Linear Parabolic Evolution Equations in L^p-Spaces (*) (**).

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Summary. – *Sia A il generatore di un semigruppo analitico in uno spazio di Banach. In questo lavoro si studia in modo sistematico la regolarità delle soluzioni dell'equazione parabolica astratta $u'(t) = Au(t) + f(t)$, $u(0) = u_0$, in funzione della regolarità dei fati f e u_0.*

0. – Introduction.

In this paper we want to develop a theory for evolution equations in Banach spaces to derive regularity results for the solutions of initial-boundary value problems of parabolic type. More precisely we shall be concerned with boundary value problems of the form

$$\text{(P)} \qquad \begin{cases} u_t(t, x) = A(x; D)u(t, x) + f(t, x)\,, & 0 < t \leqslant T\,,\ x \in \Omega\,, \\ B_i(x; D)u(t, x) = 0\,, \quad 1 \leqslant i \leqslant m\,, & 0 < t \leqslant T\,,\ x \in \partial\Omega\,, \\ u(0, x) = u_0(x)\,, \end{cases}$$

where Ω is a bounded open set of R^n with regular boundary $\partial\Omega$, $T > 0$, $A(x; D)$ is an elliptic linear differential operator in Ω of order $2m$ and $B_i(x; D)$ is a given system of m linear differential operators in $\partial\Omega$ verifying the conditions of AGMON, DOUGLIS and NIRENBERG [2, sect. 1]. In general we suppose that f and u_0 are not continuous but only in L^p (and accordingly the derivatives which appear in (P) must be understood in the distributional sense). This L^p setting has been extensively used and there exists a classical existence and uniqueness theorem when $f \in L^p \cdot \cdot (0, T; L^p(\Omega))$ with $1 < p < \infty$ due to LADYZENSKAJA, SOLONNIKOV and URAL'CEVA [12]. Here we want to investigate again this problem to derive new regularity results. To this end we rewrite problem (P) as a Cauchy problem in a Banach space E

$$\text{(P')} \qquad \begin{cases} u'(t) = Au(t) + f(t)\,, & 0 < t \leqslant T\,, \\ u(0) = u_0\,, \end{cases}$$

(*) Entrata in Redazione il 15 luglio 1983.
(**) Work done under the auspices of the G.N.A.F.A. of the C.N.R.

by taking e.g. $E = L^q(\Omega)$, $Au = A(x; D)u$ for $u \in D_A$ = the completion in $W^{2m,q}(\Omega)$ of the class of functions in $C^{2m}(\bar{\Omega})$ satisfying the boundary conditions $(\mathrm{P})_2$. For more details see [2, sect. 1]. In the same paper it is proved that A generates an analytic semigroup in $L^q(\Omega)$ when $1 < q < \infty$. For this reason we are led to study problem (P′) assuming that E is a general Banach space, $f \in L^p(0, T; E)$ $(1 \leqslant p < \infty)$ $u_0 \in E$ and A is the infinitesimal generator of an analytic semigroup in E. A solution of (P′) in general verifies $(\mathrm{P}')_1$ almost everywhere in $]0, T[$ because its derivative belongs to $L^p(0, T; E)$ (see section 3 for a more precise definition of solutions).

The study of the abstract problem (P′) is well developed when E is a Hilbert space: see for instance the LIONS and MAGENES's book for the case $p = 2$ [14] and the paper of DE SIMON [7] for the case $1 < p < \infty$. If E is a (possibly nonreflexive) Banach space several existence and regularity results are proved by DA PRATO and GRISVARD in [6] as an application of a general theory on the sum of two operators acting on functions spaces. Here we want to find again these results as well as new ones by using a direct approach based on a suitable definition of Sobolev spaces of fractional order $W^{\theta,p}(0, T; E)$ and of a class of interpolation spaces between D_A and E, which will be denoted $D_A(\theta, p)$. In this way we can give a self-contained exposition and we can describe sistematically how the regularity of the solution depends on that of the data f and u_0. Using this approach and recent results on elliptic operators in $L^1(\Omega)$ (see Amann [3] and the references therein) we are able to study the parabolic problem (P) also in some cases not previously considered (for example when $f \in L^p(0, T; L^q(\Omega))$ with p and/or $q = 1$). Moreover we can obtain further regularity of the solutions of (P) when e.g. f is more regular with respect to time or space. This direct approach proves to be useful for the developments of the theory (such as nonautonomous problems, nonlinear problems, ..., etc.) which we will pursue in future papers.

The plan of the paper is as follows. In sections 1 and 2 we give the definitions and the properties of the spaces $W^{\theta,p}$ and $D_A(\theta, p)$ which are needed in our paper. In section 3 we introduce various kind of solutions for problem (P′) and in sections 4, 5, 6 and 7 we study their regularity under various assumptions on f and u_0. In section 8 we use the results of sections 4-7 to describe in a complete way the properties of the solutions of (P′). Next we shall apply these results, in section 9, to the concrete parabolic initiall-boundary value problem (P). Finally in the Appendix we collect the proofs of the properties of the spaces $W^{\theta,p}$ and $D_A(\theta, p)$ that are used in this paper. Some of these properties are known but we give here direct proofs in order to make the paper self-contained.

1. – Analytic semigroups and interpolation spaces.

Let E be a Banach space with norm $|\cdot|$ and let $A\colon D_A \subset E \to E$ generate a bounded analytic semigroup $S(t)$ in E. By this we mean that $\bar{D}_A = E$ and there exist $M > 0$ and $\alpha \in]\pi/2, \pi[$ such that if $z \in C$, $z \neq 0$ and $|\arg z| < \alpha$, then z is in the resolvent set of A and we have

$$|z(z-A)^{-1}x|\leq M|x|$$

for each $x\in E$. It is known (see e.g. KATO [11]) that if $t>0$ then $S(t)x\in D_A$ for each $x\in E$. Moreover there exist M_0 and M_1 such that for each $x\in E$ we have

(1) $$|S(t)x|\leq M_0|x|\,,\qquad t\geq 0$$

and

(2) $$|tAS(t)x|\leq M_1|x|\,,\qquad t>0\,.$$

Now let $0<\theta<1$, let $1\leq p<\infty$ and set

$$|x|_{\theta,p}=\left(\int_0^{+\infty}|t^{1-\theta}AS(t)x|^p\,t^{-1}\,dt\right)^{1/p}.$$

We shall denote by $D_A(\theta,p)$ the following intermediate space between D_A (endowed with the graph norm) and E

$$D_A(\theta,p)=\{x\in E,\ |x|_{\theta,p}<+\infty\}$$

$D_A(\theta,p)$ will be given the norm

$$|x|_{D_A(\theta,p)}=\left(|x|^p+|x|^p_{\theta,p}\right)^{1/p}.$$

The spaces $D_A(\theta,p)$ are extensively studied by BUTZER and BERENS in [4] where it is also proved an isomorphism between $D_A(\theta,p)$ and the interpolation spaces introduced by LIONS in [13]. We refer to [4] for a more detailed description of these spaces and quote now the properties that are used in this paper.

(3) $$D_A\hookrightarrow D_A(\theta,p)\hookrightarrow D_A(\theta',p)\hookrightarrow E\qquad 0<\theta'<\theta<1\,,$$

(4) $$|S(t)|_{D_A(\theta,p)}\leq M_0|x|_{D_A(\theta,p)}\qquad 0<\theta<1\,.$$

For convenience we shall denote by $D_A(\theta+1,p)$, for $0<\theta<1$, the space

$$D_A(\theta+1,p)=\{x\in D_A\colon\ Ax\in D_A(\theta,p)\}\,.$$

In what follows we shall also consider the following intermediate space between D_{A^2} and E

$$D_{A^2}(\tfrac{1}{2},p)=\{x\in E\colon\ |x|_{2,\frac{1}{2},p}<+\infty\}$$

where

$$|x|_{2,\frac{1}{2},p} = \left(\int_0^{+\infty} |tA^2 S(t)x|^p t^{-1}\, dt\right)^{1/p}$$

endowed with the norm

$$|x|_{D_{A^2}(\frac{1}{2},p)} = \left(|x|^p + |x|^p_{2,\frac{1}{2},p}\right)^{1/p}.$$

We refer to [4] once again for a detailed description of the space $D_{A^2}(\frac{1}{2}, p)$ and limit our interest here in the properties that are used in this paper.

$$D_{A^2} \hookrightarrow D_{A^2}(\tfrac{1}{2}, p) \hookrightarrow D_A(\theta, p), \quad 0 < \theta < 1, \tag{5}$$

$$|S(t)x|_{D_{A^2}(\frac{1}{2},p)} \leqslant M_0 |x|_{D_{A^2}(\frac{1}{2},p)}. \tag{6}$$

For the reader's convenience we shall give in the Appendix the proofs of (3)-(6).

2. – Sobolev spaces of fractional order.

Let $a < b$ be real numbers and let B be any of the spaces E, $D_A(\theta, p)$, $D_{A^2}(\frac{1}{2}, p)$ and D_A. We shall be concerned with the following spaces of B-valued functions defined on $[a, b]$

— $L^p(a, b; B)$, $1 \leqslant p < \infty$, is the space of measurable functions u such that $|u(\cdot)|_B^p$ is integrable in $]a, b[$;

— $C(a, b; B)$ is the space of continuous functions on $[a, b]$;

— $W^{1,p}(a, b; B)$ is the space of functions u of $L^p(a, b; B)$ having distributional derivative u' in $L^p(a, b; B)$.

In what follows we shall set

$$L^p(B) = L^p(0, T; B), \quad C(B) = C(0, T; B) \quad \text{and} \quad W^{1,p}(B) = W^{1,p}(0, T; B),$$

where $T > 0$ is a given number. Moreover we shall denote by $\|\cdot\|$ the norm in $L^p(E)$ and by $\|\cdot\|_{L^p_{\theta,p}}$ the norm in $L^p(D_A(\theta, p))$

$$\|u\|_{L^p_{\theta,p}} = \left(\|u\|^p + H^p_{\theta,p}(u)\right)^{1/p}$$

where we have set

$$H_{\theta,p}(u) = \left(\int_0^T |u(t)|^p_{\theta,p}\, dt\right)^{1/p}.$$

Moreover we shall denote by $\|\cdot\|_{C_{\theta,p}}$ and $\|\cdot\|_{C_{2,\frac{1}{2},p}}$ the norms in $C(D_A(\theta, p))$ and $C(D_{A^2}(\frac{1}{2}, p))$, respectively.

Finally we shall denote by $L^p_+(B)$ and $W^{1,p}_+(B)$ the following spaces

$$L^p_+(B) \;\;= \{u \in L^p(\varepsilon, T; B) \text{ for each } \varepsilon > 0\}\,,$$
$$W^{1,p}_+(B) = \{u \in W^{1,p}(\varepsilon, T; B) \text{ for each } \varepsilon > 0\}\,.$$

Furthermore let $0 < \theta < 1$ and set

$$N_{\theta,p}(u) = \left(\int_0^T\!\!\int_0^T |u(t) - u(s)|^p\,|t-s|^{-1-\theta p}\,ds\,dt\right)^{1/p}.$$

We shall consider the following intermediate space between $W^{1,p}(E)$ and $L^p(E)$

$$W^{\theta,p}(E) = \{u \in L^p(E)\colon\; N_{\theta,p}(u) < +\infty\}$$

endowed with the norm

$$\|u\|_{W^{\theta,p}} = (\|u\|^p + N^p_{\theta,p}(u))^{1/p}\,.$$

The space $W^{\theta,p}(D_A(\varepsilon, p))$ is defined analogously and its norm will be denoted by $\|\cdot\|_{W^{\theta,p}_{\varepsilon,p}}$

$$\|u\|_{W^{\theta,p}_{\varepsilon,p}} = (\|u\|^p + N^p_{\theta,\varepsilon,p})^{1/p}$$

where

$$N_{\theta,\varepsilon,p}(u) = \left(\int_0^T\!\!\int_0^T |u(t) - u(s)|^p_{\varepsilon,p}\,|t-s|^{-1-\theta p}\,ds\,dt\right)^{1/p}.$$

The spaces $W^{\theta,p}$ are extensively studied by many authors from different points of view (see e.g. Adams [2] and the references therein). We refer to [2] for a detailed treatment of these spaces and quote now the properties that are used in this paper.

$$W^{\theta,p}(E) \hookrightarrow W^{\theta',p}(E) \hookrightarrow L^p(E)\,, \qquad 0 < \theta' < \theta \leqslant 1\,, \tag{7}$$

$$W^{\theta,p}(E) \hookrightarrow C(E)\,, \qquad \theta > 1/p\,, \tag{8}$$

$$\left(\int_0^T t^{-p\theta}|u(t)|^p\,dt\right)^{1/p} \leqslant c'\,\|u\|_{W^{\theta,p}}\,, \qquad 0 < \theta < 1/p\,, \tag{9}$$

$$\left(\int_0^T t^{-p\theta}|u(t)|^p\,dt\right)^{1/p} \leqslant c''\,\|u\|_{W^{\theta,p}}\,, \qquad \theta > 1/p\,,\; u(0) = 0\,, \tag{10}$$

$$(11) \qquad W^{\theta,p}(E) \cap L^p(D_A(\theta, p)) \hookrightarrow W^{\varepsilon,p}(D_A(\theta-\varepsilon, p)), \qquad \varepsilon<\theta<1,$$

$$(12) \qquad W^{\theta,p}(D_A) \cap W^{1,p}(D_A(\theta, p)) \hookrightarrow W^{\alpha,p}(D_A(\beta, p)), \qquad 0<\alpha, \ \beta<1, \ \alpha+\beta=1+\theta,$$

$$(13) \qquad W^{\theta,p}(E) \cap L^p(D_A(\theta, p)) \hookrightarrow C(D_A(\theta-1/p, p)), \qquad 1/p<\theta<1,$$

$$(14) \qquad W^{1,p}(D_A(\theta, p)) \cap W^{\theta,p}(D_A) \hookrightarrow C(D_A(\theta+1-1/p, p)), \qquad \theta<1/p,$$

$$(15) \qquad W^{1,p}(D_A(1/p, p)) \cap L^p(D_A(1+1/p, p)) \hookrightarrow C(D_{A^2}(\tfrac{1}{2}, p)),$$

$$(16) \qquad W^{1,p}(D_A(1/p, p) \cap \hat{W}^{1/p,p}(D_A) \hookrightarrow C(D_{A^2}(\tfrac{1}{2}, p)),$$

where

$$\hat{W}^{1/p,p}(B) = \left\{u \in W^{1/p,p}(B) : \int_0^T t^{-1}|u(t)|_B^p \, dt < +\infty\right\}.$$

For the reader convenience we shall collect in the Appendix the proofs of (7)-(16).

3. – Abstract parabolic equations in L^p.

Let $A: D_A \subset E \to E$ be the infinitesimal generator of a bounded analytic semigroup $S(t)$ in E. Given $f \in L^p(E)$ and $x \in E$ we shall be concerned with the following problem

$$(16) \qquad \begin{cases} u' = Au + f, & 0<t\leqslant T, \\ u(0) = x. \end{cases}$$

To study problem (16) we shall use the following definitions. A function u is called a strict solution of (16) if $u \in L^p(D_A) \cap W^{1,p}(E)$ and we have $u'(t) = Au(t) + f(t)$ a.e. on $]0, T[$ and $u(0) = x$. Moreover we shall say that u is a classical solution of (16) if $u \in C(E) \cap L^p_+(D_A) \cap W^{1,p}_+(E)$ and we have $u'(t) = Au(t) + f(t)$ a.e. on $]0, T[$ and $u(0) = x$. It is obvious that any strict solution of (16) is also classical. The converse in general is not true.

The following representation formula is well known.

THEOREM 1. – Let u be classical solution of (16). Then for each $t \in [0, T]$ we have

$$(17) \qquad u(t) = S(t)x + (S * f)(t)$$

where we have set $(S * f)(t) = \int_0^t S(t-s) f(s) \, ds$.

PROOF. – For $0<\varepsilon\leqslant s\leqslant t$ set $v(s)=S(t-s)u(s)$. We have

$$v'(s)=-AS(t-s)u(s)+S(t-s)u'(s)$$

from which it follows that

$$u(t)-S(t-\varepsilon)u(\varepsilon)=v(t)-v(\varepsilon)=\int_\varepsilon^t v'(s)\,ds=\int_\varepsilon^t S(t-s)f(s)\,ds$$

and the result follows by letting $\varepsilon\to 0$.

As a consequence of Theorem 1 we have that the classical (and hence the strict) solution of (16) is unique if it exists. Also it turns out that it is useful to study the properties of the function given by (17). We shall call such a function the mild solution of (16). In the following sections we shall study the dependence of the regularity of the mild solutions of (16) on the regularity of f and x. As a consequence we shall obtain existence and regularity results for the classical and strict solutions of (16).

4. – Properties of $u_0(t)=S(t)x$.

