Cross-Ratio Theorems on Generalized Polars of Abstract Homogeneous Polynomials (*).

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Summary. – *The present paper deals with certain new aspects of the theory of algebra-valued generalized polars of the product of abstract homogeneous polynomials, opening up new avenues to the existing (and still very young) theory put forward by Marden and Zaheer (see Pacific J. Math.,* 74 (1978), n. 2, pp. 535-557). *Our main theorem here leads, on the one hand, to an improved version of Walsh's cross-ratio theorem on critical points of functions of the form* $f_1 f_2/f_3$ *(*f_i *being complex-valued polynomials) and, on the other hand, it offers a two-fold generalization of one of the main theorems of Zaheer in the paper cited above.*

1. – Introduction.

During the last twentyfive years a number of well-known classical results on the zeros of polynomials have been generalized to abstract homogeneous polynomials (a.h.p.) by Hörmander, Zervos, Marden and Zaheer. First in the series being the paper due to HÖRMANDER [5] in 1954 which deals with Laguerre's theorem (cf. [6]; [7], p. 49), followed in 1966 and 1969 by the ones due to MARDEN ([8] and [9]) on the theorems of BÔCHER [1], GRACE [3], and SZEGÖ [11]. The memorable work of ZERVOS [22] in 1960 includes, among so many other variety of results, also the generalizations of these classical results in terms of ordinary polynomials from K to K (K being an algebraically closed field of characteristic zero). However, the studies made by ZAHEER in [17], [18] and [19] present a more general, systematic, and abstract theory of a.h.p.'s which puts all these scattered pieces in proper perspective and which offers much more general theorems that include in them the corresponding results on each of the aspects considered by Hörmandes, Zervos and Marden. The work in [9] and [18] deals with composite a.h.p.'s, while the ones in [8] and [19] with generalized polars (a variety of composite a.h.p.'s) in particular.

ZAHEER [19] has made a detailed and comparative study of certain aspects of the theory of generalized polars, obtaining generalizations of the results in [8] and providing, in addition, improved versions of Bôcher's theorem (cf. [1]; [7], Theorem (20,2)) and of Walsh's two-circle theorems (see [13] and [14], or [7], Theorems (19,1) and (20,1)). Inspired by the scope of this theory we developed a strong

(*) Entrata in Redazione il 14 febbraio 1980.

(**) Supported by the UGC Fellowship, Government of India.

feeling for pursuing other meaningful aspects (if any) in this subject area. The present paper, which is the result of such study, deals with that aspect of the theory of algebra-valued generalized polars which has access to the well-known Walsh's cross-ratio theorem (cf. [13]; [7], Theorem (22,2)) on the critical points of rational functions of the form $f_1 f_2/f_3$. It may be noted that this theorem does not come in the purview of any of the theorems in [19], whereas Theorem 2.5 in [19] does reduce to a special case of one of the corollaries of our main theorem in this paper.

The scheme of the present paper is as follows: Section 2 on preliminaries is intended to serve as a back-up material for the sections that follow and most of the discussion is based on the notations and results discussed in [19]. In Section 3 we prove our main theorem on generalized polars of a.h.p.'s and deduce an important corollary as application. A more general formulation (to algebra-valued a.h.p.'s) of our main theorem of Section 3 has been proved in Section 4. This is the most general result of this paper and, as a by-product, its application leads to a main theorem of Zaheer in [19].

2. – Preliminaries.

Given vector spaces E and V over the same field K of characteristic zero, we define (cf. [5], pp. 55, 59; [4], pp. 760-763; [10], p. 303; [16], pp. 52-61) a *vector-valued a.h.p.* of degree n to be a mapping $P: E \to V$ such that (for each $x, y \in E$)

$$P(sx + ty) = \sum_{k=0}^{n} A_k(x, y)\, s^k t^{n-k} \qquad \forall s,\ t \in K,$$

where the coefficients $A_k(x, y) \in V$ depend only on x and y. If $V = K$, we shall call P simply as an *a.h.p.* and, if V is an algebra, P will be termed an *algebra-valued* a.h.p. We shall denote by $\boldsymbol{P}_n^*$ the class of all vector-valued a.h.p.'s of degree n from E to V (even if V is an algebra) and by $\boldsymbol{P}_n$ the class of all a.h.p.'s of degree n from E to K. The n-th *polar* is the unique symmetric n-linear form $P(x_1, x_2, \ldots, x_n): E^n \to V$ such that $P(x, x, \ldots, x) = P(x)$ for every $x \in E$ (see [5], pp. 55, 59; [4], pp. 762-763 for its existence and uniqueness). We may then specify the k-th *polar* of P by the relation

$$P(x_1, \ldots, x_k, x) = P(x_1, \ldots, x_k, x, \ldots, x)\,.$$

BLANKET ASSUMPTION. – Throughout the rest of this paper K denotes an algebraically closed field of characteristic zero and V an algebra with identity over K. In particular, the field of complex numbers will be denoted by $\boldsymbol{C}$.

A *scalar homomorphism* (cf. [15], p. 253) on V is a mapping $L: V \to K$ such that

(i) $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)\ \forall u, v \in V,\ \alpha, \beta \in K,$

(ii) $L(uv) = L(u)L(v)\ \forall u, v \in V.$

Ideal maximal subspaces ([15], p. 252) of V are characterized in a one-one manner by sets of the form

$$\{v \in V | L(v) = 0\}, \tag{2.1}$$

where L is a nontrivial scalar homomorphism on V (cf. [15], Theorem 2, p. 254). The following material is borrowed from Zaheer [20].

