Fixed Point Theorems for Multivalued Weighted maps (*).

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Sunto. – *Si dimostra un teorema di punto unito per una classe abbastanza ampia di mappe plurivalenti con pesi fra gli* ANR *metrici. Questo teorema contiene tanto i classici teoremi di punto fisso di Eilenberg-Montgomery, O' Neill, Darbo come i risultati di connettività di insiemi di punti fissi di famiglie parametrizzate di mappe dovuti a Leray e Browder.*

Introduction.

It is well known that a multivalued uppersemicontinuous map which is *acyclic* in positive dimension $\big(i.e.\ H_n(F(x)) = 0,\ n > 0,\ \forall x\big)$ but not necessarily connected-valued may not carry in general a nontrivial homology homomorphism. In fact R. DUNN [7] has shown that given any set S of natural numbers different from the following ones: $\{n, 2\}$, $\{n, 1\}$, $\{n\}$, there exists a continuous finitevalued *fixed point free* map F of the 2-cell B into itself such that the number of points of $F(x)$ belongs to S for each x. Clearly, such a map cannot carry a nontrivial homomorphism in $H_0(B)$. In order to avoid this difficulties, G. DARBO in [3] introduced the concept of weighted carrier (w-carrier) which is, roughly speaking, an uppersemicontinuous multivalued map F such that to each piece (*i.e.* clopen subset) of $F(x)$ a multiplicity (or weight) that is an additive function of pieces is assigned. Moreover, this multiplicity verifies a local invariance property of the same type as the multiplicity of polynomial roots. Namely; if the boundary of an open set U in the range of F does not intersect $F(x)$, then the multiplicity of $F(x)$ in U equals the multiplicity of $F(x')$ in U whenever x' is close enough to x.

In the same paper he showed that an acyclic (in positive dimension) w-carrier from the 2-cell into itself having non zero index admits a fixed point. This result has been improved to n-cell by DAL SOGLIO ([2]).

In [4] G. DARBO introduced the category of weighted maps (that is finite-valued w-carriers) and defined a homology functor $\mathcal{H}$ in this category such that whenever restricted to the category of continuous maps between ANR's it coincides with the singular homology. By means of this functor he gave a generalisation of the Lefschetz fixed point theorem for weighted maps defined in compact metric ANR's (see [6]).

In this note we shall improve the Darbo's result to a Lefschetz fix-point theorem for Lefschetz w-carriers between ANR's. This theorem generalizes several

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well known fixed point theorems for set valued maps [8], [10], [19], and improves our earlier result [15].

To this aim we shall show that acyclic w-carriers between compact polyhedra induce a *well defined* homomorphism in the homology $\mathcal{H}$ constructed in [4]. Moreover this homomorphism is invariant by acyclic w-carrier homotopies. In the context of set-valued maps such a property in general cannot be stated (see [19], [13], [11]).

The main difference with the classical approach in defining the induced homomorphism is that we approximate the acyclic w-carrier F by w-maps instead constructing chain maps. Using the fact that w-maps close enough to F induce the same homomorphism in Darbo's homology, we define the homomorphism induced by F as the homomorphism induced by a w-map φ close enough to F. In the proof of this fact we use several deep properties of the homotopy theory of w-maps proved in [16].

This paper is divided as follows.

In Section 1 we define w-carriers and we give some of their elementary properties. Sections 2 and 3 are devoted to showing that several set valued maps which appear in concrete geometric problems are w-carriers. In Section 4 we state our main result (Theorem 4.1) and define the homomorphiem induced in homology by an acyclic w-carrier by means of which a Lefschetz fixed point theorem for acyclic w-carriers from a finite polyhedron into itself is proved. In Section 5 we extend this theorem to compact w-carriers of a complete metric ANR of the form $f \circ F$ with f a single-valued continuous map and F an acyclic w-carrier. Several consequences of this result are deduced. Furthermore we apply our results to derive the existence of a closed trajectory of a vector field in a full torus following an approach due to Fuller ([9]). Section 6 is entirely devoted to the proof of Theorem 4.1. The appendix contains the construction of the intersection index used in Sections 2 and 3. There is a variety of equivalent definitions of such an index. The main difference in our approach consists in dropping compactness assumptions

1. – Preliminaries and definitions.

Let X be a regular topological space. A *piece* of X is any open and closed subset of X. Clearly, the family $\mathcal{P}(X)$ of all pieces of X is closed under finite unions and intersections.

If K is a subset of X and U is an open subset of X such that $\partial U \cap K = \emptyset$, then $U \cap K$ is a piece of K. Conversely, if K is compact, by the regularity of X, it follows that any piece C of K is of the form $C = U \cap K$, with U open in X and $\partial U \cap K = \emptyset$.

Let us recall that a multivalued map $F: X \to Y$ is called *uppersemicontinuous* if

i) $F(x)$ is a compact subset of Y for any $x \in X$,

ii) if $C \subset Y$ is closed, then $F^{-1}(C) = \{x \in X: F(x) \cap C \neq \emptyset\}$ is closed.

DEFINITION 1.1. – A multivalued uppersemicontinuous map $F\colon X \to Y$ will be called a *weighted carrier* (*w-carrier*) if, for any $x \in X$, to any piece C of $F(x)$ is assigned a *weight* or *multiplicity* $\tilde{\imath}(C, F(x))$ belonging to some commutative ring $\boldsymbol{R}$ verifying the following conditions

a) $\tilde{\imath}(\cdot, F(x))$ is an additive function on the set $\mathcal{T}(F(x))$, *i.e.* $\tilde{\imath}(C_1 \cup C_2, F(x)) = \tilde{\imath}(C_1, F(x)) + \tilde{\imath}(C_2, F(x))$ whenever $C_1 \cap C_2 = \emptyset$;

b) if U is open in Y, with $F(x) \cap \partial U = \emptyset$, we have that $\tilde{\imath}(F(x) \cap U, F(x)) = \tilde{\imath}(F(x') \cap U, F(x'))$ whenever x' is close enough to x.

REMARK 1.1. – We say that an open subset U of Y is *admissible* for $F(x)$ if $F(x) \cap \partial U = \emptyset$. When this occurs, the multiplicity $\tilde{\imath}(F(x) \cap U, F(x))$ of the piece $F(x) \cap U$ of $F(x)$ will be called the *multiplicity* or the *index* of $F(x)$ in U and will be denoted by $\tilde{\imath}(U, F(x))$.

Notice that by the uppersemicontinuity of F we have that if U is admissible for $F(x)$, then it is also admissible for $F(x')$ with x' close enough to x. Thus property b) can be seen as a local invariance condition of the multiplicity $\tilde{\imath}(U, F(x))$. It states that if U is admissible for $F(x)$, the multiplicity of $F(x)$ in U equals the multiplicity of $F(x')$ in U whenever x' is close enough to x.

Notice also that the multiplicity $\tilde{\imath}(Y, F(x))$ does not depend on $x \in X$ if the space X is connected. In this case this element will be called the *index* of F and we denote it by $\tilde{\imath}F$.

The following properties of w-carriers can be easily verified

a) Let $F\colon X \to Y$ be uppersemicontinuous and suppose that $F(x)$ is connected for every $x \in X$. Then F becomes a w-carrier by assigning multiplicity $1 \in \boldsymbol{R}$ to $F(x)$. In particular any continuous singlevalued map is a w-carrier.

b) Let $F\colon X \to Y$ be a w-carrier and $f\colon Z \to X$ (resp. $f'\colon Y \to W$) be a singlevalued continuous map. Then $F \circ f$ (resp. $f' \circ F$) is a w-carrier.

c) Let $F\colon X \to Y$ be a w-carrier and $f\colon Z \to W$ be a continuous single-valued map. Then $F \times f\colon X \times Z \to Y \times W$ is a w-carrier.

d) If $F\colon X \to Y$ is a w-carrier, then the graph map $G_F\colon X \to X \times Y$, $G_F(x) = \{(x, y)\colon y \in F(x)\}$ is a w-carrier.

e) Let F, $T\colon X \to Y$ be w-carriers such that $F(x) \cap T(x) = \emptyset$, $\forall x \in X$. The sum $F \oplus T\colon X \to Y$ is defined as the multivalued map $x \rightsquigarrow F(x) \cup T(x)$ with multiplicities assigned as follows: if C is a piece of $F(x) \cup T(x)$ then

$$\tilde{\imath}(C, F \oplus T(x)) = \tilde{\imath}(C \cap F(x), F(x)) + \tilde{\imath}(C \cap T(x), T(x)) .$$

f) Let $F\colon X \to Y$ be a w-carrier and let W be an open set such that $\partial W \cap F(x) = \emptyset$, $\forall x \in X$. Then F can be decomposed as $F = F_1 \oplus F_2$ with

$$F_1(x) = W \cap F(x) \quad \forall x \in X; \qquad F_2(x) = W^c \cap F(x) \quad \forall x \in X.$$

2. – Multiplicity function versus index.