In this section we shall investigate whether $u_0\in L^p(D_A(\theta,p))$ or $u_0\in W^{\theta,p}(E)$ for some θ. For convenience we shall call spatial regularity the first situation and temporal regularity the latter one. These definitions are motivated from the fact that in the cases where A is associated with an elliptic operator the functions of $L^p(D_A(\theta,p))$ are more regular with respect to the «spatial» variables while the functions of $W^{\theta,p}(E)$ are more regular with respect to time. Finally we shall investigate whether $u_0\in W^{\varepsilon,p}(D_A(\theta,p))$ for some ε and θ; such a regularity will be called mixed regularity.

a) Spatial regularity.

We will examine the regularity of u_0 when $x\in D_A(\theta,p)$ and θ increases.

THEOREM 2. – For each $x\in E$ we have that $u_0\in L^p(D_A(\theta,p))$ with $\theta<1/p$. Moreover there exists a_1 verifying

$$\|u_0\|_{L^p_{\theta,p}}\leqslant a_1|x|\,.$$

PROOF. – Using (1) and (2) we have

$$\|u_0\|^p\leqslant M_0^p T|x|^p$$

and

$$H^p_{\theta,p}(u) = \int_0^T dt \int_0^{+\infty} |AS(t+s)x|^p \frac{ds}{s^{1-p+p\theta}} \leqslant$$

$$\leqslant M_1^p |x|^p \int_0^T dt \int_0^{+\infty} \frac{ds}{(s+t)^p s^{1-p+p\theta}} = M_1^p |x|^p \int_0^T dt \int_0^{+\infty} \frac{ds}{t^{\theta p}(s+1)^p s^{1-p+p\theta}}$$

where we use the transformation of variables $s = st$. Therefore the result is proved with

$$a_1 = \left(M_0^p T + M_1^p \int_0^T t^{-p\theta} dt \int_0^{+\infty} (s+1)^{-p} s^{-1+p-p\theta} ds \right)^{1/p}.$$

THEOREM 3. – Let $x \in D_A(\theta, p)$ with $\theta < 1 - 1/p$. Then $u_0 \in L^p(D_A(\theta + 1/p, p))$ and we have

$$\|u_0\|_{L^p_{\theta+1/p,p}} \leqslant \left(M_0^p T + (p - 1 - p\theta)^{-1} \right)^{1/p} |x|_{D_A(\theta,p)}.$$

PROOF. – We have

$$H^p_{\theta+1/p,p}(u) = \int_0^T dt \int_t^{+\infty} |AS(s)x|^p \frac{ds}{(s-t)^{1-p+p\theta+1}}$$

therefore, interchanging the order of integration and using the fact that $1 - p + p\theta < 0$, we obtain

$$H^p_{\theta+1/p,p}(u) = \int_0^T |AS(s)x|^p ds \int_0^s (s-t)^{-1+p-p\theta-1} dt +$$

$$+ \int_T^{+\infty} |AS(s)x|^p ds \int_0^T (s-t)^{-1+p-p\theta-1} dt \leqslant (p - p\theta - 1)^{-1} |x|^p_{\theta,p}$$

and the conclusion follows.

THEOREM 4. – Let $x \in D_A(1 - 1/p, p)$. Then $u_0 \in L^p(D_A)$ and we have $\|Au_0\| \leqslant |x|_{1-1/p,p}$.

PROOF. – The result is an immediate consequence of the definition of $|\cdot|_{1-1/p,p}$.

THEOREM 5. – Let $x \in D_A(\theta + 1 - 1/p, p)$ with $0 < \theta < 1/p$. Then $Au_0 \in L^p \cdot (D_A(\theta, p)$ and there exists a_2 verifying

$$\|Au_0\|_{L^p_{\theta,p}} \leqslant \left(a_2 |x|_{D_A(\theta+1-1/p,p)} \right).$$

PROOF. – From Theorem 4 and property (3) we have

$$\|Au_0\| \leq |x|_{1-1/p,p} \leq c_1 |x|_{D_A(\theta+1-1/p,p)} .$$

Moreover by a computation similar to that of Theorem 3 we get

$$H^p_{\theta,p}(Au_0) = \int_0^T dt \int_0^{+\infty} |A^2 S(s+t)x|^p s^{-1+p-p\theta} ds \leq (p-p\theta)^{-1} \int_0^{+\infty} |A^2 S(s)x|^p s^{p-p\theta} ds .$$

Now (2) implies that

$$|A^2 S(s)x| = |AS(s/2)AS(s/2)x| \leq M_1 2s^{-1} |AS(s/2)x|$$

and hence we obtain

$$H^p_{\theta,p}(Au_0) \leq M_1^p 2^{1+p-p\theta}(p-p\theta)^{-1} |x|^p_{\theta+1-1/p,p} .$$

Therefore the result follows with

$$a_2 = (c_1^p + M_1^p 2^{1-p+p\theta}(p-p\theta)^{-1})^{1/p}$$

where c_1 is given by (3).

THEOREM 6. – Let $1 < p < \infty$ and let $x \in D_{A^2}(\frac{1}{2}, p)$. Then $Au_0 \in L^p(D_A(1/p, p))$ and we have

$$\|Au_0\|^p_{L^p_{1/p,p}} \leq (c_2 + (p-1)^{-1})^{1/p} |x|_{D_{A^2}(\frac{1}{2},p)}$$

where c_2 is given by property (5).

PROOF. – From (5) and Theorem 4 we have

$$\|Au_0\|^p \leq |x|^p_{1-1/p,p} \leq c_2 |x|^p_{D_{A^2}(\frac{1}{2},p)}$$

moreover, interchanging the order of integration, we find

$$H^p_{1/p,p}(Au_0) = \int_0^T dt \int_t^{+\infty} |AS(s)x|^p (s-t)^{p-2} ds =$$

$$= \int_0^T |AS(s)x|^p ds \int_0^s (s-t)^{p-2} dt + \int_T^{+\infty} |AS(s)x|^p ds \int_0^T (s-t)^{p-2} dt \leq (p-1)^{-1} |x|^p_{2,\frac{1}{2},p}$$

and the result follows.

b) Temporal regularity.

As before we shall study the regularity of u_0 with respect to t when $x \in D_A(\theta, p)$ and θ increases.

THEOREM 7. – For each $x \in E$ we have that $u_0 \in W^{\theta,p}(E)$ with $\theta < 1/p$. Moreover there exists a_3 verifying

$$\|u_0\|_{W^{\theta,p}} \leqslant a_3 |x| .$$

PROOF. – We have

$$\begin{aligned} N^p_{\theta,p}(u_0) &= 2\int_0^T dt \int_0^t |S(t)x - S(s)x|^p (t-s)^{-1-\theta p}\, ds \\ &= 2\int_0^T dt \int_0^t \left| \int_s^t AS(\tau)x\, d\tau \right|^p (t-s)^{-1-\theta p}\, ds \\ &= 2\int_0^T ds \int_s^T \left| \int_0^{t-s} AS(\tau+s)x\, d\tau \right|^p (t-s)^{-1-\theta p}\, dt \\ &= 2\int_0^T ds \int_0^{T-s} \left| \int_0^t AS(\tau+s)x\, d\tau \right|^p t^{-1-\theta p}\, dt \\ &\leqslant 2\int_0^T ds \int_0^{+\infty} \left(\int_0^t |AS(\tau+s)x| d\tau \right)^p t^{-1-\theta p}\, dt \end{aligned}$$

where $S(t) = 0$ if $t \geqslant T$. Therefore, using Hardy inequality (see Appendix), we obtain

$$N^p_{\theta,p}(u_0) \leqslant 2\theta^{-p} \int_0^T ds \int_0^{T-s} |AS(t+s)x|^p t^{-1+p-p\theta}\, dt \tag{18}$$

and hence

$$N^p_{\theta,p}(u_0) \leqslant 2\theta^{-p} H^p_{\theta,p}(u_0) . \tag{19}$$

Therefore the result follows from Theorem 2 with

$$a_3 = (M_0^p T + 2\theta^{-p} a_1^p)^{1/p}$$

where a_1 is given by Theorem 2.

THEOREM 8. – Let $x \in D_A(\theta, p)$ with $\theta < 1 - 1/p$. Then $u_0 \in W^{\theta+1/p,p}(E)$ and there exists a_4 verifying

$$\|u_0\|_{W^{\theta+1/p,p}} \leqslant a_4 |x|_{D_A(\theta,p)} .$$

PROOF. – From (19) we get

$$N^p_{\theta+1/p,p}(u_0) \leqslant 2(\theta + 1/p)^{-p} H^p_{\theta+1/p,p}(u_0)$$

therefore the result follows from Theorem 3 with

$$a_4 = \left(M_0^p T + 2(\theta + 1/p)^{-p}(p - p\theta - 1)^{-1}\right)^{1/p} .$$

THEOREM 9. – Let $x \in D_A(1 - 1/p, p)$. Then $u_0 \in W^{1,p}(E)$ and we have $\|u_0'\| \leqslant \leqslant |x|_{1-1/p,p}$.

PROOF. – The result follows from Theorem 4 and from the fact that $u_0' = Au_0$.

THEOREM 10. – Let $x \in D_A(\theta + 1 - 1/p, p)$ with $0 < \theta < 1/p$. Then $u_0' \in W^{\theta,p}(E)$ and there exists a_5 verifying

$$\|u_0'\|_{W^{\theta,p}} \leqslant a_5 |x|_{D_A(\theta+1-1/p,p)} .$$

PROOF. – By a computation similar to the one used to prove (19) we get

$$N^p_{\theta,p}(u_0') \leqslant 2\theta^{-p} H^p_{\theta,p}(Au_0) . \tag{20}$$

Therefore the result follows from Theorems 5 and 9 with

$$a_5 = (c_1^p + 2\theta^{-p} a_p^2)^{1/p}$$

where c_1 and a_2 are given by (3) and Theorem 5, respectively.

THEOREM 11. – Let $1 < p < \infty$ and let $x \in D_{A^2}(\frac{1}{2}, p)$. Then $u_0' \in W^{1/p,p}(E)$ and we have

$$\|u_0'\|_{W^{1,p,p}} \leqslant \left(c_2 + 2(p - 1)^{-1} p^p\right)^{1/p} |x|_{D_A(\frac{1}{2},p)}$$

where c_2 is given by (5).

PROOF. – From (20) we have

$$N^p_{1/p,p}(u_0') \leqslant p^p H^p_{1/p,p}(Au_0) .$$

Therefore the result follows from the proof of Theorem 6 and from the fact that $u_0' = Au_0$.

c) *Mixed regularity.*

THEOREM 12. – For each $x \in E$ we have that $u_0 \in W^{\varepsilon,p}(D_A(\theta - \varepsilon, p))$ with $\varepsilon < < \theta < 1/p$. Moreover there exists a_6 such that

$$\|u_0\|_{W^{\varepsilon,p}_{\theta-\varepsilon,p}} \leqslant a_6 |x| .$$

PROOF. – The result follows from (11) and from Theorems 2 and 7 with

$$a_6 = c_3(a_1 + a_3)$$

where c_3 is given by (11).

THEOREM 13. – Let $x \in D_A(\theta, p)$ with $\theta < 1 - 1/p$. Then for each $\varepsilon < \theta + 1/p$ $u_0 \in W^{\varepsilon,p}(D_A(\theta + 1/p - \varepsilon, p))$. Moreover there exists a_7 verifying

$$\|u_0\|_{W^{\varepsilon,p}_{\theta+1/p-\varepsilon,p}} \leqslant a_7 |x|_{D_A(\theta,p)} .$$

PROOF. – The result follows from (11) and Theorems 3 and 8 with

$$a_7 = c_3\big((M_0^p T + (p - 1 - p\theta)^{-1})^{1/p} + a_4\big)$$

where c_3 is given by (11).

THEOREM 14. – Let $x \in D_A(1 - 1/p, p)$. Then $u_0 \in W^{\varepsilon,p}(D_A(1 - \varepsilon, p))$ for each $\varepsilon < 1$ and there exists a_8 such that

$$\|u_0\|_{W^{\varepsilon,p}_{1-\varepsilon,p}} \leqslant a_8 |x|_{D_A(1-1/p,p)} .$$

PROOF. – By a computation similar to the one used to prove (18) we obtain

$$N^p_{\varepsilon,1-\varepsilon,p}(u) = 2\int_0^T dt \int_0^t |S(t)x - S(s)x|^p_{1-\varepsilon,p}(t-s)^{-1-\varepsilon p}\, ds \leqslant$$
$$\leqslant 2\varepsilon^{-p}\int_0^T ds \int_0^{T-s} |AS(t+s)x|^p_{1-\varepsilon,p}\, t^{-1+p-\varepsilon p}\, dt$$

therefore we get

$$N^p_{\varepsilon,1-\varepsilon,p} \leqslant 2\varepsilon^{-p}\int_0^T ds \int_s^T |AS(t)x|^p_{1-\varepsilon,p}(t-s)^{-1+p-\varepsilon p}\, dt =$$

$$= 2\varepsilon^{-p}\int_0^T |AS(t)x|^p_{1-\varepsilon,p}\,dt\int_0^t (t-s)^{-1+p-\varepsilon p}\,ds =$$

$$= 2\varepsilon^{-p}(p-\varepsilon p)^{-1}\int_0^T |AS(t)x|^p_{1-\varepsilon,p}\,t^{p-p\varepsilon}\,dt =$$

$$= 2\varepsilon^{-p}(p-\varepsilon p)^{-1}\int_0^T t^{p-p\varepsilon}\,dt\int_0^{+\infty} |AS(s+t)x|^p s^{-1+p\varepsilon}\,ds \leqslant$$

$$\leqslant 2M_1^p\varepsilon^{-p}(p-\varepsilon p)^{-1}\int_0^T t^{p-p\varepsilon}|AS(t/2)x|^p\,dt\int_0^{+\infty}(s+t/2)^{-p}s^{-1+p\varepsilon}\,ds =$$

$$= 2M_1^p\varepsilon^{-p}(p-\varepsilon p)^{-1}\int_0^T |AS(t/2)x|^p\,dt\int_0^{+\infty}(s+\tfrac{1}{2})^{-p}s^{-1+p\varepsilon}\,ds$$

where we have set $s = st$. Therefore the result follows from Theorem 4 with

$$a_8 = \left(M_0^p T + 4M_1^p\varepsilon^{-p}(p-\varepsilon p)^{-1}\int_0^{+\infty}(s+\tfrac{1}{2})^{-p}s^{-1+p\varepsilon}\,ds\right)^{1/p}.$$

THEOREM 15. – Let $x \in D_A(\theta + 1 - 1/p, p)$ with $0 < \theta < 1/p$. Then we have

(i) $Au_0 \in W^{\theta,p}(E)$, $\|Au_0\|_{W^{\theta,p}} \leqslant a_5|x|_{D_A(\theta+1-1/p,p)}$;

(ii) $u_0' \in L^p(D_A(\theta,p))$, $\|u_0'\|_{L^p_{\theta,p}} \leqslant a_2|x|_{D_A(\theta+1-1/p,p)}$;

(iii) $u_0 \in W^{\alpha,p}(D_A(\beta,p))$, $0 < \alpha$, $\beta < 1$, $\alpha + \beta = 1 + \theta$;

(iv) $Au_0, u_0' \in W^{\varepsilon,p}(D_A(\theta-\varepsilon,p))$, $\varepsilon < \theta$.

Moreover there exists a_9 verifying

(v) $\|u_0\|_{W^{\alpha,p}_{\beta,p}}, \ \|Au_0\|_{W^{\varepsilon,p}_{\theta-\varepsilon,p}}, \ \|u_0'\|_{W^{\varepsilon,p}_{\theta-\varepsilon,p}} \leqslant a_9|x|_{D_A(\theta+1-1/p,p)}$.

PROOF. – Assertions (i) and (ii) follow from Theorem 9 and 5 and from the fact that $Au_0 = u_0'$. Assertions (iii)-(v) follow from Theorems 9 and 5, from properties (i) and (ii) and from (11) and (12) with

$$a_9 = c(a_2 + a_5)$$

where c is given by (11) or (12).

THEOREM 16. – Let $1 < p < \infty$ and let $x \in D_{A^2}(\frac{1}{2}, p)$. Then we have

(i) $Au_0 \in W^{1/p,p}(E)$, $\|Au_0\|_{W^{1,p,p}} \leqslant (c_2 + 2(p-1)^{-1}p^p)^{1/p} |x|_{D_{A^2}(\frac{1}{2},p)}$;

(ii) $u_0' \in L^p(D_A(1/p, p))$, $\|u_0'\|_{L^p_{1/p,p}} \leqslant (c_2 + (p-1)^{-1})^{1/p} |x|_{D_{A^2}(\frac{1}{2},p)}$;

(iii) $u_0 \in W^{\alpha,p}(D_A(\beta, p))$, $0 < \alpha$, $\beta < 1$, $\alpha + \beta = 1 + 1/p$;

(iv) Au_0, $u_0' \in W^{\varepsilon,p}(D_A(1/p - \varepsilon, p))$, $\varepsilon < 1/p$.

Moreover there exists a_{10} verifying

(v) $\|u_0\|_{W^{\alpha,p}_{\beta,p}}$, $\|Au_0\|_{W^{\varepsilon,p}_{1/p-\varepsilon,p}}$, $\|u_0'\|_{W^{\varepsilon,p}_{1/p-\varepsilon,p}} \leqslant a_{10}|x|_{D_{A^2}(\frac{1}{2},p)}$.

