Nonoscillatory Solutions of Forced Second Order Linear Equations. - II (*).

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Summary. – *The number of nonoscillatory solutions of a forced second order linear differential equation is studied under the hypothesis that the homogeneous equation is oscillatory. The main technique involves expressing a general solution of the forced equation in terms of two parameters, given a pair of independent solutions of the homogeneous equation* (*see* (2.4) *below*).

1. – Introduction and preliminary remarks.

We present in this paper some results dealing with the oscillation and nonoscillation of solutions of the forced second-order linear equation

$$(p(t)y')' + q(t)y = f(t)\,, \qquad a \leqslant t < \infty\,, \tag{1.1}$$

under the standing hypothesis that the associated homogeneous equation

$$(p(t)x')' + q(t)x = 0\,, \qquad a \leqslant t < \infty\,, \tag{1.2}$$

is oscillatory. The bulk of the paper is devoted to results which limit the collection of non-oscillatory solutions of (1.1); we conclude, however, with a sufficient condition for such a solution to exist.

By a non-oscillatory solution of (1.1) we mean, as usual, a solution which is either ultimately positive, or else ultimately negative as $t \to \infty$. One can also discuss, as in [4], solutions of « Z-type », which have arbitrarily large zeros but ultimately do not change sign. For want of a better term, let us say that a solution is of NC-type, if it is either ultimately non-negative, or else ultimately non-positive.

It is a basic observation that, if (1.2) is oscillatory, then the non-oscillatory solutions have the same ultimate sign, and form a convex set. Similarly, solutions of NC-type are either all ultimately non-negative, or else all ultimately non-positive, and again form a convex set.

The convexity of these sets of solutions may be realized geometrically by various maps from the set solutions of (1.1) to the plane. One method will be to associate

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with a solution y of (1.1) its initial data $(y(a), y'(a))$; a second method is given by expressing the general solution of (1.1) in terms of two parameters, given a pair of independent solutions of (1.2).

As an example of the first method, non-oscillatory solutions of

$$y''+y=1\,, \quad t\geqslant 0\,, \tag{1.3}$$

are given by taking $(y(0), y'(0))$ to be in the open unit disc, and the set of NC-type solutions corresponds in this way to the closed disc. Again,

$$y''+y=t^{-1}+2t^{-3}\,, \quad t\geqslant 1\,, \tag{1.4}$$

has a unique non-oscillatory (or NC-type) solution, corresponding under this map to the point $(1, -1)$.

In [5], an example was presented where the initial data of the non-oscillatory solutions formed a line, and other examples can be easily constructed. We refer to DOLAN [3] for a more detailed investigation concerning the structure of sets of initial data for non-oscillatory solutions of (1.1), when (1.2) is either non-oscillatory or oscillatory.

With such a mapping from non-oscillatory solutions to the plane, we can recognize the following four possibilities for the convex sets in question, associated with the non-oscillatory solutions (or alternatively the NC-type) solutions:

(i) the empty set,

(ii) a single point,

(iii) a segment of a line,

(iv) a set in the plane with non-empty interior.

We shall give mainly conditions ensuring that one of (i), (ii) is the case, or again that one of (i), (ii) or (iii) is the case.

We assume throughout that $p(t)$ is positive, and that p, q and f are continuous; the latter requirement can often be weakened to measurability. Sign-restrictions on q, f will be made only as and when needed. We often take the case $p(t)=1$, which may be regarded as covering the case that $\int\limits_a^\infty dt/p(t)=\infty$, by way of a change of independent variable.

2. – Variation of parameters.

We introduce the modified Wronskian

$$W(u, v)(t)=(upv'-vpu')(t)\,, \tag{2.1}$$

and denote by x_1, x_2 a pair of solutions of (1.2) such that

$$W(x_1, x_2) = 1 . \tag{2.2}$$

We represent a solution of (1.1) in the form

$$y = x_1 W(y, x_2) - x_2 W(y, x_1) , \tag{2.3}$$

and note that, for some constants A_1, A_2,

$$W(y, x_j) = A_j - \int_a^t f(s) x_j(s)\, ds , \quad j = 1, 2 . \tag{2.4}$$

We can substitute (2.4) in (2.3), and obtain a mapping from solutions of (1.1) to points (A_1, A_2) in the plane; this mapping will associate convex sets with non-oscillatory or NC-type solutions and was the second method referred to earlier.

3. – Preliminary results.

Suppose for definiteness that y is a non-oscillatory solution of (1.1), and that

$$y(t) > 0 , \quad t \geq T . \tag{3.1}$$

Let t', t'' be consecutive zeros of $x_1(t)$ in $[T, \infty)$, and suppose that $x_1(t) > 0$ in (t', t''). Then $x_1'(t') > 0$, $x_1'(t'') < 0$, so that, by (2.2), $x_2(t') < 0$, $x_2(t'') > 0$. From (3.1) and (2.3) we then have

$$W(y, x_1)(t') > 0 , \quad W(y, x_1)(t'') < 0 , \tag{3.2}$$

and so, by (2.4),

$$\int_{t'}^{t''} f(s) x_1(s)\, ds > 0 . \tag{3.3}$$

This inequality must be reversed if (3.1) holds but $x_1(t) < 0$ in (t', t''), or again if (3.1) is reversed. Summing up, and allowing for these variations, we have

Proposition 1. – *Let* (1.1) *have a non-oscillatory solution* $y(t)$, *with* $y(t) \neq 0$ *for* $t \geq T$. *Then for any non-trivial solution* x *of* (1.2), *there is a sequence of consecutive zeros* t_n, $n = 0, 1, \dots$, $t_0 \geq T$, *and*

$$(-1)^n x(t) > 0 , \quad t_n < t < t_{n+1} , \tag{3.4}$$

$$(-1)^n y(T) \int_{t_n}^{t_{n+1}} f(s) x(s)\, ds > 0 , \quad n = 0, 1, \dots . \tag{3.5}$$

In particular, $y(t)f(t)$ *must take a positive value somewhere between two consecutive zeros of* x *in* (T, ∞). *Furthermore,* $W(y, x)$ *must have a zero between two consecutive zeros of* x *in* (T, ∞).