Let X be a regular space and let $C \subset X$ be a compact subset. We denote by $\mathcal{U}_C$ the family of all open subsets of X which are admissible for C. By a multiplicity function on C we mean any additive function $\tilde{\imath}$ defined on the family $\mathcal{T}(C)$ of all pieces of C with values in $\boldsymbol{R}$ (alternatively $\tilde{\imath}$ can be viewed as an element of the Čech-cohomology module $\check{\mathrm{H}}^0(C, \boldsymbol{R})$).

Each multiplicity function $\tilde{\imath}$ on C can be extended to an additive function $\tilde{\imath}$ defined on the family $\mathcal{U}_C$ of all admissible open sets for C by defining

$$\tilde{\imath}(U) = \tilde{\imath}(C \cap U) \qquad \forall U \in \mathcal{U}_C .$$

Moreover, $\tilde{\imath}$ satisfies the following *excision* property:

If U_1 is an open subset of U with $C \cap (\overline{U} - U_1) = \emptyset$, then

$$U_1, \quad U \in \mathcal{U}_C \quad \text{and} \quad \tilde{\imath}(U_1) = \tilde{\imath}(U) .$$

Conversely, any additive function $\tilde{\imath}$ defined on $\mathcal{U}_C$ that verifies the excision property (such a $\tilde{\imath}$ is usually called «index») induces a multiplicity function on C as follows.

If S is any piece of C, by the regularity of X, S is of the form $S = C \cap U$ with $U \in \mathcal{U}_C$. Therefore we take $\tilde{\imath}(S) = \tilde{\imath}(U)$. By the additivity and excision properties $\tilde{\imath}(S)$ does not depend on the choice of U. Hence, $\tilde{\imath}$ is well defined and it is clearly additive. As a concluding remark we observe that if C is a finite subset of X, a multiplicity function on C is actually a function from C to $\boldsymbol{R}$.

Moreover, if $U \in \mathcal{U}_C$ we have that

$$\tilde{\imath}(U) = \sum_{x \in C \cap U} \tilde{\imath}(x) .$$

We shall see now how the indexes appear in the manifold setting.

Let M, N be topological manifolds of dimension m and n respectively. Let $L \subset N$ be a closed connected submanifold of dimension p with $m - p = n$. Suppose that M, N, L are oriented over $\boldsymbol{R}$. Let $f\colon M \to N$ be a continuous map such that the set $f^{-1}(L) = \{m : f(m) \in L\}$ is compact. For any open subset U of M which is admissible for $f^{-1}(L)$ we can assign an element $\tilde{\imath}(f, L, U)$ of R called the *intersection index* of f with L in U, such that the following properties are verified:

1) additivity if $U_1 \cap U_2 = \emptyset$ then

$$\tilde{\imath}(f, L, U_1 \cup U_2) = \tilde{\imath}(f, L, U_1) + \tilde{\imath}(f, L, U_2) ;$$

2) excision: if $U_1 \subset U$, $f(\overline{U} - U_1) \cap L = \emptyset$ then $\tilde{\imath}(f, L, U_1) = \tilde{\imath}(f, L, U)$;

3) homotopy invariance: if $h: [0,1] \times M \to N$ is such that $h^{-1}(L)$ is compact and U is admissible for h_t $\forall t \in [0,1]$ then

$$\tilde{\imath}(h_0, L, U) = \tilde{\imath}(h_1, L, U).$$

As consequence of 1), 2) we get

4) if $\tilde{\imath}(f, L, U) \neq 0$ then $f(U) \cap L \neq \emptyset$.

The construction of such an index will be sketched in the Appendix.

Examples:

a) If M, N are of the same dimension and L is a point $p \in N$, then $\tilde{\imath}(f, L, U)$ is the local degree of f at p (see [7]).

b) If $N = M \times M$ and L is the diagonal Δ of $M \times M$, then the set $f^{-1}(L)$ is the fixed point set of f and $\tilde{\imath}(f, \Delta, U)$ coincides with the usual fix-point index $i(f, U)$ (see [7]).
More generally, the coincidence index of two maps f, $g: M \to N$ can be described in the same way.

c) Let S be any orientable C^1-manifold. Let $\mathcal{T}(S)$ be its tangent bundle. Given any vector field $X: S \to \mathcal{T}(S)$, let $K(X)$ be the set ofc ritical points of X (*i.e.* $K(X) = \{s: X(s) = 0\}$). Let L be the zero section of $\mathcal{T}(S)$. For any admissible open set U the index $\tilde{\imath}(X, L, U)$ is just the index of critical points of X in U.

3. – Examples of w-carriers.

By a parametrized family of mappings from M to N we mean a continuous map $f: M \times P \to N$, where P is any locally path connected space. The following theorem shows that the solution set of a parametrized family of equations on manifolds is the graph of a w-carrier.

THEOREM 3.1. – *Let M^m, N^n, L^p as in 2. Let $f: M \times P \to N$ be a parametrized family of maps such that for any compact set $B \subset P$ we have that $\bigcup_{p \in B} f_p^{-1}(L)$ is relatively compact in M.*

Then the multivalued map S given by $S(p) = f_p^{-1}(L)$ is a w-carrier from P into M.

PROOF. – It is well known that a multivalued map is uppersemicontinuous if and only if it has a closed graph and it sends compact subsets of the domain into relatively compact subsets of the range. Now S has a closed graph because its graph is $\{(m, p): f(m, p) \in L\} = f^{-1}(L)$. The second condition is supplied by the hypothesis, hence S is uppersemicontinuous.

Fix $p \in P$. If U is an open admissible set for $S(p)$, we can define $\tilde{\imath}(U, S(p))$ as the intersection index $\tilde{\imath}(f_p, L, U)$ of Section 2.

Since this index verifies the additivity and excision properties, it follows from the discussion made in Section 2 that this index induces a multiplicity function on $S(p)$ for each $p \in P$. Thus *a*) of Definition 1.1 is verified.

To show that this multiplicity satisfies also *b*) let us observe that if U is admissible for $S(p)$ then by uppersemicontinuity of S and since P is locally connected, there exists a path connected neighborhood W of p such that U is admissible for $S(q)$ for each $q \in W$. For a given $q \in W$ let us take a path $\xi(t)$ in W with $\xi(0) = p$, $\xi(1) = q$. Let us consider the homotopy $h_t = f_{\xi(\,)}: M \to N$ between f_p and f_q.

Since $\xi(t) \in W$ for each t, it follows that U is admissible for h_t, $\forall t \in [0, 1]$. From this fact, by the homotopy invariance of the index $\tilde{\imath}(h_t, L, U)$ we get

$$\tilde{\imath}(U, S(p)) = \tilde{\imath}(f_p, L, U) = \tilde{\imath}(f_q, L, U) = \tilde{\imath}(U, S(q)) .$$

This achieves *b*) of Definition 1.1.

As a consequence of Theorems 3.1 we get the following examples of w-carriers.

3.1. Let M, N be of the same dimension. If $f: M \times P \to N$ verifies the assumption of Th. 3.1 with respect to a point $n \in N$, then the equation $f(m, p) = n$ defines a unique w-carrier $S: P \to N$ with graph $= \{(m, p): f(m, p) = n\}$.

3.2. If $f: M \times P \to M$ is uniformly continuous and each f_p is compact, then the multivalued map $p \to \operatorname{Fix}(f_p) = \{m: f(m, p) = m\}$ is a w-carrier.

Using the Leray fixed point index we can extend the above example to parametric families of compact maps between ANR's.

3.3. Let M, N be of the same dimension and $f: M \to N$ be a proper map. Then the multivalued map $f^{-1}: N \to M$ is a w-carrier such that $f \circ f^{-1} = \deg(f)\, \mathrm{id}_N$. In fact, let $P = N$ and let us take the parametrized family $g: N \times M \to N \times N$, $g(n, m) = (n, f(m))$. Then, if L is the diagonal in $N \times N$ we get that $g_n^{-1}(L) = f^{-1}(n)$. Now the assertion follows from 3.1.

3.4. Let S be a compact C^1-manifold, $\mathcal{T}(S)$ be its tangent bundle, $\Gamma(S)$ be the space of all C^0 vector fields on S endoved with the compact-open topology. Then the multivalued map that assigns to each vector field X the set $K(X)$ of the critical points of X is a w-carrier from $\Gamma(S)$ into S. This follows by taking $\Gamma(S)$ as parameter space and using the critical point index defined in *c*) of Section 2.