PROOF. – Assertions (i)-(v) follow by a computation similar to that of Theorem 15 with

$$a_{10} = 2c(c_2 + 2(p-1)^{-1})^{1/p}$$

where c is given by (11) or (12).

REMARK 1. – As A generates an analytic semigroup it is known that for each $x \in E$ the function u_0 belongs to $W^{1,p}(\varepsilon, T; E) \cap C(\varepsilon, T; D_A)$, for each $\varepsilon > 0$. Therefore for each $x \in E$ we have that $u_0 \in C(E) \cap L^p_+(D_A) \cap W^{1,p}_+(E)$ and that $u_0'(t) = Au_0(t)$, for each $t > 0$.

REMARK 2. – It follows easily from properties (11)-(15) that the spatial and temporal regularity results established in this section cannot be improved.

5. – Properties of $u_1 = S * f$ with $f \in L^p(E)$.

In what follows we shall set $f(t) = 0$ if $t < 0$.

a) Spatial regularity.

THEOREM 17. – Let $f \in L^p(E)$. Then $u_1 \in L^p(D_A(\theta, p))$ for each $0 < \theta < 1$. Moreover there exists b_1 verifying

$$\|u_1\|_{L^p_{\theta,p}} \leqslant b_1 \|f\| .$$

PROOF. – We have

$$\|u_1\|^p = \int_0^T \left| \int_0^t S(t-s) f(s)\, ds \right|^p dt \leqslant M_0^p T^p p^{-1} \|f\| .$$

Furthermore

$$H^p_{\theta,p}(u_1) = \int_0^T dt \int_0^{+\infty} \left| AS(\tau) \int_0^t S(s) f(t-s)\, ds \right|^p \tau^{-1+p-p\theta}\, d\tau \leqslant$$

$$\leqslant M_1^p \int_0^T dt \int_0^{+\infty} \tau^{-1+p-p\theta}\, d\tau \left(\int_0^t (\tau+s)^{-1} |f(t-s)|\, ds \right)^p \leqslant$$

$$\leqslant 2^{p-1} M_1^p \int_0^T dt \int_0^{+\infty} \tau^{-1}\, d\tau \left(\tau^{-\theta} \int_0^\tau |f(t-s)|\, ds \right)^p +$$

$$+ 2^{p-1} M_1^p \int_0^T dt \int_0^{+\infty} \tau^{-1}\, d\tau \left(\tau^{1-\theta} \int_\tau^{+\infty} |f(t-s)|\, s^{-1}\, ds \right)^p .$$

Therefore using Hardy inequality (see Appendix) we get

$$H^p_{\theta,p}(u_1) \leqslant 2^{p-1} M_1^p (\theta^{-p} + (1-\theta)^{-p}) \int_0^T dt \int_0^t |f(t-s)|^p s^{-1+p-p\theta}\, ds \leqslant$$

$$\leqslant 2^{p-1} M_1^p (\theta^{-p} + (1-\theta)^{-p}) T^{p-p\theta} (p-p\theta)^{-1} \|f\| .$$

Hence the result follows with

$$b_1 = \left(M_0^p T^p p^{-1} + 2^{p-1} M_1^p (\theta^{-p} + (1-\theta)^{-p}) T^{p-p\theta} (p-p\theta)^{-1} \right)^{1/p} .$$

b) Temporal regularity.

THEOREM 18. – Let $f \in L^p(E)$. Then $u_1 \in W^{\theta,p}(E)$ for each $0<\theta<1$. Moreover there exists b_2 verifying

$$\|u_1\|_{W^{\theta,p}} \leqslant b_2 \|f\| .$$

PROOF. – We have

$$N^p_{\theta,p}(u_1) = 2\int_0^T dt \int_0^t \left| \int_0^t S(t-\sigma) f(\sigma)\, d\sigma - \int_0^s S(s-\sigma) f(\sigma)\, d\sigma \right|^p (t-s)^{-1-\theta p}\, ds \leqslant$$

$$\leqslant 2^p \int_0^T dt \int_0^t \left| \int_0^s (S(t-\sigma) - S(s-\sigma)) f(\sigma)\, d\sigma \right|^p (t-s)^{-1-\theta p}\, ds +$$

$$+ 2^p \int_0^T dt \int_0^t \left| \int_s^t S(t-\sigma) f(\sigma)\, d\sigma \right|^p (t-s)^{-1-\theta p}\, ds = J_1 + J_2 .$$

Now

$$J_1 = 2^p \int_0^T dt \int_0^t \left| (S(t-s)-I) \int_0^s S(s-\sigma) f(\sigma)\, d\sigma \right|^p (t-s)^{-1-\theta p}\, ds =$$

$$= 2^p \int_0^T dt \int_0^t \left| \int_0^{t-s} AS(\tau) \int_0^s S(s-\sigma) f(\sigma)\, d\sigma\, d\tau \right|^p (t-s)^{-1-\theta p}\, ds =$$

$$= 2^p \int_0^T ds \int_s^T \left| \int_0^{t-s} AS(\tau) \int_0^s S(s-\sigma) f(\sigma)\, d\sigma\, d\tau \right|^p (t-s)^{-1-\theta p}\, dt \leqslant$$

$$\leqslant 2^p \int_0^T ds \int_0^{+\infty} \left(\int_0^t \left| AS(\tau) \int_0^s S(s-\sigma) f(\sigma)\, d\sigma \right| d\tau \right)^p t^{-1-\theta p}\, dt .$$

Therefore using Hardy inequality we get

$$J_1 \leqslant 2^p \theta^{-p} \int_0^T ds \int_0^{+\infty} \left| AS(t) \int_0^s S(s-\sigma) f(\sigma)\, d\sigma \right|^p t^{-1+p-p\theta}\, dt \tag{21}$$

so that

$$J_1 \leqslant 2^p \theta^{-p} H^p_{\theta,p}(u_1) .$$

Furthermore

$$J_2 = 2^p \int_0^T dt \int_0^t (t-s)^{-1-\theta p}\, ds \left| \int_0^{t-s} S(\sigma) f(t-\sigma)\, d\sigma \right|^p = 2^p \int_0^T dt \int_0^t s^{-1-\theta p}\, ds \left| \int_0^s S(\sigma) f(t-\sigma)\, d\sigma \right|^p \leqslant$$

$$\leqslant 2^p \int_0^T dt \int_0^{+\infty} s^{-1-\theta p}\, ds \left(\int_0^s |S(\sigma) f(t-\sigma)|\, d\sigma \right)^p .$$

Therefore using Hardy inequality we obtain

$$J_2 \leqslant 2^p \theta^{-p} \int_0^T dt \int_0^t |S(s) f(t-s)|^p s^{-1+p-p\theta}\, ds \leqslant 2^p \theta^{-p} M_0^p T^{p-p\theta} (p-p\theta)^{-1} \|f\|^p . \tag{22}$$

Hence the result follows from Theorem 17 with

$$b_2 = \left(M_0^p T^p p^{-1} + 2^p M_0^p \theta^{-p} T^{p-p\theta} (p-p\theta)^{-1} + 2^p \theta^{-p} b_1^p \right)^{1/p} .$$

c) Mixed regularity.

THEOREM 19. – Let $f \in L^p(E)$. Then $u_1 \in W^{\varepsilon,p}(D_A(\theta-\varepsilon, p)) \cap C(D_A(\alpha - 1/p, p))$ for each $\varepsilon < \theta < 1$ and $1/p < \alpha < 1$. Moreover there exists b_3 verifying

$$\|u_1\|_{W^{\varepsilon,p}_{\theta-\varepsilon,p}},\ \|u_1\|_{C_{\alpha-1/p,p}} \leqslant b_3 \|f\| .$$

PROOF. – The result follows from Theorems 17 and 18 with

$$b_3 = c(b_1 + b_2)$$

where c is given by (11) or (13).

REMARK 3. – Theorems 17 and 18 are proved by DA PRATO and GRISVARD in [6] as an application of their theory on the sum of operators.

6. – Properties of $u_1 = S * f$ with $f \in L^p(D_A(\theta, p))$, $0 < \theta < 1$.

a) Spatial regularity.

We begin with the following preliminary lemma.

LEMMA 1. – Let $f \in C(D_A)$. Then $Au_1 \in L^p(D_A(\theta, p))$ for each $0 < \theta < 1$ and moreover there exists b_4 verifying

$$\|Au_1\|_{L^p_{\theta,p}} \leqslant b_4 \|f\|_{L^p_{\theta,p}} .$$

PROOF. – We have

$$\|Au_1\|^p \leqslant \int_0^T dt \left(\int_0^t |AS(t-s) f(s)| \right)^p ds$$

therefore if $p = 1$ we have

$$\|Au_1\| \leqslant T^\theta \int_0^T dt \int_0^t |AS(t-s) f(s)| (t-s)^{-\theta} ds =$$

$$= T^\theta \int_0^T ds \int_s^T |AS(t-s) f(s)| (t-s)^{-\theta} dt = T^\theta \int_0^T ds \int_0^{T-s} |AS(t) f(s)| (t-s)^{-\theta} dt$$

whereas if $p > 1$ we have

$$\|Au_1\|^p \leqslant (1-1/p)^{p-1}\theta^{1-p}\int_0^T t^{p\theta}\,dt\int_0^t |AS(t-s)f(s)|^p(t-s)^{-1+p-p\theta}\,ds \leqslant$$

$$\leqslant T^{p\theta}\theta^{1-p}\int_0^T dt\int_0^t |AS(t-s)f(s)|^p(t-s)^{-1+p-p\theta}\,ds =$$

$$= T^{p\theta}\theta^{1-p}\int_0^T ds\int_s^T |AS(t-s)f(s)|^p(t-s)^{-1+p-p\theta}\,dt = T^{p\theta}\theta^{1-p}\int_0^T ds\int_0^{T-s} |AS(t)f(s)|^p t^{-1+p-p\theta}\,dt.$$

Summarizing we have, for each $p \geqslant 1$

$$\|Au_1\|^p \leqslant T^{p\theta}\theta^{1-p}H^p_{\theta,p}(f)\,.$$

Furthermore

$$H^p_{\theta,p}(Au_1) = \int_0^T dt\int_0^{+\infty}\left|\int_0^t AS((\tau+s)/2)AS((\tau+s)/2)f(t-s)\,ds\right|^p \tau^{-1+p-p\theta}\,d\tau \leqslant$$

$$\leqslant 2^p M_1^p\int_0^T dt\int_0^{+\infty}\left(\int_0^t (\tau+s)^{-1}\,|AS((\tau+s)/2)f(t-s)|\,ds\right)^p \tau^{-1+p-p\theta}\,d\tau \leqslant$$

$$\leqslant 2^p M_0^p M_1^p\int_0^T dt\int_0^{+\infty}\left(\int_0^t (\tau+s)^{-1}\,|AS(s/2)f(t-s)|\,ds\right)^p \tau^{-1+p-p\theta}\,d\tau\,.$$

Therefore by a calculation similar to that of Theorem 17, with $f(t-s)$ replaced by $AS(s/2)f(t-s)$, we obtain

$$H^p_{\theta,p}(Au_1) \leqslant 2^{2p-1}M_0^p M_1^p(\theta^{-p}+(1-\theta)^{-p})\int_0^T dt\int_0^t |AS(s/2)f(t-s)|^p s^{-1+p-p\theta}\,ds =$$

$$= 2^{2p-1}M_0^p M_1^p(\theta^{-p}+(1-\theta)^{-p})\int_0^T s^{-1+p-p\theta}\,ds\int_s^T |AS(s/2)f(t-s)|^p\,dt =$$

$$= 2^{2p-1}M_0^p M_1^p(\theta^{-p}+(1-\theta)^{-p})\int_0^T s^{-1+p-p\theta}\,ds\int_0^{T-s} |AS(s/2)f(t)|^p\,dt \leqslant$$

$$\leqslant 2^{3p-p\theta-1}M_0^p M_1^p(\theta^{-p}+(1+\theta)^{-p})H^p_{\theta,p}(f)\,.$$

Therefore the result follows with

$$b_4 = \left(T^{p\theta}\theta^{1-p} + 2^{3p-p\theta-1}M_0^p M_1^p(\theta^{-p}+(1-\theta)^{-p})\right)^{1/p}.$$

We are now able to prove the following result.

THEOREM 20. – Let $f \in L^p(D_A(\theta, p))$. Then $Au_1 \in L^p(D_A(\theta, p))$ and moreover

$$\|Au_1\|_{L^p_{\theta,p}} \leqslant b_4 \|f\|_{L^p_{\theta,p}}$$

where b_4 is given by Lemma 1.

PROOF. – Let $f_n \in C(D_A)$ be such that $f_n \to f$ in $L^p(D_A(\theta, p))$ and set $u_{1,n} = S * f_n$. We first prove that for each $0 \leqslant t \leqslant T$ we have

$$u_{1,n}(t) \to u_1(t) \quad \text{in } D_A(\theta, p) . \tag{23}$$

We have

$$|u_{1,n}(t) - u_1(t)|^p \leqslant M_0^p T^{p-1} \|f_n - f\|^p$$

and

$$|u_{1,n}(t) - u_1(t)|^p_{\theta,p} \leqslant M_0^p T^{p-1} H^p_{\theta,p}(f_n - f)$$

and (23) is proved. Using Lemma 1 we further obtain

$$\|Au_{1,n} - Au_1\|_{L^p_{\theta,p}} \leqslant b_4 \|f_n - f\|_{L^p_{\theta,p}} .$$

Therefore $\{Au_{1,n}\}$ is Cauchy in $L^p(D_A(\theta, p))$ and hence there exists $v \in L^p \cdot (D_A(\theta, p))$ such that $Au_{1,n} \to v$ in $L^p(D_A(\theta, p))$. This in turn implies that there exists a subsequence $\{Au_{1,n_k}\}$ satisfying

$$Au_{1,n_k}(t) \to v(t) \tag{24}$$

for a.e. $0 \leqslant t \leqslant T$. Finally using (23), (24) and the closedness of A we have that $u_1(t) \in D_A$ and $Au_1(t) = v(t)$ a.e. on $]0, T[$. Therefore we find that $Au_{1,n} \to Au_1$ in $L^p(D_A(\theta, p))$ and the result follows using Lemma 1 once again.

b) Mixed regularity.

THEOREM 21. – Let $f \in L^p(D_A(\theta, p))$. Then $u_1' \in L^p(D_A(\theta, p))$ and he have $u_1' = Au_1 + f$ and

$$\|u_1'\|_{L^p_{\theta,p}} \leqslant (b_4^p + 1)^{1/p} \|f\|_{L^p_{\theta,p}}$$

where b_4 is given by Lemma 1.

PROOF. – Let f_n and $u_{1,n}$ be defined as in the proof of Theorem 20. It is known that $u_{1,n}$ belongs to $W^{1,p}(E)$ and verifies $u_{1,n}' = Au_{1,n} + f_n$. Moreover from the proof of Theorem 20 we have that $u_{1,n} \to u_1$, $Au_{1,n} \to Au_1$ and $f_n \to f$ in $L^p(D_A(\theta, p))$. Therefore the result follows from Lemma 1.

THEOREM 22. – Let $f\in L^p(D_A(\theta,p))$. Then $Au_1\in W^{\theta p}(E)$ and there exists b_5 verifying

$$\|Au_1\|_{W^{\theta,p}}\leqslant b_5\|f\|_{L^p_{\theta,p}}.$$

PROOF. – From the proof of Lemma 1 we have

$$\|Au_1\|^p\leqslant T^{p\theta}\theta^{1-p}\,\|f\|^p_{L^p_{\theta,p}}$$

moreover

$$\begin{aligned}N^p_{\theta,p}(Au_1)&=2\int_0^T dt\int_0^t(t-s)^{-1-\theta p}\,ds\left|A\int_0^t S(t-\sigma)f(\sigma)\,ds-A\int_0^s S(s-\sigma)f(\sigma)\,d\sigma\right|^p\leqslant\\&\leqslant 2^p\int_0^T dt\int_0^t(t-s)^{-1-\theta p}\,ds\left|A\int_0^s\big(S(t-\sigma)-S(s-\sigma)\big)f(\sigma)\right|^p+\\&+2^p\int_0^T dt\int_0^t(t-s)^{-1-\theta p}\,ds\left|\int_s^t AS(t-\sigma)f(\sigma)\,d\sigma\right|^p=J_1+J_2\,.\end{aligned}$$

Now a calculation similar to that of Theorem 18, formulas (21) and (22), shows that

$$J_1\leqslant 2^p\theta^{-p}\int_0^T ds\int_0^{+\infty}\left|A^2S(t)\int_0^s S(s-\sigma)f(\sigma)\,d\sigma\right|^p t^{-1+p-p\theta}\,dt\leqslant 2^p\theta^{-p}H^p_{\theta,p}(Au_1)$$

and that

$$\begin{aligned}J_2&\leqslant 2^p\theta^{-p}\int_0^T dt\int_0^t|AS(s)f(t-s)|^p s^{-1+p-p\theta}\,ds=2^p\theta^{-p}\int_0^T ds\int_s^T|AS(s)f(t-s)|^p s^{-1+p-p\theta}\,dt=\\&=2^p\theta^{-p}\int_0^T ds\int_0^{T-s}|AS(s)f(t)|^p s^{-1+p-p\theta}\,dt\leqslant 2^p\,\theta^{-p}H^p_{\theta,p}(f)\,.\end{aligned}$$

Therefore the result follows from Theorem 20 with

$$b_5=\big(T^{p\theta}\theta^{1-p}+2^p\theta^{-p}(b_4+1)\big)^{1/p}\,.$$

THEOREM 23. – Let $f\in L^p(D_A(\theta,p))$. Then

(i) $u_1\in W^{\alpha,p}(D_A(\beta,p))$, $0<\alpha,\ \beta<1,\ \alpha+\beta=1+\theta$,

(ii) $Au_1\in W^{\varepsilon,p}(D_A(\theta-\varepsilon,p))$, $\varepsilon<\theta$,

(iii) if $\theta<1/p$ then $u_1\in C(D_A(\theta+1-1/p,p))$;

(iv) if $\theta = 1/p$ then $u_1 \in C(D_{A^2}(\frac{1}{2}, p))$;

(v) if $\theta > 1/p$ then $u_1 \in C(D_A(\theta - 1/p, p))$.

