Double Adjoints of Hammerstein Functionals Defined on Continuous Functions (*).

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Summary. — *One of the main reasons for studying Hammerstein operators is their possible application to the analytic study of nonlinear differential equations. In fact, the kernel φ used to represent such operators has properties similar to ones that insure the existence of solutions to equations of the form $\dot{x}(t) = \varphi(x(t), t)$ or $\dot{x}(t) = \varphi(x(t))$. The main purpose of the present paper is to study limits of sequences of Hammerstein functionals. While these functionals fail to be linear, their adjoints (in the sense of J. Batt, see [3] and [4]) are linear operators. The Vitali-Hahn-Saks theorem is one of the main tools used to show that if the double adjoint of T_n converges on simple functions, then T_n converges to T uniformly over an appropriate set of continuous functions. Moreover, T is Hammerstein. Other results use the kernel representation of T_n and T . This situation is of particular interest when the double adjoints of T_n are dominated by probability measures.*

1. — Introduction.

Recently many extensions of Riesz type theorems to nonlinear operators have been obtained; see for example [1], [2], [3], [8], and [11]. In [2] operators on $M(X, \mathfrak{R})$, the space of X valued functions that are uniform limits of $\mathcal{B}$ -simple functions are represented, while in [11] non linear operators on L^p spaces are considered. Essentially there are two main types of representation: The first type represents the operator as an integral with respect to a positive measure involving an appropriate kernel function, for example $T(f) = \int \varphi[f(t), t] d\mu(t)$ as in [11]. The second type of representation is via an integral with respect to a measure that maps a measurable set into an appropriate non linear operator—such an approach is taken in [1], [2], and [3].

In [3] J. BATT studies operators defined on $C(K, X)$ satisfying

$$(1) \quad T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f)$$

whenever f_1 and f_2 have disjoint supports. In this case K denotes a compact set and $C(K, X)$ denotes the space of X valued continuous functions (under the sup norm). Operators satisfying (1) were called Hammerstein operators by J. BATT and

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additive operators by FRIEDMAN and TONG [8]. Hammerstein operators have direct applications to the theory of generalized random processes (see [9]) and to the theory of fading memory in continuum mechanics (see [5]). One of the main reasons for studying Hammerstein operators, however, is the possible application to the functional analytic study of non linear differential equations. In fact the kernel φ used to represent T as in [11] has properties very similar to the kernel used in the study of the existence of solutions to the equation $\dot{x}(t) = \varphi(x(t), t)$ or the equation $\dot{x}(t) = \varphi[x(t)]$. For additional references on applications of this type the reader is referred to [11].

The main purpose of the present article is to obtain limit theorems for sequences of Hammerstein functionals. The main technical tool for these results will be the existence of the double adjoint for these functionals. While each functional fails to be linear it has a linear adjoint in the sense of [4]. J. BART in [4] characterizes the compact and weakly compact mappings among $\mathcal{A}$ -bounded (not necessarily linear) maps. One of the main points there, is the introduction of the space E^β in which the adjoint takes values and which plays the role of the traditional dual space E' . The properties of the double adjoint (for the linear case) was used in [10] where the authors study the w. c. a. property.

Let T be a mapping from $C(K, X)$ into C where X represents a Banach space and C represents the scalar field. Assume that T satisfies (1). It is shown in [3] that under additional continuity conditions $T(f) = \int f dm$ where m maps $\mathcal{B}$ into $M(X, C)$. There $\mathcal{B}$ denotes the Borel field of K and $M(X, C)$ denotes certain maps from X into C . Of course, $\int f dm$ is additive in f (but not linear). Our first result is a Egoroff type theorem—that is, if s_n is a sequence of simple functions converging m a.e. (this will be defined later) to f , then K may be partitioned into a sequence A_k and a set N such that s_n converges to f uniformly on A_k and $\int_N s_n dm = 0$ for all n . Next we consider simple functions of the form $\sum \chi_{C_i} x_i$ where C_i are disjoint sets, χ_{C_i} is the characteristic function of C_i and $x_i \in X$. Naturally $\sum \chi_{C_i} x_i$ fails to be in $C(K, X)$. The question is « How far removed is it from $C(K, X)$? » We show that there exists a map $\hat{}$ that is additive on $M(X, \mathcal{B})$ such that $\sum \hat{\chi}_{C_i} x_i$ is in the dual of $M_H[C(K, X), C]$ (the space of Hammerstein functionals on $C(K, X)$ satisfying some continuity conditions). Moreover