3.5. Let S be a C^1-manifold, $X: S \to \mathcal{T}(S)$ be a complete vector field and let $\varphi: S \times R \to S$ be the flow of X.

Let $L \subset S$ be a submanifold of codimension one. We shall denote by $\Gamma(s)$ the trajectory of X passing by s at the time 0.

Suppose that $\{t: \varphi(s, t) \in L, s \in K\}$ is compact for any compact set K. The multivalued map $s \rightsquigarrow \Gamma(s) \cap L$ is a w-carrier from S to L. In fact, by Theorem 3.1 the multivalued map $\tau(s) = \{t: \varphi(s, t) \in L\}$ is a w-carrier from S to R. On the other hand $\Gamma(\cdot) \cap L$ coincides with

$$S \xrightarrow{\mathrm{id} \times \tau} S \times R \xrightarrow{\varphi} S .$$

4. – Approximation of w-carrier by w-maps and induced homomorphism.

By a *weighted map* (*w-map*) we mean any w-carrier $F: X \to Y$ such that $F(x)$ is a finite subset of Y for any $x \in X$.

Actually, as defined by G. DARBO ([4]), a w-map is an equivalence class of finite valued w-carriers, but the above definition is more adeguate to our purposes and it does not give any substantial difference with those of [4] (see also [18]). In particular all the results of [16] holds.

In order to avoid any possibl e confusion, w-maps will be denoted with greek letter s $\varphi, \psi, \ldots$. The composition of two w-maps is a w-map. Moreover, w-maps can be added and multiplied by scalars in $\boldsymbol{R}$. Actually, the set of all w-maps from X to Y have an $\boldsymbol{R}$-module structure compatible with the composition of w-maps. Hence the w-maps between Hausdorff spaces form an additive category which contains as a subcategory the topological one.

For any Hausdorff space X, G. DARBO has defined in [4] a graded homology module $\mathcal{H}(X) = \{\mathcal{H}_q(X)\}_{q \geqslant 0}$ such any w-map $\varphi: X \to Y$ induces in a functorial way a homomorphism $\varphi_*: \mathcal{H}_q(X) \to \mathcal{H}_q(Y)$. Moreover, φ_* is a σ-homotopy invariant of φ.

The functor $\mathcal{H}$ verifies all the axioms of a homology theory with compact supports. Therefore, it coincides with the singular homology with coefficients in $\boldsymbol{R}$ at least when X is a compact metric ANR.

Let X be a metric space and let D be a subset of X. We shall denote by εD its ε-neighborhood in X, that is

$$\varepsilon D = \{x \in X: d(x, D) < \varepsilon\} .$$

Let us denote by $\hat{\mathcal{H}}_q(D)$ the inverse limit over the family of ε-neighborhoods of D of $\mathcal{H}_q(\varepsilon D)$.

In general $\hat{\mathcal{H}}_q(D)$ depends on the shape of D in X, but, if X is an ENR, $D \subset X$ is a compact subset, then $\hat{\mathcal{H}}_q$ is just the q-th Čech homology module of D.

To see this we observe that $\forall \varepsilon > 0$, εD, being an open subset of an ENR, is an ENR. Hence, by the above discussion we have that $\forall \varepsilon > 0$ $\mathcal{H}_q(\varepsilon D) = H_q^{\mathrm{sing}}(\varepsilon D)$. Since, as it was already observed in [7], the Čech homology of a compact subset of an ENR can be obtained as the inverse limit of the singular homology of its neighborhoods, we get

$$\hat{\mathcal{H}}_q(D) = \lim.\mathrm{inv.}\ \mathcal{H}_q(\varepsilon D) = \lim.\mathrm{inv.}\ H_q^{\mathrm{sing}}(\varepsilon D) = \check{H}_q(D)\,.$$

DEFINITION 4.1. – A w-carrier $F: X \to Y$ will be called *acyclic* if $\forall x \in X\ \hat{\mathcal{H}}_q(F(x)) = 0$, $\forall q > 0$.

REMARK 4.1. – Notice that $\hat{\mathcal{H}}_0(F(x))$ need not be $\boldsymbol{R}$. For example, if each component of $F(x)$ is an acyclic set then F is an acyclic w-carrier. If $\boldsymbol{R} = \boldsymbol{Q}$, $\hat{\mathcal{H}}_q(F(x), \boldsymbol{Z})$ are finite groups $\forall q \geqslant 1$, $\forall x \in X$ then F is an acyclic w-carrier over the ring $\boldsymbol{Q}$. Also, since any open subset of the real line is a disjoint union of open intervals, it follow that $\hat{\mathcal{H}}_q(C) = 0$ for each $q > 0$, $C \subset R$. Thus any w-carrier with values in the real line is acyclic.

DEFINITION 4.2. – We say that a w-map $\varphi: X \to Y$ is an ε-approximation of a w-carrier $F: X \to Y$ if

i) $\varphi(x) \subset \varepsilon F(\varepsilon x)$ $\quad\quad \forall x \in X$;

ii) $\tilde{i}(C, \varphi(x)) = \tilde{i}(C, F(x))$ $\quad$ for any piece C of $\varepsilon F(\varepsilon x)$.

We are now in a position of stating our main approximation result.

THEOREM 4.1. – *Let $X_0 \subset X$ be a finite polyhedral pair, let Y be a metric* ANR *and let $F: X \to Y$ be an acyclic w-carrier.*

Given any $\varepsilon > 0$ there exists a $\delta > 0$ such tath any δ-approximation $\varphi: X_0 \to Y$ of F restricted to X_0 can be extended to an ε-approximation $\tilde{\varphi}: X \to Y$ of F.

The proof will given in Section 6.

COROLLARY 4.1. – *Let $F: X \to Y$ be an acyclic. For each $\varepsilon > 0$ there exists an ε-approximation $\varphi: X \to Y$ of F.*

PROOF. – Take $X_0 = \emptyset$ in the above theorem.

COROLLARY 4.2. – *Let $S: X \times [0, 1] \to Y$ be an acyclic w-carrier. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\varphi_i: X \to Y$, $i = 0, 1$ are δ-approximations of S restricted to $X \times \{i\}$, $i = 0, 1$, then there exists an ε-approximation $\psi: X \times [0, 1] \to Y$ of S such that ψ restricted to $X \times \{i\}$ coincides with φ_i.*

PROOF. – Let us take $X_0 = X \times \{0\} \cup X \times \{1\}$ and let $\varphi: X_0 \to Y$ be defined by $\varphi(x, 0) = \varphi_0(x)$; $\varphi(x, 1) = \varphi_1(x)$, $\forall x \in X$. Now Theorem 4.1 applies.

COROLLARY 4.3. – *Let $F: X \to Y$ be an acyclic w-carrier. Then there exists $\delta > 0$ such that any two δ-approximations φ_0, φ_1 of F are σ-homotopic (i.e. homotopic as w-maps).*

PROOF. – Let us take $S: X\times[0,1]\to Y$ defined by $S(x,t)=F(x)$ and apply Corollary 4.2 with $\varepsilon=1$.

THEOREM 4.2. – *Any acyclic w-carrier $F: X\to Y$ induces a graded homomorphism $F^q_*: \mathcal{H}_q(X)\to\mathcal{H}_q(Y)$, $\forall_q\geqslant 0$ such that*

a) If two acyclic w-carriers F_1, $F_2: X\to Y$ are joined by an acyclic w-carrier homotopy, then

$$F^q_{1*}=F^q_{2*}: \mathcal{H}_q(X)\to\mathcal{H}_q(Y);$$

b) when the sum of F and T is defined we have

$$(F\oplus T)^q_*=F^q_*+T^q_*: \mathcal{H}_q(X)\to\mathcal{H}_q(Y);$$

c) if $v: Z\to X$ is a continuous single-valued map and $F: X\to Y$ is an acyclic w-carrier, then

$$(F\circ v)^q_*=F^q_*\circ v^q_*: \mathcal{H}_q(Z)\to\mathcal{H}_q(Y);$$

d) when $F: X\to Y$ is a w-map, F coincides with the homomorphism defined in [4].

PROOF. – Given any w-carrier $F: X\to Y$ and $\delta>0$ as in Corollary 4.3, if $\varphi: X\to Y$ is any δ-approximation of F then the homomorphism $\varphi^q_*: \mathcal{H}_q(X)\to\mathcal{H}_q(Y)$ does not depend on such a φ. Hence, we define $F^q_*\equiv\varphi^q_*: \mathcal{H}_q(X)\to\mathcal{H}_q(Y)$.

