Oscillatory and Asymptotic Behavior of Strongly Superlinear Differential Equations with Deviating Arguments (*).

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Summary. – *This paper deals with the oscillation and the asymptotic behavior of the solutions of superlinear differential equations with general (retarded, advanced or mixed type) deviating arguments. The equations considered involve a damping term. The results obtained extend known fundamental oscillation criteria for superlinear differential equations without damping terms and especially the recent basic reults of Kitamura and Kusano* [5], *and Staikos* [19, 20].

This paper is concerned with the oscillation and the asymptotic behavior of the solutions of n-th order $(n > 1)$ differential equations with deviating arguments of the form

$$(E, \delta) \qquad [r(t)x^{(n-1)}(t)]' + \delta f(t; x[g_1(t)], \dots, x[g_m(t)]) = 0\,, \quad t \geqq t_0 \ (\delta = \pm 1)$$

where r and g_j $(j = 1, \dots, m)$ are continuous real-valued functions on the interval $[t_0, \infty)$ and f is a continuous real-valued function defined at least on $R_+^m \cup R_-^m$, $R_+ = (0, \infty)$ and $R_- = (-\infty, 0)$. The following assumptions are made:

(i) *r is positive on $[t_0, \infty)$ and such that*

$$\int^{\infty} \frac{dt}{r(t)} = \infty\,.$$

(ii) *For every $t \geqq t_0$,*

$$f(t; y) \geqq 0 \quad \textit{for all } y \in R_+^m\,, \qquad f(t; y) \leqq 0 \quad \textit{for all } y \in R_-^m$$

and $f(t; y)$ is increasing with respect to y in $R_+^m \cup R_-^m$:

(iii) $$\lim_{t\to\infty} g_j(t) = \infty \qquad (j = 1, \dots, m)\,.$$

Note that the increasing character of real-valued functions defined on subsets of R^m is considered with respect to the usual order in R^m defined by the positive cone $\{y = (y_1, \dots, y_m) \in R^m\colon (\forall j = 1, \dots, m)\, y_j \geqq 0\}$, i.e. as follows

$$y \leqq z \Leftrightarrow (\forall j = 1, \dots, m)\, y_j \leqq z_j\,.$$

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Throughout the paper, p will be a continuous real-valued function on the interval $[t_0, \infty)$ and φ, ψ will be continuous real-valued functions defined at least on $R - \{0\}$, R_+ respectively. For these functions the following conditions are introduced:

(I) *p is nonnegative on $[t_0, \infty)$.*

(II) *φ is increasing on $R - \{0\}$ and has the sign property $y\varphi(y) > 0$ for all $y \neq 0$; ψ is positive and increasing on R_+;*

$$\int^{\infty} \frac{dy}{\varphi(y)\psi(y)} < \infty \quad \textit{and} \quad \int^{-\infty} \frac{dy}{\varphi(y)\psi(y)} < \infty .$$

We consider only such solutions $x(t)$ of (E, δ) which are defined for all large t. Sufficient smoothness for the existence of such solutions are assumed. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*.

In the last ten years there has been an increasing interest in studying theo scillatory and asymptotic behavior of the solutions of differential equations with deviating arguments. Most of the literature, however, is concerned with equations involving retarded arguments. For general interest on typical oscillation results concerning such equations we refer to an excellent survey article of MITROPOL'SKIĬ and ŠEVELO [10].

The oscillatory and asymptotic behavior of the solutions of *superlinear* differential equations with retarded or general deviating arguments has been the object of intensive studies. We choose to refer to GRAMMATIKOPOULOS, SFICAS and STAIKOS [1], KITAMURA [4], KITAMURA and KUSANO [5], KUSANO [6], KUSANO and ONOSE [7, 8, 9], RYDER and WEND [13], ŠEVELO and VAREH [14, 15], SFICAS [16], SFICAS and STAIKOS [17, 18], and STAIKOS [19, 20]. The purpose here is to give oscillation results for superlinear equations including a damping term.

The oscillatory and asymptotic behavior of the bounded solutions of the differential equation (E, δ) is well described by the following theorem, which is a special case of a result of the author [11].

THEOREM 0. – *Let* (i)-(iii) *be satisfied and suppose that:*

(C_0) *For every nonzero constant c either*

$$\int^{\infty} |f(t; c, \ldots, c)| dt = \infty$$

or

$$\int^{\infty} \frac{t^{n-2}}{r(t)} \int_t^{\infty} |f(s; c, \ldots, c)| ds\, dt = \infty .$$

Then for n even [resp. odd] all bounded solutions of the differential equation $(E, +1)$ [resp. of the equation $(E, -1)$] are oscillatory, while for n odd [resp. even] every bounded *solution x of the differential equation $(E, +1)$ [resp. of the equation $(E, -1)$] is either oscillatory or such that $x^{(i)}$ $(i = 0, 1, \dots, n-2)$ and $rx^{(n-1)}$ tend monotonically to zero at ∞.*

The purpose here is to study the oscillatory character and the asymptotic behavior of all solutions of the differential equation (E, δ). For this purpose, we need the following lemma which is originated in two well-known lemmas due to KIGURADZE [2, 3]. This lemma is obtained here as a special case of a lemma given by the same author in [12].