Moreover there exists b_6 such that

(vi) $\|u_1\|_{W^{\alpha,p}_{\beta,p}}$, $\|Au_1\|_{W^{c,p}_{\theta-c,p}} \leqslant b_6 \|f\|_{L^p_{\theta,p}}$;

(vii) $\|u_1\|_{C_{\theta+1-1/p,p}} \leqslant b_6 \|f\|_{L^p_{\theta,p}}$, if $\theta < 1/p$;

(viii) $\|u_1\|_{C_{2,\frac{1}{2},p}} \leqslant b_6 \|f\|_{L^p_{1/p,p}}$, if $\theta = 1/p$;

(ix) $\|Au_1\|_{C_{\theta-1/p,p}} \leqslant b_6 \|f\|_{L^p_{\theta,p}}$, if $\theta > 1/p$.

PROOF. – Assertions (i)-(ix) follow from Theorems 20, 21, 22 with

$$b_6 = c(2b_4 + b_5 + 1)$$

where c is given by (11)-(15), depending on the context.

REMARK 4. – Theorems 20 and 21 were proved previously in [6] by different methods.

7. – Properties of $u_1 = S * f$ with $f \in W^{\theta,p}(E)$, $0 < \theta < 1$.

a) Temporal regularity.

We begin with the following preliminary lemma.

LEMMA 2. – Let $f \in W^{1,p}(E)$. Then $u_1' \in W^{\theta,p}(E)$ for each $0 < \theta < 1$. Moreover there exist b_7 and b_8 such that

$$\|u_1'\|_{W^{\theta,p}} \leqslant b_7 \|f\|_{W^{\theta,p}} + b_8 \Big(\int_0^T t^{-p\theta} |f(t)|^p \, dt \Big)^{1/p} .$$

PROOF. – It is known (see DA PRATO [5]) that if $f \in W^{1,p}(E)$ then $u_1' \in C(E)$, $u_1 \in C(D_A)$ and we have

$$u_1'(t) = A \int_0^t S(t-\sigma) f(\sigma) \, d\sigma + f(t) .$$

Now

$$A \int_0^t S(t-\sigma) f(\sigma) \, d\sigma = A \int_0^t S(t-\sigma)\big(f(\sigma) - f(t)\big) \, d\sigma + A \int_0^t S(t-\sigma) f(t) \, d\sigma =$$

$$= A \int_0^t S(t-\sigma)\big(f(\sigma) - f(t)\big) \, d\sigma + S(f) f(t) - f(t)$$

where we used the fact that for each $x \in E$ we have

$$\int_0^t AS(s)x\,ds = S(t)x - x\,.$$

Hence u_1' can be written as

$$u_1'(t) = A\int_0^t S(t-\sigma)\big(f(\sigma)-f(t)\big)\,d\sigma + S(t)f(t) = v_1(t) + v_2(t)\,.$$

Therefore we have

$$\|u_1'\|^p_{W^{\theta,p}} \leqslant 2^{p-1}\big(\|v_1\|^p + \|v_2\|^p + N^p_{\theta,p}(v_1) + N^p_{\theta,p}(v_2)\big)\,.$$

Now

$$\|v_1\|^p = \int_0^T dt\,\left|A\int_0^t S(t-\sigma)(f(\sigma)-f(t))\,d\sigma\right|^p \leqslant$$
$$\leqslant M_1^p\int_0^T dt\left(\int_0^t |f(\sigma)-f(t)|(t-\sigma)^{-1}\,d\sigma\right)^p \leqslant M_1^p T^{p\theta}\theta^{1-p}\|f\|^p_{W^{\theta,p}}$$

and

$$\|v_2\|^p \leqslant M_0^p\|f\|^p\,.$$

Furthermore

$$N^p_{\theta,p}(v_1) = 2\int_0^T dt\int_0^t |v_1(t)-v_1(s)|^p\,(t-s)^{-1-\theta p}\,ds \leqslant$$
$$\leqslant 2^p\int_0^T dt\int_0^t \left|A\int_0^s \big(S(t-\sigma)-S(s-\sigma)\big)\big(f(\sigma)-f(s)\big)\,d\sigma\right|^p (t-s)^{-1-\theta p}\,ds +$$
$$+ 2^{2p-1}\int_0^T dt\int_0^t \left|A\int_0^s S(t-\sigma)\big(f(s)-f(t)\big)\,d\sigma\right|^p (t-s)^{-1-\theta p}\,ds +$$
$$+ 2^{2p-1}\int_0^T dt\int_0^t \left|A\int_s^t S(t-\sigma)\big(f(\sigma)-f(t)\big)\,d\sigma\right|^p (t-s)^{-1-\theta p}\,ds = J_1' + J_2' + J_3'\,.$$

Now a calculation similar to the one used in Theorem 18, formula (21), shows that

$$J_1' \leqslant 2^p \theta^{-p} \int_0^T ds \int_0^{+\infty} t^{-1+p-p\theta}\, dt \left| A^2 S(t) \int_0^s S(s-\sigma)\big(f(\sigma)-f(s)\big)\, d\sigma \right|^p . \tag{25}$$

Furthermore

$$J_1' \leqslant M_1^{2p} 2^{3p} \theta^{-p} \int_0^T ds \int_0^{+\infty} t^{-1+p-p\theta}\, dt \left(\int_0^s (t+s-\sigma)^2 |f(\sigma)-f(s)|\, d\sigma \right)^p =$$

$$= M_1^{2p} 2^{3p} \theta^{-p} \int_0^T ds \int_0^{+\infty} t^{-1+p-p\theta}\, dt \left(\int_0^s (t+\sigma)^2 |f(s-\sigma)-f(s)|\, d\sigma \right)^p .$$

Therefore using Hardy inequality we get (here $f(t) = 0$ if $t < 0$)

$$J_1' \leqslant M_1^{2p} 2^{4p-1} \theta^{-p} \left(\int_0^T ds \int_0^{+\infty} t^{-1}\, dt \left(t^{-\theta} \int_0^t |f(s-\sigma)-f(s)| \sigma^{-1}\, d\sigma \right)^p + \right.$$

$$\left. + \int_0^T ds \int_0^{+\infty} t^{-1}\, dt \left(t^{1-\theta} \int_t^{+\infty} |f(s-\sigma)-f(s)| \sigma^{-2}\, d\sigma \right)^p \right) \leqslant$$

$$\leqslant M_1^{2p} 2^{4p-1} \theta^{-p} \big(\theta^{-p} + (1-\theta)^{-p}\big) \int_0^T ds \int_0^{+\infty} |f(s-t)-f(s)|^p t^{-1-\theta p}\, dt =$$

$$= M_1^{2p} 2^{4p-1} \theta^{-p} \big(\theta^{-p} + (1-\theta)^{-p}\big) \left(\int_0^T ds \int_0^s |f(s-t)-f(s)|^p t^{-1-\theta p}\, dt + \right.$$

$$\left. + \int_0^T ds \int_s^{+\infty} |f(s)|^p t^{-1-\theta p}\, dt \right)$$

so that

$$J_1' \leqslant M_1^{2p} 2^{4p-1} \theta^{-p} \big(\theta^{-p} + (1-\theta)^{-p}\big) \left(N_{\theta,p}^p(f) + (\theta p)^{-1} \int_0^T t^{-p\theta} |f(t)^p|\, dt \right) .$$

Conferning J_2' we have

$$J_2' = 2^{2p-1} \int_0^T dt \int_0^t |(S(t) - S(t-s)\big(\big(f(s)-f(t)\big)|^p (t-s)^{-1-\theta p}\, ds \leqslant 2^{3p-1} M_0^p N_{\theta,p}^p(f) .$$

Finally, by a slight modification of the proof of (22), we get

$$J_3' \leqslant 2^{2p-1}\theta^{-p}\int_0^T dt \int_0^{+\infty} |AS(s)(f(t-s)-f(t))|^p s^{-1+p-p\theta}\, ds$$

so that

$$J_3' \leqslant M_1^p 2^{2p-1}\theta^{-p}\Big(\int_0^T dt \int_0^t |f(t-s)-f(t)|^p\, ds + \int_0^T dt \int_t^{+\infty} |f(t)|^p s^{-1-\theta p}\, ds\Big) \leqslant$$
$$\leqslant M_1^p 2^{2p-1}\theta^{-p}\Big(N_{\theta,p}^p(f) + (\theta p)^{-1}\int_0^T t^{-p\theta}|f(t)|^p\, dt\Big)\,.$$

It remains to consider the term concerning v_2. We have

$$N_{\theta,p}^p(v_2) = 2\int_0^T dt \int_0^t |v_2(t)-v_2(s)|^p (t-s)^{-1-\theta p}\, ds \leqslant 2^p \int_0^T dt \int_0^t |S(t)(f(t)-f(s))|^p (t-s)^{-1-\theta p}\, ds +$$
$$+ 2^p \int_0^T dt \int_0^t |(S(t)-S(s))f(s)|^p (t-s)^{-1-\theta p}\, ds = J_1'' + J_2''\,.$$

Now

$$J_1'' \leqslant 2^p M_0^p N_{\theta,p}^p(f)$$

and

$$J_2'' \leqslant 2^p \int_0^T dt \int_0^t (t-s)^{-1-\theta p}\, ds \Big|\int_s^t AS(\sigma)f(s)\, d\sigma\Big|^p \leqslant M_1^p 2^p \int_0^T dt \int_0^t |f(s)|^p (t-s)^{-1-\theta p}(\log(t/s))^p\, ds =$$
$$= M_1^p 2^p \int_0^T |f(s)|^p\, ds \int_s^T (\log(t/s))^p (t-s)^{-1-\theta p}\, dt =$$
$$= M_1^p 2^p \int_0^T s^{-p\theta}|f(s)|^p\, ds \int_1^{+\infty} (\log t)^p (t-1)^{-1-p\theta}\, dt$$

where we have set $t = st$.

Summarizing we find that the result follows with

$$b_7 = \Big(2^{p-1}M_1^p T^{p\theta}\theta^{1-p} + M_0^p + 2^{p-1}M_1^{2p}2^{4p-1}\theta^{-p}(\theta^{-p} + (1-\theta)^{-p}) +$$
$$+ 2^{4p-2}M_0^p + 2^{3p-2}M_1^p\theta^{-p}\Big)^{1/p}$$

and

$$b_8 = \Big(2^{5p-2} M_1^{2p} \theta^{-p}(\theta^{-p} + (1-\theta)^{-p})(\theta p)^{-1} + 2^{3p-2} M_1^p \theta^{-p}(\theta p)^{-1} + \\ + 2^{2p-1} M_1^p \int_1^{+\infty} (\log t)^p (t-1)^{-1-\theta p}\, dt\Big)^{1/p}.$$

In what follows we shall set

$$\hat{W}^{\theta,p}(E) = W^{\theta,p}(E), \quad \text{if } \theta < 1/p,$$

$$\hat{W}^{1/p,p}(E) = \Big\{ f \in W^{1/p,p}(E) \colon \int_0^T t^{-1} |f(t)|^p\, dt < +\infty \Big\},$$

$$\hat{W}^{\theta,p}(E) \;\; \{ f \in W^{\theta,p}(E) \colon f(0) = 0, \text{ if } \theta > 1/p \}.$$

We are now able to prove the following result.

THEOREM 24. – Let $f \in \hat{W}^{\theta,p}(E)$. Then $u_1' \in \hat{W}^{\theta,p}(E)$ and we have

(i) $\|u_1'\|_{W^{\theta,p}} \leqslant (b_7 + cb_8)\|f\|_{W^{\theta,p}}$, if $\theta \neq 1/p$,

(ii) $\|u_1'\|_{W^{1/p,p}} \leqslant b_7 \|f\|_{W^{1/p,p}} + b_8 \Big(\int_0^T t^{-1} |f(t)|^p\, dt \Big)^{1/p}$.

where b_7 and b_8 are given by Lemma 2 and $c = c'$ if $\theta < 1/p$, $c = c''$ if $\theta > 1/p$, and c', c'' are given by (9) and (10).

PROOF. – Let $f \in W^{\theta,p}(E)$ with $\theta < 1/p$ and let $f_n \in W^{1,p}(E)$ be such that $f_n \to f$ in $W^{\theta,p}(E)$. Moreover set $u_{1,n} = S * f_n$. By (9) and Lemma 2 we find that $\{u_{1,n}'\}$ is Cauchy in $W^{\theta,p}(E)$. Furthermore from (7) we have that $\{u_{1,n}\}$ is Cauchy in $W^{\theta,p}(E)$ since $\{u_{1,n}'\}$ is. Hence we have that $u_1' \in W^{\theta,p}(E)$ and that $u_{1,n}' \to u_1'$ in $W^{\theta,p}(E)$. Therefore assertion (i) is a consequence of Lemma 2. Finally $u_{1,n}' = Au_{1,n} + f_n$, so that using the closedness of A and the fact that $f_n \to f$ in $W^{\theta,p}(E)$ we find that $u_1' = Au_1 + f$, which in turn implies that

$$u_1'(t) = A \int_0^t S(t-s)\big(f(s) - f(t)\big)\, ds + S(t) f(t). \tag{26}$$

Therefore we have proved that if $f \in W^{\theta,p}(E)$ with $\theta < 1/p$ then $u_1' \in W^{\theta,p}(E)$ and satisfies (i) and (26). Now let $f \in W^{1/p,p}(E)$ be such that

$$\int_0^T t^{-1} |f(t)|^p\, dt < +\infty.$$

Using (7) and the preceding results we find that u_1' belongs to $W^{\theta,p}(E)$ with $\theta < 1/p$ and satisfies (26). Therefore by a calculation similar to that of Lemma 2 we find that $u_1' \in W^{1/p,p}(E)$ and satisfies (ii). Moreover we have

$$\int_0^T t^{-1}|u_1(t)|^p\,dt \leqslant 2^{p-1}\int_0^T t^{-1}\,dt\left|A\int_0^t S(t-s)(f(s)-f(t))\,ds\right|^p dt + 2^{p-1}M_0^p\int_0^T t^{-1}|f(t)|^p\,dt\,.$$

Now

$$\int_0^T t^{-1}\left|A\int_0^t S(t-s)(f(s)-f(t))\,ds\right|^p dt \leqslant M_1^p\int_0^T t^{-1}\,dt\left(\int_0^t |f(s)-f(t)|(t-s)^{-1}\,ds\right)^p \leqslant M_1^p N_{1/p,p}^p(f)\,.$$

Therefore we have proved that if $f \in \hat{W}^{1/p,p}(E)$ then $u_1' \in \hat{W}^{1/p,p}(E)$ and satisfies (ii). Finally let $f \in W^{\theta,p}(E)$ with $\theta > 1/p$ and let $f(0) = 0$. By (10) and a calculation similar to the one used above we find that $u_1' \in W^{\theta,p}(E)$ and satisfies (i). Since $u_1(0) = 0$ the proof is complete.

b) Mixed regularity.

THEOREM 25. – Let $f \in \hat{W}^{\theta,p}(E)$. Then $Au_1 \in \hat{W}^{\theta,p}(E)$ and we have $u_1' = Au_1 + f$ and

(i) $\|Au_1\|_{W^{\theta,p}} \leqslant (1 + b_7 + cb_8)\|f\|_{W^{\theta,p}}$, if $\theta \neq 1/p$,

(ii) $\|Au_1\|_{W^{1/p,p}} \leqslant (1 + b_7)\|f\|_{W^{1/p,p}} + b_8\left(\int_0^T t^{-1}|f(t)|^p\,dt\right)^{1/p}$

where b_7, b_8 and c are defined as in Theorem 24.

PROOF. – From the proof of Theorem 24 we have $u_1' = Au_1 + f$. Therefore the result follows from Theorem 24.