Here «between» is meant in the strict sense.

COROLLARY 1. – *If* $f(t)$ *is ultimately non-negative, then any non-oscillatory solution of* (1.1) *is ultimately positive.*

This is noted by KEENER [11, Theorem 1], but with additional hypotheses.

COROLLARY 2. – *Let* $f(t)$ *change sign on every half-axis, and be such that if* $f(t)$ *changes sign at* t', t'' *then every solution of* (1.2) *has a zero in* $[t', t'']$. *Then* (1.1) *has only oscillatory solutions.*

This is essentially Theorem 1 of RANKIN [13], to which we refer for other similar results.

COROLLARY 3. – *If for some non-trivial solution* $x(t)$ *of* (1.2) *and for some sequence* (α_k, β_k), $k = 1, 2, \ldots$, *of consecutive zeros of* x *we have*

$$\int_{\alpha_k}^{\beta_k} f(s)x(s)\, ds = 0\,, \qquad k = 1, 2, \ldots, \tag{3.6}$$

then (1.1) *does not have a non-oscillatory solution (though it may have one of NC-type).*

EXAMPLE 1. – The equation $y'' + y = 1 - 3\cos 2t$ has the NC-type solution $1 + \cos 2t$, but no non-oscillatory solution (we take $x(t) = \cos t$ in this example).

We now use another aspect of this argument. Let x_1, x_2 be a pair of solutions of (1.2), satisfying (2.2), and write

$$F_j(t) = \int_a^t f(s)x_j(s)\, ds\,, \qquad j = 1, 2\,. \tag{3.7}$$

Let S_j denote the set of values of A_j in (2.4), corresponding to non-oscillatory solutions y of (1.1). Then S_j is necessarily an interval, possibly a single point, unless it is empty. Excluding the latter case, we have

PROPOSITION 2. – *Let the successive zeros of* x_j *be* t_{nj}, $n = 0, 1, \ldots$, *starting from a point such that* $x_j > 0$ *in* (t_{0j}, t_{1j}). *Then every point of* S_j *lies ultimately in the intervals*

$$\big(F_j(t_{2n,j}), F_j(t_{2n+1,j})\big)\,, \qquad \big(F_j(t_{2n+2,j}), F_j(t_{2n+1,j})\big)\,, \qquad n = 0, 1, \ldots, \tag{3.8}$$

where ultimately

$$F_j(t_{2n,j}) < F_j(t_{2n+1,j})\,, \qquad F_j(t_{2n+2,j}) < F_j(t_{2n+1,j})\,. \tag{3.9}$$

For a corresponding result for the wider class of NC-type solutions we replace (3.8) by the closed intervals, and allow equality in (3.9).

COROLLARY 4. – *If for some solution x of* (1.2) *we have*

$$\int_a^\infty f(s)x(s)\,ds = \infty \qquad (\text{or } -\infty)\,, \tag{3.10}$$

then (1.1) *does not have a non-oscillatory solution.* (HAMMETT [6].)

Slightly more generally, we have

COROLLARY 5. – *A necessary condition for* (1.1) *to have a non-oscillatory* (*or an NC-type*) *solution is that for every non-trivial solution x of* (1.2) *there be a sequence $t_0, t_1, \dots$ of its successive zeros such that, writing*

$$F(t) = \int_a^t f(s)x(s)\,ds\,, \tag{3.11}$$

we have

$$\limsup F(t_{2n}) \leqslant \liminf F(t_{2n+1})\,. \tag{3.12}$$

PROOF. – Suppose $y(t)$ is a non-oscillatory solution which is >0 (<0) for $t \geqslant T$. Given a solution $x(t)$ of (1.2), let $t_0, t_1, \dots$ be a sequence of consecutive zeros of $x(t)$ such that $t_0 \geqslant T$ and $x(t) > 0$ (<0) for $t \in (t_0, t_1)$. Let x_2 be a solution of (1.2) such that $W(x_1, x_2) = 1$. Repeating the argument that led to (3.3), we can actually show that

$$F(t_{2n}) < F(t_{2k+1})\,, \qquad \text{for any } n \text{ and } k\,.$$

The result follows.

4. – Uniqueness conditions.

If (1.1) has more than one non-oscillatory solution, and so a continuum of such solutions, then at least one of the sets S_j in Proposition 2 would be an interval rather than a point. We obtain sufficient conditions for uniqueness by negating this eventuality.