$$q_\alpha(\sum \hat{\chi}_{C_i} x_i) \leq \frac{\|\sum \chi_{C_i} x_i\|}{\alpha}$$

q_α is a family of semi-norms defined on the dual of $M_H[C(K, X), C]$. The Egoroff proposition is used to show this. Again, using the Egoroff proposition, we show that if a sequence s_n of simple functions converges m a.e. to f then there exists a partition B_k and N of K such that $T''[\chi_{B_k}(s_n - f)]$ converges to 0 and $\int_N s_n dm = 0$.

Here T'' denotes the double adjoint of $T \in M_H[C(K, X), C]$ in the sense of [4]. Our first theorem asserts that if we have a sequence of functionals $T_n \in M_H[C(K, X), C]$

and if the double adjoints of T_n behave appropriately, then we can conclude that T_n converges to $T \in M_H[C(K, X), C]$. This convergence will be made precise later on and will involve functions in the set $C(K, X)$ of the form $V(\cdot)x$ where x is fixed in X and V is a real valued continuous function on K with $0 \leq V \leq 1$. Next we look at the kernel representation of the functionals in $M_H[C(K, X), C]$. This involves the use of a non linear Radon Nikodym type theorem shown in [3]. We can write

$$T_n(f) = \int u_n(t, f(t)) d\mu_n(t), \quad T(f) = \int u(t, f(t)) d\mu(t)$$

(where T is as in the first theorem). Moreover, we show that

$$\int_B u_n(t, x) d\mu_n(t) \rightarrow \int_B u(t, x) d\mu(t) \quad \text{uniformly for } \|x\| \leq \alpha$$

where u has the same properties as u_n . Finally the third theorem yields conditions under which μ_n may be taken equal to μ . These conditions are similar to the ones imposed on dominated operations as defined in [6] where T is dominated by μ if $|T(f)| \leq \int |f| d\mu$ where μ is a positive measure.

2. - Results.

We start by establishing some basic notations. Let K be a compact set, X a Banach space, $\mathfrak{B}$ the Borel field of K and $C(K, X)$ and $M(X, \mathfrak{B})$ as defined previously. Let u be a function from X into C . For $\alpha > 0$, u_α denotes the restriction of u to the closed α -ball of X . Let $\|u_\alpha\| = \sup \|u(x)\|$ where the sup is over $\|x\| \leq \alpha$. For $\delta > 0$ define

$$D_\delta u_\alpha = \sup \|u(x) - u(y)\|$$

where the sup is over $\|x\| \leq \alpha$, $\|y\| \leq \alpha$, $\|x - y\| \leq \delta$. $M(X, C)$ will denote the space of all functions from X into C bounded on the α -balls of X , uniformly continuous on bounded sets of X and 0 at 0. That is, if $u \in M(X, C)$, then

$$\begin{aligned} u(0) &= 0 \\ \|u_\alpha\| &< \infty \quad \text{for } \alpha > 0 \\ \lim_{\delta \rightarrow 0} D_\delta u_\alpha &= 0 \end{aligned}$$

Let m be a finitely additive function from $\mathfrak{B}$ into $M(X, C)$. We set $m_\alpha(B) = m(B_\alpha)$ and define

$$sv[m_\alpha, B] = \sup \left\| \sum m(B_i)x_i \right\|$$

where the sup is over partitions B_i of B with $\|x_i\| \leq \alpha$.

Also define

$$sv_\delta(m_\alpha, B) = \sup \left\| \sum m(B_i)x_i - \sum m(B_i)y_i \right\|$$

where the sup is over partitions B_i of B with $\|x_i\| \leq \alpha$, $\|y_i\| \leq \alpha$, $\|x_i - y_i\| \leq \delta$.