THEOREM 26. – Let $f \in \hat{W}^{\theta,p}(E)$. Then $u_1' \in L^p(D_A(\theta, p))$. Moreover there exist b_9 and b_{10} such that

(i) $\|u_1'\|_{L^p_{\theta,p}} \leqslant (b_9 + cb_{10})\|f\|_{W^{\theta,p}}$, if $\theta \neq 1/p$,

(ii) $\|u_1'\|_{L^p_{1/p,p}} \leqslant b_9\|f\|_{W^{1/p,p}} + b_{10}\left(\int_0^T t^{-1}|f(t)|^p\,dt\right)^{1/p}$

where $c = c'$ if $\theta < 1/p$, $c = c''$ if $\theta > 1/p$ and c', c'' are given by (9) and (10).

PROOF. – From the proof of Lemma 2 we have

$$\|u_1'\|^p \leqslant 2^{p-1}(M_1^p T^{p\theta}\theta^{1-p} + M_0^p)\|f\|^p_{W^{\theta,p}}.$$

Furthermore from (25) we get

$$H^p_{\theta,p}(u_1) \leqslant 2^{p-1}\int_0^T dt \int_0^{+\infty} s^{-1+p-p\theta}\, ds \left| A^2 S(t)\int_0^t S(s-\sigma)(f(\sigma)-f(t))\, d\sigma\right|^p + \\ + 2^{p-1}\int_0^T dt \int_0^{+\infty} |AS(s+t)f(t)|^p s^{-1+p-p\theta}\, ds = J_4 + J_5 .$$

Now J_4 can be estimated by the same procedure used for (25) and so we obtain

$$J_4 \leqslant 2^{4p-2}M_1^{2p}(\theta^{-p} + (1-\theta)^{-p})\Big(N^p_{\theta,p}(f) + (\theta p)^{-1}\int_0^T t^{-p\theta}|f(t)|^p\, dt\Big).$$

Furthermore, concerning J_5, we have

$$J_5 \leqslant 2^{p-1}M_1^p\int_0^T |f(t)|^p\, dt \int_0^{+\infty}(s+t)^{-p}s^{-1+p-p\theta}\, ds = 2^{p-1}M_1^p\int_0^T t^{-p\theta}|f(t)|^p\, dt\int_0^{+\infty}(s+1)^{-p}s^{-1+p-p\theta}\, ds$$

where we have set $s = st$. Therefore the result follows from (9) and (10) with

$$b_9 = \big(2^{p-1}(M_1 T^{p\theta}\theta^{1-p} + M_0^p) + 2^{4p-2}M_1^{2p}(\theta^{-p} + (1-\theta)^{-p})\big)^{1/p}$$

and

$$b_{10} = \left(2^{4p-2}M_1^{2p}(\theta^{-p} + (1-\theta)^{-p})(\theta p)^{-1} + 2^{p-1}M_1^p\int_0^{+\infty}(s+1)^{-p}s^{-1+p-p\theta}\, ds\right)^{1/p}.$$

THEOREM 27. – Let $f \in \hat{W}^{\theta,p}(E)$. Then

(i) $u_1 \in W^{\alpha,p}(D_A(\beta, p))$, $0 < \alpha$, $\beta < 1$, $\alpha + \beta = 1 + \theta$,

(ii) $u_1' \in W^{\varepsilon,p}(D_A(\theta - \varepsilon, p))$, $\varepsilon < \theta$,

(iii) if $\theta < 1/p$ then $u_1 \in C(D_A(\theta + 1 - 1/p, p))$,

(iv) if $\theta = 1/p$ then $u_1 \in C(D_{A^2}(\frac{1}{2}, p))$,

(v) if $\theta > 1/p$ then $u_1' \in C(D_A(\theta - 1/p, p))$.

Moreover there exist b_{11} and b_{12} such that

(vi) $\|u_1\|_{W^{\alpha,p}_{\beta,p}},\ \|u_1'\|_{W^{\varepsilon,p}_{\theta-\varepsilon,p}} \leqslant (b_{11} + (c' + c'')b_{12})\|f\|_{W^{\theta,p}},\ \theta \neq 1/p,$

(vii) $\|u_1\|_{C_{\theta+1-1/p,p}} \leqslant (b_{11} + (c' + c'')b_{12})\|f\|_{W^{\theta,p}},\ \theta < 1/p,$

(viii) $\|u_1'\|_{C_{\theta-1/p,p}} \leqslant (b_{11} + (c' + c'')b_{12})\|f\|_{W^{\theta,p}},\ \theta > 1/p,$

(ix) $\|u_1\|_{W^{\alpha,p}_{\beta,p}},\ \|u_1'\|_{W^{\varepsilon,p}_{1/p-\varepsilon,p}},\ \|u_1\|_{C_{2,1/2,p}} \leqslant b_{11}\|f\|_{W^{1/p,p}} + b_{12}\left(\int_0^T t^{-1}|f(t)|^p\,dt\right)^{1/p}.$

PROOF. – Assertions (i)-(ix) follow from Theorems 24-26 and properties (9)-(14) and (16) with

$$b_{11} = c(2b_7 + 1 + b_9)\,, \qquad b_{12} = c(2b_8 + 1 + b_{10})$$

where c is given by (11)-(14) and (16), depending on the context.

REMARK 5. – Theorems 24 and 25 were previously proved in [6] by different methods.

8. – Solutions of the abstract Cauchy problem.

We are now able to study the properties of the solutions of the following abstract parabolic initial value problem

$$\begin{cases} u'(t) = Au(t) + f(t)\,, \\ u(0) = x\,. \end{cases} \tag{27}$$

The following theorem describes the properties of the solutions of (27) when $f \in L^p(0, T; E)$ and $x \in D_A(\theta, p)$ and θ increases.

THEOREM 28. – Let $f \in L^p(0, T; E)$. Then we have

(i) if $x \in E$ then (27) admits a unique mild solution u and we have

$$u \in C(0, T; E) \cap L^p(0, T; D_A(\theta, p)) \cap W^{\theta,p}(0, T; E) \cap W^{\varepsilon,p}(0, T; D_A(\theta - \varepsilon, p)),\quad \text{for } \varepsilon < \theta < 1/p\,,$$

(ii) if $x \in D_A(\theta, p)$ with $\theta < 1 - 1/p$ then

$$u \in C(0, T; D_A(\theta, p)) \cap L^p(0, T; D_A(\theta + 1/p, p)) \cap W^{\theta+1/p,p}(0, T; E) \cap \cap W^{\varepsilon,p}(0, T; D_A(\theta + 1/p - \varepsilon, p))\,, \quad \text{for } \varepsilon < \theta + 1/p\,,$$

(iii) if $x \in D_A(1-1/p, p)$ then

$$u \in C(0, T; D_A(\theta - 1/p, p)) \cap L^p(0, T; D_A(\theta, p)) \cap W^{\theta,p}(0, T; E) \cap$$
$$\cap W^{\varepsilon,p}(0, T; D_A(\theta - \varepsilon, p)), \quad \text{for } \varepsilon < \theta < 1.$$

Moreover u depens continuously upon the data f and x with respect to the norms given by (i)-(iii).

PROOF. – The result follows from properties (1), (4), (6) and from Theorems 2, 3, 4, 7, 8, 9, 12, 13, 14, 17, 18 and 19.

The following theorem describes the properties of the solutions of (27) when $f \in L^p(0, T; D_A(\theta, p))$, $x \in D_A(\theta', p)$ and θ' increases.

THEOREM 29. – Let $f \in L^p(0, T; D_A(\theta, p))$ for some $\theta < 1$. Then

(i) if $x \in E$ then (27) admits a unique classical solution $u \in C(E) \cap L^p_+(0, T; D_A) \cap W^{1,p}_+(0, T; E)$ and we have

$$u \in L^p(0, T; D_A(\theta', p)) \cap W^{\theta',p}(0, T; E) \cap W^{\varepsilon,p}(0, T; D_A(\theta' - \varepsilon, p))$$
$$\text{for } \varepsilon < \theta' < 1/p,$$

(ii) if $x \in D_A(\theta', p)$ with $\theta' < 1 - 1/p$ then the classical solution satisfies

$$u \in C(0, T; D_A(\theta', p)) \cap L^p(0, T; D_A(\theta' + 1/p, p)) \cap W^{\theta'+1/p,p}(0, T; E) \cap$$
$$\cap W^{\varepsilon,p}(0, T; D_A(\theta' + 1/p - \varepsilon, o)), \quad \text{for } \varepsilon < \theta' + 1/p,$$

(iii) if $x \in D_A(1 - 1/p, p)$ then (27) admits a unique strict solution u and we have

$$u \in C(0, T; D_A(1 - 1/p, p)) \cap L^p(0. T; D_A) \cap W^{1,p}(0, T; E) \cap$$
$$\cap W^{\varepsilon,p}(0, T; D_A(1 - \varepsilon, p)), \quad \text{for } \varepsilon < 1,$$

(iv) if $x \in D_A(\theta + 1 - 1/p, p)$ and $\theta < 1/p$ then the strict solution satisfies

1) $u \in C(0, T; D_A(\theta + 1 - 1/p, p)) \cap W^{\alpha,p}(0, T; D_A(\beta, p))$,

$$\text{for } 0 < \alpha, \ \beta < 1, \ \alpha + \beta = 1 + \theta,$$

2) $Au \in L^p(0, T; D_A(\theta, p)) \cap W^{\theta,p}(0, T; E) \cap W^{\varepsilon,p}(0, T; D_A(\theta - \varepsilon, p))$,

$$\text{for } \varepsilon < \theta,$$

3) $u' \in L^p(0, T; D_A(\theta, p))$,

(v) if $x \in D_{A^2}(\frac{1}{2}, p)$ and $1/p \leqslant \theta < 1$ then the strict solution satisfies

1) $u \in C(0, T; D_{A^2}(\frac{1}{2}, p)) \cap W^{\alpha,p}(0, T; D_A(\beta, p))$,

$$\text{for } 0 < \alpha, \ \beta < 1, \ \alpha + \beta = 1 + 1/p,$$

2) $Au \in L^p(0, T; D_A(1/p, p)) \cap W^{1/p,p}(0, T; E) \cap W^{\varepsilon,p}(0, T; D_A(1/p - \varepsilon, p))$,

$$\text{for } \varepsilon < 1/p,$$

3) $u' \in L^p(0, T; D_A(1/p, p))$,

(vi) if $x \in D_A(\theta + 1 - 1/p, p)$ and $1/p < \theta < 1$ then the strict solution satisfies

1) $u \in C(0, T; D_A(\theta + 1 - 1/p, p)) \cap W^{\alpha,p}(0, T; D_A(\beta, p))$,

$$\text{for } \alpha + \beta = 1 + \theta, \ 0 < \alpha, \ \beta < 1,$$

2) $Au \in C(0, T; D_A(\theta - 1/p, p)) \cap L^p(0, T; D_A(\theta, p)) \cap W^{\theta,p}(0, T; E) \cap$

$$\cap \ W^{\varepsilon,p}(0, T; D_A(\theta - \varepsilon, p)), \quad \text{for } \varepsilon < \theta,$$

3) $u' \in L^p(0, T; D_A(\theta, p))$.

Moreover u depends continuously upon the data f and x with respect to the norms given by (i)-vi).

PROOF. - To prove (i) it suffices to show that the function $u = u_0 + u_1$ is a classical solution of (27) and then use Theorems 2,7, 12, 20, 21 and 23. Now we have (see Remark 1 of sect. 4) that $u_0 \in C(0, T; E) \cap L^p_+(0, T; D_A) \cap W^{1,p}_+(0, T; E)$ and that $u_0'(t) = Au_0(t)$ for $t > 0$. Moreover from Theorem 21 it follows that $u_1' = Au_1 + f$. Therefore $u'(t) = Au(t) + f(t)$ a.e. on $]0, T[$ and $u(0) = x$. Assertion (ii) follows from (i), from (4) and from Theorems 3, 8, 13, 20, 21 and 23. Assertion (iii) follows from Theorems 4, 9, 14, 20, 21 and 23. Assertion (iv) follows from (4) and from Theorems 5, 15, 20, 21, 22 and 23. Assertion (v) follows from (6) and from Theorems 6, 16, 20, 21, 22 and 23. Finally assertion (vi) follows from (4), from Theorems 3, 8 and 13 with x replaced by Ax and from Theorems 20, 21, 22 and 23.

Finally the following theorem describes the properties of the solutions of (27) when $f \in W^{\theta,p}(0, T; E)$, $x \in D_A(\theta', p)$ and θ' increases.

THEOREM 30. - Let $f \in W^{\theta,p}(0, T; E)$ for some $\theta < 1$. Then

(i) if $x \in E$ then (27) admits a unique classical solution $u \in C(0, T; E) \cap \cap L^p_+(0, T; D_A) \cap W^{1,p}_+(0, T; E)$ and we have

$$u \in L^p(0, T; D(\theta', \mathrm{p})) \cap W^{\theta',p}(0, T; E) \cap W^{\varepsilon,p}(0, T; D_A(\theta' - \varepsilon, p))$$

$$\text{for } \varepsilon < \theta' < 1/p,$$

(ii) if $x \in D_A(\theta', p)$ with $\theta' < 1 - 1/p$ then the classical solution satisfies

$$u \in C(0, T; D_A(\theta', p)) \cap L^p(0, T; D_A(\theta' + 1/p, p)) \cap W^{\theta'+1/p, p}(0, T; E) \cap$$
$$\cap W^{\varepsilon, p}(D_A(\theta' + 1/p - \varepsilon, p)) , \quad \text{for } \varepsilon < \theta' + 1/p ,$$

(iii) if $x \in D_A(1 - 1/p, p)$ then (27) admits a unique strict solution u and we have

$$u \in C(0, T; D_A(1 - 1/p, p)) \cap L^p(0, T; D_A) \cap W^{1,p}(0, T; E) \cap$$
$$\cap W^{\varepsilon, p}(0, T; D_A(1 - \varepsilon, p)) , \quad \text{for } \varepsilon < 1 ,$$

(iv) if $x \in D_A(\theta + 1 - 1/p, p)$ and $\theta < 1/p$ then the strict solution satisfies

1) $u \in C(0, T; D_A(\theta + 1 - 1/p, p)) \cap W^{\alpha, p}(0, T; D_A(\beta, p))$,
$$\text{for } 0 < \alpha ,\ \beta < 1 ,\ \alpha + \beta = 1 + \theta ,$$

2) $u' \in L^p(0, T; D_A(\theta, p)) \cap W^{\theta, p}(0, T; E) \cap W^{\varepsilon, p}(0, T; D_A(\theta - \varepsilon, p))$,
$$\text{for } \varepsilon < \theta ,$$

3) $Au \in W^{\theta, p}(0, T; E)$,

(v) if $x \in D_{A^2}(\frac{1}{2}, p)$ and $1/p \leqslant \theta < 1$ then if $\int_0^T t^{-1} |f(t)|^p \, dt < +\infty$ the strict solutions satisfies

1) $u \in C(0, T; D_{A^2}(\frac{1}{2}, p)) \cap W^{\alpha, p}(0, T; D_A(\beta, p))$,
$$\text{for } 0 < \alpha ,\ \beta < 1 ,\ \alpha + \beta = 1 + \theta ,$$

2) $u' \in L^p(0, T; D_A(1/p, p)) \cap W^{1/p, p}(0, T; E) \cap W^{\varepsilon, p}(0, T; D_A(1/p - \varepsilon, p))$,
$$\text{for } \varepsilon < 1/p ,$$

3) $Au \in W^{1/p, p}(0, T; E)$,

(vi) if $x \in D_A$ and $1/p < \theta < 1$ then if $Ax + f(0) \in D_A(\theta - 1/p, p)$ the strict solution satisfies

1) $u \in C(0, T; D_A) \cap W^{\alpha, p}(0, T; D_A(\beta, p))$,
$$\text{for } 0 < \alpha ,\ \beta < 1 ,\ \alpha + \beta = 1 + \theta ,$$

2) $u' \in C(0, T; D_A(\theta - 1/p, p)) \cap L^p(0, T; D_A(\theta, p)) \cap$
$$\cap W^{\theta, p}(0, T; E) \cap W^{\varepsilon, p}(0, T; D_A(\theta - \varepsilon, p)) , \quad \text{for } \varepsilon < \theta ,$$

3) $Au \in W^{\theta, p}(0, T; E)$.

Moreover u depends continuously upon the data f and x with respect to the norms given by (i)-(vi).

PROOF. – Assertion (i) follows by a computation similar to the one used in Theorem 29 (i) and from Theorems 2, 7, 12, 24, 25 and 27. Assertion (ii) follows from (4) and from Theorems 3, 8, 13, 24, 25 and 27. Assertion (iii) follows from (4) and from Theorems 4, 9, 14, 24, 25 and 27. Assertion (iv) follows from (4) and from Theorems 5, 15, 24, 25, 26 and 27. Assertion (v) follows from (6) and from Theorems 6, 16, 24, 25, 26 and 27. To prove (vi) let us first note that by (iv) there exists a unique strict solution u of (27). Moreover u can be written as

$$u(t) = S(t)\,x + (S * f)(t) = S(t)x + \int_0^t S(t-s)\big(f(s) - f(0)\big)\,ds + \int_0^t S(s)\,f(0)\,ds = u_0 + v_1 + v_2 .$$

Now Theorems 24, 25, 26 and 27 imply that v_1 satisfies properties 1)-3) of assertion (vi). Moreover we have $u_0'(t) + v_2'(t) = S(t)\big(Ax + f(0)\big)$ and $Au_0(t) + Av_2(t) = S(t)\big(Ax + f(0)\big) - f(0)$ so that the function $u_0 + v_2$ satisfies 1)-3) of assertion (vi) by virtue of (4) and Theorens 3, 8 and 13. The proof is complete.