THEOREM 1. – *Let x_1, x_2 be linearly independent solutions of* (1.2). *Let $(\alpha_{nj}, \beta_{nj})$ be a sequence of pairs of successive zeros of x_j, with $\alpha_{nj} \to \infty$ as $n \to \infty$, such that*

$$\int_{\alpha_{nj}}^{\beta_{nj}} f(s)x_j(s)\,ds \to 0 \qquad \text{as} \qquad n \to \infty,\ j = 1, 2\,. \tag{4.1}$$

Then (1.1) *has at most one non-oscillatory* (*or NC-type*) *solution.*

This follows because if the lengths of some sequence of the intervals (3.8) tend to 0, then the set S_j is at most a single point.

COROLLARY 6. – *The conclusion of Theorem* 1 *holds if for every solution* x *of* (1.2) *the improper Riemann integral*

$$\int_a^\infty f(s)x(s)\,ds \tag{4.2}$$

converges.

This extends Theorems 1 and 2 of [5], and also Theorem 3.2 of WALLGREN [15].

COROLLARY 7. – *Let, for every non-trivial solution* x *of* (1.2),

$$\{N(T)\}^{-1}\int_a^T |f(s)x(s)|\,ds \to 0\,, \qquad \text{as } T\to\infty\,, \tag{4.3}$$

where $N(T)$ *denotes the number of zeros of* x *in* (a, T). *Then the conclusion of Theorem* 1 *holds.*

It is easily seen that (4.3) implies the hypotheses of Theorem 1.

We offset these results by one in the opposite direction.

THEOREM 2. – *Let, for some non-trivial solution* x *of* (1.2),

$$\int_{t_n}^{t_{n+1}} |f(s)x(s)|\,ds \to 0\,, \tag{4.4}$$

where $t_0, t_1, \ldots$ *denote the successive zeros of* x. *Then if* (1.1) *has any non-oscillatory* (*or* NC-*type*) *solutions, the integral* (4.2) *converges.*

PROOF. – Suppose the integral (4.2) does not converge. If the integral diverges to ∞ or $-\infty$, Corollary 4 implies that any solution eventually oscillates and actually assumes positive and negative values. Since we are assuming the existence of an NC-type solution, the integral (4.2) must not diverge to ∞ or $-\infty$.

From (3.11), we have $F(t_{2n+1}) = F(t_{2n}) + \int_{t_{2n}}^{t_{2n+1}} f(s)x(s)\,ds$, which means $\limsup F(t_{2n+1}) \leqslant \leqslant \limsup F(t_{2n}) + \limsup \int_{t_{2n}}^{t_{2n+1}} fx\,ds \leqslant \limsup F(t_{2n}) + 0$, by (4.4). Using (3.12), we have $\limsup F(t_{2n+1}) \leqslant \limsup F(t_{2n}) \leqslant \liminf F(t_{2n+1})$. But this implies $\lim_{n\to\infty} F(t_n)$ exists; call it L. Whether L is finite or infinite, using (4.4) we can show $F(t)\to L$, a contradiction to our original assumptions. This proves the theorem.

We can check the precision of Corollary 7 by use of such examples as (1.3)-(1.4), which are also covered by results of the next section. A less trivial illustration is given by the following example, where $p(t) = 1$.

EXAMPLE 2. – Consider

$$y'' + \sin^2(t^{\frac{1}{2}})y = e^{-\lambda t}, \qquad \lambda > 0 . \tag{4.5}$$

The essence of this example is that we have $q \geqslant 0$, $q' \to 0$, so that solutions of the homogeneous equation cannot grow exponentially; a relevant investigation is given by KAUFFMAN [10]. Corollary 6 is thus satisfied, with absolute convergence.

To be more explicit, we note that suitable estimates for solutions of $x'' + \sin^2(t^{\frac{1}{2}})x = 0$ may be obtained by using

$$\tfrac{1}{2}x'^2 + \tfrac{1}{2}x^2\{\sin^2(t^{\frac{1}{2}}) + t^{-\frac{1}{3}}\}$$

as a Lyapunov-type function. Details are left to the reader.

We conclude that (4.5) has at most one non-oscillatory solution. If $\lambda > 1$, it may be seen that such a solution exists by means of a criterion given at the end of this paper.

We pass to uniqueness conditions derivable from Theorem 1 for special classes of equations.

5. – The case that the homogeneous equation has only bounded solutions.

Developing Theorem 1 of [5], we have

THEOREM 3. – *Let the solutions of*

$$x'' + q(t)x = 0, \qquad t \geqslant a, \tag{5.1}$$

be all bounded. Then

$$x'' + q(t)x = f(t), \qquad t \geqslant a, \tag{5.2}$$

has at most one non-oscillatory solution if one of the following sets of hypotheses is fulfilled:

(i) *For every* $h > 0$ *there is a sequence* $\{t_k\}$, *with* $t_k \to \infty$, *such that*

$$\int_{t_k}^{t_k+h} |f(s)|\, ds \to 0, \tag{5.3}$$

(ii) $q > 0$, *and for every* $h > 0$ *there is a sequence* $\{t_k\}$, *with* $t_k \to \infty$, *such that*

$$\max_{t_k \leqslant \alpha \leqslant \beta \leqslant t_k + h} \int_{\alpha}^{\beta} f(s)\, ds \to 0 . \tag{5.4}$$

(iii) $q \to \infty$, *and for some* $h > 0$ *there is a sequence* $\{t_k\}$, $t_k \to \infty$, *such that the function*

$$g(t) = \int_a^t f(s)\, ds \tag{5.5}$$

is uniformly continuous on the union of the intervals $[t_k, t_k + h]$.