LEMMA. – *Suppose that (i) holds and let h be a positive and $(n-1)$-times differentiable function on an interval $[\tau, \infty)$, $\tau \geqq t_0$, such that the function $rh^{(n-1)}$ is differentiable with its derivative of constant sign on $[\tau, \infty)$ and not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$.*

Then there exist a $T \geqq \tau$ and an integer l, $0 \leqq l \leqq n$, with $n + l$ odd for $[rh^{(n-1)}]'$ nonpositive or $n + l$ even for $[rh^{(n-1)}]'$ nonnegative so that

$$\begin{cases} l \leqq n-1 \Rightarrow (-1)^{l+j} h^{(j)} > 0 & \text{on } [T, \infty) \ (j = l, \dots, n-1) \\ l > 1 \Rightarrow h^{(i)} > 0 & \text{on } [T, \infty) \ (i = 1, \dots, l-1). \end{cases}$$

In order to formulate our results we introduce here the functions g, R_1 and R_2 defined on $[t_0, \infty)$ by

$$g(t) = \min\{t, g_1(t), \dots, g_m(t)\},$$

$$R_1(t) = \int_{t_0}^{t} \frac{(t-s)^{n-2}}{r(s)}\, ds \quad \text{and} \quad R_2(t) = \int_{0}^{t} \frac{(s-t_0)^{n-2}}{r(s)}\, ds.$$

THEOREM 1. – *Let the conditions* (i)-(iii), (I)-(II) *and the following ones be satisfied:*

$$(H) \qquad |f(t; y, \dots, y)| \geqq p(t)|\varphi(y)| \quad \textit{for all } (t, y) \in [t_0, \infty) \times (R - \{0\}).$$

$$(C) \qquad \int^{\infty} \frac{R_k[g(t)]}{\varphi(R_1[g(t)])}\, p(t)\, dt = \infty \quad (k = 1, 2).$$

Then we have:

α) For n even, all solutions of the differential equation $(E, +1)$ are oscillatory.

β) For n odd, every solution x of the equation $(E, +1)$ is oscillatory or satisfies

$$(X_0) \qquad \begin{cases} \lim_{t\to\infty} x^{(i)}(t) = 0 & \textit{monotonically } (i = 0, 1, \dots, n-2) \\ \lim_{t\to\infty} r(t)x^{(n-1)}(t) = 0 & \textit{monotonically}. \end{cases}$$

THEOREM 2. – *Let the conditions* (i)-(iii), (I)-(II), (*H*) *and* (*C*) *be satisfied. Then we have*:

α) *For n even, every solution x of the differential equation* $(E, -1)$ *is oscillatory or satisfies one of* (X_0),

$$(X_\infty) \quad \lim_{t\to\infty} x^{(i)}(t) = \infty \qquad (i = 0, 1, \dots, n-2) \text{ and } \lim_{t\to\infty} r(t)x^{(n-1)}(t) = \infty,$$

$$(X_{-\infty}) \quad \lim_{t\to\infty} x^{(i)}(t) = -\infty \qquad (i = 0, 1, \dots, n-2) \text{ and } \lim_{t\to\infty} r(t)x^{(n-1)}(t) = -\infty.$$

β) *For n odd, every solution x of the equation* $(E, -1)$ *is oscillatory or satisfies one of* (X_∞), $(X_{-\infty})$.

PROOF OF THEOREM 1. – The substitution $w = -x$ transforms the differential equation $(E, +1)$ into the equation

$$[r(t)\,w^{(n-1)}(t)]' + \hat{f}(t; w[g_1(t)], \dots, w[g_m(t)]) = 0 ,$$

where $\hat{f}(t; y) = -f(t; -y)$ for all $(t; y)$ in the domain of f. The transformed equation is subject to the same conditions posed for the equation $(E, +1)$ with the function $\hat{\varphi}$ in place of φ, where $\hat{\varphi}(y) = -\varphi(-y)$ for all $y \in \operatorname{dom} \varphi$. Hence, with respect to the nonoscillatory solutions of $(E, +1)$ we can restrict our attention only to the positive ones.

Since, by (i) and (iii), $\lim_{t\to\infty} R_1(t) = \infty$ and $\lim_{t\to\infty} g(t) = \infty$, we have

$$R_1[g(t)] \geqq 1 \qquad \text{for all large } t$$

and therefore, in view of (II), we obtain

$$\psi(R_1[g(t)]) \geqq \psi(1) > 0 \qquad \text{for all large } t.$$

This and conditions (I) and (*C*) give

$$\int^\infty R_2[g(t)]\,p(t)\,dt = \infty$$

and consequently

$$\int^\infty R_2(t)\,p(t)\,dt = \infty .$$

Furthermore, we consider an arbitrary constant $c \neq 0$ and we suppose that

$$\int^\infty |f(t; c, \dots, c)|\,dt < \infty .$$

In view of (H), we have

$$|f(t; c, \ldots, c)| \geqq p(t)|\varphi(c)|, \qquad t \geqq t_0,$$

where $\varphi(c) \neq 0$. So, we get

$$\int^{\infty} R_2(t)|f(t; c, \ldots, c)|dt = \infty .$$

Therefore, it is a matter of elementary calculus to derive that

$$\int^{\infty} \frac{t^{n-2}}{r(t)} \int_t^{\infty} |f(s; c, \ldots, c)| ds\, dt = \infty .$$

We have thus proved that the condition (C) implies (C_0). Hence, by Theorem 0, we can confine our discussion only to the unbounded solutions of the differential equation $(E, +1)$.

Let, now, x be a positive unbounded solution on an interval $[\tau_0, \infty)$, $\tau_0 > t_0$, of the differential equation $(E, +1)$. Moreover, let $\tau \geqq \tau_0$ be chosen, by (iii), so that

$$g_j(t) \geqq \tau_0 \quad \text{for every } t \geqq \tau \ (j = 1, \ldots, m).$$

Then, by taking into account (ii), (H), (I) and (II), from $(E, +1)$ we obtain that for every $t \geqq \tau$

$$\begin{aligned} -[r(t)x^{(n-1)}(t)]' &= f(t; x[g_1(t)], \ldots, x[g_m(t)]) \\ &\geqq f\big(t; \min_{1 \leqq j \leqq m} x[g_j(t)], \ldots, \min_{1 \leqq j \leqq m} x[g_j(t)]\big) \\ &\geqq p(t)\varphi\Big(\min_{1 \leqq j \leqq m} x[g_j(t)]\Big) \geqq 0 . \end{aligned}$$

Thus, the function $[rx^{(n-1)}]'$ is nonpositive on $[\tau, \infty)$. Moreover, $[rx^{(n-1)}]'$ is not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$, since, because of (C), the same holds for the function p. Hence, by Lemma, there exist a $T \geqq \tau$ and an integer l, $0 \leqq l \leqq n-1$, with $n + l$ odd so that

$$\begin{cases} (-1)^{l+j} x^{(j)}(t) > 0 & \text{for every } t \geqq T \ (j = l, \ldots, n-1) \\ x^{(i)}(t) > 0 & \text{for every } t \geqq T \ (i = 1, \ldots, l-1), \text{ provided that } l > 1. \end{cases}$$

Because of the unboundedness of x, we always have $l \geqq 1$.