Given $P \in \boldsymbol{P}^*_{n_k}$ and scalars m_k $(k = 1, 2, \ldots, q)$, let us write

$$Q(x) = P_1(x) \cdot P_2(x) \ldots P_q(x), \tag{2.2}$$

$$Q_k(x) = P_1(x) \ldots P_{k-1}(x) \cdot P_{k+1}(x) \ldots P_q(x), \tag{2.3}$$

and define (cf. [19], p. 539)

$$\Phi(Q; x_1, x) = \sum_{k=1}^{q} m_k Q_k(x) \cdot P_k(x_1, x) \qquad \forall x, x_1 \in E.$$

We shall call $\Phi(Q; x_1, x)$ as an *algebra-valued generalized polar* of the product $Q(x)$. For $V = K$ it will be termed simply a *generalized polar*, to be in conformity with the notation in [19], p. 539. If we write $n = n_1 + n_2 + \ldots + n_q$, we note (cf. [4], Theorem 26.2.3) that $Q \in \boldsymbol{P}^*_n$, $Q_k \in \boldsymbol{P}^*_{n-n_k}$ and $P_k(x_1, x)$ is an algebra-valued a.h.p. of degree $n_k - 1$ in x and of degree 1 in x_1, $1 \leqslant k \leqslant q$. Therefore, $\Phi(Q; x_1, x)$ is an algebra-valued a.h.p. of degree $n - 1$ in x and of degree 1 in x_1. Given a nontrivial scalar homomorphism L on V and a polynomial $P \in \boldsymbol{P}^*_n$, we define the mapping $LP\colon E \to K$ by $(LP)(x) = L(P(x))$ for all $x \in E$. It is obvious that $LP \in \boldsymbol{P}_n$. In the notations of (2.2) and (2.3) the product of the polynomials $LP_k \in \boldsymbol{P}_{n_k}$ is given by LQ and the corresponding partial product $(LQ)_k$ (got by deleting the k-th factor in the expression for LQ) is given by LQ_k. In view of this, we record the following

REMARK 2.1. – The algebra-valued generalized polar $\Phi(Q; x_1, x)$ of the product $Q(x)$ and the generalized polar $\Phi(LQ; x_1, x)$ of the corresponding product $(LQ)(x)$, with the same m_k's, satisfy the relation

$$L(\Phi(Q; x_1, x)) = \Phi(LQ; x_1, x)$$

for every nontrivial scalar homomorphism L on V.

The concepts of generalized circular regions, K_0-convexity, and circular cones have appeared at many places in the literature (cf. [17]; [18]; [19]). However, we shall explain these only briefly as follows. We know (cf. [5]; [2], pp. 38-40; [12], pp. 248-255) that K must have a maximal ordered subfield K_0 such that $K = K_0(i)$, where $-i^2$ is the unity element of K. Since every element z in K has the form $z = a + ib$ with $a, b \in K_0$, we define $\bar{z} = a - ib$, Re $(z) = (z + \bar{z})/2$, and $|z| = +(a^2 + b^2)^{\frac{1}{2}}$

in analogy with the complex plane. Let K_ω denote the projective field (see [19]; [22], pp. 353, 373) got by adjoining to K an element ω having the properties of scalar *infinity*. A subset $A \subseteq K$ is K_0-*convex* if $\sum_{i=1}^{n} \mu_i a_i \in A$ for every $a_i \in A$ and $\mu_i \in K_{0+}$ (the set of all nonnegative elements of K_0) such that $\sum_{i=1}^{n} \mu_i = 1$. Given $\zeta \in K$, we denote by φ_ζ the permutation of K_ω given by

$$\varphi_\zeta(z) = (z - \zeta)^{-1} \quad \forall z \in K_\omega . \tag{2.4}$$

Obviously, $\varphi_\zeta(\omega) = 0$ and $\varphi_\zeta(\zeta) = \omega$. The following concept is due to ZERVOS [22]. We call a subset A of K_ω a *generalized circular region* (*g.c.r.*) of K_ω if A is either one of the sets ϕ, K, K_ω or A satisfies the following two conditions:

(i) $\varphi_\zeta(A)$ is K_0-convex for every $\zeta \in K - A$;

(ii) $\omega \in A$ if A is not K_0-convex.

$D(K_\omega)$ denotes the class of all g.c.r.'s of K_ω. Some properties and details regarding g.c.r.'s, relevant to our present needs, are stated in [19].

The terms « nucleus », « circular mapping », and « circular cone » are due to Zaheer [17]. Given a nucleus N of E^2 and a circular mapping $G: N \to D(K_\omega)$, we define the *circular cone* $E_0(N, G)$ by

$$E_0(N, G) = \bigcup_{(x,y)\in N} T_G(x, y) ,$$

where

$$T_G(x, y) = \{sx + ty \neq 0 \,|\, s, t \in K;\ s/t \in G(x, y)\} .$$

REMARK 2.2. – (I) If G is a mapping from N into the class of all subsets of K_ω, the resulting set $E_0(N, G)$ will be termed only a *cone*.

(II) If $\dim E = 2$, then (cf. [17], Remark (2.1), p. 117) every circular cone $E_0(N, G)$ is of the form

$$E_0(N, G) = \{sx_0 + ty_0 \neq 0 \,|\, s, t \in K;\ s/t \in A\}$$

for some $A \in D(K_\omega)$, where x_0, y_0 are any two linearly independent elements of E, $N = \{(x_0, y_0)\}$, and $G(x_0, y_0) = A$.

The following proposition due to Zaheer (cf. [19], Proposition 2.1) tells us that any two circular cones can always be expressed relative to a common nucleus.

PROPOSITION 2.3. – *Given a circular cone $E_0(N, G)$ and an arbitrary nucleus $N' \subseteq E^2$, there exists a circular mapping $G': N' \to D(K_\omega)$ such that $E_0(N, G) = E_0(N', G')$.*

The following definition and the related discussion has been taken from [19], p. 550. The term « *homographic transformation* » has been used by Zervos for the usual linear fractional transformation. Such transformations preserve the class $D(K_\omega)$ (see [22], p. 353).

DEFINITION 2.4. – Given distinct elements $\varrho_1, \varrho_2, \varrho_3 \in K$, we define the *cross-ratio mapping* (with respect to $\varrho_1, \varrho_2, \varrho_3$) to be the homographic transformation $h\colon K_\omega \to K_\omega$ given by

$$h(\varrho) = \frac{\varrho - \varrho_2}{\varrho - \varrho_3}\cdot\frac{\varrho_1 - \varrho_3}{\varrho_1 - \varrho_2} = (\varrho, \varrho_1, \varrho_2, \varrho_3)\,, \qquad \text{say } \forall \varrho \in K_\omega. \tag{2.5}$$

We call $(\varrho, \varrho_1, \varrho_2, \varrho_3)$ as the *cross-ratio* of ϱ with $\varrho_1, \varrho_2, \varrho_3$. In case any one of the ϱ_i's is taken as ω, we define the corresponding cross-ratio to be the expression got by deleting in (2.5) the factors which thereby involve ω. E.g. (for $\varrho_1 = \omega$)

$$(\varrho, \omega, \varrho_2, \varrho_3) = (\varrho - \varrho_2)/(\varrho - \varrho_3) \qquad \text{etc.} \tag{2.6}$$

Obviously, the homographic transformation h maps $\varrho_1, \varrho_2, \varrho_3$ to 1, 0, ω, respectively, and there is no other homographic transformation with this property. Consequently, identity mapping is the only homograrphic transformation which can map 1, 0 ω to 1, 0, ω, respectively.