PROOF. – We recall first that the boundedness of the solutions of (5.1) implies an upper bound for the distance between consecutive zeros of non-trivial solutions, by an argument of HARTMAN [7, p. 349]. In (i), we may therefore suppose $h > 0$ to be such that in any interval of length h, every solution of (5.1) has at least two zeros. The sufficiency of (i) then follows from Theorem 1.

Passing to (ii), we again use Theorem 1. However, in considering integrals of the form appearing in (4.1), we note that the interval between two zeros of x_j falls into two parts, in each of which x_j is monotone. The result then follows on use of the second mean-value theorem.

The case of (iii) is dealt with similarly. We now note that, since $q \to \infty$, the distance between consecutive zeros tends to 0.

The hypothesis (i) has the obvious specializations $f(t) \to 0$, or $f \in L(a, \infty)$. More generally, it would be sufficient that

$$T^{-1}\int_a^T |f(s)|\, ds \to 0 \qquad \text{as } T \to \infty\,. \tag{5.6}$$

This could be viewed as special case of Corollary 7.

From case (ii) we have

COROLLARY 8. – *If* $q > 0$, *and solutions of* (5.1) *are bounded, and the improper Riemann integral*

$$\int_a^\infty f(s)\, ds \tag{5.7}$$

converges, then (5.2) *has at most one non-oscillatory solution.*

The hypothesis that (5.1) have only bounded solutions is of course ensured if q is positive and non-decreasing (see e.g., [7, p. 510]). Thus, case (iii) yields

COROLLARY 9. – *If* $q(t) \to \infty$, $q'(t) \geqslant 0$, *and* $f(t)$ *is bounded on an infinite sequence of disjoint intervals of fixed length, then* (5.2) *has at most one non-oscillatory solution.*

The above results apply equally to solutions of NC-type.

The boundedness of solutions of (5.1) is also ensured in certain cases in which $q(t)$ is periodic. We refer to [5], Corollaries 1 and 2, and to [12], Theorems 4.4, 4.1 and 2.1.

We can use Theorem 2 to link conditions similar to those of Theorem 3 with Corollary 6.

THEOREM 4. – *Let* (5.1) *have only bounded solutions, and let for some* $h > 0$

$$\int_t^{t+h} |f(s)|\, ds \to 0 \quad \text{as } t \to \infty . \tag{5.8}$$

Then if (5.2) *has a non-oscillatory solution, the integral*

$$\int_a^\infty f(s)x(s)\, ds \tag{5.9}$$

must converge, for every solution x *of* (5.1); *the solution in question is then unique.*

6. – The limit-circle case.

The relevance of this to non-oscillation was brought out in Theorem 2 of [5]. We prove here

THEOREM 5. – *Let all solutions of* (5.1) *be in* $L^2(a, \infty)$. *Let*

$$\int_a^T f^2(t)\, dt = O(T^4 \log T) , \tag{6.1}$$

as $T \to \infty$. *Then* (5.2) *has at most one non-oscillatory solution.*

PROOF. – We show that Theorem 1 applies. For some non-trivial solution x_1 of (5.1) we denote its successive zeros by $t_0, t_1, \ldots$, and claim that

$$\liminf \int_{t_n}^{t_{n+1}} |f(t)x_1(t)|\, dt = 0 . \tag{6.2}$$

It will be sufficient for this to show that

$$\liminf n^{-1} \int_{t_n}^{t_{2n}} |f(t)x_1(t)|\, dt = 0 . \tag{6.3}$$

Now

$$\left\{ \int_{t_n}^{t_{2n}} |f(t)x_1(t)| dt \right\}^2 \leqslant \left\{ \int_{t_n}^{t_{2n}} f^2(t)\, dt \right\} \left\{ \int_{t_n}^{t_{2n}} x_1^2(t)\, dt \right\} ,$$

and since $x_1 \in L^2(a, \infty)$, there will be an infinite sequence of n-values such that

$$\int_{t_n}^{t_{2n}} x_1^2(t)\, dt < \{\log n \log\log n\}^{-1} .$$

If the above inequality is not true for some sequence of n values then by considering the sequence $n = 2^k$, we can show the right side represents terms from a divergent series with respect to k. This means $x_1(t) \notin L^2(a, \infty)$, a contradiction. Thus for (6.3) it will be sufficient to show that

$$\int_a^{t_n} f^2(t)\, dt = O(n^2 \log n) .$$

This will follow from (6.1) if we show that $t_n^2 = O(n)$.

Let now x_2 be a solution of (5.1) satisfying (2.2), with $p = 1$. By Hartman's method (loc. cit.) we have

$$\pi = \int_{t_m}^{t_{m+1}} dt/(x_1^2 + x_2^2) ,$$

and so

$$(t_{m+1} - t_m)^2 \leqslant \pi \int_{t_m}^{t_{m+1}} (x_1^2 + x_2^2)\, dt .$$

Hence

$$\sum (t_{m+1} - t_m)^2 < \infty . \tag{6.5}$$

Hence

$$(t_n - t_0)^2 = \left\{\sum_0^{n-1} (t_{m+1} - t_m)\right\}^2 \leqslant n \sum_0^{n-1} (t_{m+1} - t_m)^2 .$$

The desired result that $t_n^2 = O(n)$ now follows from (6.5). This completes the proof.