Next, by (iii), we choose a $T_1 \geqq T$ so that

$$g_j(t) \geqq T \quad \text{for every } t \geqq T_1 \ (j = 1, \ldots, m).$$

Then, by taking into account (ii) and the fact that x is increasing on $[T, \infty)$, from $(E, +1)$ we obtain

$$- [r(t)x^{(n-1)}(t)]' = f(t; x[g_1(t)], \dots, x[g_m(t)]) \geqq f(t; x[g(t)], \dots, x[g(t)])$$

for all $t \geqq T_1$. Therefore, in view of (H), we get

$$- [r(t)x^{(n-1)}(t)]' \geqq p(t)\varphi(x[g(t)]), \qquad t \geqq T_1 .$$

Thus, for every t, t^* with $T_1 \leqq t \leqq t^*$ we derive

$$r(t)x^{(n-1)}(t) \geqq r(t^*)x^{(n-1)}(t^*) + \int_t^{t^*} p(u)\varphi(x[g(u)])\,du \geqq \int_t^{t^*} p(u)\varphi(x[g(u)])\,du .$$

Furthermore, if $l < n - 1$, by using the Taylor formula with integral remainder, for every t, t^* with $T_1 \leqq t \leqq t^*$ we obtain

$$\begin{aligned} x^{(l)}(t) &= \sum_{j=l}^{n-2} \frac{(t - t^*)^{j-l}}{(j-l)!} x^{(j)}(t^*) + \frac{1}{(n-2-l)!} \int_{t^*}^{t} (t-s)^{n-2-l} x^{(n-1)}(s)\,ds \\ &= \sum_{j=l}^{n-2} \frac{(t^* - t)^{j-l}}{(j-l)!} [(-1)^{l+j} x^{(j)}(t^*)] + \frac{1}{(n-2-l)!} \int_t^{t^*} \frac{(s-t)^{n-2-l}}{r(s)} [r(s)x^{(n-1)}(s)]\,ds \\ &\geqq \frac{1}{(n-2-l)!} \int_t^{t^*} \frac{(s-t)^{n-2-l}}{r(s)} \int_s^{t^*} p(u)\varphi(x[g(u)])\,du\,ds . \end{aligned}$$

Hence, for $T_1 \leqq t \leqq t^*$ we have

$$x^{(l)}(t) \geqq \begin{cases} \dfrac{1}{r(t)} \displaystyle\int_t^{t^*} p(u)\varphi(x[g(u)])\,du, & \text{if } l = n-1 \\ \dfrac{1}{(n-2-l)!} \displaystyle\int_t^{t^*} \frac{(s-t)^{n-2-l}}{r(s)} \int_s^{t^*} p(u)\varphi(x[g(u)])\,du\,ds, & \text{if } l < n-1 . \end{cases}$$

But, if $l > 1$, by the Taylor formula with integral remainder, we derive that for $t \geqq T_1$

$$\begin{aligned} x'(t) &= \sum_{i=1}^{l-1} \frac{(t - T_1)^{i-1}}{(i-1)!} x^{(i)}(T_1) + \frac{1}{(l-2)!} \int_{T_1}^{t} (t-s)^{l-2} x^{(l)}(s)\,ds \\ &\geqq \frac{1}{(l-2)!} \int_{T_1}^{t} (t-s)^{l-2} x^{(l)}(s)\,ds \end{aligned}$$

$$= \begin{cases} \dfrac{1}{(n-3)!}\displaystyle\int_{T_1}^{t} \frac{(t-s)^{n-3}}{r(s)}[r(s)x^{(n-1)}(s)]ds, & \text{if } l = n-1 \\ \dfrac{1}{(l-2)!}\displaystyle\int_{T_1}^{t} (t-s)^{l-2}x^{(l)}(s)ds, & \text{if } l < n-1 \end{cases}$$

$$\geqq \begin{cases} \dfrac{1}{(n-3)!}\left[\displaystyle\int_{T_1}^{t} \frac{(t-s)^{n-3}}{r(s)} ds\right] r(t)x^{(n-1)}(t), & \text{if } l = n-1 \\ \dfrac{1}{(l-2)!}\left[\displaystyle\int_{T_1}^{t} (t-s)^{l-2}ds\right] x^{(l)}(t) = \dfrac{1}{(l-1)!}(t-T_1)^{l-1}x^{(l)}(t), & \text{if } l < n-1. \end{cases}$$

We have thus proved that for all t, t^* with $T_1 \leqq t \leqq t^*$

$$(*) \qquad x'(t) \geqq \begin{cases} \dfrac{1}{r(t)}\displaystyle\int_{t}^{t^*} p(u)\varphi(x[g(u)])du, & \text{if } 1 = l = n-1 \\ \dfrac{1}{(n-3)!}\left[\displaystyle\int_{T_1}^{t} \frac{(t-s)^{n-3}}{r(s)} ds\right]\displaystyle\int_{t}^{t^*} p(u)\varphi(x[g(u)])du, & \text{if } 1 < l = n-1 \\ \dfrac{1}{(l-1)!\,(n-2-l)!}(t-T_1)^{l-1}\displaystyle\int_{t}^{t^*} \frac{(s-t)^{n-2-l}}{r(s)} \int_{s}^{t^*} p(u)\varphi(x[g(u)])du\,ds, & \\ \qquad\qquad \text{if } l < n-1. \end{cases}$$