REMARK 2.5. – Let us note that (2.6) follows naturally from (2.5) and that the two together cover all the cases when $\varrho_1, \varrho_2, \varrho_3$ are distinct elements of K_ω. However, even in the case when $\varrho_1, \varrho_2, \varrho_3$ are not distinct elements of K_ω, we agree to denote the expression $(\varrho - \varrho_2)(\varrho_1 - \varrho_3)/(\varrho - \varrho_3)(\varrho_1 - \varrho_2)$ by $(\varrho, \varrho_1, \varrho_2, \varrho_3)$. But, in this case, h is not a homographic transformation even if h is a function.

3. – The main theorems on generalized polars.

As a first step towards obtaining our most general theorem of this paper concerning algebra-valued generalized polars from E to V, we study first the corresponding simpler situation when $V = K$. In fact, we are able to establish the following result on the generalized polars of a.h.p.'s from E to K. This result will be shown to include Walsh's cross-ratio theorem (cf. [13]; [7], Theorem (22,2)) on critical points of rational functions of the form $f_1 f_2/f_3$, f_i being polynomials from C to C. In the following theorem we take circular cones with a common nucleus (cf. Proposition 2.3) and denote by $Z_P(x, y)$ the *null-set* of P (relative to $x, y \in E$) defined by

$$Z_P(x, y) = \{sx + ty \neq 0 | s, t \in K;\ P(sx + ty) = 0\}\,.$$

$\mathfrak{L}[x, y]$ denotes the subspace of E generated by x and y.

THEOREM 3.1. – *Let $E_0^{(i)} \equiv E_0(N, G_i)$, $i = 1, 2, 3$, be circular cones with $E_0^{(1)} \cap E_0^{(2)} \cap \cap E_0^{(3)} = \emptyset$ and let $P_k \in \boldsymbol{P}_{n_k}$ $(k = 1, 2, \ldots, q)$ such that*

$$Z_{P_k}(x, y) \subseteq \begin{cases} T_{G_1}(x, y) & \text{for } k = 1, 2, \ldots, r (< p < q) \\ T_{G_2}(x, y) & \text{for } k = r + 1, \ldots, p \\ T_{G_3}(x, y) & \text{for } k = p + 1, \ldots, q \end{cases} \tag{3.1}$$

for all $(x, y) \in N$. Let $\Phi(Q; x_1, x)$ be the generalized polar of the product $Q(x)$ (cf. (2.2)) *with $m_k > 0$ for $k = 1, 2, \ldots, p$ and $m_k < 0$ for $k = p + 1, \ldots, q$ such that $\sum_{k=1}^{q} m_k = 0$. If $A_1 = \sum_{k=1}^{r} m_k$ and $A_2 = \sum_{k=r+1}^{p} m_k$, then $\Phi(Q; x_1, x) \neq 0$ for all linearly independent elements x, x_1 of E such that $x \in E - \bigcup_{i=1}^{4} E_0^{(i)}$, where $E_0^{(4)} \equiv E_0(N, G)$ is the cone defined by*

$$G(x, y) = \{\varrho \in K_\omega | (\varrho, \varrho_3, \varrho_2, \varrho_1) = -A_2/A_1;\ \varrho_i \in G_i(x, y),\ i = 1, 2, 3\} \quad \forall (x, y) \in N.$$

PROOF. – Let x, x_1 be linearly independent elements such that $x \notin \bigcup_{i=1}^{4} E_0^{(i)}$. These elements determine a unique element $(x_0, y_0) \in N \cap \mathfrak{L}^2[x, x_1]$, a unique set of scalars α, β, γ, δ (with $\alpha\delta - \beta\gamma \neq 0$) such that $x = \alpha x_0 + \beta y_0$ and $x_1 = \gamma x_0 + \delta y_0$. This means that α/β does not belong to any of the sets $G_i(x_0, y_0)$ or to $G(x_0, y_0)$. If P_k is given by

$$P_k(sx + tx_1) = \prod_{j=1}^{n_k} (\delta_{jk} s - \gamma_{jk} t) ,$$

then $P_k(x) \neq 0$ for $1 \leqslant k \leqslant q$ and so $\delta_{jk} \neq 0$ for $1 \leqslant j \leqslant n_k$, $k = 1, 2, \ldots, q$. Put $\varrho_{jk} = \gamma_{jk}/\delta_{jk}$. Since (for each $k = 1, 2, \ldots, q$)

$$P_k(\varrho_{jk} x + x_1) = P_k[(\alpha\varrho_{jk} + \gamma) x_0 + (\beta\varrho_{jk} + \delta) y_0] = 0 \qquad \forall 1 \leqslant j \leqslant n_k ,$$

the hypotheses (3.1) imply that

$$\frac{\alpha\varrho_{jk} + \gamma}{\beta\varrho_{jk} + \delta} \in \begin{cases} G_1(x_0, y_0) & \forall 1 \leqslant j \leqslant n_k,\ k = 1, 2, \ldots, r \\ G_2(x_0, y_0) & \forall 1 \leqslant j \leqslant n_k,\ k = r + 1, \ldots, p \\ G_3(x_0, y_0) & \forall 1 \leqslant j \leqslant n_k,\ k = p + 1, \ldots, q . \end{cases}$$