Example 3. – Consider, with $\varepsilon > 0$,

$$y'' + t^{2+\varepsilon} y = t^{2+\varepsilon+n} + n(n-1)\, t^{n-2} , \tag{6.6}$$

which has the non-oscillatory solution $y = t^n$. The homogeneous equation has oscillatory solutions, with amplitude asymptotic to a multiple of $t^{-\frac{1}{2}-\frac{1}{4}\varepsilon}$. It may be shown from this that (6.6) has a unique non-oscillatory solution if $n < -\frac{1}{2} - \frac{1}{4}\varepsilon$, but an infinity of them if $n > -\frac{1}{2} - \frac{1}{4}\varepsilon$. Application of Theorem 5 indicates that there is at most one non-oscillatory solution if

$$2(2 + \varepsilon + n) + 1 \leqslant 4 ,$$

and so if $n \leqslant -\frac{1}{2} - \varepsilon$. While this is not precise, it shows, since $\varepsilon > 0$ is arbitrary, that the index 4 in (6.1) cannot be raised.

7. – Use of asymptotic integration.

If q is sufficiently well-behaved, we can use asymptotic formulae for the solutions of the homogeneous equation to obtain further explicit criteria.

THEOREM 6. – *Let q be positive, bounded from zero and in $C''[a, \infty)$. Let also $q'q^{-\frac{3}{2}}$ be of bounded variation over $[a, \infty)$. Let*

$$q^{-\frac{3}{4}}f \to 0 \qquad \text{as } t \to \infty \,. \tag{7.1}$$

Then (5.2) *has at most one non-oscillatory solution.*

PROOF. – The assumption that $q'q^{-\frac{3}{2}}$ is of bounded variation implies that $(q'q^{-\frac{3}{2}})' \in L^1(a, \infty)$. This fact and the assumption that q is bounded below by a positive constant imply that $q'(t) = o\big(q^{\frac{3}{2}}(t)\big)$. This enables us to apply Corollary 5.3 in [7, p. 348] and conclude that $\pi N(T) \sim \int_0^T q^{\frac{1}{2}}(t)\, dt$ as $T \to \infty$.

We can also refer to Theorem 1 in [1] and observe that any solution $x(t) = O\big(q^{-\frac{1}{4}}(t)\big)$. We make the assumption

$$\int_a^T |q^{-\frac{1}{4}}(t) f(t)|\, dt = o\Big(\int_a^T q^{\frac{1}{2}}(t)\, dt\Big) \,. \tag{7.2}$$

Criterion (7.1) actually represents a simpler special case of (7.2). Then

$$[N(T)]^{-1}\int_a^T |f(s)\, x(s)| ds = [N(T)]^{-1}\Big[\int_a^T |f(s)\, q^{\frac{1}{4}}(s)\, x(s)\, q^{-\frac{1}{4}}| ds\Big]\Big[\int_a^T q^{\frac{1}{2}}(t)\, dt\Big]\Big[\int_a^T q^{\frac{1}{2}}(t)\, dt\Big]^{-1} .$$

However, we can conclude that the right side of the above equation approaches zero as T grows large. Thus (4.3) is satisfied and the result follows.

Note that we may apply Theorem 6 to Example 3 to obtain the correct value of n.

8. – The case of a line-segment.

We pass now from conditions for the uniqueness of any non-oscillatory solution, to conditions which do no more than exclude the case when the initial data of such solutions form a set with non-empty interior, as in (iv) of § 1. The set of non-oscillatory solutions of (1.1) can then be represented in the form

$$y(t) = y_0(t) + cx_1(t) \,, \qquad c \in I \,, \tag{8.1}$$

where y_0 is a particular such solution, x_1 is a non-trivial solution of (1.2), and the range I of the parameter c is an interval on the real line. One must admit the eventualities that I might reduce to a single point, or be empty.

We start with a general test, which will then be simplified.

THEOREM 7. – *Let there be a sequence of solutions $u_m(t)$ of* (1.2), *with*

$$|u_m(a)|^2 + |u_m'(a)|^2 = 1\,, \qquad m = 1, 2, \ldots\,, \tag{8.2}$$

and for each u_m a pair γ_m, δ_m of its consecutive zeros, such that

$$\gamma_m \to \infty\,, \tag{8.3}$$

and

$$\int_{\gamma_m}^{\delta_m} f(s)\, u_m(s)\, ds \to 0\,. \tag{8.4}$$

Then non-oscillatory solutions of (1.1), *if they exist, are given as in* (8.1).

PROOF. – We suppose the contrary, that the set of initial data $(y(a), y'(a))$ of non-oscillatory solutions contains a non-empty open set. Without loss of generality, we may suppose this set to be generated by convex linear combinations of a finite number of points of the set, sot that the associated solutions will have the same fixed sign on a common half-axis $[T, \infty)$. We may therefore suppose that there is a non-oscillatory solution y^*, and an $\varepsilon > 0$ such that any solution of (1.1) satisfying

$$|y(a) - y^*(a)|^2 + |y'(a) - y^{*\prime}(a)|^2 \leqslant \varepsilon^2 \tag{8.5}$$

is also non-oscillatory, and has fixed sign for $t \geqslant T$.