By Corollary 4.2 if F_1, F_2 are joined by an acyclic homotopy, then any two sufficiently close approximations are σ-homotopic and therefore they induce the same homomorphism in homology. This proves a). b) follows from the fact that if φ, ψ are δ-approximations of F and T, then $\varphi+\psi$ is a 2δ-approximation of $F\oplus T$. The last two properties are trivial.

Let $\boldsymbol{R}$ be a field (or a principal integral domain).

Let X be a finite polyhedron, $F: X\to X$ be an acyclic w-carrier from X into itself. The *Lefschetz number* of F is defined as $\mathcal{L}(F)=\sum_{q=0}^{\infty}(-1)^q \operatorname{trace} F^q_*$.

With our definition of F_* the following Lefschetz fix-point theorem for acyclic w-carriers is an immediate consequences of those for w-maps.

THEOREM 4.3. – *Let X and $F: X\to X$ as above. If $\mathcal{L}(F)\neq 0$, then there exists $x\in X$ such that $x\in F(x)$.*

PROOF. – Suppose that $\forall x\in X$, $x\notin F(x)$. Then by the uppersemicontinuity of the distance function $d(x, F(x))$ and since X is compact, there exists $\delta>0$ such that $x\notin\delta F(\delta x)$ $\forall x\in X$. Therefore, each δ-approximation $\varphi: X\to X$ of F, with

$\delta' \leqslant \delta$, is fix-point free. Hence, by a result of [6] we have that $\mathfrak{L}(\varphi) = 0$. Thus also $\mathfrak{L}(F) = 0$ which proves the theorem.

Let us observe that if F is an acyclic uppersemicontinuous map, then F is an acyclic w-carrier. More generally, suppose that $F: X \to X$ is a multivalued continuous map such that

i) $F(x)$ has n acyclic components $\quad \forall x \in X$;

ii) $F(x)$ has n or 1 acyclic components.

Let us assigne multiplicity 1 to each component of $F(x)$ in the first case. In the second case if $F(x)$ has n components we assinge to each component the multiplicity 1 and, if F has 1 component, then we take it with multiplicity n. It is easy to see that F, endowed with this multiplicity, becomes an acyclic w-carrier.] Hence our result generalizes the well known Eilenberg-Montgomery [8], O'Neill [19] fixed point theorems.

Notice, also, that F_* is uniquely determined by the w-carrier F and it is invariant under homotopy. This property is not satisfied by O'Neill's construction [19].

5. – Fixed point theorem for Lefschetz w-carriers.

In what follows we assume that $\boldsymbol{R}$ is a field or a P.I.D.

Let us recall that if $h = \{h_i\}_{i \geqslant 0}$ is an endomorphism of a graded $\boldsymbol{R}$-module E with finitely generated image then we can define the Lefschetz number of h as follows:

Let $L = \{L_i\}$ be any finitely generated submodule of E such that $\operatorname{Im} h \subset L$. We define $\mathfrak{L}(h) = \sum (-1)^j \operatorname{trace} (h_j / L_j)$. It is not difficult to see that $\mathfrak{L}(h)$ does not depend upon the choice of L.

DEFINITION 5.1. – A w-carrier from a complete metric ANR X into itself will be called a *Lefschetz w-carrier* if

a) it is compact (*i.e.* $\overline{F(X)}$ is a compact subset of X);

b) F can be factorized in the form

$$\begin{array}{ccc} X & \xrightarrow{\;F\;} & X \\ & {\scriptstyle G}\searrow \quad \nearrow{\scriptstyle r} & \\ & Y & \end{array}$$

where $G: X \to Y$ is an acyclic w-carrier from X into a complete metric ANR space Y and $r: Y \to X$ is a singlevalued map.

THEOREM 5.1. – *Let $F: X \to X$ be a Lefschetz w-carrier of the form $F = r \circ G$. Let $h: \mathcal{H}(X) \to \mathcal{H}(X)$ be defined by $h = r_* \circ G_*$. Then* $\operatorname{Im} h$ *is finitely generated (hence $\mathcal{L}(h)$ is defined). Moreover if $\mathcal{L}(h) \neq 0$ there is $x \in X$ such that $x \in F(x)$.*

PROOF. – Firstly suppose that X is a compact polyhedron. Suppose that $x \notin F(x)$ for every $x \in X$. Then by the compactness of X there exists $\varepsilon > 0$ such that $x \notin \varepsilon F(\varepsilon x)$, $\forall x \in X$.

From Lemma 7.1 of [10] it follows that there exists a compact ANR $Z \subset Y$ such that $Z \supset G(X)$. We have that the restriction of r to Z is uniformly continuous and hence there exists a δ_0; $\varepsilon > \delta_0 > 0$ such that

$$d(r(z), r(z')) < \varepsilon\,, \qquad \forall z, z' \qquad \text{with} \qquad d(z, z') < \delta_0\,.$$

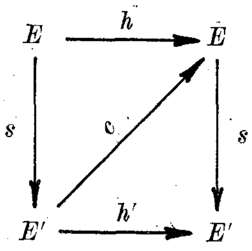
For any $\delta < \delta_0$ let $\varphi: X \to Z \subset Y$ be a δ-approximation of the acyclic w-carrier G. Since $\varphi(x) \subset \delta G(\delta x)$ $\forall x \in X$, we get that

$$\psi(x) = r \circ \varphi(x) \subset \varepsilon F(\delta x) \subset \varepsilon F(\varepsilon x)\,.$$

Thus ψ is fixed point free and $\mathcal{L}(\psi_*) = \mathcal{L}(r_* \circ \varphi_*) = 0$. From definition of G_* we get that $\mathcal{L}(r_* \circ G_*) = \mathcal{L}(h) = 0$ which proves the assertion.

Before going further we recall the following well known result ([10]).

PROPOSITION 5.1. – *Let*



be a commutative diagram of graded homomorphism of graded modules. Then $\mathcal{L}(h)$ is defined if and only if $\mathcal{L}(h')$ is defined and $\mathcal{L}(h) = \mathcal{L}(h')$.

Suppose now that X is a compact ANR. Let $\varepsilon > 0$ be such that $d(x, F(x)) > \varepsilon$. By Corollary 6.2 of [12] there exists a compact polyhedron Z and two singlevalued maps $g: X \to Z$, $f: Z \to X$ such that

i) $d(f \circ g(x), x) < \varepsilon$;

ii) $f \circ g$ is homotopic to id_X

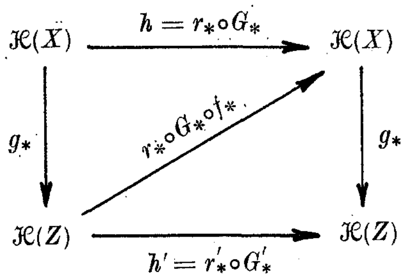
Let us call F' the composition

$$Z \xrightarrow{\ f\ } X \xrightarrow{\ F\ } X \xrightarrow{\ g\ } Z .$$

Then $F' = r' \circ G'$, where $r' = g \circ r$, $G' = G \circ f$.

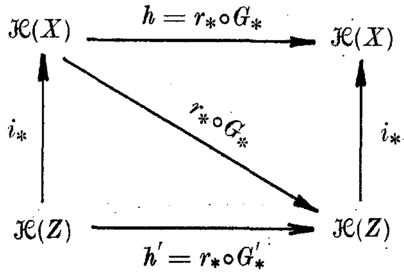
Furthermore F' is fixed point free. In fact, if $z \in F'(z) = g \circ F \circ f(z)$, then $f(z) \in f \circ g \circ F \circ f(z)$. For $x = f(z)$ by i) we have that $x \in \varepsilon F(x)$ contradicting the assumption. By the first step of the proof $\mathcal{L}(r'_* \circ G'_*) = 0$.

Applying the above proposition to the diagram



which is commutative since by ii) $f_* \circ g_* = \mathrm{id}$, we get that $\mathcal{L}(h)$ is defined and $\mathcal{L}(h) = \mathcal{L}(h') = 0$.

Finally, if X is any metric ANR, let us take Z to be any compact ANR containing $\overline{(FX)}$. Let $G' : Z \to Y$ be the restriction of G to Z and let $F' = r \circ G'$. It is clear that F and F' have the same fixed points. Furthermore, the diagram



commutes. Therefore, $\mathcal{L}(h)$ is defined and $\mathcal{L}(h) = \mathcal{L}(h')$. This completes the proof.