If we put $\varrho'_{jk} = (\alpha\varrho_{jk} + \gamma)/(\beta\varrho_{jk} + \delta)$ then $\varrho_{jk} = (\delta\varrho'_{jk} - \gamma)/(-\beta\varrho'_{jk} + \alpha) = U(\varrho'_{jk})$, say, so that

$$\varrho_{jk} \in \begin{cases} U\big(G_1(x_0, y_0)\big) & \forall 1 \leqslant j \leqslant n_k,\ k = 1, 2, \ldots, r \\ U\big(G_2(x_0, y_0)\big) & \forall 1 \leqslant j \leqslant n_k,\ k = r + 1, \ldots, p \\ U\big(G_3(x_0, y_0)\big) & \forall 1 \leqslant j \leqslant n_k,\ k = p + 1, \ldots, q , \end{cases} \tag{3.2}$$

where U is the homographic transformation of K_ω given by $U(\varrho) = (\delta\varrho - \gamma)/(-\beta\varrho + \alpha)$. Since $G_i(x_0, y_0) \in D(K_\omega)$ and U preserves the class $D(K_\omega)$ (cf. [22], p. 353), we note that $U(G_i(x_0, y_0)) \in D(K_\omega)$ for $i = 1, 2, 3$. Due to the fact that $\alpha/\beta \notin \bigcup_{i=1}^{3} G_i(x_0, y_0)$ and $U(\alpha/\beta) = \omega$, we see that $\omega \notin U(G_i(x_0, y_0))$ for any i. From definition of g.c.r., $U(G_i(x_0, y_0))$ are K_0-convex g.c.r.'s of K_ω and (hence) relations (3.2) give

$$\mu_k(\text{say}) = \sum_{j=1}^{n_k} \frac{1}{n_k} \varrho_{jk} \in \begin{cases} U(G_1(x_0, y_0)) & \forall 1 \leqslant k \leqslant r \\ U(G_2(x_0, y_0)) & \forall r+1 \leqslant k \leqslant p \\ U(G_3(x_0, y_0)) & \forall p+1 \leqslant k \leqslant q \,. \end{cases} \tag{3.3}$$

Observe that each of the sets of scalars m_k/A_1 (for $k = 1, 2, \ldots, r$), m_k/A_2 (for $k = r+1, \ldots, p$), and m_k/A_3 (for $k = p+1, \ldots, q$) consists of positive elements of K_0 with sum as 1 $\left(A_3 \text{ denoting the sum } \sum_{k=p+1}^{q} m_k\right)$. Therefore (cf. (3.3)), there exist elements

$$\varrho_i \in G_i(x_0, y_0) \qquad \text{for } i = 1, 2, 3$$

such that

$$\nu_i/A_i = U(\varrho_i) = (\delta\varrho_i - \gamma)/(-\beta\varrho_i + \alpha) \neq \omega\,, \qquad i = 1, 2, 3, \tag{3.4}$$

where

$$\nu_1 = \sum_{k=1}^{r} m_k \mu_k\,, \qquad \nu_2 = \sum_{k=r+1}^{p} m_k \mu_k\,, \qquad \nu_3 = \sum_{k=p+1}^{q} m_k \mu_k\,.$$

First, we claim that the ν_i's cannot vanish simultaneously. For, otherwise, γ/δ would be common to all the g.c.r.'s $G_i(x_0, y_0)$ (cf. (3.4)), implying that $x_1 = \gamma x_0 + \delta y_0$ is common to all the sets $T_{G_i}(x_0, y_0)$. This would mean that x_1 belongs to all the circular cones $E_0^{(i)}$, contradicting the hypothesis that $E_0^{(1)} \cap E_0^{(2)} \cap E_0^{(3)} = \emptyset$.

Therefore, in order to show that $\nu_1 + \nu_2 + \nu_3 \neq 0$, we may assume (without loss of generality) that at most one of the ν_i's is zero. Now, with this assumption, if on the contrary $\nu_1 + \nu_2 + \nu_3$ were to vanish, equation (3.4) would imply that $A_1 U(\varrho_1) + A_2 U(\varrho_2) + A_3 U(\varrho_3) = 0$. Since $A_1 + A_2 + A_3 = 0$ (cf. hypotheses on m_k), we have

$$A_1[U(\varrho_1) - U(\varrho_3)] + A_2[U(\varrho_2) - U(\varrho_3)] = 0\,. \tag{3.5}$$

A routine computation then provides:

$$\begin{aligned} U(\varrho_1) - U(\varrho_3) &= \Delta(\varrho_1 - \varrho_3)/(-\beta\varrho_1 + \alpha)(-\beta\varrho_3 + \alpha) \\ U(\varrho_2) - U(\varrho_3) &= \Delta(\varrho_2 - \varrho_3)/(-\beta\varrho_2 + \alpha)(-\beta\varrho_3 + \alpha)\,, \end{aligned} \tag{3.6}$$

where $\Delta = \alpha\delta - \beta\gamma \neq 0$. Since α/β is not a member of any $G_i(x_0, y_0)$, we can say that $\alpha/\beta \neq \varrho_i$, i.e. $-\beta\varrho_i + \alpha \neq 0$ for $i = 1, 2, 3$. Equation (3.5) says that $U(\varrho_1) - U(\varrho_3) = 0$ if and only if $U(\varrho_2) - U(\varrho_3) = 0$, which can happen if and only if $\varrho_1 = \varrho_2 = \varrho_3 = \varrho$ (say) (Recall that U is one-one and onto). This would mean that $\varrho \in G_i(x_0, y_0)$ for all i, contradicting that $E_0^{(1)} \cap E_0^{(2)} \cap E_0^{(3)} = \emptyset$. Therefore, $\varrho_1, \varrho_2, \varrho_3$ must all be distinct elements of K_ω. First, we take up the case when no ϱ_i is ω. In view of this and the fact that $U(\varrho_i) \neq \omega$ for any i (cf. (3.4)), we conclude that none of the expressions in (3.6) can take the value 0 or ω. Consequently, equations (3.5) and (3.6) give

$$\frac{U(\varrho_1) - U(\varrho_3)}{U(\varrho_2) - U(\varrho_3)} = \frac{\varrho_1 - \varrho_3}{\varrho_2 - \varrho_3} \cdot \frac{-\beta\varrho_2 + \alpha}{-\beta\varrho_1 + \alpha} = -\frac{A_2}{A_1}$$

or,

$$\frac{(\alpha/\beta - \varrho_2)}{(\alpha/\beta - \varrho_1)} \cdot \frac{\varrho_3 - \varrho_1}{\varrho_3 - \varrho_2} = -A_2/A_1\,,$$