By using the Taylor formula with integral remainder, for every $t \geqq T_1$ we obtain

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-2} \frac{x^{(k)}(T_1)}{k!}(t-T_1)^k + \frac{1}{(n-2)!}\int_{T_1}^{t} (t-s)^{n-2}x^{(n-1)}(s)ds \\ &= \sum_{k=0}^{n-2} \frac{x^{(k)}(T_1)}{k!}(t-T_1)^k + \frac{1}{(n-2)!}\int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)}[r(s)x^{(n-1)}(s)]ds \\ &\leqq \sum_{k=0}^{n-2} \frac{x^{(k)}(T_1)}{k!}(t-T_1)^k + \frac{r(T_1)x^{(n-1)}(T_1)}{(n-2)!}\int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)}ds. \end{aligned}$$

But, by applying the L'Hospital rule, we can easily derive that

$$\lim_{t\to\infty} \frac{(t-T_1)^k}{R_1(t)} = 0 \qquad (k = 0, 1, \dots, n-2),$$

$$\lim_{t\to\infty} \frac{1}{R_1(t)}\int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)} ds = 1$$

and consequently

$$\limsup_{t\to\infty} \frac{x(t)}{R_1(t)} \leqq \frac{r(T_1)x^{(n-1)}(T_1)}{(n-2)!}.$$

So, there exists a constant α with $\alpha \geqq 1$ so that

$$x(t) \leqq \alpha R_1(t) \quad \text{for every } t \geqq T_1.$$

Next, for $t \geqq T_1$ we define

$$R_1(t; T_1) = \int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)}\, ds \quad \text{and} \quad R_2(t; T_1) = \int_{T_1}^{t} \frac{(s-T_1)^{n-2}}{r(s)}\, ds.$$

By applying the L'Hospital rule, we immediately obtain

$$\lim_{t\to\infty} \frac{R_k(t; T_1)}{R_k(t)} = 1 \quad (k = 1, 2)$$

and hence there exist a positive constant β and a $\hat{T}_1 > T_1$ so that

$$R_k(t; T_1) \geqq \beta R_k(t) \quad \text{for all } t \geqq \hat{T}_1 \ (k = 1, 2).$$

Furthermore, we choose a $T_2 \geqq \hat{T}_1$ such that

$$g_j(t) \geqq \hat{T}_1 \quad \text{for every } t \geqq T_2 \ (j = 1, \dots, m).$$

Then we have

$$R_k[g(t); T_1] \geqq \beta R_k[g(t)], \quad t \geqq T_2 \ (k = 1, 2).$$

Now, let t^* be an arbitrary number with $t^* \geqq T_2$. We divide both sides of $(*)$ by $\varphi[x(t)/\alpha]\psi[x(t)/\alpha]$, where $T_1 \leqq t \leqq t^*$, and integrate it over $[T_1, t^*]$ obtaining

$$\int_{T_1}^{t^*} \frac{x'(t)}{\varphi[x(t)/\alpha]\psi[x(t)/\alpha]}\, dt = \alpha \int_{x(T_1)/\alpha}^{x(t^*)/\alpha} \frac{dy}{\varphi(y)\psi(y)} \geqq$$

$$\geqq \begin{cases} \displaystyle\int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\psi[x(t)/\alpha]} \cdot \frac{1}{r(t)} \int_{t}^{t^*} p(u)\varphi(x[g(u)])\, du\, dt, & \text{if } 1 = l = n-1 \\ \displaystyle\frac{1}{(n-3)!}\int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\psi[x(t)/\alpha]} \left[\int_{T_1}^{t} \frac{(t-s)^{n-3}}{r(s)}\, ds\right] \int_{t}^{t^*} p(u)\varphi(x[g(u)])\, du\, dt, & \text{if } 1 < l = n-1 \\ \displaystyle\frac{1}{(l-1)!(n-2-l)!}\int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\psi[x(t)/\alpha]} (t-T_1)^{l-1} \int_{t}^{t^*} \frac{(s-t)^{n-2-l}}{r(s)} \cdot \\ \qquad \cdot \displaystyle\int_{s}^{t^*} p(u)\varphi(x[g(u)])\, du\, ds\, dt, & \text{if } l < n-1 \end{cases}$$

$$=\begin{cases}\displaystyle\int_{T_1}^{t^*} p(u)\int_{T_1}^{t}\frac{1}{r(t)}\cdot\frac{\varphi(x[g(u)])}{\varphi[x(t)/\alpha]}\cdot\frac{1}{\psi[x(t)/\alpha]}\,dt\,du\,, & \text{if } 1=l=n-1\\[2ex] \displaystyle\frac{1}{(n-3)!}\int_{T_1}^{t^*} p(u)\int_{T_1}^{u}\left[\int_{T_1}^{t}\frac{(t-s)^{n-3}}{r(s)}\,ds\right]\cdot\frac{\varphi(x[g(u)])}{\varphi[x(t)/\alpha]}\cdot\frac{1}{\psi[x(t)/\alpha]}\,dt\,du\,, & \text{if } 1<l=n-1\\[2ex] \displaystyle\frac{1}{(l-1)!\,(n-2-l)!}\int_{T_1}^{t^*} p(u)\int_{T_1}^{u}(t-T_1)^{l-1}\left[\int_{t}^{u}\frac{(s-t)^{n-2-l}}{r(s)}\,ds\right]\cdot\frac{\varphi(x[g(u)])}{\varphi[x(t)/\alpha]}\cdot\frac{1}{\psi[x(t)/\alpha]}\,dt\,du\,, & \text{if } l<n-1\end{cases}$$