COROLLARY 5.1. – *Let C be an acyclic metric* ANR.
Then any Lefschetz w-carrier F from C into itself with $\tilde{i}(F) \neq 0$ has a fixed point.

PROOF. – Since C is acyclic we get $\mathcal{L}(h) = \operatorname{trace} r^0_* \circ G^0_*$: But $r^0_* \circ G^0_* : \mathcal{H}_0(C) \to \mathcal{H}_0(C)$ maps each class $\xi \in \mathcal{H}_0(C) \simeq \boldsymbol{R}$ into $\tilde{i}(F) \cdot \xi$. Thus $\mathcal{L}(h) = \tilde{i}(F) \neq 0$, hence F has a fixed point.

COROLLARY 5.2. – *Let* $B = \{x: \|x\| \leqslant \varrho\}$ *be a closed ball in a Banach space* E. *Let* $F: B \to E$ *be a Lefschetz* w*-carrier with* $\tilde{\imath}(F) \neq 0$. *If* $x \in \partial B$ *and* $\lambda x \in F(x)$ *implies that* $\lambda \leqslant 1$, *then* F *has a fixed point.*

PROOF. – Let $\varrho: E \to B$ be the radial retraction defined by

$$\varrho(x) = \begin{cases} x & \text{when } \|x\| \leqslant \varrho \\ \dfrac{\varrho x}{\|x\|} & \text{when } \|x\| \geqslant \varrho \,. \end{cases}$$

$\varrho \circ F$ is a Lefschetz w-carrier from B into itself with index different from zero. Hence there exists $x \in B$ such that $x \in \varrho F(x)$. Thus $x = \varrho(y)$ for some $y \in F(x)$.

If $\|y\| > \varrho$, $x = \varrho y/\|y\|$ and hence $\lambda x \in F(x)$ with $\lambda = \|x\|/\varrho > 1$, which contradicts the hypothesis. Thus $\|y\| \leqslant \varrho$ and so $x = \varrho(y) = y \in F(x)$.

COROLLARY 5.3. – *Let* $F: E \to E$ *be a Lefschetz* w*-carrier with* $\tilde{\imath}(F) \neq 0$. *Then either the set* $\{x: \lambda x \in F(x)$ *for some* $\lambda > 1\}$ *is bounded or* F *has a fixed point.*

PROOF. – If $\{x: \lambda x \in F(x)$ for some $\lambda > 1\}$ is contained in some ball B, we get that the restriction of F to ∂B satisfies the hypothesis of the above corollary.

COROLLARY 5.4. – *Let* $\partial B = \{x \in E: \|x\| = \varrho\}$ *be the boundary of a ball in an infinite dimensional Banach space* E. *Let* $F: \partial B \to E$ *be a Lefschetz* w*-carrier with* $\tilde{\imath}(F) \neq 0$. *If* $\inf\{\|y\|: x \in \partial B, y \in F(x)\} > 0$, *then there exists* $\lambda > 0$ *such that* $\lambda x \in F(x)$ *for some* $x \in \partial B$.

PROOF. – $F(\partial B) \subset \{x: \varepsilon \leqslant \|x\| \leqslant r\} = D$, for some ε, $r > 0$, $0 < \varepsilon < r$. Let $g: D \to \partial B$ be defined by $g(x) = \varrho x/\|x\|$. We have that $F \circ g$ is a Lefschetz w-carrier from D into itself with index different from zero. But D is acyclic, being deformable to the boundary of a ball. Hence, by Corollary 5.1, there exists some $x \in D$ such that $x \in F(g(x)) = F(\varrho x/\|x\|)$. Putting $y = \varrho x/\|x\|$, $\lambda = \|x\|/\varrho$ we get $\lambda y \in F(y)$.

COROLLARY 5.5. – *Let* $E = R^n$, $B \subset E$ *be a ball and let* $F: B \to E$ *be a Lefschetz* w*-carrier with* $\tilde{\imath}(F) \neq 0$. *Suppose that* $\inf\{\langle x, y\rangle: x \in \partial B, y \in F(x)\} \geqslant 0$. *Then for some* $x \in B$, $0 \in F(x)$.

PROOF. – Let $D: B \to E$ be defined by $D(x) = \{y: x - y \in F(x)\}$. D is a Lefschetz w-carrier since it can be decomposed in the form

$$D \xrightarrow{G_F} E \times E \xrightarrow{d} E$$

where G_F is the graph of F and $d(x, y) = x - y$. We shall see that D satisfies the hypothesis of Corollary 5.2.

Suppose that $x \in \partial B$ with $\lambda x \in D(x)$. Let $y \in F(x)$ such that $\lambda x = x - y$, then $y = (1-\lambda)x$. By hypothesis we have $\langle x, y \rangle = (1-\lambda)\langle x, x\rangle \geqslant 0$, hence $\lambda \leqslant 1$. Therefore, by Corollary 5.2, D has a fixed point x in B. Thus $0 \in F(x)$.

By Remark 4.1 we have that any w-carrier with values in the real line is a Lefschetz w-carrier. Thus from the above corollary we get

COROLLARY 5.6. – *Let $F\colon [a, b] \to R$ be a w-carrier from the interval $[a, b]$ into R with $\tilde{\imath}(F) \neq 0$. Suppose that $F(a) \subset R^-$, $F(b) \subset R^+$. Then for some $x \in [a, b]$, $0 \in F(x)$.*

PROOF. – It sufficies to apply Corollary 5.5 to the w-carrier $F'\colon [-1, 1] \to R$ defined by $F'(x) = F\ (x\cdot(b-a)/2 + (b+a)/2)$.

COROLLARY 5.7. – *Let X be path connected, $F\colon X \to Y$ be a w-carrier with $\tilde{\imath}(F) \neq 0$. Then for each $a, b, \in X$ there is a connected compact subset $C \subset F(X)$ joining $F(a)$ with $F(b)$.*

PROOF. – Let $\gamma\colon [0, 1] \to X$ a path between a and b and let $\bar{F} = F\circ\gamma$. Now $D = \bar{F}([0, 1])$ is a compact subset of Y. If does not exist a compact connected set $C \subset D$ joining $\bar{F}(0)$ with $\bar{F}(1)$, there is a continuous function $f\colon D \to \{-1, 1\}$ that maps $\bar{F}(0)$ into $\{-1\}$ and $\bar{F}(1)$ into $\{1\}$. But this contradicts Corollary 5.6 since the w-carrier $\bar{\bar{F}}\colon [0, 1] \to \{-1, 1\} \subset R$ defined by $\bar{\bar{F}} = f\circ\bar{F}$ satisfies all the assumptions of 5.6 and $0 \notin \bar{\bar{F}}(x)$, $\forall x \in X$.

Jointly with Theorem 3.1 this gives

COROLLARY 5.8. – *Let M, N, L, as in Theorem 3.1, $f\colon M \times [0, 1] \to N$ be a homotopy such that $f^{-1}(L)$ is a compact subset of $M \times [0, 1]$. Suppos e that $\tilde{\imath}(f_0, M, L) \neq 0$. Then there exists a compact connected set $C \subset f^{-1}(L)$ such that $M \times \{0\} \cap C \neq \emptyset$ and $M \times \{1\} \cap C \neq \emptyset$.*

PROOF. – Apply Corollary 5.7 to the w-carrier $F(t) = \{(m, t)\colon f(m, t) \in L\}$.

REMARK 5.1. – For fixed point sets of parametrized family of compact mappings between ANR's the above result is well known ([1]). It was also used by P. RABINOWITZ in proving the existence of unbounded branches of solutions for nonlinear Sturm Liouville problems [20]. In the form stated as in Corollary 5.7 it has been used by H. SHAW for nonlinear partial differential equations ([21]).

Hence Corollary 5.7 can be viewed as an extension of this connection property to Lefschetz w-carriers. Actually the same result holds for all w-carriers defined as solution sets of equations depending on parameters (see [17]).

We shall give now an example firstly due to Fuller which states a sufficient condition for existence of closed trajectories of vector fields in a full torus.

Let $C = B^2 \times S^1$ be the full 2-torus that is the product of the 2-ball B^2 with the circle S^1. Let X be a vector field on C pointing inward on the boundary $S^1 \times S^1$ of C.

This implies that the trajectory of a point is defined for all time $t \geqslant 0$. Hence, X generates a semiflow $\Phi(c, t)$ defined in $C \times R^+$. The universal covering of C is a cylinder $D = B^2 \times R$. The covering map $\pi: D \to C$ is defined by $\pi(x, \theta) = (x, e^{i\theta})$. The angular coordinate $e^{i\theta}$ defines a 1-form ω on C such that $\pi^*(\omega) = d\tilde{\theta}$ where π^* is the induced map on contangent bundle and $\tilde{\theta}: D \to R$ is given by $\tilde{\theta}\ (x, \theta) = \theta$.