irrespective of whether β is or is not zero (because $\beta = 0$ and $\Delta \neq 0$ imply that $\alpha \neq 0$). That is, by Definition 2.4 we have

$$(\alpha/\beta, \varrho_3, \varrho_2, \varrho_1) = -A_2/A_1 . \tag{3.7}$$

In case, however, any one of ϱ_1, ϱ_2 or ϱ_3 (say, ϱ_i) is ω, equation (3.4) gives $U(\varrho_i) = -\delta/\beta \neq \omega$ (so that $\beta \neq 0$). Starting with this change in equation (3.4) and making the corresponding changes all along in the above analysis, we still arrive at the same relation (3.7) (cf. the definition of cross-ratio for the case when any one $\varrho_i = \omega$). Since (3.7) holds in both the cases considered above, the element $\alpha/\beta \in G(x_0, y_0)$, which implies that $x = \alpha x_0 + \beta y_0 \in T_Q(x_0, y_0)$. This means that $x \in E_0^{(4)}$, contradicting the choice of x already made. We have, therefore, shown that

$$\sum_{k=1}^{q} \sum_{j=1}^{n_k} \frac{m_k}{n_k} \varrho_{jk} = \nu_1 + \nu_2 + \nu_3 \neq 0 . \tag{3.8}$$

But, we know (cf. [19], p. 543) that

$$\Phi(Q; x_1, x) = -\left[\sum_{k=1}^{q} \sum_{j=1}^{n_k} \frac{m_k}{n_k} \varrho_{jk}\right]\left(\prod_{k=1}^{q} P_k(x)\right). \tag{3.9}$$

Since $P_k(x) \neq 0$ for $k = 1, 2, \ldots, q$, relations (3.8) and (3.9) imply that $\Phi(Q; x_1, x) \neq 0$. This completes the proof

The following corollary is essentially an improved form of a result due to Walsh [13] (cf. also [7], Theorem (22,2)). Here we take g.c.r.'s of $\boldsymbol{C}_\omega$ whereas Walsh considers only the *classical* c.r.'s (a proper subclass of g.c.r.'s—see [19], p. 536). We shall use the notation « $Z(f)$ » for the set of all zeros of a polynomial f.

COROLLARY 3.2. – *For each $i = 1, 2, 3$, let f_i be a polynomial (from $\mathbf{C}$ to $\mathbf{C}$) of degree n_i and let $C_i \in D(\mathbf{C}_\infty)$ such that $Z(f_i) \subseteq C_i$. If $n_1 + n_2 = n_3$ and if $C_1 \cap C_2 \cap C_3 = \emptyset$, then every finite zero of the derivative of the function $f_1(z)f_2(z)/f_3(z)$ lies in $\bigcup_{i=1}^{4} C_i$, where*

$$C_4 = \{\varrho \in \mathbf{C}_\infty | (\varrho, \varrho_3, \varrho_2, \varrho_1) = -n_2/n_1;\ \varrho_i \in C_i,\ i = 1, 2, 3\}.$$

PROOF. – For $i = 1, 2, 3$, let us take the circular cones

$$E_0^{(i)} \equiv E_0(N, G_i) = \{sx_0 + ty_0 \neq 0 | s, t \in \mathbf{C};\ s/t \in C_i\}$$

in the 2-dimensional vector space $E = \mathbf{C}^2$ (cf. Remark 2.2(II)), where $x_0 = (1, 0)$, $y_0 = (0, 1)$, $N = \{(x_0, y_0)\}$, and $G_i(x_0, y_0) = C_i$. Letting $f_i(z) = \sum_{k=0}^{n_i} a_{ki} z^k$, observe that the mappings $P_i\colon \mathbf{C}^2 \to \mathbf{C}$, defined by

$$P_i(x) \equiv P_i(sx_0 + ty_0) = \sum_{k=0}^{n_i} a_{ki} s^k t^{n_i - k} \qquad \forall x = (s, t) \in \mathbf{C}^2$$

are a.h.p.'s of degree n_i such that $Z_{P_i}(x_0, y_0) \subseteq T_{G_i}(x_0, y_0)$. (Because $P_i(sx_0 + ty_0) = t^{n_i} f_i(s/t)$ for all nonzero elements $x = (s, t)$ and because $Z(f_i) \subseteq C_i = G_i(x_0, x_0)$.). Now, let us consider the generalized polar of the product $Q(x) = P_1(x)P_2(x)P_3(x)$, defined by

$$\Phi(Q; x_1, x) = n_1 P_1(x_1, x) P_2(x) P_3(x) + n_2 P_1(x) P_2(x_1, x) P_3(x) - n_3 P_1(x) P_2(x) P_3(x_1, x) \tag{3.10}$$

for all elements $x = (s, t)$ and $x_1 = (s_1, t_1)$ of $\mathbf{C}^2$. Then the polynomials P_i, the circular cones $E_0^{(i)}$, and the generalized polar $\Phi(Q; x_1, x)$ satisfy all the hypotheses of Theorem 3.1 with $r = 1$, $p = 2$, $q = 3$, $m_1 = n_1 = A_1 > 0$, $m_2 = n_2 = A_2 > 0$, and $m_3 = -n_3 < 0$. We know (cf. [8], equation (2.4) or [19], p. 545) that

$$P_i(x_1, x) = \frac{1}{n_i}\left(s_1 \frac{\partial P_i}{\partial s} + t_1 \frac{\partial P_i}{\partial t}\right), \qquad i = 1, 2, 3, \tag{3.11}$$

and that $P_i(x) = t^{n_i} f_i(s/t)$, $\partial P_i/\partial s = t^{n_i - 1} f_i'(s/t)$ for all i and for all nonzero elements x. If we now take $x_1 = x_0$ (so that $s_1 = 1$ and $t_1 = 0$), from (3.10) and (3.11) we can easily obtain (setting $f = f_1 f_2/f_2$)

$$\Phi(Q; x_0, x) = t^{n_1+n_2+n_3-1}[f_1'(s/t) f_2(s/t) f_3(s/t) + f_1(s/t) f_2'(s/t) f_3(s/t) - f_1(s/t) f_2(s/t) f_3'(s/t)] = t^{n_1+n_2+n_3-1} \cdot [f_3(s/t)]^2 \cdot f'(s/t). \tag{3.12}$$

