each f_k: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ is a nonnegative, locally Lipschitz continuous function, verifying one of the assumptions* (54), (55).

Then Pb. (53) *is globally solvable in $X_{0^+}(\boldsymbol{B})$.*

More precisely, there exists a constant $M > 0$ such that the conclusion of Th. 5.1 *holds.*

PROOF. – Set $F_k(s) := \int_0^s f_k(r)\, dr$. If $u(t)$ is a solution of (53) on the interval $[0, T[$, we define its *energy* as

$$E_u(t) := \tfrac{1}{2}\{F_1(\langle A_1 u(t), u(t)\rangle) + F_2(\langle A_2 u(t), u(t)\rangle) + \|u'(t)\|_H^2\,.$$

It is easily verified that $E'_u = 0$, so that

$$E_u(t) = E_u(0) \quad \text{if } t \in [0, T[\,.$$

Note that the nonnegativity of f_k implies the nonnegativity of F_k, therefore $F_k(\langle A_k u(t), u(t)\rangle)$ $(k = 1, 2)$ is bounded on $[0, T[$.

Suppose now that f_k verifies (55): then $F_k(r) \to +\infty$ if and only if $r \to +\infty$, then $\langle A_k u(t), u(t)\rangle$ is bounded, too. As f_k is a continuous function, it follows that $f_k(\langle A_k u(t), u(t)\rangle)$ is bounded on $[0, T[$.

If f_k is itself bounded, the same result holds.

We can now proceed as in the proof of Th. 5.1: the only difference is that, when linearizing the equation, the boundedness of $a_k(t)$ follows from the argument above (instead of being an immediate consequence of the assumptions). ■

REMARK 5.4. – All of the preceeding results can be extended to the more general problem

$$(56)\qquad \begin{cases} u'' + \varphi_1(u)A_1 u + \ldots + \varphi_n(u)A_n u = 0 \\ u(0) = u_0\,, \qquad u'(0) = u_1 \end{cases}$$

with no particular difficulty. Note also that it's possible to perform any sort of «redistribution» of the assumptions: each φ_i can satisfy any one of the conditions stated in Ths. 5.1, 5.3, and Cor. 5.2, the thesis remaining the same.

REMARK 5.5. – A slightly more general result can be proved: namely, in Pb. (56) we can suppose that, for some (one or more) K,

$$A_K \in \mathcal{L}(V, H) \text{ is of } \omega\text{-order } (1, \Lambda) \text{ with respect to } \boldsymbol{B}$$

(with no assumption of nonnegativity, symmetry or commutativity) while the corresponding φ_K satisfy

$$\varphi_K \text{ is bounded, locally Lipschitz continuous on } X_r(\boldsymbol{B}) \text{ for every } r > 0$$

(with no assumption of nonnegativity, or even to be realvalued).

This is not surprising, as we can see such terms as «perturbations» of order one to an equation of order 2.

6. – Applications.

Let Ω be a parallelepiped in $\boldsymbol{R}^n$. With $\mathcal{A}_{\text{per}}(\Omega)$ we will denote the space of analytic Ω-periodic functions on $\boldsymbol{R}^n$.

Consider the problem, for real $p, q \geq 1$,

$$(16)\qquad \begin{cases} u_{tt} - \left(\int\limits_\Omega u_x^2\,dx\,dy\right)^p u_{xx} - \left(\int\limits_\Omega u_y^2\,dx\,dy\right)^q u_{yy} = 0 \\ u(x, y, 0) = u_0(x, y) \\ u_t(x, y, 0) = u_1(x, y) \end{cases}$$

where Ω is a rectangle in $\boldsymbol{R}^2 = \boldsymbol{R}_x \times \boldsymbol{R}_y$.

THEOREM 6.1. – *Pb.* (16) *is globally solvable in* $\mathcal{A}_{\rm per}(\Omega)$. *More precisely, given any* $u_0, u_1 \in \mathcal{A}_{\rm per}(\Omega)$, *there exists a unique solution* u *in* $C^2(\boldsymbol{R}^2 \times [0, +\infty[)$, *and for every* $t \geq 0$, $u(\cdot, \cdot, t)$ *and* $u_t(\cdot, \cdot, t)$ *are in* $\mathcal{A}_{\rm per}(\Omega)$.

PROOF. – Set $H = L^2(\Omega)$; $V = \{v \in H^1_{\rm loc}(\Omega)$: v is Ω-periodic$\}$, with the norm of $H^1(\Omega)$; V' will result to be the space of $H^{-1}_{\rm loc}$ Ω-periodic functions.

We choose $\boldsymbol{B} = (\partial/\partial x, \partial_k \partial y)$; with this assumption, $D_\infty(\boldsymbol{B})$ is the space of C^∞ Ω-periodic functions, while $X_r(\boldsymbol{B})$ are spaces of analytic Ω-periodic functions with uniform radius of convergence. Moreover, $X_{0^+}(\boldsymbol{B}) = \mathcal{A}_{\rm per}(\Omega)$.