$$\geqq\begin{cases}\displaystyle\int_{T_2}^{t^*} p(u)\int_{T_1}^{g(u)}\frac{1}{r(t)}\cdot\frac{\varphi(x[g(u)])}{\varphi[x(t)/\alpha]}\cdot\frac{1}{\psi[x(t)/\alpha]}\,dt\,du\,, & \text{if } 1=l=n-1\\[2ex] \displaystyle\frac{1}{(n-3)!}\int_{T_2}^{t^*} p(u)\int_{T_1}^{g(u)}\left[\int_{T_1}^{t}\frac{(t-s)^{n-3}}{r(s)}\,ds\right]\cdot\frac{\varphi(x[g(u)])}{\varphi[x(t)/\alpha]}\cdot\frac{1}{\psi[x(t)/\alpha]}\,dt\,du\,, & \text{if } 1<l=n-1\\[2ex] \displaystyle\frac{1}{(l-1)!\,(n-2-l)!}\int_{T_2}^{t^*} p(u)\int_{T_1}^{g(u)}(t-T_1)^{l-1}\left[\int_{t}^{g(u)}\frac{(s-t)^{n-2-l}}{r(s)}\,ds\right]\cdot\frac{\varphi(x[g(u)])}{\varphi[x(t)/\alpha]}\cdot\frac{1}{\psi[x(t)/\alpha]}\,dt\,du\,, & \text{if } l<n-1\,.\end{cases}$$

But, for every u, t with $T_2 \leqq u \leqq t^*$ and $T_1 \leqq t \leqq g(u)$ we have

$$x[g(u)] \geqq x(t) \geqq x(t)/\alpha\,, \qquad R_1[g(u)] \geqq R_1(t) \geqq x(t)/\alpha$$

and therefore, by (II), we obtain

$$\varphi(x[g(u)]) \geqq \varphi[x(t)/\alpha]\,, \qquad \psi(R_1[g(u)]) \geqq \psi[x(t)/\alpha]\,.$$

Thus, we get

$$\alpha\int_{x(T_1)/\alpha}^{x(t^*)/\alpha}\frac{dy}{\varphi(y)\psi(y)} \geqq$$

$$\geqq \begin{cases} \displaystyle\int_{T_2}^{t^*} \frac{1}{\psi(R_1[g(u)])} p(u) \left[\int_{T_1}^{g(u)} \frac{1}{r(t)} dt\right] du, & \text{if } 1 = l = n-1 \\ \displaystyle\frac{1}{(n-3)!}\int_{T_2}^{t^*} \frac{1}{\psi(R_1[g(u)])} p(u) \left[\int_{T_1}^{g(u)} \int_{T_1}^{t} \frac{(t-s)^{n-3}}{r(s)} ds\,dt\right] du, & \text{if } 1 < l = n-1 \\ \displaystyle\frac{1}{(l-1)!\,(n-2-l)!}\int_{T_2}^{t^*} \frac{1}{\psi(R_1[g(u)])} \cdot \\ \displaystyle\qquad \cdot\, p(u) \left[\int_{T_1}^{g(u)} (t-T_1)^{l-1} \int_{t}^{g(u)} \frac{(s-t)^{n-2-l}}{r(s)} ds\,dt\right] du, & \text{if } l < n-1 \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{(n-2)!}\int_{T_2}^{t^*} \frac{R_1[g(u);\, T_1]}{\psi(R_1[g(u)])} p(u)\,du, & \text{if } l = n-1 \\ \displaystyle\frac{1}{(n-2)!}\int_{T_2}^{t^*} \frac{R_2[g(u);\, T_1]}{\psi(R_1[g(u)])} p(u)\,du, & \text{if } l < n-1 \end{cases}$$

$$\geqq \begin{cases} \displaystyle\frac{\beta}{(n-2)!}\int_{T_2}^{t^*} \frac{R_1[g(u)]}{\psi(R_1[g(u)])} p(u)\,du, & \text{if } l = n-1 \\ \displaystyle\frac{\beta}{(n-2)!}\int_{T_2}^{t^*} \frac{R_2[g(u)]}{\psi(R_1[g(u)])} p(u)\,du, & \text{if } l < n-1. \end{cases}$$

We have thus proved that

$$\alpha \int_{x(T_1)/\alpha}^{x(t^*)/\alpha} \frac{dy}{\varphi(y)\,\psi(y)} \geqq \frac{\beta}{(n-2)!} \int_{T_2}^{t^*} \frac{R[g(u)]}{\psi(R_1[g(u)])} p(u)\,du$$

where $R = R_1$ for $l = n-1$, $R = R_2$ for $l < n-1$. Finally, since t^* is arbitrary with $t^* \geqq T_2$, by taking the limits as $t^* \to \infty$, we get

$$\alpha \int_{x(T_1)/\alpha}^{\infty} \frac{dy}{\varphi(y)\,\psi(y)} \geqq \frac{\beta}{(n-2)!} \int_{T_2}^{\infty} \frac{R[g(t)]}{\psi(R_1[g(t)])} p(t)\,dt.$$

This contradicts the conditions (II) and (C).

PROOF OF THEOREM 2. – Condition (C) implies (C_0) and hence, by Theorem 0, it suffices to prove that every unbounded nonoscillatory solution x of $(E, -1)$ satisfies one of (X_∞), $(X_{-\infty})$. Furthermore, the substitution $w = -x$ transforms $(E, -1)$

into an equation of the same form satisfying the assumptions of the theorem with the function $\hat{\varphi}$ in place of φ, where $\hat{\varphi}(y) = -\varphi(-y)$ for all $y \in \operatorname{dom}\varphi$. Thus, with respect to the nonoscillatory solutions of $(E, -1)$ we can confine our discussion only to the positive ones.