We consider in particular solutions of the form

$$y = y^* + \varrho v_m\,, \tag{8.6}$$

where v_m is a solution of (1.2) such that

$$v_m(a) = u_m'(a)/p(a)\,, \qquad v_m'(a) = - u_m(a)/p(a)\,. \tag{8.7}$$

We then have $W(v_m, u_m) = 1$, and so, as in § 2,

$$y^* = v_m W(y^*, u_m) - u_m W(y^*, v_m)\,. \tag{8.8}$$

Now solutions of the form (8.6) will satisfy (8.5) if $|\varrho| \leqslant \varepsilon p(a) = \varepsilon_1$, say. It thus follows that

$$v_m\{W(y^*, u_m) \pm \varepsilon_1\} - u_m W(y^*, v_m)$$

will have the same fixed sign for $t \geqslant T$. Arguing as in § 3, and supposing m so large that $\gamma_m > T$, we have that $W(y^*, u_m) \pm \varepsilon_1$ have the same sign at $t = \gamma_m$, while they have the same (but opposite) sign at $t = \delta_m$. It follows that

$$|W(y^*, u_m)(\delta_m) - W(y^*, u_m)(\gamma_m)| \geqslant 2\varepsilon_1 ,$$

for large m. This contradicts (8.4). The theorem is thus proved.

In particular, the solutions u_m could be all the same. We give a fuller result for this case.

THEOREM 8. – *Let there be a non-trivial solution x_1 of* (1.2) *and a sequence α_k, β_k of pairs of successive zeros of x_1, with $\alpha_k \to \infty$, such that*

$$\int_{\alpha_k}^{\beta_k} f(s) x_1(s)\, ds \to 0 . \tag{8.9}$$

Then the set of solutions y of (1.1) *such that, for all large k,*

$$y(\alpha_k) y(\beta_k) \geqslant 0 \tag{8.10}$$

has the form (8.1), *where y_0 is some solution with this property, if one exists, and c has any real value.*

The proof is the same as that of Theorem 1; we argue now that S_1 reduces to a single point A_1.

The set of solutions in question will of course include all non-oscillatory solutions.

COROLLARY 9. – *Let x_1 be a non-trivial solution of* (1.2), *and let $t_0, t_1, \ldots$ be a sequence of its successive zeros. Let*

$$\int_{t_n}^{t_{n+1}} f(s) x_1(s)\, ds \to 0 . \tag{8.11}$$

Then the set of non-oscillatory solutions of (1.1) *has the form* (8.1). *If y is a solution of* (1.1) *which is not of the form $y_0 + cx_1$, where y_0 is a non-oscillatory solution of* (1.1), *then beyond some point between two successive zeros of x_1 there lies a zero of y.*

More specially, if $\int_a^\infty f(s) x_1(s)\, ds$ converges, then the last statement can be reversed;

between two successive zeros of y there lies a zero of x_1. We deduce this easily once we have shown that $W(y, x_1)$ tends to a non-zero limit, which we show as follows.

First, $W(y, x_1)(t) = W(y, x_1)(a) - \int_a^t f(s)x_1(s)\,ds$. Thus $W(y, x_1)(t)$ has a limit as $t \to \infty$. We must show it is nonzero. If y is not of the form $y_0 + cx_1$, we may assume y has the form $y = y_0 + cx_3$, for some $c \neq 0$, where x_3 is a solution of (1.2) which is linearly independent from x_1. Then

$$W(y, x_1) = W(y_0 + cx_3, x_1) = W(y_0, x_1) + cW(x_3, x_1)\,. \tag{8.12}$$

Since y_0 is nonoscillatory, $W(y_0, x_1)(t)$ has a sequence of zeros $\{t_k\}$ where $t_k \to \infty$ as $k \to \infty$. See [5, Lemma 2] or the remarks following (3.5). Since $W(y, x_1)(t)$ has a limit, (8.12) implies that $W(y, x_1)(t) \to cW(x_3, x_1) \neq 0$, as $t \to \infty$. This means that for t large enough, say $t \geqslant T$, $y(t)$ and $x_1(t)$ cannot have any common zeros.

Suppose $t_1 \geqslant T$ and $t_2 > t_1$ are two consecutive zeros of y. If $x_1(t)$ did not have a zero in (t_1, t_2), Rolle's Theorem applied to (y/x_1) would imply that $W(y, x_1)(t)$ has a zero in (t_1, t_2), a contradiction. This establishes the separation of zeros between y and x_1.

For other results on separation, see [14].

9. - Some special line-segment conditions.

In Theorem 6 we gave conditions based on methods of asymptotic integration, which limited non-oscillatory solutions to, at most, a single solution. We give now some similar arguments, with less demanding hypotheses, which limit the set of non-oscillatory solutions to, at most, a set with initial data forming a line-segment.

THEOREM 9. – *Let q be positive and locally L^1 on $(0, \infty)$, and let f be locally L^1. Let for some $T > a$ and some $A \in [0, 2/(\pi e)]$,*

$$\ln q(t'') - \ln q(t') \geqslant -A\int_{t'}^{t''} q^{\frac{1}{2}}(s)\,ds\,, \qquad T \leqslant t' \leqslant t'' < \infty\,, \tag{9.1}$$

and let

$$q^{-\frac{3}{4}}(t)f(t) \to 0 \qquad \text{as } t \to \infty\,. \tag{9.2}$$

Then non-oscillatory solutions of (5.1), *if more than one exists, are given in the form* (8.1).