For $c \in C$ and $t \in R^+$ let us consider the integral of the form ω over the part of the trajectory going from c to $\Phi(c, t)$, *that is*

$$\eta(c, t) = \int_0^t \omega(X_{\Phi(c,\tau)})\, d\tau .$$

If $\lim_{t\to\infty} \eta(c, t) = \infty$ *uniformly in* c, *then there exists a closed trajectory of the field* X.

PROOF. - The vector field X induces a vector field $\tilde{X}$ on D such that the semiflow $\tilde{\Phi}(d, t)$ covers the semiflow $\Phi(c, t)$ under the covering projection π. If $d \in D$, $c = \pi(d)$, $t \geqslant 0$, we have that the integral of the form ω over the trajectory from c to $\Phi(c, t)$ coincides with the integral of the form $d\tilde{\theta} = \pi^*\omega$ over the path of trajectory from d to $\tilde{\Phi}(d, t)$. Hence we get $\eta(c, t) = \eta(d, t) = \tilde{\theta}(\Phi(d, t)) - \tilde{\theta}(\Phi(d, 0))$. Therefore the coordinate $\tilde{\theta}(\tilde{\Phi}(d, t))$ of the trajectory passing by d must go to infinity with t. Let h_0, h_1 be the imbeddings of the ball B^2 in D given by $h_0(x) = (x, 0)$, $h_1(x) = (x, 1)$. Thus, $S_i = h_i(B^2) = \tilde{\theta}^{-1}(i)$, $i = 0, 1$ are 2-dimensional submanifolds of D.

By the above discussion the set $\{t: \tilde{\theta}(\tilde{\Phi}(h_0(x), t)) = 1,\ x \in B^2\}$ is compact. Therefore, exactly as in example 3.5, the multivalued map $\tau(x) = \{t: \tilde{\Phi}(h_0(x), t) \in S_1\}$ is a w-carrier which is acyclic by Remark 4.1. Since $\tilde{\theta}(\tilde{\Phi}(h_0(x), t)) \to \infty$ as $t \to \infty$, it is not difficult to see that the index $\tilde{\imath}(\tau) = 1$.

For each $d \in D$, let us denote by $\Gamma(d) = \{\tilde{\Phi}(d, t): t \geqslant 0\}$ the trajectory of the vector field $\tilde{X}$ passing trough d. Since the multivalued map $G(x) = \Gamma(h_0(x)) \cap S_1$ can be decomposed as

$$B^2 \xrightarrow{h_0 \times \tau} D \times R \xrightarrow{\tilde{\Phi}} D ,$$

it follows that G is a Lefschetz w-carrier. Since $G(B^2) \subset S_1$, it follows that $F = h_1^{-1} \circ G$ is a Lefschetz w-carrier from the ball B^2 into itself. Furthermore $\tilde{\imath}(F) = \tilde{\imath}(\tau) \neq 0$. Hence by Corollary 5.1 F has a fixed point. But the fixed points of the map F correspond to the closed trajectories of the field X since $\pi h_1 = \pi h_0$.

6. - Proof of the main theorem.

We start with some auxiliary propositions.

Let U, V be open subsets of X such that $U \subset V$. We say that U is *banal* in V if the homomorphism $\mathcal{H}_q(U) \xrightarrow{i} \mathcal{H}_q(V)$ induced in homology by the inclusion $i: U \to V$ is equal to zero $\forall q \geqslant 1$.

PROPOSITION 6.1. – *Let X be a compact polyhedron, let Y be a metric* ANR *and let $F: X \to Y$ be an acyclic w-carrier. Then for any $\varepsilon > 0$ there exists $\delta > 0$, $\delta < \varepsilon$, such that $\delta F(\delta x)$ is banal in $\varepsilon F(\varepsilon x)$, $\forall x \in X$.*

PROOF. – *Since* $\forall x \in X$, $\tilde{\mathcal{H}}_q(F(x)) = 0$, $\forall q > 0$, by the construction of $\tilde{\mathcal{H}}_q$ we have that there exists ε_x, $0 < \varepsilon_x < \varepsilon$, such that $2\varepsilon_x F(x)$ is banal in $\varepsilon F(x)$, $\forall x \in X$.

By the uppersemicontinuity of F, $\forall x$ there exists $\delta_x \leqslant \varepsilon_x$ such that $F(\delta_x x) \subset \varepsilon_x F(x)$. Let δ be the Lebesgue number of the covering $\{\frac{1}{2}\delta_x x\}_{x \in X}$. Then for every $x \in X$ there exists $x' \in X$ such that $\delta x \subset \delta_{x'} x'$ and hence $F(\delta x) \subset \varepsilon_{x'} F(x')$. Since $\delta \leqslant \delta_{x'} \leqslant \varepsilon_{x'}$ we have that $\delta F(\delta x) \subset 2\varepsilon_x F(x') \subset \varepsilon F(x')$. Moreover, since $x' \in \varepsilon x$, we get that $\delta F(\delta x) \subset \subset \varepsilon F(x') \subset \varepsilon F(\varepsilon x)$.

The banality of $\delta F(\delta x)$ in $\varepsilon F(\varepsilon x)$ follows now from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{H}_q(2\varepsilon_{x'} F(x')) & \xrightarrow{\ 0\ } & \mathcal{H}_q(\varepsilon F(x')) \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle i} \\ \mathcal{H}_q(\delta F(\delta x)) & \xrightarrow{\ i\ } & \mathcal{H}_q(\varepsilon F(\varepsilon x)) \end{array}$$

PROPOSITION 6.2. – *Let Y be as in Theorem* 4.1. *A w-map $\varphi: \{0, 1\} \to Y$ can be extended to whole of* $[0, 1]$ *if and only if for any component C of Y we have that*

$$\tilde{i}(C, \varphi(1)) = \tilde{i}(C, \varphi(0)) .$$

PROOF. – That this condition is necessary follows from *b*) of Definition 1.1.

To show that it is also sufficient, let us notice that $\varphi(0)$, $\varphi(1)$ are singular 0-chains that can be decomposed in a sum of 0-chains with support in each connected component. Thus we can assume that $Y = C$ is connected. Since $\tilde{i}(C, \varphi(0) - \varphi(1)) = 0$, the singular chain $\varphi(0) - \varphi(1)$ is a reduced singular 0-cycle. Hence it is a boundary of some singular 1-chain $\sum \lambda_i \sigma_i$.

The w-map $\psi(x) = \sum \lambda_i \sigma_i(x)$ is the desired extension.

PROPOSITION 6.3. – *Let $U \subset V$ be an open subset of Y such that U is banal in V. Any w-map φ from the q-sphere S^q into U can be extended to a w-map ψ from the $q+1$-ball B^{q+1} into V, $q \geqslant 1$.*

PROOF. – Let us denote with $\sigma[S^q; Y]$ the $\boldsymbol{R}$-module of all σ-homotopy classes of w-mappings from S^q into Y with index 0.

By Theorem 3.5 of [16] we have that, if $j \in \mathcal{H}_q(S^q)$ is a generator of $\mathcal{H}_q(S^q) \simeq \boldsymbol{R}$ and Y has the homotopy type of a CW-complex, then the Hurewicz map $h: \sigma[S^q; Y] \to \to \mathcal{H}_q(Y)$ given by $h(\varphi) = \varphi_*(j)$ is an isomorphism for every q. From this, by the commutativity of the diagram

$$\begin{array}{ccc} \sigma[S^q;\, U] & \xrightarrow{\;i^\#\;} & \sigma[S^q;\, V] \\ h\wr\downarrow & & \wr\downarrow h \\ \mathcal{H}_q(U) & \xrightarrow{\;i_*=0\;} & \mathcal{H}_q(V) \end{array}$$

it follows that $i^\# = 0$. Thus, any w-map of index 0 from S^q into U considered as w-map into V is null homotopic. Therefore, it can be extended to B^{q+1}. When ψ has arbitrary index the problem can be reduced to the above one taking $\varphi = \psi - i(\psi)y$ with $y \in U$. This achieves the proof.

PROOF OF THEOREM 4.1. – Suppose that $\dim(X \setminus X_0) = n$. Using Proposition 6.1 we shall construct a sequence $\{\varepsilon_i\}_{0\leqslant i\leqslant n}$, $\varepsilon_i > 0\ \forall_i$, in the following way.