Let now x be a positive unbounded solution on an interval $[\tau_0, \infty)$, $\tau_0 > t_0$, of the equation $(E, -1)$. Moreover, let $\tau \geqq \tau_0$ be chosen so that

$$g_j(t) \geqq \tau_0 \quad \text{for every } t \geqq \tau \ (j = 1, ..., m).$$

Then, as in the proof of Theorem 1, we conclude that $[rx^{(n-1)}]'$ is nonnegative on $[\tau, \infty)$ and not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$. Thus, by Lemma, there exists a $T \geqq \tau$ and an integer l, $0 \leq l \leq n$, with $n + l$ even so that

$$\begin{cases} l < n-1 \Rightarrow (-1)^{l+j} x^{(j)}(t) > 0 & \text{for every } t \geqq T \ (j = l, ..., n-1) \\ l > 1 \Rightarrow x^{(i)}(t) > 0 & \text{for every } t \geqq T \ (i = 1, ..., l-1)\,. \end{cases}$$

The unboundedness of x ensures that $l \geqq 1$. Also, since $n + l$ is even, we always have $l \neq n-1$. So, we consider the following two cases.

Case 1. $1 \leqq l < n-1$. Let $T_1 \geqq T$ be chosen so that

$$g_j(t) \geqq T \quad \text{for every } t \geqq T_1 \ (j = 1, ..., m).$$

Then, following the arguments used in the proof of Theorem 1, for every t, t^* with $T_1 \leqq t \leqq t^*$ we obtain

$$(*) \qquad x'(t) \geqq \frac{1}{(l-1)!\,(n-2-l)!}(t - T_1)^{l-1} \int_t^{t^*} \frac{(s-t)^{n-2-l}}{r(s)} \int_s^{t^*} p(u)\varphi(x[g(u)])\,du\,ds\,.$$

Furthermore, by using the Taylor formula with integral remainder, for $t \geqq T_1$ we get

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-2} \frac{x^{(k)}(T_1)}{k!}(t - T_1)^k + \frac{1}{(n-2)!}\int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)}[r(s)x^{(n-1)}(s)]ds \\ &\leqq \sum_{k=0}^{n-2} \frac{x^{(k)}(T_1)}{k!}(t - T_1)^k + \frac{1}{(n-2)!}\int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)}[-r(s)x^{(n-1)}(s)]\,ds \\ &\leqq \sum_{k=0}^{n-2} \frac{x^{(k)}(T_1)}{k!}(t - T_1)^k + \frac{-r(T_1)x^{(n-1)}(T_1)}{(n-2)!}\int_{T_1}^{t} \frac{(t-s)^{n-2}}{r(s)}ds\,. \end{aligned}$$

Thus, as in the proof of Theorem 1, there exists a constant $\alpha \geqq 1$ so that

$$x(t) \leqq \alpha R_1(t) \quad \text{for every } t \geqq T_1.$$

Next, we define

$$R_2(t;T_1)=\int_{T_1}^{t}\frac{(s-T_1)^{n-2}}{r(s)}ds,\qquad t\geqq T_1$$

and, as in the proof of Theorem 1, we conclude that for some $\hat{T}_1>T_1$

$$R_2(t;T_1)\geqq\beta R_2(t)\qquad\text{for all } t\geqq\hat{T}_1,$$

where β is a positive constant. Furthermore, we choose a $T_2\geqq\hat{T}_1$ such that

$$g_j(t)\geqq\hat{T}_1\qquad\text{for every } t\geqq T_2\ (j=1,\ ,m).$$

Then the same arguments as in the proof of Theorem 1 lead to

$$\alpha\int_{x(T_1)/\alpha}^{x(t^*)/\alpha}\frac{dy}{\varphi(y)\psi(y)}\geqq\frac{\beta}{(n-2)!}\int_{T_2}^{t^*}\frac{R_2[g(u)]}{\psi(R_1[g(u)])}p(u)du$$

for all $t^*\geqq T_2$. Letting $t^*\to\infty$ in this inequality, we obtain

$$\alpha\int_{x(T_1)/\alpha}^{\infty}\frac{dy}{\varphi(y)\psi(y)}\geqq\frac{\beta}{(n-2)!}\int_{T_1}^{\infty}\frac{R_2[g(t)]}{\psi(R_1[g(t)])}p(t)dt,$$

which, by (II) and (*C*), is a contradiction.

Case 2. $l=n$. By using the Taylor formula with integral remainder, for $t\geqq T$ we obtain

$$x(t)=\sum_{i=0}^{n-2}\frac{(t-T)^i}{i!}x^{(i)}(T)+\frac{1}{(n-2)!}\int_{T}^{t}\frac{(t-s)^{n-2}}{r(s)}[r(s)x^{(n-1)}(s)]ds$$
$$\geqq\frac{r(T)x^{(n-1)}(T)}{(n-2)!}\int_{T}^{t}\frac{(t-s)^{n-2}}{r(s)}ds.$$

Thus, for some $\tau_1>T$ we have

$$x(t)\geqq\gamma R_1(t)\qquad\text{for all } t\geqq\tau_1,$$

where γ is a positive constant. Furthermore, we choose a $\tau_2\geqq\tau_1$ so that

$$g_j(t)\geqq\tau_1\qquad\text{for every } t\geqq\tau_2\ (j=1,\ldots,m).$$

Then, by taking into account (ii) and the fact that x is increasing on $[T, \infty)$, for $t \geqq \tau_2$ we get

$$[r(t)x^{(n-1)}(t)]' = f(t; x[g_1(t)], \dots, x[g_m(t)]) \geqq f(t; x[g(t)], \dots, x[g(t)]) \geqq f(t; \gamma R_1[g(t)], \dots, \gamma R_1[g(t)]) .$$

Therefore, because of (H), we obtain

$$[r(t)x^{(n-1)}(t)]' \geqq p(t)\varphi(\gamma R_1[g(t)]) , \qquad t \geqq \tau_2 .$$