PROOF. – Let $t_0, t_1, \ldots$ denote successive zeros of some non-trivial solution of (5.1) in (T, ∞). We claim that for some $B \in (0, 1)$, we have

$$q(t) > Bq(t_n)\,, \qquad t_n \leqslant t < t_{n+1}\,. \tag{9.3}$$

Supposing this false for some B, we choose t'' as the first value in (t_n, t_{n+1}) for which $q(t'') = Bq(t_n)$, and t' as the last value in $[t_n, t'')$ for which $q(t') = q(t_n)$. Then (9.1) yields

$$\int_{t'}^{t''} q^{\frac{1}{2}}(t)\, dt \geqslant A^{-1} \ln B^{-1} .$$

Now in (t', t'') we have

$$q(t'') < q(t) < q(t') = q(t_n) = B^{-1} q(t'') ,$$

and so we deduce that

$$(t'' - t')\, q^{\frac{1}{2}}(t'') \geqslant A^{-1} B^{\frac{1}{2}} \ln B^{-1} . \tag{9.4}$$

With A restricted as above, we can choose B so that the right of (9.4) is not less than π. The Sturm comparison theorem then shows that a solution of (5.1) must have a zero in $(t', t'']$. This gives a contradiction, so that (9.3) is established.

Using the Sturm comparison theorem again, we then have

$$(t_{n+1} - t_n)\{Bq(t_n)\}^{\frac{1}{2}} \leqslant \pi . \tag{9.5}$$

Since the zeros have no finite limit-point, it follows that

$$\sum \{q(t_n)\}^{-\frac{1}{2}} = \infty .$$

We deduce that we must have

$$q(t_{n+2}) \leqslant 2q(t_n) \tag{9.6}$$

for an infinite sequence of n-values.

We now apply Theorem 7. We consider the set of solutions of (5.1) such that

$$x^2(a) + x'^2(a) \leqslant 1 . \tag{9.7}$$

We note that the area in the (x, x')-plane in this case will be the same as that described by the set of initial data given by

$$q^{\frac{1}{2}}(t_n) x^2(t_n) + q^{-\frac{1}{2}}(t_n) x'^2(t_n) \leqslant 1 . \tag{9.8}$$

Since the map $\big(x(a), x'(a)\big) \mapsto \big(x(t_n), x'(t_n)\big)$ has unit Jacobian (see e.g. [7, p. 96] or [16, p. 88]), we deduce that there must be a point of the boundary of (9.7) which is mapped into a point of the boundary of (9.8). In other words, there is a solution x_n say of (5.1) which satisfies (9.7)(9.8) with equality in both cases. We restrict attention to n for which (9.6) holds, and denote by α_n, β_n the first two zeros of x_n in $[t_n, t_{n+2})$. We claim that, in fulfillment of (8.4),

$$\int_{\alpha_n}^{\beta_n} f(s) x_n(s)\, ds \to 0 \tag{9.9}$$

as $n \to \infty$, subject to (9.6). This will follow from the results

$$x_n(t) = O\{(q(t_n))^{-\frac{1}{4}}\}\,, \qquad t_n \leqslant t \leqslant t_{n+2}\,, \tag{9.10}$$

and

$$\beta_n - \alpha_n = O\{(q(t_n))^{-\frac{1}{2}}\}\,, \tag{9.11}$$

together with (9.2).

We first note that, by two applications of (9.3),

$$q(t) \geqslant B^2 q(t_n)\,, \qquad t_n \leqslant t \leqslant t_{n+2}\,. \tag{9.12}$$

We have also, by an interchange of the roles of t, t_n in (9.3),

$$q(t_{n+1}) \geqslant B q(t)\,, \qquad t_n \leqslant t \leqslant t_{n+1}\,,$$

and so by using the above inequality twice and (9.6)

$$q(t) \leqslant 2B^{-2}\, q(t_n)\,, \qquad t_n \leqslant t \leqslant t_{n+2}\,. \tag{9.13}$$

We thus get (9.11) with the aid of (9.12) and the Sturm theorem.

To establish (9.10) we note that (9.1) implies that the function

$$q(t) \exp\left\{A \int_a^t q^{\frac{1}{2}}(s)\, ds\right\}$$

is non-decreasing in $[T, \infty)$. It is then seen that the function

$$\{x^2(t) + x'^2(t)/q(t)\} \exp\left\{- A \int_a^t q^{\frac{1}{2}}(s)\, ds\right\}$$

is non-increasing in $[T, \infty)$, if x is a solution of (5.1). If

$$h_n(t) = \left(x_n^2(t) + x_n'^2(t)/q(t)\right) \exp\left(- A \int_{t_n}^t q^{\frac{1}{2}}(s)\, ds\right)$$

then $h_n(t) \leqslant h_n(t_n)$ for $t \in [t_n, t_{n+2}]$. It follows now from (9.8) that

$$x_n^2(t) \leqslant q^{-\frac{1}{2}}(t_n)\left\{\exp A\int_{t_k}^{t} q^{\frac{1}{2}}(s)\, ds\right\}, \qquad t_n \leqslant t \leqslant t_{n+2}\,.$$

Here the last factor is bounded, by a further use of (9.12)-(9.13) and the Sturm comparison theorem. This proves (9.10), and completes the proof of Theorem 9.

One special case will be that in which q is positive and non-decreasing, so that (9.1) is trivially satisfied. In another, we assume q locally absolutely continuous, with $q'q^{-\frac{3}{2}} \geqslant -2/(\pi e)$ almost everywhere. This covers the situation of Theorem 6, since if $q'q^{-\frac{3}{2}}$ is of bounded variation and so tends to a limit, this limit must be 0; however, the conclusion of Theorem 9 is weaker, along with the assumptions.

10. – A remark on the two-dimensional case.