Now, Theorem 3.1 comes in to say that $\Phi(Q; x_0, x) \neq 0$ whenever the element $x = (s, t)$ is linearly independent to x_0 such that $x \notin \bigcup_{i=1}^{4} E_0^{(i)}$, where

$$E_0^{(4)} \equiv E_0(N, G) = \{sx_0 + ty_0 \neq 0 | s, t \in \boldsymbol{C};\ s/t \in G(x_0, y_0)\}$$

and

$$\begin{aligned} G(x_0, y_0) &= \{\varrho \in \boldsymbol{C}_\omega | (\varrho, \varrho_3, \varrho_2, \varrho_1) = -A_2/A_1;\ \varrho_i \in G_i(x_0, y_0),\ i = 1, 2, 3\} \\ &= C_4 \quad (\text{since } A_1 = n_1,\ A_2 = n_2). \end{aligned}$$

That is, $\Phi(Q; x_0, x) \neq 0$ for all elements $x = (s, t) = sx_0 + ty_0$ for which $t \neq 0$ and for which $s/t \notin \bigcup_{i=1}^{4} C_i$. Finally, (3.12) says that $f'(s/t) \neq 0$ for all $s, t \in \boldsymbol{C}$ such that $t \neq 0$ and $s/t \notin C_1 \cup C_2 \cup C_3 \cup C_4$. This establishes our corollary.

REMARK 3.3. – (I) If we define the formal derivative $f'(z)$ of a polynomial $f(z) = \sum_{k=0}^{n} a_k z^k$ (from K to K) to be the polynomial $f'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$ (cf. [19], p. 553; [21]), we can easily verify that the formal derivative of the product $f_1 f_2$ of two polynomials f_1 and f_2 is given by (cf. [21], Proposition (1.2)) $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$. If we now define the formal derivative of the quotient f_1/f_2 to be given by $(f_1/f_2)' = (f_1' f_2 - f_1 f_2')/f_2^2$, then the formal derivative of the quotient $f_1(z) f_2(z)/f_3(z)$ (f_i being polynomials) is easily seen to be given by

$$[f_1'(z) f_2(z) f_3(z) + f_1(z) f_2'(z) f_3(z) - f_1(z) f_2(z) f_3'(z)]/\{f_3(z)\}^2 . \tag{3.13}$$

(II) In view of the definition of formal derivatives $f'(z)$ of a polynomial $f(z)$ from K to K and the definition of formal partial derivatives $\partial P/\partial s$ of a polynomial $P(s, t)$ from K^2 to K (cf. [19], p. 553), we remark that *Corollary* 3.2 *can be easily extended to the field K in general.* The proof being exactly the same as that of Corollary 3.2 except only that we replace $\boldsymbol{C}$ by K all along. Note that the expression (3.13), which is precisely the formal derivative of the function $f(z) = f_1(z) f_2(z)/f_3(z)$, justifies the validity of step (3.12) in the proof of Corollary 3.2.

4. – The main theorem on algebra-valued generalized polars.

Before we come to our main cross-ratio theorem for algebra-valued generalized polars, we define the concept of fully supportable subsets of V and establish a lemma, which we shall need in the proofs of our results in this section. As regards the concept of supportable subsets of a vector space we refer to Hörmander [5], p. 59. The analogous concept in an algebra V is given in the following (originally due to ZAHEER [20], Definition 2.2).

DEFINITION 4.1. – A subset M of V is called *fully supportable* if every point ξ outside M is contained in some ideal maximal subspace of V which does not intersect M. That is, for every $\xi \in V - M$, there exists (cf. (2.1)) a unique nontrivial scalar homomorphism L on V such that $L(\xi) = 0$ but $L(v) \neq 0$ for every $v \in M$.

If M is a fully supportable subset of V, then M is naturally a supportable subset of V (regarded as a vector space), but not conversely. In view of the statement (2.1) concerning ideal maximal subspaces, it is easy to see that the complement in V of every ideal maximal subspace of V is a fully supportable subset of V (cf. [20], Proposition 2.3).

Given $P \in \boldsymbol{P}_n^*$ and a fully supportable subset M of V, we shall write (for any given x, $y \in E$)

$$E_P(x, y) = \{sx + ty \neq 0 | s, t \in K;\ P(sx + ty) \notin M\} . \tag{4.1}$$

REMARK 4.2. – Since K is a field and the identity map from K to K is the only nontrivial scalar homomorphism on K, it is obvious that $M = K - \{0\}$ is the only fully supportable subset of K (Take $V = K$ in Definition 4.1) and that the corresponding set $E_P(x, y)$ reduces essentially to the null-set $Z_P(x, y)$ of P defined earlier in Section 3.

LEMMA 4.3. – *Given $\lambda > 0$ and g.c.r.'s $G_i \in D(K_\omega)$ for $i = 1, 2, 3$, let us define*

$$G = \{\varrho \in K_\omega | (\varrho, \varrho_2, \varrho_3, \varrho_1) = -\lambda;\ \varrho_i \in G_i,\ i = 1, 2, 3\} .$$

If $G_1 = G_3$ and $G_1 \cap G_2 = \emptyset$, then $G \subseteq G_1 \cup G_2$.

PROOF. – Suppose on the contrary that $G \not\subseteq G_1 \cup G_2$. Then there exists an element $\varrho \in G$ such that $\varrho \notin G_1 \cup G_2$ and $(\varrho, \varrho_2, \varrho_3, \varrho_1) = -\lambda$ for some $\varrho_1, \varrho_3 \in G_1$ and some $\varrho_2 \in G_2$, where $\varrho_1, \varrho_3 \neq \varrho_2$. Let us also note that no element of G can come from two coincident elements ϱ_1 and ϱ_3 of G_1 (cf. Remark 2.5). That is, $\varrho_1, \varrho_2, \varrho_3$ must all be distinct. Now, since the transformation $h(\varrho) = (\varrho, \varrho_2, \varrho_3, \varrho_1)$ maps $\varrho_2, \varrho_3, \varrho_1$ to $1, 0, \omega$ (respectively) and since $\lambda > 0$, we have that $\varrho \neq \varrho_1, \varrho_2, \varrho_3$.