But, by (II), it is easy to derive that

$$\lim_{y\to\infty} \frac{\varphi(y)\psi(y)}{y} = \infty$$

and hence

$$\varphi(y) \geqq \frac{y}{\psi(y)} \qquad \text{for all large } y .$$

Thus, for some $\tau_3 \geqq \tau_2$ we have

$$\varphi(\gamma R_1[g(t)]) \geqq \frac{\gamma R_1[g(t)]}{\psi(\gamma R_1[g(t)])} \geqq \gamma \frac{R_1[g(t)]}{\psi(R_1[g(t)])}$$

for all $t \geqq \tau_3$, where the constant γ is considered to be chosen so that $0 < \gamma \leqq 1$. So, for every $t \geqq \tau_3$

$$[r(t)x^{(n-1)}(t)]' \geqq \gamma \frac{R_1[g(t)]}{\psi(R_1[g(t)])} p(t)$$

and therefore, by integration, we obtain that for $t \geqq \tau_3$

$$r(t)x^{(n-1)}(t) \geqq r(\tau_3)x^{(n-1)}(\tau_3) + \gamma \int_{\tau_3}^{t} \frac{R_1[g(s)]}{\psi(R_1[g(s)])} p(s)ds .$$

Hence, by condition (C), we get

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t) = \infty .$$

This, in view of (i), gives

$$\lim_{t\to\infty} x^{(i)}(t) = \infty \qquad (i = 0, 1, \dots, n-2)$$

and consequently the solution x satisfies (X_∞).

Next, let us consider the special case where $r = 1$, i.e. the differential equation

$$(\tilde{E}, \delta) \qquad x^{(n)}(t) + \delta f(t; x[g_1(t)], \ldots, x[g_m(t)]) = 0\,, \quad t \geqq t_0,$$

where there is no loss of generality to suppose $t_0 > 0$. Then we have

$$R_1(t) = R_2(t) = \frac{1}{n-1}(t - t_0)^{n-1}\,, \quad t \geqq t_0$$

and consequently there exist two constants c_1, c_2 with $0 < c_1 \leqq c_2 \leqq 1$ so that

$$0 < c_1 t^{n-1} \leqq R_1(t) = R_2(t) \leqq c_2 t^{n-1} \quad \text{for all large } t.$$

Thus, by (iii) and (II), we obtain that for all large t

$$R_1]g(t)] = R_2[g(t)] \geqq c_1[g(t)]^{n-1}\,, \quad \psi(R_1[g(t)]) \leqq \psi(c_2[g(t)]^{n-1}) \leqq \psi([g(t)]^{n-1})\,.$$

Therefore for $k = 1, 2$

$$\frac{R_k[g(t)]}{\psi(R_1[g(t)])} \geqq c_1 \frac{[g(t)]^{n-1}}{\psi([g(t)]^{n-1})} \quad \text{for all large } t\,.$$

Hence, in the considered case the condition (C) follows from the following one

$$(\tilde{C}) \qquad \int^{\infty} \frac{[g(t)]^{n-1}}{\psi([g(t)]^{n-1})}\, p(t)\, dt = \infty\,.$$

From Theorem 1, by applying it for the differential equation $(\tilde{E}, +1)$, we obtain the main result of a recent paper by KITAMURA and KUSANO [5]. The method used in proving Theorems 1 and 2 patterns after that of Kitamura and Kusano in the paper mentioned above..

We now turn our attention to differential equations of the form

$$(D) \qquad [r(t)\, x^{(n-1)}(t)]' + a(t)\, \Phi(x[g_1(t)], \ldots, x[g_m(t)]) = 0\,,$$

where a is a continuous real-valued function on the interval $[t_0, \infty)$ and Φ is a continuous real-valued function defined at least on $R_+^m \cup R_-^m$. Foı the functions a, Φ we introduce the conditions:

(iv) *a is of constant sign on $[t_0, \infty)$.*

(iv) *Φ is increasing on $R_+^m \cup R_-^m$ and has the sign property*

$$\Phi(y) > 0 \quad \textit{for all } y \in R_+^m\,, \qquad \Phi(y) < 0 \quad \textit{for all } y \in R_-^m\,.$$

We have the following corollary.

COROLLARY. – *Let the conditions* (i), (iii), (iv) *and* (v) *be satisfied. Moreover, let the differential equation* (*D*) *be strongly superlinear in the sense that*

$$\text{(III)} \qquad \int^{\infty} \frac{dy}{\Phi(y, \ldots, y)} < \infty \quad \textit{and} \quad \int^{-\infty} \frac{dy}{\Phi(y, \ldots, y)} < \infty .$$

Then, under the condition

$$(A) \qquad \int^{\infty} R_k[g(t)]|a(t)|dt = \infty \qquad (k = 1, 2),$$

we have the following:

α_1) *For a nonnegative and n even, all solutions of* (*D*) *are oscillatory.*

β_1) *For a nonnegative and n odd, every solution x of* (*D*) *is oscillatory or satisfies* (X_0).

α_2) *For a nonpositive and n even, every solution x of* (*D*) *is oscillatory or satisfies one of* (X_0), (X_∞), ($X_{-\infty}$).

β_2) *For a nonpositive and n odd, every solution x of* (*D*) *is oscillatory or satisfies one of* (X_∞), ($X_{-\infty}$).

PROOF. – The differential equation (*D*) is of the form (*E*, δ) with $\delta = +1$ for $a \geqq 0$ or $\delta = -1$ for $a \leqq 0$, and $f(t; y) = |a(t)|\Phi(y)$ for $(t; y) \in [t_0, \infty) \times \operatorname{dom} \Phi$. By (v), the function f satisfies (ii). Next, we define

$$\begin{aligned} p(t) &= |a(t)| && \text{for } t \geqq t_0, \\ \psi(y) &= 1 && \text{for } y > 0, \\ \varphi(y) &= \Phi(y, \ldots, y) && \text{for } y \neq 0 . \end{aligned}$$

Then, by taking into account (v) and (III), we can see that (I), (II) and (*H*) are satisfied. Also, (*C*) reduces to (*A*). Hence, the corollary follows immediately from Theorems 1 and 2.