Writing $A_1 = -\partial^2/\partial x^2$, $A_2 = -\partial^2/\partial y^2$, our equation becomes

$$u'' + (\langle A_1 u, u\rangle)^p A_1 \mathrm{J} + (\langle A_2 u, u\rangle)^q A_2 u = 0\,,$$

satisfying the assumptions of Th. 5.3 (note that A_1, A_2 commute with $\boldsymbol{B}^\alpha$, so that (23) is trivially verified, and are of order $(2, \Lambda)$ for any $\Lambda > 0$).

Finally, the scale $X_r(\boldsymbol{B})$ is dense in itself: apply Remark 1.8 (noting that, for $r > 0, \tilde{X}_r(\boldsymbol{B})$ contains the trigonometric polynomials). ■

Consider now the problem in $\boldsymbol{R}^n_x \times \boldsymbol{R}^+_t$

$$(57)\qquad \begin{cases} u_{tt} - \left(\displaystyle\int_\Omega |\nabla_x u|^2\, dx\right) \Delta u - \displaystyle\sum_{i,j=1}^n (a_{ij}(x) u_{x_j})_{x_i} = 0 \\ u(x, 0) = u_0(x)\,, \qquad u_t(x, 0) = u_1(x) \end{cases}$$

under the assumptions

$$(58)\qquad \begin{cases} a_{ij} \in \mathcal{A}_{\rm per}(\Omega)\,, & i, j = 1, \ldots, n \\ |D^\alpha a_{ij}(x)| \leq C_0 \Lambda_0^{|\alpha|}\,, & \forall \alpha \in \boldsymbol{N}^n_{k},\ x \in \Omega \\ a_{ij} = \overline{a_{ji}} \quad \text{and} \quad \sum a_{ij} \xi_i \xi_j \geq 0 & \text{for every } (\xi_1, \ldots, \xi_n) \in \boldsymbol{R}^n\,. \end{cases}$$

THEOREM 6.2. – *Pb.* (57) *under the assumptions* (58) *is globally solvable in* $\mathcal{A}_{\rm per}(\Omega)$. *More precisely, given any* $u_0, u_1 \in \mathcal{A}_{\rm per}(\Omega)$, *there exists an unique solution* u *in* $C^2(\boldsymbol{R}^n \times [0, +\infty[)$, *and for every* $t \geq 0$, $u(\cdot, t)$ *and* $u_t(\cdot, t)$ *are in* $\mathcal{A}_{\rm per}(\Omega)$.

PROOF. – The proof follows the lines of the preceding one, with $A_1 = -\Delta_x$, $A_2 = -\sum (a_{ij}(x) v_{x_j})_{x_i}$, $f_1(r) = r$, $f_2 = 1$, $\boldsymbol{B} = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. The only difference is the verification of hypotheses (49), (50), that follows, under assumptions (58), from Lemmas 4.1, 4.2 of [AS2]. ■

In a similar way, an application of Corollary 5.2 yields an analogous global existence result in $\mathcal{A}_{\rm per}(\Omega)$ for the problem

$$(59)\qquad \begin{cases} u_{tt} - \Delta u - \psi\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right) \sum (a_{ij}(x) u_{x_j})_{x_i} = 0 \\ u(x, 0) = u_0(x)\,, \qquad u_t(x, 0) = u_1(x) \end{cases}$$

under hypotheses (58), and assuming that

(60) $\psi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is nonnegative, bounded and locally Lipschitz continuous.

7. – Appendix.

LEMMA. – *Let* $\sigma > 0$, $t \in [0, \sigma[$, $m \geqslant 2$; *then*

$$I(t) = \int_0^t dt_2 \dots \int_0^{t_m} ds \frac{\sigma + s}{(\sigma - s)^{m+1}} \leqslant \frac{2(m-1)}{m!}\left(1 - \frac{t}{\sigma}\right)^{-1}.$$

PROOF. – As

$$\frac{d^m}{ds^m}\left[\frac{1}{m!}\frac{1}{\sigma - s}\right] = (\sigma - s)^{-m-1}, \qquad \frac{d^m}{ds^m}\left[\frac{1}{(m-1)!}\ln(\sigma - s)\right] = -(\sigma - s)^{-m},$$

writing

$$\frac{\sigma + s}{(\sigma - s)^{m+1}} = 2\sigma(\sigma - s)^{-m-1} - (\sigma - s)^{-m},$$

we obtain

$$I(t) = \left[\frac{1}{m!}\frac{2\sigma}{\sigma - s} + \frac{1}{(m-1)!}\cdot \ln(\sigma - s)\right]_{s=0}^{s=t} = \frac{2t}{m!(\sigma - t)} + \frac{1}{(m-1)!}\ln\frac{\sigma - t}{\sigma}.$$

As $\ln x \leqslant x - 1$,

$$I(t) \leqslant \frac{1}{m!}\left(m\frac{t^2}{\sigma^2} - (m-2)\frac{t}{\sigma}\right)\frac{\sigma}{\sigma - t} \leqslant \frac{2(m-1)}{m!}\,\frac{\sigma}{\sigma - t}. \quad \blacksquare$$

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