9. – Examples.

We shall use our abstract results to study the regularity of solutions of linear parabolic equations.

Let Ω be an open bounded set in R^n with regular boundary $\partial\Omega$ and consider a second order uniformly elliptic differential operator in Ω.

$$A(x, D) = \sum_{i,j}^{1,n} a_{i,j}(x)\,\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} a_i(x)\,\frac{\partial}{\partial x_i} + a_0(x)$$

where $x = (x_1, \ldots, x_n)$ and $a_{j,i}$, a_i are regular functions defined in $\bar{\Omega}$ (see [2] for the precise assumptions on Ω and the functions $a_{i,j}$, etc.).

Now let $1 < q < \infty$ and consider the realization of $A(x, D)$ in $L^q(\Omega)$

$$D_{A_q} = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)\,,$$

$$A_q u = A(x, D)\,u\,.$$

It is known (see [2]) that A_q generates an analytic semigroup. Therefore we can use the results of sect. 8 with $E = L^q(\Omega)$ and $A = A_q$ to study the following initial-boundary value problem

(28.1) $$u_t(t,x) = A(x,D)u(t,x) + f(t,x)\,, \quad \text{a.e. on }]0,T[\times\Omega\,,$$

(28.2) $$u(t,x) = 0\,, \quad \text{a.e. on }]0,T[\times\partial\Omega\,,$$

(28.3) $$u(0,x) = u_0(x)\,, \quad \text{a.e. on } \Omega\,.$$

With the given definitions of E and A it is obvious in which sense problem (27) is the abstract version of (28.1)-(28.3).

In a subsequent paper we will sistematically use all the results of Theorems 28-30 to obtain existence, uniqueness and regularity for the solutions of (28.1)-(28.3). Here we will limit ourselves to point out those results which seem to be less familiar and more interesting. One of these can obtained if we suppose f regular only with respect to t and use Theorem 30 (i).

Theorem 31. Let $t \to f(t,\cdot)$ belong to $W^{\theta,p}(0,T; L^q(\Omega))$ for some $1 \leqslant p < \infty$ and $\theta < 1/p$. Then for each $u_0 \in L^q(\Omega)$ there exists a unique $u(t,x)$ such that $t \to u(t,\cdot)$ belongs to

$$C(0,T; L^q(\Omega)) \cap L^p_+(0,T; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)) \cap W^{1,p}_+(0,T; L^q(\Omega))\,,$$

satisfying (28.1), (28.3). Moreover the function $t \to u(t,\cdot)$ belongs to

$$L^p(0,T; D_{A_q}(\theta',p)) \cap W^{\theta',p}(0,T; L^q(\Omega)) \cap W^{\varepsilon,p}(0,T, D_{A_q}(\theta'-\varepsilon,p))\,, \text{ for } \varepsilon < \theta' < 1/p\,.$$

If we use Theorem 30 (iv) we obtain the following result.

Theorem 32. Let $t \to f(t,\cdot)$ belong to $W^{\theta,p}(0,T; L^q(\Omega))$ for some $1 \leqslant p < \infty$ and $\theta < 1/p$ and let $u_0 \in D_{A_q}(\theta + 1 - 1/p, p)$. Then the solution of (28.1)-(28.3) is such that $t \to u(t,\cdot)$ belongs to

$$C(0,T; D_{A_q}(\theta + 1 - 1/p, p)) \cap W^{\alpha,p}(0,T; D_{A_q}(\beta,p))\,,$$
$$\text{for } 0 < \alpha\,,\ \beta < 1 \text{ and } \alpha + \beta = 1 + \theta\,.$$

Moreover the function $t \to u(t,\cdot)$ belongs to

$$L^p(0,T; D_{A_q}(\theta,p)) \cap W^{\theta,p}(0,T; L^q(\Omega)) \cap W^{\varepsilon,p}(0,T; D_{A_q}(\theta-\varepsilon,p))\,, \quad \text{for } \varepsilon < \theta$$

and the function $t \to A_q u(t,\cdot)$ belongs to $W^{\theta,p}(0,T; L^q(\Omega))$;

Here $D_{A_q}(\theta; p)$ is the intermediate space between $W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ and $L^q(\Omega)$ defined in sect. 1.

In order to describe in a simpler way the spaces $D_{A_q}(\theta,p)$ let now assume $p = q$ (the case $p \neq q$ would need to use of Besov spaces with three parameters). If $p = q$

these spaces have been completely characterized (also when $A(x, D)$ is a higher order elliptic operator with general boundary conditions, according to [2]) by GRISVARD in [9] in this way

$$D_{A_p}(\theta, p) = \begin{cases} B^{2\theta,p}(\Omega)\,, & \text{if } 2\theta < 1/p\,, \\ u \in B^{2\theta,p}(\Omega)\,,\ \int\limits_{\Omega} (d(x, \partial\Omega))^{-1} |u(x)|^p\, dx < \infty\,, & \text{if } 2\theta = 1/p\,, \\ u \in B^{2\theta,p}(\Omega)\,,\ u(x) = 0 \quad \text{for a.e. } x \in \partial\Omega\,, & \text{if } 2\theta > 1/p \end{cases}$$

where $B^{2\theta,p}(\Omega)$ denotes a Besov space and $d(x, \partial\Omega)$ is the distance from x to $\partial\Omega$. Hence $B^{2\theta,p}(\Omega) = W^{2\theta,p}(\Omega)$ (Sobolev space) if $p = 2$ or $\theta \neq \frac{1}{2}$, whereas if $p \neq 2$ we have

$$B^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int\limits_{\Omega} \int\limits_{\Omega} |u(x) + u(y) - 2u((x+y)/2)|^p\, |x - y|^{1+p}\, dx\, dy < \infty \right\}$$

(for further references see [1], [9]).

Furthermore let us note that if $p > q$ we can still obtain simple regularity properties for the solutions of (28.1)-(18.3) by using the characterization of $D_{A_q}(\theta, q)$ and the inclusion $D_{A_q}(\theta, q) \subset D_{A_q}(\theta, p)$ which follows from the definition given in sect. 1.

REMARK 6. – Some of the results of Theorems 31 and 32 are contained in those of LADYZENSKAJA, SOLONNIKOV and URAL'CEVA [12] if $p = q$ and in those of VON WAHL [15] if $p > 1$.

Finally let us show how we can study problem (28.1)-(28.3) choosing f such that $f(t, \cdot) \in L^1(\Omega)$. The realization of $A(x, D)$ in $L^1(\Omega)$ is defined as follows

$$D_{A_1} = \{u \in W_0^{1,1}(\Omega) : A(x, D)u \in L^1(\Omega)\}\,,$$
$$A_1 u = A(x, D)u\,,$$

where $A(x, D)u$ is understood in the sense of distributions. It has been proved by AMANN [3] that if $A(x, D)$ is in divergence form then A_1 generates an analytic semigroup. Therefore using Theorem 30 (i) we get the following result.

THEOREM 33. – *Let $t \to f(t, \cdot)$ belongs to $W^{\theta,p}(0, T; L^1(\Omega))$ for some $1 \leqslant p < \infty$ and $\theta < 1/p$. Then for each $u_0 \in L^1(\Omega)$ there exists a unique $u(t, x)$ such that $t \to u(t, \cdot)$ belongs to $C(0, T; L^1(\Omega)) \cap L^p_+(0, T; D_{A_1}) \cap W^{1,p}_+(0, T; L^1(\Omega))$ satisfying (28.1), (28.3). Moreover $t \to u(t, \cdot)$ belongs to*

$$L^p(0, T; D_{A_1}(\theta', p)) \cap W^{\theta',p}(0, T; L^1(\Omega)) \cap W^{\varepsilon,p}(0, T; D_{A_1}(\theta' - \varepsilon, p)),, \quad \text{for } \varepsilon < \theta' < 1/p\,.$$

Finally, using Theorem 30 (ii), we obtain the following

THEOREM 34. – Let $t \to f(t, \cdot)$ belong to $W^{\theta,p}(0, T; L^1(\Omega))$ for some $1 \leq p < \infty$ and $\theta < 1/p$ and let $u_0 \in D_{A_1}(\theta + 1 - 1/p, p)$. Then the solution u of (28.1), (28.3) is such that $t \to u(t, \cdot)$ belongs to

$$C(0, T; D_{A_1}(\theta + 1 - 1/p, p)) \cap W^{\alpha,p}(0, T; D_{A_1}(\beta, p)) ,\qquad \text{for } 0 < \alpha ,\ \beta < 1 ,\ \alpha + \beta = 1 + \theta .$$

Moreover the function $t \to u_t(t, \cdot)$ belongs to

$$L^p(0, T; D_{A_1}(\theta, p)) \cap W^{\theta,p}(0, T; L^1(\Omega)) \cap W^{\varepsilon,p}(0, T; D_{A_1}(\theta - \varepsilon, p)) ,\qquad \text{for } \varepsilon < \theta$$

and the function $t \to A_1 u(t, \cdot)$ belongs to $W^{\theta,p}(0, T; L^1(\Omega))$.

Analogous results can be obtained for the solutions of (28.1)-(28.3) if f is assumed to be regular with respect to x by using Theorem 29. Also we could replace $A(x, D)$ by an uniformly elliptic operator of order $2m$ and/or the boundary condition (28.2) by a system of m linear differential operators satisfying the assumptions of [2].

Appendix.

We want to summarize here the results concerning the intermediate spaces that are used in this paper.

a) The spaces $D_A(\theta, p)$.

Let $A: D_A \subset E \to E$ generate a bounded analytic semigroup $S(t)$ in a Banach space E with norm $|\cdot|$. Hence there exist $M, M_0, M_1 > 0$ and $\alpha \in]\pi/2, \pi[$ such that if $z \in C$, $z \neq 0$, $|\arg z| < \alpha$, we have that z is in the resolvent set of A and moreover

$$|z(z - A)^{-1}x| \leq M|x| , \tag{1}$$

$$|S(t)x| \leq M_0|x| , \quad t \geq 0 , \tag{2}$$

$$|tAS(t)x| \leq M_1|x| , \quad t > 0 . \tag{3}$$

For fixed $0 < \theta < 1$ and $1 \leq p < \infty$ we shall denote by $D_A(\theta, p)$ the following intermediate space between D_A and E (for more details see [4])

$$D_A(\theta, p) = \{x \in E: |x|_{\theta,p} < +\infty\}$$

where

$$|x|_{\theta,p} = \left(\int_0^{+\infty} |t^{1-\theta} AS(t)\,x|^p\, t^{-1}\, dt\right)^{1/p}$$

endowed with the norm

$$|x|_{D_A(\theta,p)} = \left(|x|^p + |x|^p_{\theta,p}\right)^{1/p}$$

If $0 < \theta < 1$ we shall also set

$$D_A(\theta + 1, p) = \{x \in D_A \colon Ax \in D_A(\theta, p)\}$$

with norm

$$|x|_{D_A(\theta+1,p)} = \left(|Ax|^p + |Ax|^p_{\theta,p}\right)^{1/p} .$$

Finally we denote by $D_{A^2}(\frac{1}{2}, p)$ the following intermediate space between D_A^2 and E

$$D_{A^2}(\tfrac{1}{2}, p) = \{x \in E \colon |x|_{2,\frac{1}{2},p} < +\infty\}$$

where

$$|x|_{2,\frac{1}{2},p} = \left(\int_0^{+\infty} |tA^2 S(t)x|^p t^{-1}\, dt\right)^{1/p}$$

endowed with the norm

$$|x|_{D_{A^2}(\frac{1}{2},p)} = \left(|x|^p + |x|^p_{2,\frac{1}{2},p}\right)^{1/p} .$$

We now collect some important properties of $D_A(\theta, p)$ and $D_{A^2}(\frac{1}{2}, p)$. To this end we recall the following inequality (see [4], p. 199).

LEMMA 1 (Hardy-Yound inequality). – Let $\alpha > 0$ and $1 \leqslant p < \infty$. If f is a non negative measurable function on $]0, +\infty[$, then

$$\int_0^{+\infty} \left(t^{-\alpha} \int_0^t f(s)\, s^{-1}\, ds\right)^p t^{-1}\, dt \leqslant \alpha^{-p} \int_0^{+\infty} (s^{-\alpha} f(s))^p s^{-1}\, ds , \tag{4}$$

$$\int_0^{+\infty} \left(t^{\alpha} \int_t^{+\infty} f(s)\, s^{-1}\, ds\right)^p t^{-1}\, dt \leqslant \alpha^{-p} \int_0^{+\infty} (s^{\alpha} f(s))^p s^{-1}\, ds . \tag{5}$$

The following imbedding results hold.

LEMMA 2. – For each $1 \leqslant p < \infty$ we have

(i) $D_A \hookrightarrow D_A(\theta, p) \hookrightarrow D_A(\theta', p) \hookrightarrow E$, $\theta' < \theta < 1$,

(ii) $D_{A^2} \hookrightarrow D_{A^2}(\frac{1}{2}, p) \hookrightarrow D_A(\theta, p)$, $\theta < 1$.

PROOF. – Let $x \in D_A$; using (2) and (3) we have

$$|x|^p_{\theta,p} = \int_0^1 |S(t)Ax|^p t^{-1+p-p\theta}\,dt + \int_1^{+\infty} |AS(t)x|^p t^{-1+p-p\theta}\,dt \leqslant$$
$$\leqslant M_0^p |Ax|^p \int_0^1 t^{-1+p-p\theta}\,dt + M_1^p |x|^p \int_1^{+\infty} t^{-1-p\theta}\,dt\,.$$

Therefore we have that $D_A \subset D_A(\theta, p)$ and that $|x|_{D_A(\theta,p)} \leqslant \text{const}\, |x|_{D_A}$. To prove that D_A is dense in $D_A(\theta, p)$ let $x \in D_A(\theta, p)$ and let $x_n = n(n - A)^{-1}x$. Then we have that $x_n \in D_A$ and that $x_n \to x$ in E from (1) and from the fact that D_A is dense in E. Hence using (1) and the dominated convergence theorem it is easy to see that $x_n \to x$ in $D_A(\theta, p)$. Therefore we have proved that $D_A \hookrightarrow D_A(\theta, p)$. Since $D_A \hookrightarrow E$ and since the imbedding $D_A(\theta, p) \hookrightarrow D_A(\theta', p)$ is obvious, the proof of assertion (i) is complete. Concerning (ii) let us note that the imbedding $D_{A^2} \hookrightarrow D_{A^2}\cdot$ $\cdot(\frac{1}{2}, p)$ can be proved by a computation similar to the one used above. To prove that $D_{A^2}(\frac{1}{2}, p) \hookrightarrow D_A(\theta, p)$ let us note that for $0 < t < T$ we have

$$AS(T)x - AS(t)x = \int_t^T A^2 S(s)x\,ds$$

so that from (3)

$$AS(t)x = -\int_t^{+\infty} A^2 S(s)x\,ds\,.$$

Therefore if $x \in D_{A^2}(\frac{1}{2}, p)$ we have

$$|x|^p_{\theta,a} = \int_0^{+\infty} |AS(s)x|^p s^{-1+p-p\theta}\,ds \leqslant \int_0^{+\infty} \left(\int_s^{+\infty} |A^2 S(\sigma)x|\,d\sigma \right)^p s^{-1+p-p\theta}\,ds =$$
$$= \int_0^{+\infty} \left(s^{1-\theta} \int_s^{+\infty} |A^2 S(\sigma)x|\,d\sigma \right)^p s^{-1}\,ds \leqslant (1-\theta)^{-p} \int_0^{+\infty} |A^2 S(s)\,x|^p s^{2p-p\theta-1}\,ds$$

where we used (5). Therefore using (3) we obtain

$$|x|^p_{\theta,p} \leqslant \left(\int_0^1 |sA^2 S(s)x|^p s^{p-p\theta-1}\,ds + \int_1^{+\infty} s^{-1-p\theta}\,ds\; M_1^{2p} 2^{2p} |x|^p \right)(1-\theta)^{-p} \leqslant$$
$$\leqslant \left(|x|^p_{2,\frac{1}{2},p} + M_1^{2p} 2^{2p} (p\theta)^{-1} |x|^p \right)(1-\theta)^{-p}$$

from which it follows that $D_{A^2}(\frac{1}{2}, p) \subset D_A(\theta, p)$. Since D_{A^2} is dense in D_A the conclusion follows.

LEMMA 3. – For each $\theta < 1$ and $1 \leqslant p < \infty$ we have

(i) $|S(t)|x|_{D_A(\theta,p)} \leqslant M_0 |x|_{D_A(\theta,p)}$, $t \geqslant 0$,

(ii) $|S(t)\,x|_{D_{A^2}(\theta,p)} \leqslant M_0 |x|_{D_{A^2}(\frac{1}{2},p)}$, $t \geqslant 0$.