First, we claim that $\varrho \neq \omega$. Because, $\varrho = \omega$ would imply that $\varrho_1, \varrho_2, \varrho_3 \neq \omega$, and (in view of (2.5)) we have $h(\omega) = (\varrho_1 - \varrho_2)/(\varrho_3 - \varrho_2) = -\lambda$. That is (see the notation in (2.4)),

$$(\varrho_3 - \varrho_2)^{-1} + \lambda(\varrho_1 - \varrho_2)^{-1} = [\varphi_{\varrho_2}(\varrho_3) + \lambda\varphi_{\varrho_2}(\varrho_1)] = 0 , \tag{4.2}$$

where $\varphi_{\varrho_2}(\varrho_3)$ and $\varphi_{\varrho_2}(\varrho_1)$ are elements of $\varphi_{\varrho_2}(G_1)$. Since $G_1 \in D(K_\omega)$ and $\varrho_2 \notin G_1$, from the definition of a g.c.r. we see that $\varphi_{\varrho_2}(G_1)$ is K_0-convex and (hence) that

$$\text{(cf. (4.2))} \qquad [\varphi_{\varrho_2}(\varrho_3) + \lambda\varphi_{\varrho_2}(\varrho_1)]/(1 + \lambda) = 0 \in \varphi_{\varrho_2}(G_1) .$$

Therefore, there exists a $\varrho^* \in G_1$ such that $1/(\varrho^* - \varrho_2) = 0$. Since $\varrho_2 \neq \omega$, we must have $\varrho^* = \omega = \varrho$, contradicting that $\varrho \notin G_1$.

Having shown that $\varrho \neq \omega$ and that the elements ϱ, ϱ_1, ϱ_2, ϱ_3 are all distinct, we now claim that

$$(4.3) \qquad [\varphi_\varrho(\varrho_1) + \lambda\varphi_\varrho(\varrho_3)]/(1+\lambda) = \varphi_\varrho(\varrho_2)\,,$$

where φ_ϱ is as defined in (2.4). To this effect, we consider the following cases:

Case 1. ϱ_1, ϱ_2, $\varrho_3 \neq \omega$. In this case ϱ, ϱ_1, ϱ_2, ϱ_3 are all distinct elements of K and the expression of cross-ratio (cf. (2.5)) gives $(\varrho - \varrho_3)(\varrho_2 - \varrho_1)/(\varrho - \varrho_1)(\varrho_2 - \varrho_3) = -\lambda$. Hence

$$[1 + (\varrho_1 - \varrho_2)/(\varrho - \varrho_1)] + \lambda[1 + (\varrho_3 - \varrho_2)/(\varrho - \varrho_3)] = 1 + \lambda\,,$$

or,

$$[(\varrho - \varrho_1)^{-1} + \lambda(\varrho - \varrho_3)^{-1}]/(1+\lambda) = (\varrho - \varrho_2)^{-1}\,.$$

This is precisely the equation (4.3).

Case 2. One of the ϱ_i's is ω. The definition of cross-ratio (cf. (2.6)) implies

$$-\lambda = \begin{cases} (\varrho - \varrho_3)/(\varrho_2 - \varrho_3) & \text{if } \varrho_1 = \omega \qquad (4.4)\\ (\varrho - \varrho_3)/(\varrho - \varrho_1) & \text{if } \varrho_2 = \omega \qquad (4.5)\\ (\varrho_2 - \varrho_1)/(\varrho - \varrho_1) & \text{if } \varrho_3 = \omega \qquad (4.6)\end{cases}$$

From (4.4) and (4.6), we get

$$-(\lambda + 1) = \begin{cases} (\varrho - \varrho_2)/(\varrho_2 - \varrho_3) & \text{if } \varrho_1 = \omega \qquad (4.7)\\ (\varrho_2 - \varrho)/(\varrho - \varrho_1) & \text{if } \varrho_3 = \omega\,. \qquad (4.8)\end{cases}$$

On taking the quotient of the expressions in (4.4) and (4.7), and on rewriting the equations (4.5) and (4.8), we have:

$$\lambda(\varrho - \varrho_3)^{-1}/(\lambda + 1) = (\varrho - \varrho_2)^{-1}; \qquad (\varrho - \varrho_1)^{-1} + \lambda(\varrho - \varrho_3)^{-1} = 0\,;$$
$$(\varrho - \varrho_1)^{-1}/(\lambda + 1) = (\varrho - \varrho_2)^{-1}$$

according as $\varrho_1 = \omega$, $\varrho_2 = \omega$, or $\varrho_3 = \omega$. Or, in terms of the function φ_ϱ, these equations are respectively given by:

$$\lambda\varphi_\varrho(\varrho_3)/(\lambda + 1) = \varphi_\varrho(\varrho_2)\,; \qquad \varphi_\varrho(\varrho_1) + \lambda\varphi_\varrho(\varrho_3) = 0\,; \qquad \varphi_\varrho(\varrho_1)/(\lambda + 1) = \varphi_\varrho(\varrho_2)\,.$$

Since $\varphi_\varrho(\varrho_i) = (\varrho_i - \varrho)^{-1} = 0 \in \varphi_\varrho(G_i)$ if and only if $\varrho_i = \omega$, we see that the last three equalities are, essentially, the special cases of equation (4.3) in the respective cases when ϱ_1, ϱ_2, or ϱ_3 is taken as ω.

Now since equation (4.3) has been established for all cases and since $\varrho \notin G_1 \in D(K_\omega)$, the set $\varphi_\varrho(G_1)$ is K_0-convex and (hence) equation (4.3) implies that $\varphi_\varrho(\varrho_2) \in \varphi_\varrho(G_1)$. That is, $\varrho_2 \in G_1$, which contradicts the fact that $G_1 \cap G_2 = \emptyset$. Therefore, our assumption in the beginning must be false, and the proof is complete.

Now, we stablish the most general theorem of this paper. We make use of Theorem 3.1 in the proof of this theorem.