Let $\varrho(\varepsilon) = \sup\{\delta\colon \delta < \varepsilon,\ \delta F(\delta x)$ is banal in $\varepsilon F(\varepsilon x)\ \forall x \in X,\ \forall q > 0\}$. We take $\varepsilon_0 = \varepsilon$ and define $\varepsilon_i = \frac{1}{2}\varrho(\frac{1}{2}\varepsilon_{i-1})$.

Clearly, we have

1) $4\varepsilon_i \leqslant \varepsilon_{i+1}$, $0 \leqslant i \leqslant n-1$;

2) $2\varepsilon_i F(2\varepsilon_i x)$ is banal in $\frac{1}{2}\varepsilon_{i+1}F(\frac{1}{2}\varepsilon_{i+1}x)$, $\forall x \in X$.

Since F is uppersemicontinuous, for every $x \in X$, there exists an open neighborhood U_x with $\operatorname{diam} U_x < \varepsilon_0$ such that $F(U_x) \subset \varepsilon_0 F(x)$, $\forall x \in X$.

Let (K, L) be a triangulation of (X, X_0) with mesh less than the Lebesgue number of the covering $\{U_x\}_{x\in X}$.

To prove our theorem it is enough to show that any ε_0-approximation $\varphi\colon L \to Y$ of the w-carrier F restricted to L can be extended to an ε_r-approximation $\varphi\colon K^{(r)} \cup L \to Y$ of $F|_{K^{(r)}\cup L}$ (where $K^{(r)}$ denotes the r-skeleton of the simplicial complex K).

Let us choose for each simplex σ of K not in L a point x_σ such that $\sigma \subset U_{x_\sigma}$ (this is always possible by the above construction). Also there are not restrictions in assuming that if σ is actually a vertex v of K not in L then $x_\sigma = v$.

We extended now φ from L to $K^{(0)} \cup L$ in the following way.

Let v be a vertex of K not in L. Since $F(v)$ is compact it meets only a finite number of components of the open set $\varepsilon_0 F(\varepsilon_0 v)$, say $C_1, \ldots, C_r$. Let us choose a point y_i in each component C_i. We define φ by

$$\varphi(v) = \sum_{i=1}^{r} \tilde{i}(C_i, F(v))\, y_i\,.$$

Clearly, φ is an ε_0-approximation of $F|_{K^{(0)}\cup L}$.

We extend now φ to $K^{(1)} \cup L$. Let $\sigma = \langle v_0, v_1\rangle$ be a 1-simplex of K not in L. Since $\sigma \subset U_{x_\sigma}$ and $\operatorname{diam} U_{x_\sigma} < \varepsilon_0$, we have that

3) $\varepsilon_0 v_i \subset \varepsilon_0 \sigma \subset 2\varepsilon_0 x_\sigma$.

Let $W = 2\varepsilon_0 F(2\varepsilon_0 x_\sigma)$. Since φ is an ε_0-approximation of $F|_{K^{(0)}\cup L}$ we have that $\varphi(v_i) \subset \varepsilon_0 F(\varepsilon_0 v_i) \subset \varepsilon_0 F(2\varepsilon_0 x_\sigma) \subset W$ by 3).

We will extend $\varphi\colon \partial\sigma \to W$ to whole of σ. By Proposition 6.2 it sufficies to prove that $\tilde{\imath}(C, \varphi(v_0)) = \tilde{\imath}(C, \varphi(v_1))$ for each component C of W. In fact, if C is any such component we have that

$$C_i = C \cap \varepsilon_0 F(\varepsilon_0 v_i) \quad \text{are pieces of} \quad \varepsilon_0 F(\varepsilon_0 v_i)\,, \quad i = 1, 2\,.$$

By excision and since φ is an ε_0-approximation of $F|_{\partial\sigma}$ we obtain

$$\tilde{\imath}(C, \varphi(v_i)) = \tilde{\imath}(C_i, \varphi(v_i)) = \tilde{\imath}(C_i, F(v_i)) = \tilde{\imath}(C, F(v_i)) \quad i = 0, 1\,.$$

But $F\colon \sigma \to W$ is a w-carrier; therefore by connectedness of σ we have that $\tilde{\imath}(C, F(v_0)) = \tilde{\imath}(C, F(v_1))$. Hence $\tilde{\imath}(C, \varphi(v_0)) = \tilde{\imath}(C, \varphi(v_1))$.

Let $\varphi\colon \sigma \to W$ be any extension of $\varphi\colon \partial\sigma \to W$. We will see that φ is actually an ε_1-approximation of $F|_\sigma$.

From 1) it follows that $2\varepsilon_0 x_\sigma \subset \varepsilon_1 x$, $\forall x \in \sigma$. Hence,

4) $W = 2\varepsilon_0 F(2\varepsilon_0 x_\sigma) \subset \varepsilon_1 F(\varepsilon_1 x)$, $\forall x \in \sigma$.

Thus, we obtain that $\varphi(\sigma) \subset \varepsilon_1 F(\varepsilon_1 x)$, $\forall x \in \sigma$. This shows i) of Definition 4.2.

In order to prove ii), let us observe that by 4) if C is any piece of $\varepsilon_1 F(\varepsilon_1 x)$ then $C_1 = C \cap W$ is a piece of W. Since $\varphi(\sigma) \subset W$, $F(\sigma) \subset W$, by *b*) of Definition 1.1 we have that

$$\tilde{\imath}(C_1, \varphi(v_0)) = \tilde{\imath}(C_1, \varphi(x))\,; \qquad \tilde{\imath}(C_1, F(v_0)) = \tilde{\imath}(C_1, F(x))\,, \qquad \forall x \in \sigma\,.$$

Hence by excision

$$\tilde{\imath}(C, \varphi(x)) = \tilde{\imath}(C_1, \varphi(x)) = \tilde{\imath}(C_1, \varphi(v_0)) = \tilde{\imath}(C_1, F(v_0) = \tilde{\imath}(C_1, F(x)) = \tilde{\imath}(C, F(x))\,.$$

This proves ii).

By glueing together each approximation $\varphi_\sigma\colon \sigma \to Y$ with those defined on simplexes that meet σ, we get an ε_1-approximation φ of F restricted to $K^{(1)} \cup L$.

Suppose that $\varphi\colon K^{(r)} \cup L \to Y$ is an ε_r-approximation of F restricted to $K^{(r)} \cup L$. Let σ be a $(r+1)$-simplex of K not in L. We have that φ is defined on the boundary $\partial\sigma$ of σ and $\forall x \in \partial\sigma$, $\varphi(x) \subset \varepsilon_r F\ (\varepsilon_r x)$. Since $\sigma \subset \varepsilon_0 x_\sigma \subset \varepsilon_r x_\sigma$, we have that $\varphi(\partial\sigma) \subset$ $\subset \varepsilon_r F(2\varepsilon_r x_\sigma) \subset 2\varepsilon_r F(2\varepsilon_r x_\sigma)$.

Let $W = \frac{1}{2}\varepsilon_{r+1} F(\frac{1}{2}\varepsilon_{r+1} x_\sigma)$. By 2) we have that $2\varepsilon_r F(2\varepsilon_r x_\sigma)$ is banal in W. Thus by Proposition 6.3 we can extend φ to a w-map $\varphi\colon \sigma \to W$.

In order to show that φ is an ε_{r+1}-approximation of $F|_\sigma$, let us observe that $\frac{1}{2}\varepsilon_{r+1} x_\sigma \subset \varepsilon_{r+1} x$, $\forall x \in \sigma$. This implies that $W \subset \varepsilon_{r+1} F(\varepsilon_{r+1} x)$ $\forall x \in \sigma$. Since $\varphi(\sigma) \subset W$ it follows that i) holds.

In oder to prove ii) let C be a piece of $\varepsilon_{r+1}F(\varepsilon_{r+1}x)$. Let $C^1 = C \cap W$. By the above argument C^1 is a piece of W. Take some x_0 belonging to the boundary $\partial\sigma$. Since $F\colon \sigma \to W$ is a w-carrier, by the connectedness of σ we get for every $x \in \sigma$

$$\tilde{\imath}(C, \varphi(x)) = \tilde{\imath}(C^1, \varphi(x)) = \tilde{\imath}(C^1, \varphi(x_0)) = \tilde{\imath}(C^1, F(x_0)) = \tilde{\imath}(C^1, F(x)) = \tilde{\imath}(C, F(x)) .$$

This achieves the proof of the theorem.

7. – Appendix.

We shall give here a sketch of the construction of the intersection index of Section 3.

We shall use the singular homology theory.