PROOF. – We have

$$|S(t)x|^p_{\theta,p} = \int_0^{+\infty} |AS(s)\,S(t)x|^p\, s^{-1+p-p\theta}\, ds \leqslant M_0^p\, |x|^p_{\theta,p}$$

and (i) follows. Assertion (ii) can be proved in a similar way.

b) The spaces $W^{\theta,p}$.

For each $0 < \theta < 1$ and $1 \leqslant p < \infty$ we denote by $W^{\theta,p}(E)$ the following intermediate space between $W^{1,p}(E)$ and $L^p(E)$

$$W^{\theta,p}(E) = \{u \in L^p(E) \colon N_{\theta,p}(u) < +\infty\}$$

where

$$N_{\theta,p}(u) = \int_0^T dt \int_0^T |u(t) - u(s)|^p\, |t-s|^{-1-\theta p}\, ds$$

endowed with the norm

$$\|u\|_{W^{\theta,p}} = \left(\|u\|^p + N^p_{\theta,p}(u)\right)^{1/p}.$$

The space $W^{\theta,p}(D_A(\varepsilon, p))$ is defined analogously and its norm will be noted by

$$\|u\|_{W^{\theta,p}_{\varepsilon,p}} = \left(\|u\|^p + N^p_{\theta,\varepsilon,p}\right)^{1/p}$$

where

$$N_{\theta,\varepsilon,p}(u) = \left(\int_0^T dt \int_0^T |u(t) - u(s)|_{\varepsilon,p}\, |t-s|^{-1-\theta p}\, ds\right).$$

The space $W^{\theta,p}$ are extensively studied by many authors (see e.g. ADAMS [1] and the reference therein). For the reader's convenience we give here direct proofs (that is proofs that use only the preceding definitions) of some known results concerning these spaces. Also we prove some properties of these spaces which seem to be new and which are used in the preceding sections (see Lemmas 9, 10 and 14).

In what follows we shall set $u(t) = 0$ for $t \notin [0, T]$.

We begin with the following inclusion result.

LEMMA 4. – For each $\theta<\theta'\leqslant 1$ and $1\leqslant p<\infty$ we have

$$W^{\theta',p}(E)\hookrightarrow W^{\theta,p}(E)\hookrightarrow L^p(E)\,.$$

PROOF. – Let $u\in W^{1,p}(E)$; we have

$$N^p_{\theta,p}(u)=2\int_0^T dt\int_0^t\left|\int_s^t u'(\sigma)\,d\sigma\right|^p(t-s)^{-1-\theta p}\,ds=2\int_0^T dt\int_0^t\left|\int_0^{t-s}u'(\sigma+s)\,d\sigma\right|^p(t-s)^{-1-\theta p}\,ds=$$

$$=2\int_0^T ds\int_s^T\left|\int_0^{t-s}u'(\sigma+s)\,d\sigma\right|^p(t-s)^{-1-\theta p}\,dt=2\int_0^T ds\int_0^{T-s}\left|\int_0^t u'(\sigma+s)\,d\sigma\right|^p t^{-1-\theta p}\,dt\leqslant$$

$$\leqslant 2\int_0^T ds\int_0^{+\infty}\left(\int_0^t|u'(\sigma+s)|\,d\sigma\right)^p t^{-1-\theta p}\,dt\,.$$

Therefore using (4) we have

$$N^p_{\theta,p}(u)\leqslant 2\theta^{-p}\int_0^T ds\int_0^{T-s}|u'(t+s)|^p\,t^{-1+p-p\theta}\,dt\leqslant 2\theta^{-p}\,T^{p-p\theta}(p-p\theta)^{-1}\,\|u'\|^p\,.$$

Therefore we have proved that $W^{1,p}(E)\subset W^{\theta,p}(E)$. Since the other inclusions are obvious the result follows.

LEMMA 5. – For each $\theta<\theta'\leqslant 1$ and $1\leqslant p<\infty$ we have

$$W^{\theta',p}(E)\hookrightarrow W^{\theta,p}(Z)\hookrightarrow L^p(E)\,.$$

PROOF. – Let us first prove that $W^{1,p}(E)$ is dense in $W^{\theta,p}(E)$. To this end let us denore by $\varphi(t)$ the function defined by $\varphi(t)=\exp[1/(t^2-1)]$ if $-1<t<1$ and $\varphi(t)=0$ elsewhere. Moreover let $u\in W^{\theta,p}(E)$ and set for $h>0$ and $0\leqslant t\leqslant T$

$$u_h(t)=Kh^{-1}\int_0^T\varphi\big((t-s)/h\big)\,u(s)\,ds=K\int_{-1}^1\varphi(s)\,u(t-hs)\,ds$$

where

$$K=\left(\int_{-1}^1\varphi(s)\,ds\right)^{-1}.$$

It is easy to see that $u_h\in W^{1,p}(E)$. Moreover we have

$$\|u-u_h\|^p=K^p\int_0^T\left|\int_{-1}^1\varphi(s)\big(u(t-hs)-u(t)\big)\,ds\right|^p dt\leqslant \text{cont}\int_0^T dt\int_{-1}^1|u(t-hs)-u(t)|^p\,ds$$

and

$$N^{\theta}_{,p}(u-u_h)=K^p\int_0^T dt\int_0^T\left|\int_{-1}^{1}\varphi(\sigma)\big(u(t-h\sigma)-u(t)+u(s)-u(s-h\sigma)\big)\,d\sigma\right|^p|t-s|^{-1-\theta p}\,ds\leqslant$$

$$\leqslant \text{const}\int_0^T dt\int_0^T\int_{-1}^{1}|u(t-h\sigma)-u(s-h\sigma)+u(t)-u(s)|^p\,d\sigma\,|t-s|^{-1-\theta p}\,ds\leqslant$$

$$\leqslant \text{const}\int_{-1}^{1}d\sigma\int_0^T dt\int_0^T|u(t-h\sigma)-u(s-h\sigma)+u(t)-u(s)|^p|t-s|^{-1-\theta p}ds$$

so thar $u_h\to u$ in $W^{\theta,p}(E)$, since the shift operator is continuous in L^p. Therefore we have proved that $W^{1,p}(E)$ is dense in $W^{\theta,p}(E)$ so that from Lemma 4 we have that $W^{1,p}(E)\hookrightarrow W^{\theta,p}(E)$, for $\theta<1$. Since the remaining imbeddings are obvious, the conclusion follows.

LEMMA 6. – Let $\theta>1/p$. Then

$$W^{\theta,p}(E)\hookrightarrow C(E)\,.$$

PROOF. – Let $u\in W^{1,p}(E)$. We shall use the following representation formula (see [8], p. 240)

$$u(0)=T^{-1}\int_0^T u(s)\,ds-\int_0^T t^{-2}\int_0^t\big(u(t)-u(s)\big)\,ds\,.$$

Therefore

$$|u(0)|\leqslant T^{-1/p}\,\|u\|+\left(\int_0^T t^{(\theta-1)\,p/(p-1)}\,dt\right)^{(p-1)/p}N_{\theta,p}(u)$$

Hence, setting $v(t)=u(t_0+t)$, for fixed $0\leqslant t_0<T$ and each $0\leqslant t\leqslant T-t_0$, we get

$$|u(t_0)|=|v(0)|.$$

Therefore for each $u\in W^{1,p}(E)$ we have

$$|u(t)|\leqslant \text{const}\,\|u\|_{W^{\theta,p}}\,,\qquad \text{for } 0\leqslant t\leqslant T$$

and the result follows since $W^{1,p}(E)$ is dense in $W^{\theta,p}(E)$.

The following two lemmas are concerned with the «behaviour» near $t=0$ of functions of $W^{\theta,p}$.

LEMMA 7. – Let $u \in W^{\theta,p}(E)$ with $\theta < 1/p$. Then there exists c' verifying

$$\left(\int_0^T t^{-p\theta}|u(t)|^p\,dt\right)^{1/p} \leqslant c'\,\|u\|_{W^{\theta,p}}\,.$$

PROOF. – It suffices to prove that

$$\int_0^{T/4} t^{-p\theta}\,|u(t)|^p\,dt \leqslant \text{const}\,\|u\|^p_{W^{\theta,p}}$$

moreover from Lemma 5 it suffices to prove the assertion for $u \in W^{1,p}(E)$. To this end let φ be a continuously differentiable real function satisfying the following properties: $0 \leqslant \varphi(t) \leqslant 1$ for each t, $\varphi(t) = 1$ for $0 \leqslant t \leqslant T/4$ and $\varphi(t) = 0$ for $t \geqslant T/2$. Moreover let $v\colon [0, +\infty[\to E$ be defined as $v(t) = \varphi(t)u(t)$ for $0 \leqslant t \leqslant T$ and $v(t) = 0$ elsewhere. We have

$$\int_0^{T/4} t^{-p\theta}\,|u(t)|^p\,dt \leqslant \int_0^{+\infty} t^{-p\theta}\,|v(t)|^p\,dt \tag{7}$$

moreover it is easy to check that

$$\int_0^{+\infty} dt \int_0^t |v(t) - v(s)||^p (t-s)^{-1-\theta p}\,ds \leqslant \text{const}\,\|u\|^p_{W^{\theta,p}}\,. \tag{8}$$

Therefore from (7) and (8) it suffices to show that

$$\int_0^{+\infty} t^{-p\theta}\,|v(t)|^p\,dt \leqslant \text{const}\int_0^{+\infty} dt \int_0^t |v(t) - v(s)|^p (t-s)^{-1-\theta p}\,ds\,.$$

To this end we shall use the following representation formula (see [10] and [14])

$$v(t) = -\,g(t) + \int_t^{+\infty} s^{-1} g(s)\,ds$$

where

$$g(t) = t^{-1}\int_0^t \big(v(s) - v(t)\big)\,ds\,. \tag{9}$$

Now

$$\int_0^{+\infty} t^{-p\theta}|g(t)|^p\,dt \leqslant \int_0^{+\infty} t^{-p-p\theta}\,dt\left(\int_0^t \big(v(s)-v(t)\big)\,ds\right)^p \leqslant \int_0^{+\infty} t^{-1-p\theta}\,dt\int_0^t |v(s)-v(t)|^p\,ds \leqslant N^p_{\theta,p}(v)\,.$$

Moreover using (5) we get

$$\int_0^{+\infty} t^{-p\theta}\,dt\left(\int_t^{+\infty} s^{-1}\,|g(s)|\,ds\right)^p \leqslant (1/p-\theta)^{-p}\int_0^{+\infty} t^{-p\theta}\,|g(t)|^p\,dt$$

and the conclusion follows.

LEMMA 8. – Let $u\in W^{\theta,p}(E)$ with $\theta>1/p$ and let $u(0)=0$. Then there exists c'' verifying

$$\left(\int_0^T t^{-p\theta}\,|u(t)|^p\,dt\right)^{1/p} \leqslant c''\,\|u\|_{W^{\theta,p}}\,.$$

PROOF. – We shall use a computation similar to that of Lemma 7 and shall prove that

$$\int_0^{+\infty} t^{-p\theta}|v(t)|^p\,dt \leqslant \text{const}\int_0^{+\infty} dt\int_0^t |v(t)-v(s)|^p(t-s)^{-1-\theta p}\,ds$$

where v is defined as in the proof of Lemma 7. To this end we shall use the following identity (see [10] and [14])

$$v(t) = -\,g(t) - \int_0^t s^{-1} g(s)\,ds$$

where g is given by (9). From the proof of Lemma 7 we have

$$\int_0^{+\infty} t^{-p\theta}\,|g(t)|^p\,dt \leqslant \text{const}\int_0^{+\infty} dt\int_0^t |v(t)-v(s)|^p(t-s)^{-1-\theta p}\,ds$$

moreover, using (4), we get

$$\int_0^{+\infty} t^{-p\theta}\,dt\left(\int_0^t s^{-1}\,|g(s)|\,ds\right)^p \leqslant (\theta-1/p)^{-p}\int_0^{+\infty} t^{-p\theta}\,|g(t)|^p\,dt$$

and the result follows.

We now prove a series of imbedding results.

LEMMA 9. – For each $\varepsilon<\theta<1$ we have

$$W^{\theta,p}(E)\cap L^p\big(D_A(\theta,p)\big) \hookrightarrow W^{\varepsilon,p}\big(D_A(\theta-\varepsilon,p)\big)\,.$$

PROOF. – We have

$$N^p_{\varepsilon,\theta-\varepsilon,p}(u) = 2\int_0^T dt \int_0^t (t-s)^{-1-\varepsilon p}\, ds \int_0^{+\infty} |AS(\sigma)(u(t)-u(s))|^p \sigma^{-1+p-p(\theta-\varepsilon)}\, d\sigma =$$

$$= 2\int_0^T dt \int_0^t (t-s)^{-1-\varepsilon p}\, ds \int_0^{t-s} |AS(\sigma)(u(t)-u(s))|^p \sigma^{-1+p-p(\theta-\varepsilon)}\, d\sigma +$$

$$+ 2\int_0^T dt \int_0^t (t-s)^{-1-\varepsilon p}\, ds \int_{t-s}^{+\infty} |AS(\sigma)(u(t)-u(s))|^p \sigma^{-1+p-p(\theta-\varepsilon)}\, d\sigma = J_1 + J_2 .$$

Now

$$J_1 \leqslant 2^p \int_0^T dt \int_0^t (t-s)^{-1-\varepsilon p}\, ds \int_0^{t-s} |AS(\sigma)u(t)|^p \sigma^{-1+p-p(\theta-\varepsilon)}\, d\sigma +$$

$$+ 2^p \int_0^T ds \int_s^T (t-s)^{-1-\varepsilon p}\, dt \int_0^{t-s} |AS(\sigma)u(s)|^p \sigma^{-1+p-p(\theta-\varepsilon)}\, d\sigma \leqslant$$

$$\leqslant 2^{p+1} \int_0^T dt \int_0^{+\infty} s^{-1-\varepsilon p}\, ds \int_0^s |AS(\sigma)u(t)|^p \sigma^{-1+p-p(\theta-\varepsilon)}\, d\sigma .$$

Therefore, using (4), we get

$$J_1 \leqslant 2^{p+1}(\varepsilon p)^{-1} H^p_{\theta,p}(u) .$$

Moreover we have

$$J_2 \leqslant 2M_1^p \int_0^T dt \int_0^t |u(t)-u(s)|^p (t-s)^{-1-\varepsilon p}\, ds \int_{t-s}^{+\infty} \sigma^{-1-p(\theta-\varepsilon)}\, d\sigma = 2M_1^p p^{-1}(\theta-\varepsilon)^{-1} N^p_{\theta,p}(u)$$

and the result follows.

LEMMA 10. – For each $0<\theta<1$ we have

$$W^{\theta,p}(D_A) \cap W^{1,p}(D_A(\theta,p)) \hookrightarrow W^{\alpha,p}(D_A(\beta,p))$$

for $0<\alpha,\ \beta<1,\ \alpha+\beta = 1+\theta$.