THEOREM 4.4. – *Let $E_0^{(i)} \equiv E_0(N, G_i)$, $i = 1, 2, 3$, be circular cones such that $E_0^{(1)} \cap E_0^{(2)} \cap E_0^{(3)} = \emptyset$, M a fully supportable subset of V, and let $P_k \in \boldsymbol{P}^*_{n_k}$ $(k = 1, 2, \ldots, q)$ such that*

$$E_{P_k}(x, y) \subseteq \begin{cases} T_{G_1}(x, y) & \text{for } k = 1, 2, \ldots, r (< p < q) \\ T_{G_2}(x, y) & \text{for } k = r + 1, \ldots, p \\ T_{G_3}(x, y) & \text{for } k = p + 1, \ldots, q \end{cases}$$

for all $(x, y) \in N$. If $\Phi(Q; x_1, x)$ is the algebra-valued generalized polar with m_k satisfying the hypotheses of Theorem 3.1, then $\Phi(Q; x_1, x) \in M$ for all linearly independent elements x, x_1 of E such that $x \in E - \bigcup_{i=1}^{4} E_0^{(i)}$, where $E_0^{(4)} \equiv E_0(N, G)$ is the cone corresponding to G as defined in Theorem 3.1.

PROOF. Let $\xi \in V - M$. Since M is fully supportable subset, there is a unique nontrivial scalar homomorphism L on V such that $L(\xi) = 0$ but $L(v) \neq 0$ for every $v \in M$. We easily see that $Z_{LP_k}(x, y) \subseteq E_{P_k}(x, y)$. From this and the hypotheses on the P_k, we conclude that the polynomials LP_k satisfy the hypotheses (3.1). In view of Remark 2.1 and the discussion immediately preceding it, LQ is essentially the product of polynomials $LP_k \in \boldsymbol{P}_{n_k}$ and

$$L(\Phi(Q; x_1, x)) = \Phi(LQ; x_1, x) , \tag{4.9}$$

both sides using the same m_k's. Consequently, the circular cones $E_0^{(i)}$, the polynomials $LP_k \in \boldsymbol{P}_{n_k}$, and the generalized polar $\Phi(LQ; x_1, x)$ of the product

$$(LQ)(x) = \prod_{k=1}^{q} (LP_k)(x)$$

satisfy the hypotheses of Theorem 3.1. Therefore, $\Phi(LQ; x_1, x) \neq 0$ for all x, x_1 as claimed in the present theorem. Finally, the relation (4.9) then says that $\Phi(Q; x_1, x) \neq \xi$ for all x, x_1 as claimed. Since ξ is arbitrary in our arguments, the proof is complete.

In the special case when $V = K$ (so that $\boldsymbol{P}_n^* \equiv \boldsymbol{P}_n$) and $M = K - \{0\}$ is the fully supportable subset of K (cf. Remark 4.2), Theorem 4.4 reduces to Theorem 3.1, and (hence) the most general result of this paper. Furthermore, we deduce the following result (cf. [20], Theorem 2.6) as an important application of Theorem 4.4.

Corollary 4.5. – *Let $E_0^{(i)} \equiv E_0(N, G_i)$, $i = 1, 2$, be disjoint circular cones, M a fully supportable subset of V, and let $P_k \in \boldsymbol{P}_{n_k}^*$ $(k = 1, 2, \ldots, q)$ such that*

$$E_{P_k}(x, y) \subseteq \begin{cases} T_{G_1}(x, y) & \text{for } k = 1, 2, \ldots, p(< q) \\ T_{G_2}(x, y) & \text{for } k = p + 1, \ldots, q \end{cases}$$

for all $(x, y) \in N$. If $\Phi(Q; x_1, x)$ is the algebra-valued generalized polar of Theorem 4.4, then $\Phi(Q; x_1, x) \in M$ for all linearly independent elements x, x_1 of E such that $x \in E - E_0^{(1)} \cup E_0^{(2)}$.

Proof. – It is easily seen that the hypotheses in Corollary 4.5 are a restatement of the hypotheses in Theorem 4.4 after we interchange the roles of the circular cones $E_0^{(2(}$ and $E_0^{(3)}$ in the statement of Theorem 4.4 and the take $G_1 = G_3$ (so that $G_1(x, y) = G_3(x, y)$ for all $(x, y) \in N$, and $E_0^{(1)} = E_0^{(3)}$). Consequently, Theorem 4.4 implies that $\Phi(Q; x_1, x) \in M$ for all linearly independent elements x, x_1 of E such that $x \in E - E_0^{(1)} \cup E_0^{(2)} \cup E_0^{(4)}$, where $E_0^{(4)} \equiv E_0(N, G)$ and

$$G(x, y) = \{\varrho \in K_\infty | (\varrho, \varrho_2, \varrho_3, \varrho_1) = -A_2/A_1;\ \varrho_1, \varrho_3 \in G_1(x, y),\ \varrho_2 \in G_2(x, y)\}$$

for all $(x, y) \in N$. Since $G_1(x, y) = G_3(x, y)$ and $G_1(x, y) \cap G_2(x, y) = \emptyset$ for all $(x, y) \in N$, Lemma 4.3 with $\lambda = A_2/A_1$ implies that

$$G(x, y) \subseteq G_1(x, y) \cup G_2(x, y) \qquad \forall (x, y) \in N .$$

Therefore, $E_0^{(4)} \subseteq E_0^{(1)} \cup E_0^{(2)}$, and the rest of the proof is obvious.

In the case when $V = K$ and $M = K - \{0\}$, the above corollary gives a result due to Zaheer ([19], Theorem 2.5) on generalized polars of a.h.p.'s from E to K. Consequently, Marden's theorem ([19], Corollary 2.6) and Bôcher's theorem ([19], Corollaries 2.7 and 2.8) automatically follow from Corollary 4.5.

Concluding remarks. – (I) Since Theorem 2.5 in [19] becomes a special case of Corollary 4.5 of Theorem 4.4 and since Theorem 4.4 is the most general result of this paper, we conclude from Example 2.9 in Zaheer [19] that none of the present theorems can be generalized for nonalgebraically closed fields of characteristic zero. Due to same reasons as indicated above, Example 2.10 in [19] suggests that in none of our results here can we replace the g.c.r.'s $G_i(x, y)$ or C_i by g.c.r.'s adjoined with arbitrary subsets of their boundary.

(II) Since there do exist (cf. [17], pp. 123-125) circular cones $E_0(N, G)$ and a.h.p.'s $P \in \boldsymbol{P}_n$ such that $Z_P(x, y) \subseteq T_G(x, y)$ for every $(x, y) \in N$, we conclude that the hypotheses (3.1) in Theorem 3.1 are valid and that our theorems are not vacuously true.

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