We refer to case (iv) of § 1, in which the set of initial data $(y(a), y'(a))$ of non-oscillatory solutions has non-empty interior. Let us suppose that it is also bounded and so, being convex, has an area J, say. In the notation of §§ 2, 3 we write for a non-oscillatory solution

$$y = x_1(A_2 - F_2) - x_2(A_1 - F_1)\,.$$

As t increases the radius vector (x_1, x_2) will execute complete revolutions, roughly speaking once for every two zeros of x_1 or x_2. The radius vector $(F_1 - A_1, F_2 - A_2)$ will do the same (with an error of absolute value $< \pi$). We thus get

$$\begin{aligned}
\tfrac{1}{2}N(T)\,J &\leqslant \tfrac{1}{2}\int_a^T |(F_1 - A_1)\,dF_2 - (F_2 - A_2)\,dF_1| + O(1)\\
&= \tfrac{1}{2}\int_a^T |(F_1 - A_1)\,fx_2 - (F_2 - A_2)\,fx_1|dt + O(1)\\
&= \tfrac{1}{2}\int_a^T |fy|dt + O(1)\,.
\end{aligned}$$

Thus

$$J \leqslant \liminf \{N(T)\}^{-1}\int_0^T |fy|\,dt\,. \tag{10.1}$$

For example, in the case $y'' + y = 1$, we can have $y = 1$, with $f = 1$, and $N(T) \sim T/\pi$. This gives $J \leqslant \pi$. Since J is the unit disc (with parts of the boundary), (10.1) is here precise.

Of course, the positivity of the right of (10.1) is a necessary condition for case (iv) of § 1 to occur.

11. – Sufficient conditions for the existence of non-oscillatory solutions.

In previous sections of this paper, we studied the question of uniqueness of the nonoscillatory solution of (1.1) or (5.2). In this part, we briefly indicate some known results on existence. We will assume $p(t) \equiv 1$.

In ATKINSON [2], the following theorem was proven.

THEOREM 10 [2, Lemma 2]. – *Consider* (5.2) *on some* t *interval*, $t \geqslant t_0$. *Let* $q(t) > 0$ *and let* $h(t)$ *be a bounded positive function such that* $h''(t) = f(t)$. *Suppose* $h(t) > \int_t^\infty du \int_u^\infty q(s) h(s)\, ds$, $t > t_0$. *Then* (5.2) *has a solution* $y(t)$ *satisfying* $0 < y(t) < h(t)$, $t > t_0$.

As a corollary to Theorem 10, we would like to point out the following result, which sometimes might be easier to apply.

COROLLARY 10. – *Suppose* $f(t)$ *and* $q(t)$ *are positive and* $f(t) > q(t)\int_t^\infty (s-t) f(s)\, ds$. *Then* (6) *has a nonoscillatory solution* $y(t)$ *which satisfies* $0 < y(t) < \int_t^\infty (s-t) f(s)\, ds$.

PROOF. – Define $h(t) = \int_t^\infty (s-t) f(s)\, ds$. It is straight-forward to show that $h(t)$ satisfies the conditions of Theorem 10.

We remark that it can be shown that Corollary 10 is actually equivalent to Theorem 10.

As an example of Corollary 10, consider the equation

$$y'' + (1/t) y = (1 + \sin^2 t) e^{-t}, \qquad t \geqslant 4 . \tag{11.1}$$

We note that

$$(1 + \sin^2 t) e^{-t} \geqslant e^{-t} > (2/t) e^{-t} = (2/t)\int_t^\infty (s-t) e^{-s}\, ds \geqslant (1/t)\int_t^\infty (s-t)(1 + \sin^2 s) e^{-s}\, ds.$$

Thus Corollary 10 implies that a nonoscillatory solution of (11.1) exists.

We also point out that a combination of Theorem 10 or Corollary 10 with any uniqueness result, for example Corollaries 6, 7 and 8 or Theorems 3, 5 and 6 in this paper or Theorem 3 and Corollaries 2, 3 and 4 in [5], yields the existence of a unique nonoscillatory solution of (5.2).

For instance in the preceding example, equation (11.1), $q(t) = 1/t \geqslant 0$ and $q'(t) \to 0$ as $t \to \infty$. Therefore by the previously mentioned result of KAUFFMAN [10] no solution of the homogeneous equation can grow exponentially. Thus $\int^\infty x(t) f(t)\, dt$ exists, where $f(t) = (1 + \sin^2 t) e^{-t}$ and $x(t)$ is any solution of the homogeneous equation. Thus Corollaries 10 and 6 together imply that (11.1) has a unique nonoscillatory solution.

Another theorem which yields existence and uniqueness is the following known result of Hartman.

THEOREM 11 [8, p. 433]. - *Suppose* $q(t)$ *and* $f(t) > 0$, $q'(t) \geqslant 0$, $f'(t) \leqslant 0$, *and* $f(t) \to 0$ *as* $t \to \infty$. *Then* (5.2) *has a unique nonoscillatory solution.*

To prove this theorem, let $x_1(t)$ and $x_2(t)$ be linearly independent solutions of (5.1) such that $W(x_2, x_2)(t) \equiv 1$. Define $y(t)$ as follows;

$$y(t) = \int_t^\infty \big(x_1(s)x_2(t) - x_2(s)x_1(t)\big) f(s)\, ds .$$

Then it can be shown that $y(t)$ is the unique nonoscillatory solution. See [8, p. 450] for the details.

For a result similar to Theorem 11, see JONES [9, Theorem 6].

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