PROOF. – We have

$$N^p_{\alpha,\beta,p}(u) = 2\int_0^T dt \int_0^t (t-s)^{-1-\alpha p}\, ds \int_0^{t-s} |AS(\sigma)(u(t)-u(s))|^p \sigma^{-1+p-p\beta}\, d\sigma +$$

$$+ 2\int_0^T dt \int_0^t (t-s)^{-1-\alpha p}\, ds \int_{t-s}^{+\infty} |AS(\sigma)(u(t)-u(s))|^p \sigma^{-1+p-p\beta}\, d\sigma = J_1' + J_2' .$$

Furthermore

$$J_1' \leqslant M_0^p 2\int_0^T dt \int_0^t |Au(t) - Au(s)|^p (t-s)^{-1-\alpha p}\, ds \int_0^{t-s} \sigma^{-1+p-p\beta}\, d\sigma =$$

$$= M_0^p 2(p-p\beta)^{-1}\int_0^T dt\int_0^t |Au(t) - Au(s)|^p (t-s)^{-1-p(\alpha+\beta-1)}\, ds \leqslant \text{const}\, N_{\theta,p}^p(Au)\,.$$

Concerning J_2' we have

$$J_2' \leqslant 2\int_0^T dt \int_0^t (t-s)^{-1-\alpha p}\, ds \int_{t-s}^{+\infty} \left(\int_s^t |AS(\sigma)u'(\tau)|\, d\tau\right)^p \sigma^{-1+p-p\beta}\, d\sigma =$$

$$= 2\int_0^T ds \int_s^T (t-s)^{-1-\alpha p}\, dt \int_{t-s}^{+\infty} \left(\int_0^{t-s} |AS(\sigma)u'(\tau+s)|\, d\tau\right)^p \sigma^{-1+p-p\beta}\, d\sigma =$$

$$= 2\int_0^T ds \int_0^{T-s} t^{-1-\alpha p}\, dt \int_t^{+\infty} \left(\int_0^t |AS(\sigma)u'(\tau+s)|\, d\tau\right)^p \sigma^{-1+p-p\beta}\, d\sigma$$

interchanging the order of integration we further obtain

$$J_2' \leqslant 2\int_0^T ds \int_0^{T-s} \sigma^{-1+p-p\beta}\, d\sigma \int_0^\sigma t^{-1-\alpha p}\, dt \left(\int_0^t |AS(\sigma)u'(\tau+s)|\, d\tau\right)^p +$$

$$+ 2\int_0^T ds \int_{T-s}^{+\infty} \sigma^{-1+p-p\beta}\, d\sigma \int_0^{T-s} t^{-1-\alpha p}\, dt \left(\int_0^t |AS(\sigma)u'(\tau+s)|\, d\tau\right)^p$$

therefore, using (4), we get

$$J_2' \leqslant 2\alpha^{-p}\int_0^T ds \int_0^{T-s} \sigma^{-1+p-p\beta}\, d\sigma \int_0^\sigma t^{-1-\alpha p+p} |AS(\sigma)u'(t+s)|^p\, dt +$$

$$+ 2\alpha^{-p}\int_0^T ds \int_{T-s}^{+\infty} \sigma^{-1+p-p\beta}\, d\sigma \int_0^{T-s} t^{-1-\alpha p+p} |AS(\sigma)u'(t+s)|^p\, dt = J_3 + J_4\,.$$

Now

$$J_3 \leqslant 2\alpha^{-p}\int_0^T \sigma^{-1+p-p\beta}\, d\sigma \int_0^T ds \int_s^{s+\sigma} (t-s)^{-1-\alpha p+p} |AS(\sigma)\, u'(t)|^p\, dt =$$

$$= 2\alpha^{-p}\int_0^T \sigma^{-1+p-p\beta}\,d\sigma\int_0^\sigma dt\,|AS(\sigma)u'(t)|^p\int_0^t (t-s)^{+1-\alpha p+p}\,ds +$$

$$+ 2\alpha^{-p}\int_0^T \sigma^{-1+p-p\alpha}\,d\sigma\int_\sigma^T |AS(\sigma)u'(t)|^p\,dt\int_{t-\sigma}^t (t-s)^{-1-\alpha p+p}\,ds =$$

$$= 2^{-p}(p-\alpha p)^{-1}\int_0^T \sigma^{-1+p-p\beta}\,d\sigma\Big(\int_0^\sigma t^{p-\alpha p}\,|AS(\sigma)u'(t)|^p\,dt + \int_\sigma^T |AS(\sigma)u'(t)|^p\sigma^{p-\alpha p}\,dt\Big) \leqslant$$

$$\leqslant 2\alpha^{-p}(p-p\alpha)^{-1}\int_0^T \sigma^{-1+p-p(\alpha+\beta-1)}\,d\sigma\int_0^T |AS(\sigma)\,u'(t)|^p\,dt \leqslant 2\alpha^{-p}(p-\alpha p)^{-1}H^p_{\theta,p}(u')\,.$$

Finally, concerning J_4', we have

$$J_4' = 2\alpha^{-p}\Big(\int_0^T \sigma^{-1+p-p\beta}\,d\sigma\int_{T-\sigma}^T ds\int_s^T (t-s)^{-1+p-\alpha p}\,|AS(\sigma)u'(t)|^p\,dt +$$

$$+\int_T^{+\infty} \sigma^{-1+p-p\beta}\,d\sigma\int_0^T ds\int_s^T (t-s)^{-1+p-\alpha p}\,|AS(\sigma)u'(t)|^p\,dt\Big) =$$

$$= 2\alpha^{-p}\Big(\int_0^T \sigma^{-1+p-p\alpha}\,d\sigma\int_{T-\sigma}^T |AS(\sigma)u'(t)|^p\,dt\int_{T-\sigma}^t (t-s)^{-1+p-\alpha p}\,ds +$$

$$+\int_T^{+\infty} \sigma^{-1+p-p\beta}\,d\sigma\int_0^T |AS(\sigma)\,u'(t)|^p\,dt\int_0^t (t-s)^{-1+p-\alpha p}\,ds\Big) \leqslant$$

$$\leqslant 2\alpha^{-p}(p-\alpha p)^{-1}\Big(\int_0^T \sigma^{-1+p-p\beta}\,d\sigma\int_{T-\sigma}^T (t-T+\sigma)^{p-p\alpha}\,|AS(\sigma)u'(t)|^p\,dt +$$

$$+\int_T^{+\infty} \sigma^{-1+p-p\beta}\,d\sigma\int_0^T t^{p-p\alpha}\,|AS(\sigma)\,u'(t)|^p\,dt\Big) \leqslant$$

$$\leqslant 2\alpha^{-p}(p-\alpha p)^{-1}\int_0^{+\infty} \sigma^{-1+p-p(\alpha+\beta-1)}\,d\sigma\int_0^T |AS(\sigma)\,u'(t)|^p\,dt = 2\alpha^{-p}(p-\alpha p)^{-1}H^p_{\theta,p}(u')$$

and the result follows.

LEMMA 11. – Let $\theta > 1/p$. Then

$$W^{\theta,p}(E)\cap L^p(D_A(\theta,p)) \hookrightarrow C(D_A(\theta-1/p,p))\,.$$

PROOF. – Let v be the function introduced in the proof of Lemma 7. Using (6) with T replaced by σ we have

$$|u(0)|^p_{\theta-1/p,p} = |v(0)|^p_{\theta-1/p,p} \leqslant 2^{p-1}\int_0^{+\infty}\left|AS(\sigma)\sigma^{-1}\int_0^\sigma v(s)\,ds\right|^p \sigma^{p-p\theta}\,d\sigma +$$

$$+\,2^{p-1}\int_0^{+\infty}\left|AS(\sigma)\int_0^\sigma t^{-2}\int_0^t\big(v(t)-v(s)\big)\,ds\right|^p\sigma^{p-p\theta}\,d\sigma\,dt = J_3 + J_4\,.$$

Now we have from the definition of v

$$J_3 \leqslant 2^{p-1}\int_0^{+\infty}\sigma^{p-1}\int_0^\sigma |AS(\sigma)v(s)|^p\,ds\,\sigma^{-p\theta}\,d\sigma \leqslant \tag{10}$$

$$\leqslant 2^{p-1}\int_0^{+\infty}ds\int_0^{+\infty}|AS(\sigma)v(s)|^p\sigma^{-1+p-p\theta}\,d\sigma \leqslant 2^{p-1}H^p_{\theta,p}(u)\,.$$

Moreover using (3) and (4) we get

$$J_4 \leqslant 2^{p-1}M_1^p\int_0^{+\infty}\sigma^{-1}\,d\sigma\left(\sigma^{1/p-\theta}\int_0^\sigma t^{-2}\,dt\int_0^t|v(t)-v(s)|\,ds\right)^p \leqslant$$

$$\leqslant 2^{p-1}M_1^p(\theta-1/p)^{-p}\int_0^{+\infty}\sigma^{-1}\,d\sigma\left(\sigma^{1/p-\theta-1}\int_0^\sigma|v(\sigma)-v(s)|\,ds\right)^p \leqslant$$

$$\leqslant 2^{p-1}M_1^p(\theta-1/p)^{-p}\int_0^{+\infty}\sigma^{-1-p\theta}\,d\sigma\int_0^\sigma|v(\sigma)-v(s)|^p\,ds \leqslant$$

$$\leqslant 2^{p-1}M_1^p(\theta-1/p)^{-p}\int_0^{+\infty}d\sigma\int_0^\sigma|v(\sigma)-v(s)|^p(\sigma-s)^{-1-p\theta}\,ds\,.$$

Therefore, using (8) and (10), we obtain

$$|u(0)|_{\theta-1/p,p} = \text{const}\left(\|u\|_{L^p_{\theta,p}} + \|u\|_{W^{\theta,p}}\right)$$

so that, by a computation similar to that of Lemma 6, we get

$$|u(t)|_{\theta-1/p,p} \leqslant \text{const}\left(\|u\|_{L^p_{\theta,p}} + \|u\|_{W^{\theta,p}}\right), \qquad 0 \leqslant t \leqslant T$$

and the result follows.

LEMMA 12. – For each $\theta < 1/p$ we have

$$W^{1,p}(D_A(\theta, p)) \cap W^{0,p}(D_A) \hookrightarrow C(D_A(\theta + 1 - 1/p, p)) .$$

PROOF. – Let v be the function introduced in the proof of Lemma 7. From (6) with T replaced by σ we get

$$|u(0)|^p_{\theta+1-1/p,p} = |v(0)|^p_{\theta+1-1/p,p} \leqslant 2^{p-1}\int_0^{+\infty}\left|AS(\sigma)\sigma^{-1}\int_0^\sigma v(s)\,ds\right|^p \sigma^{-p\theta}\,d\sigma + \\ + 2^{p-1}\int_0^{+\infty}\left|AS(\sigma)\int_0^\sigma t^{-2}\,dt\int_0^t (v(t)-v(s))\,ds\right|^p \sigma^{-p\theta}\,d\sigma = J_5 + J_6 .$$

Now we have

$$J_5 \leqslant 2^{2p-2}\int_0^{+\infty}\sigma^{-p-p\theta}\,d\sigma\left(\int_0^\sigma |AS(\sigma)(v(s)-v(\sigma))|\,ds\right)^p + \\ + 2^{2p-2}\int_0^{+\infty}\sigma^{-p-p\theta}\,d\sigma\left(\int_0^\sigma |AS(\sigma)v(\sigma)|\,ds\right)^p \leqslant \\ \leqslant M_0^p 2^{2p-2}\int_0^{+\infty}\sigma^{-1-p\theta}\,d\sigma\int_0^\sigma |A(v(s)-v(\sigma))|^p\,ds + M_0^p 2^{2p-2}\int_0^{+\infty}|Av(s)|^p s^{-p\theta}\,ds .$$

Therefore from Lemma 7 we get

$$J_5 \leqslant M_0^p 2^{2p-2}(N^p_{\theta,p}(Au) + \text{const}\, N^p_{\theta,p}(Au)) .$$

Furthermore, concerning J_6, we have

$$J_6 \leqslant 2^{p-1}\int_0^{+\infty}\sigma^{-p\theta}\,d\sigma\left(\int_0^\sigma t^{-2}\,dt\int_0^t\left|AS(\sigma)\int_s^t v'(\tau)\,d\tau\right|ds\right)^p \leqslant \\ \leqslant 2^{p-1}\int_0^{+\infty}\sigma^{-p\theta}\,d\sigma\left(\int_0^\sigma t^{-2}\,dt\int_0^t |AS(\sigma)v'(\tau)|\,d\tau\int_0^\tau ds\right)^p \leqslant \\ \leqslant 2^{p-1}\int_0^{+\infty}\sigma^{-p\theta}\,d\sigma\left(\int_0^{+\infty} t^{-2}\,dt\int_0^t \tau|AS(\sigma)v'(\tau)|\,d\tau\right)^p$$

where $v'(t) = 0$ for $t \geqslant \sigma$. Therefore using (4) we get

$$J_6 \leqslant 2^{p-1} \int_0^{+\infty} \sigma^{-p\theta}\, d\sigma \left(\int_0^{\sigma} |AS(\sigma) v'(t)|\, dt \right)^p \leqslant 2^{p-1} H^p_{\theta,p}(v') .$$

Therefore, using the definition of v, we get

$$|u(0)|_{\theta+1-1/p,p} \leqslant \text{const} \left(\|Au\|_{W^{\theta,p}} + \|u'\|_{L^p_{\theta,p}} \right)$$

so that by a computation similar to that of Lemma 6 we have

$$|u(t)|_{\theta+1-1/p,p} \leqslant \text{const} \left(\|Au\|_{W^{\theta,p}} + \|u'\|_{L^p_{\theta,p}} \right), \qquad 0 \leqslant t \leqslant T$$

and the result follows.

LEMMA 13. – For each $1 < p < \infty$ we have

$$W^{1,p}(D_A(1/p, p)) \cap L^p(D_A(1 + 1/p, p)) \hookrightarrow C(D_{A^2}(\tfrac{1}{2}, p)) ,$$

$$W^{1,p}(D_A(1/p, p)) \cap \hat{W}^{1/p,p}(D_A) \hookrightarrow C(D_{A^2}(\tfrac{1}{2}, p)) ,$$

where $\hat{W}^{1/p,p}(D_A) = \left\{ u \in W^{1/p,p}(D_A) \colon \int_0^T t^{-1} |Au(t)|^p\, dt < +\infty \right\}$.

PROOF. – Let v be the function introduced in the proof of Lemma 7. From (6) with T replaced by σ we get

$$|u(0)|^p_{2,1/2,p} = |v(0)|^p_{2,1/2,p} \leqslant 2^{p-1} \int_0^{+\infty} \left| A^2 S(\sigma) \int_0^{\sigma} v(s)\, ds \right|^p \sigma^{-1}\, d\sigma +$$

$$+ 2^{p-1} \int_0^{+\infty} \left| \sigma A^2 S(\sigma) \int_0^{\sigma} t^{-2}\, dt \int_0^{t} (v(t) - v(s))\, ds \right|^p \sigma^{-1}\, d\sigma = J'_5 + J'_6 .$$

Now if $u \in L^p(D_A(1 + 1/p, p)$ we have, from the definition of v,

$$J'_5 \leqslant 2^{p-1} \int_0^{+\infty} \sigma^{p-2}\, d\sigma \int_0^{\sigma} |A^2 S(\sigma) v(s)|^p\, ds \leqslant 2^{p-1} \int_0^{+\infty} ds \int_0^{+\infty} |AS(\sigma) Av(s)|^p \sigma^{p-2}\, d\sigma \leqslant \text{const}\, H^p_{1/p,p}(Au) .$$

Moreover if $u \in W^{1/p,p}(D_A)$ and $\int\limits_0^T t^{-1}|Au(t)|^p\,dt < +\infty$, then

$$J_5' \leqslant 2^{p-1}\int\limits_0^{+\infty}\left(\int\limits_0^\sigma |A^2S(\sigma)(v(s)-v(\sigma))|\,ds + \int\limits_0^\sigma |A^2S(\sigma)v(\sigma)|ds\right)^p \sigma^{-1}\,d\sigma \leqslant$$

$$\leqslant 2^{2p-2}M_1^p\int\limits_0^{+\infty}\sigma^{-2}\,d\sigma\int\limits_0^\sigma |A(v(s)-v(\sigma))|^p\,ds + 2^{2p-2}M_1^p\int\limits_0^{+\infty}\sigma^{-1}|Av(\sigma)|^p\,d\sigma \leqslant$$

$$\leqslant \mathrm{const}\left(\|Au\|^p_{W^{1/p,p}} + \int\limits_0^T t^{-1}|Au(t)|^p\,dt\right).$$

Moreover if $u' \in L^p(D_A(1/p, p))$ we have

$$J_6' \leqslant 2^{p-1}\int\limits_0^{+\infty}\sigma^{p-1}\,d\sigma\left(\int\limits_0^\sigma t^{-2}\,dt\int\limits_0^t ds\int\limits_s^t |A^2S(\sigma)v'(\tau)|\,d\tau\right)^p \leqslant$$

$$\leqslant 2^{p-1}\int\limits_0^{+\infty}\sigma^{p-1}\,d\sigma\left(\int\limits_0^\sigma t^{-2}\,dt\int\limits_0^t |A^2S(\sigma)v'(\tau)|\tau\,d\tau\right)^p \leqslant$$

$$\leqslant 2^{2p-1}M_1^p\int\limits_0^{+\infty}\sigma^{-1}\,d\sigma\left(\int\limits_0^{+\infty} t^{-2}\,dt\int\limits_0^t |AS(\sigma/2)v'(\tau)|\tau\,d\tau\right)^p$$

where $v'(t) = 0$ if $t \geqslant \sigma$. Therefore using (4) we get

$$J_6' \leqslant 2^{2p-1}M_1^p\int\limits_0^{+\infty}\sigma^{-1}\,d\sigma\left(\int\limits_0^\sigma |AS(\sigma/2)v'(t)|\,dt\right)^p =$$

$$= 2^{2p-1}M_1^p\int\limits_0^{+\infty}\sigma^{p-2}\,d\sigma\int\limits_0^\sigma |AS(\sigma/2)v'(t)|^p\,dt \leqslant 2^{3p-2}M_1^p H^p_{1/p,p}(v').$$

Summarizing we have from (8) and the definition of v

$$|u(0)|_{2,\frac{1}{2},p} \leqslant \mathrm{const}\left(\|Au\|_{L^p_{1/p,p}} + \|u'\|_{L^p_{1/p,p}}\right)$$

if $u \in L^p(D_A(1+1/p, p)) \cap W^{1/p,p}(D_A(1/p, p))$, whereas if $u \in \hat{W}^{1/p,p}(D_A) \cap W^{1/p,p}\cdot(D_A(1/p, p))$ we have

$$|u(0)|_{2,\frac{1}{2},p} \leqslant \mathrm{const}\left(\|Au\|_{W^{1/p,p}} + \left(\int\limits_0^T t^{-1}|Au(t)|^p\,dt\right)^{1/p} + \|u'\|_{L^p_{1/p,p}}\right)$$

and the conclusion follows by a computation similar to that of Lemma 6.

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