For ordinary or advanced differential equations of the form (*D*) the condition (*A*) becomes

$$(A^*) \qquad \int^{\infty} R_k(t)|a(t)|dt = \infty \qquad (k = 1, 2).$$

When the equation (*D*) is an equation of retarded or mixed type, our corollary, in general, ceases to hold if the condition (*A*) is replaced by (*A**). This is illustrated by the following four examples of retarded differential equations. These equations fail to satisfy (*A*). However, they satisfy the rest of the conditions of Corollary and the condition (*A**).

EXAMPLE 1. – The equation

$$[t^{1/3}x'(t)]' + \frac{1}{12}t^{-5/3}x^3(t^{1/3}) = 0\,, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{1/2}$, a contradiction to conclusion α_1) of Corollary.

EXAMPLE 2. – The equation

$$[t^{1/3}x''(t)]' + \frac{1}{8}t^{-8/3}x^3(t^{1/3}) = 0\,, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{3/2}$ for which we have $\lim_{t\to\infty} x(t) = \infty$, a contradiction to conclusion β_1) of Corollary.

EXAMPLE 3. – The equation

$$[t^{1/2}x'''(t)]' - \frac{3}{8}t^{-7/2}x^3(t^{1/3}) = 0\,, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{3/2}$ for which we have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \infty$ while $\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} t^{1/2}x'''(t) = 0$, a contradiction to conclusion α_2) of Corollary.

EXAMPLE 4. – The equation

$$[t^{1/2}x''(t)]' - \frac{1}{4}t^{-5/2}x^3(t^{1/3}) = 0\,, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{1/2}$ for which we have $\lim_{t\to\infty} x(t) = \infty$ while $\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} t^{1/2}x''(t) = 0$, a contradiction to conclusion β_2) of Corollary.

Differential equations of the form (D) subject to the condition

$$\text{(IV)} \qquad \liminf_{t\to\infty} \frac{R_k[g(t)]}{R_k(t)} > 0 \qquad (k = 1, 2)$$

include obviously the ordinary, advanced equations and some other ones of retarded or mixed type. For such equations the condition (A) is equivalent to (A^*) and hence our corollary leads to the following result

Let the conditions (i), (iii), (iv), (v), (III) *and* (IV) *be satisfied. Then, under the condition* (A^*), *we have the conclusion of Corollary.*

Note that the condition (IV) cannot be removed from this result, as it is demonstrated by Examples 1-4 of retarded equations for which (IV) fails while all other assumptions are satisfied.

Moreover, we notice that in the special case where $r = 1$ the condition (IV) is satisfied if

$$(\widetilde{\text{IV}}) \qquad \liminf_{t\to\infty} \frac{g_j(t)}{t} > 0 \qquad (j = 1, \ldots, m)$$

and the condition (A^*) becomes

$$(\tilde{A}^*) \qquad \int^{\infty} t^{n-1}|a(t)|\,dt = \infty .$$

Indeed, in this case we have

$$R_1(t) = R_2(t) = \frac{1}{n-1}(t-t_0)^{n-1}, \qquad t \geqq t_0$$

and, provided that (IV) holds,

$$g(t) = ct \qquad \text{for all large } t ,$$

where

$$c = \min\left\{1, \frac{1}{2}\liminf_{t\to\infty}\frac{g_1(t)}{t}, \ldots, \frac{1}{2}\liminf_{t\to\infty}\frac{g_m(t)}{t}\right\} > 0 .$$

Next, let us consider the special case of the differential equation

$$(\tilde{D}) \qquad x^{(n)}(t) + a(t)\,\Phi\big(x[g_1(t)], \ldots, x[g_m(t)]\big) = 0 ,$$

which is obtained from (D) for $r = 1$. For this equation the condition (A) reduces to

$$(\tilde{A}) \qquad \int^{\infty} [g(t)]^{n-1}|a(t)|\,dt = \infty$$

and our corollary leads to the following result:

Let the conditions (iii), (iv), (v) *and* (III) *be satisfied. Then, under the condition* $(\tilde{A})$, *we have the conclusion of Corollary for the differential equation* $(\tilde{D})$.

STAIKOS [19, 20] proved the following result:

Let the conditions (iii), (iv), (v) *and* (III) *be satisfied. Moreover, let* σ *be a continuously differentiable and increasing function on the interval* $[t_0, \infty)$ *with* $\lim_{t\to\infty}\sigma(t) = \infty$ *and such that*

$$\sigma(t) \leqq \min\{t, g_1(t), \ldots, g_m(t)\} , \qquad t \geqq t_0 .$$

Then, under the condition

$$\int^{\infty} [\sigma(t)]^{n-1}|a(t)|\,dt = \infty ,$$

we have the conclusion of Corollary for the equation $(\tilde{D})$.

Our result given above is a substantial improvement of this Staikos' result. This is illustrated by the following example due to KITAMURA and KUSANO [5].

EXAMPLE 5. – For the differential equation

$$x''(t) + t^{-7/4}x^3[t + (t - t^{1/2}) \sin t] = 0 , \qquad t \geqq 1$$

we can apply our result (cf. [5]) to conclude that all solutions are oscillatory. On the other hand, the Staikos' result cannot be applied (cf. [5]) for this equation.

Finally, it remains an open question to the author if the results of this paper can be extended for more general differential equations of the form

$$\Big[r_{n-1}(t)\Big[r_{n-2}(t)\Big]\ldots\big[r_2(t)\,[r_1(t)\,x'(t)]'\big]'\ldots\Big]'\Big]' + \delta f(t;\, x[g_1(t)], \ldots, x[g_m(t)]) = 0 ,$$

where r_i $(i = 1, \ldots, n-1)$ are positive continuous functions on the interval $[t_0, \infty)$ such that

$$\int^{\infty} \frac{dt}{r_i(t)} = \infty \qquad (i = 1, \ldots, n-1) .$$

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