Let M be a topological n-manifold oriented over a ring $\boldsymbol{R}$ and let K be a compact subset of M. We denote by $O_K \in H_n(M, M-K)$ the orientation class of M around K [7]. This class is caracterized as follows: $\forall q \in K$ if $\tilde{\imath}_q\colon (M, M-K) \to (M, M-q)$ is the inclusion of pairs, we have that $\tilde{\imath}_{q*}(O_K) \in H_n(M, M-q)$ is just the orientation class O_q of M at q.

If U is an open set admissible for K we denote with $O_{K\wedge U} \in H_n(U, U-U \cap K)$ the image of O_K by the composition

$$H_n(M, M-K) \xrightarrow{i_*} H_n(M, M-K\cap U) \underset{\sim}{\xrightarrow{\text{excision}}} H_n(U, U-K\cap U) .$$

From the above characterization of orientation classes it follows that $O_{K\wedge U}$ is just the orientation class of the submanifold U of M around the compact set $K \cap U$. For this reason we shall call $O_{K\wedge U}$ *the orientation class around K in U*. Notice also that $O_{K\wedge U} = O_{K'\wedge U}$ where $K' = K \cap U$.

LEMMA 7.1. – *Let U be a disjoint union of a family $\{U_\lambda\}$ of admissible open sets. Let $i^\lambda\colon (U_\lambda, U_\lambda - K \cap U_\lambda) \to (U, U - U \cap K)$ be the inclusion of pairs. We have*

$$O_{K\wedge U} = \sum_\lambda i^\lambda_*(O_{K\wedge U_\lambda}) ,$$

the latter sum being finite since $K \cap U_\lambda = \emptyset$ for all but a finite number of $\lambda - s$.

PROOF. – Consider the composition

$$h\colon \bigoplus_\lambda H_n(U_\lambda, U_\lambda - U_\lambda \cap K) \xrightarrow{\{i^\lambda_*\}} H_n(U, U-K) \xrightarrow{j_{q*}} H_n(U, U-q)$$

where $q \in K \cap U$. Now, h sends all components of $(O_{K\wedge U_\lambda}\}$ into zero except those

λ_0 one for which $q \in K \cap U_{\lambda_0}$. The last one goes into $O_{q \wedge U}$. Therefore,

$$j_q[\{j_*^\lambda\}(\{O_{K \wedge U_\lambda}\})] = O_{q \wedge U}\,, \qquad \forall_q \in K \cap U\,.$$

Hence, by the characterization of the orientation class, we get

$$\sum_\lambda i_*^\lambda(O_{K \wedge U_\lambda}) = \{i_*^\lambda\}(\{O_{K \wedge U_\lambda}\}) = O_{K \wedge U}\,.$$

LEMMA 7.2. – *Assume that $K \subset Q$ are compact, $W \subset U$ are open sets such that $(\overline{U} - W) \cap Q = \emptyset$, If $i_*\colon H_n(W.\ W - Q \cap W) \to H_n(U, U - K \cap U)$ is the excision of pairs $(W, W - Q \cap W) \to (U, U - K \cap U)$ we have that*

$$i_*(O_{Q \wedge W}) = O_{K \wedge U}\,.$$

In particular if

a) $K = Q$ *then* $i_*(O_{K \wedge W}) = O_{K \wedge U}\,, \qquad W \subset U\,,$

b) $W = U$ *then* $i_*(O_{Q \wedge U}) = O_{K \wedge U}\,, \qquad K \subset Q\,.$

PROOF. – Notice that $K \cap U = K \cap W \subset Q \cap W$. Now, for $\forall q \in K \cap U$ consider the following diagram

$$\begin{array}{ccccc}
H_n(W, W - Q \cap W) & \xrightarrow{\;j_{q*}^W\;} & H_n(W, W - q) & & \\
\Big\downarrow i_* & & \Big\downarrow j_* & \overset{\sim}{\searrow} & H_n(M, M - q) \\
H_n(U, U - K \cap U) & \xrightarrow[\;j_{q*}^U\;]{} & H_n(U, U - q) & \overset{\sim}{\nearrow} &
\end{array}$$

Since j_* sends $O_{q \wedge W}$ into $O_{q \wedge U}$ we get that $\forall q \in K \cap U$

$$j_{q*}^U(i_*(O_{Q \wedge W})) = O_{q \wedge U}\,,$$

hence, by the definition of orientation class, we have that

$$i_*(O_{Q \wedge W}) = O_{K \wedge U}\,.$$

Let us consider an oriented $(n + k)$-manifold N.

Let L be a closed connected oriented submanifold of N with $\dim L = k$. Let $\eta \in H_n(N, N-L)$ be its transverse class in N which is a generator of $H_n(N, N-L)$ (see [7]). Let $f\colon M \to N$ be a continuous map such that $K = f^{-1}(L)$ is a compact set. For any admissible set U we denote by f^U the restriction of f to U viewed as a map of pairs $f^U\colon (U, U-K) \to (N, N-L)$.

The intersection index of f with L in U is defined as the unique $r = \tilde{i}(f, L, U) \in \mathbf{R}$ such that

$$f_*^U(O_{K\wedge U}) = r\cdot\eta \qquad \text{in } H_n(N, N-L)\,.$$

We will see that the intersection index satisfies all the properties of 2.1.

1) *Additivity.* By Lemma 7.1 we have

$$f_*^U(O_{K\wedge U}) = f_*^U\Big(\sum_\lambda i_*^\lambda(O_{K\wedge U_\lambda})\Big) = \sum_\lambda f_*^U i_*^\lambda(O_{K\wedge U_\lambda}) = \sum_\lambda f_*^{U_\lambda}(O_{K\wedge U_\lambda})\,.$$

Then by definition of $\tilde{i}(f, L, U)$ we get

$$\tilde{i}(f, L, U) = \sum_\lambda \tilde{i}(f, L, U_\lambda)\,.$$

2) *Excision.* If $W \subset U$ and $\bar{U} - W \cap K = \emptyset$ then by Lemma 7.2 with $Q = K$ $i_*\colon H_n(W, W - K\cap W) \to H_n(U, U - K \cap U)$ sends the orientation class $O_{K\wedge W}$ into $O_{K\wedge U}$. Thus we have

$$f_*^U(O_{K\wedge U}) = f_*^U \circ i_*(O_{K\wedge W}) = f_*^W(O_{K\wedge W})$$

and hence

$$\tilde{i}(f, L, U) = \tilde{i}(f, L, W)\,.$$

3) *Homotopy property.* Let $h\colon M\times I \to N$ be such that $Q = \{x \in U\colon h(x, t) \in L$ for some $t\}$ is a compact subset of U.

Let $K_0 = \{x \in U\colon h(x, 0) \in L\}$, $K_1 = \{x \in U\colon h(x, 1) \in L\}$.

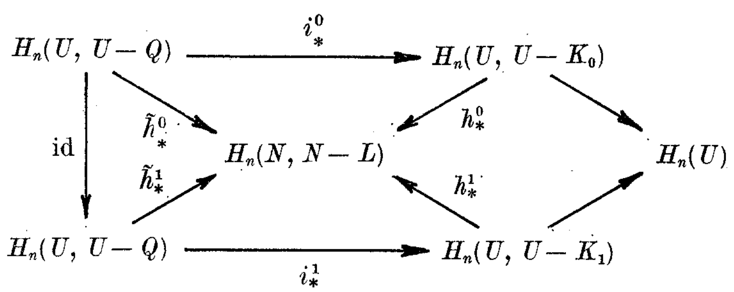
Then K_0, K_1 are compact sets contained in Q. Applying Lemma 7.2 with $U = W$ to the pairs $i^0\colon (U, U-Q) \to (U, U-K_0)$; $i^1\colon (U, U-Q) \to (U, U-K_1)$ we get

$$i_*^0(O_{Q\wedge U}) = O_{K_0\wedge U};\; i_*^1(O_{Q\wedge U}) = O_{K_1\wedge U}\,.$$

Since $\tilde{h}\colon (U, U-Q)\times I \to (N, N-L)$ is a homotopy of pairs we have that

$$\tilde{h}_*^0 = \tilde{h}_*^1\colon H_n(U, U-Q) \to H_n(N, N-L)\,.$$

The above relation, jointly with the commutativity of the diagram



implies that

$$h^0_*(O_{K_0 \wedge U}) = h^0_*(i^0_*(O_{Q \wedge U})) = \tilde{h}^0_*(O_{Q \wedge U}),$$
$$h^1_*(O_{K_1 \wedge U}) = h^1_*(i^1_*(O_{Q \wedge U})) = \tilde{h}^1_*(O_{Q \wedge U}).$$

Therefore $h(O_{K_0 \wedge U}) = h(O_{K_1 \wedge U})$ which achieves the proof.

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