Unless otherwise stated m will denote a finitely additive set function from $\mathcal{B}$ into $M(X, C)$ with

$$sv[m_\alpha, K] < \infty \quad \text{and} \quad \lim_{\delta \rightarrow 0} sv_\delta[m_\alpha, K] = 0.$$

We define $\int \sum \chi_{C_i} x_i dm = \sum m(C_i)x_i$ for disjoint C_i . By $M_H[C(K, X), C]$ we mean the set of all maps T from $C(K, X)$ into C satisfying

- (1) $T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f)$
where f_1 and f_2 have disjoint support
- (2) $T(0) = 0$,
- (3) $\|T_\alpha\| < \infty$ for all $\alpha > 0$
- (4) $\lim_{\delta \rightarrow 0} D_\delta T_\alpha = 0$.

Y' will denote the dual of the space Y and T' will denote the adjoint of T (in the sense of [4]). Let $T \in M_H[C(K, X), C]$ and let m_T be its representative measure—that is, $T(f) = \int f dm_T$. It is shown in [3] that m_T satisfies the properties stated above for m .

Finally we define a property to hold m a.e., if there exists a set A such that the property is true for all $x \notin A$ and for $B \in \mathcal{B}$ and $B \subset A$ we have $m(B) = 0$.

PROPOSITION (Egoroff). — Assume that $\{s_n\}$ is a sequence of X -valued simple functions converging pointwise to f . Then there exists a partition of K into sets $A_k \in \mathcal{B}$ and $N \in \mathcal{B}$ such that

- (1) $\{s_n\}$ converges to f uniformly on each A_k ($k = 1, 2, \dots$);
- (2) $\int_N s_n dm_K = 0$ for all n .

PROOF. — Let

$$\hat{m}(E) = \sum_{n=1}^{\infty} \frac{\int_E s_n dm_T}{2^n [1 + \sup_{A \in \mathcal{B}} |\int_A s_n dm_T|]}$$

clearly $\hat{m}$ is finitely additive and bounded. Moreover since $m_K(\cdot)x$ is countably additive for each x (see [3]) it follows that $\hat{m}$ is a bounded countably additive

measure. Thus since s_n converges to $f \hat{m}$ a.e. it follows from the Egoroff theorem [6, p. 95] that there exists sets $A_k \in \mathfrak{B}$ and $N \in \mathfrak{B}$ such that s_n converges to f uniformly on A_k and N is a $\hat{m}$ null set i.e. $\int_N s_n dm_T = 0$.

Note. — The above proof goes through if pointwise convergence is replaced by m_T a.e. convergence.

We now define some semi-norms of interest. For every $\alpha > 0$ define p_α on

$$M_H[C(K, X), C] \quad \text{by} \quad p_\alpha(T) = \|T_\alpha\|.$$

Then we can define the dual topology on $M_H[C(K, X), C]'$ by

$$q_\alpha(V) = \sup |\langle V, T \rangle|$$

where the sup is over $p_\alpha(T) \leq 1$ and $V \in M_H[C(K, X), C]'$. Let $T \in M_H[C(K, X), C]$. Since $\|T_\alpha\| < \infty$, the adjoint of T exists by [4]. (Here $\mathcal{A}$ consists of all α -balls of $C(K, X)$ and $E^{\mathfrak{B}}$ is $M_H[C(K, X), C]$ while E is $C(K, X)$.) Let T' be the adjoint of T .

COROLLARY 1. — There exists a map $\hat{\cdot}$ from $M(X, \mathfrak{B})$ into $M_H[C(K, X), C]'$ that is additive on functions of disjoint supports in $M(X, \mathfrak{B})$ and has the following property:

$$q_\alpha(\sum \chi_{C_i} \hat{x}_i) \leq \frac{\|\sum \chi_{C_i} x_i\|}{\alpha}$$

where C_i are disjoint sets of $\mathfrak{B}$.

PROOF. — Let s_n be a sequence of X -valued simple functions converging to f pointwise. Since T maps bounded sets into bounded sets T is a A -weakly compact in the terminology of [4]. Thus T' maps C into $M_H[C(K, X), C]$ and T'' maps $M_H[C(K, X), C]'$ into C (Theorem 1 of [4]). Let N and B_k be sets as in the previous proposition. Then $\chi_{B_k} s_n$ converges uniformly to $\chi_{B_k} f$. Thus $\chi_{B_k} f \in M(X, \mathfrak{B})$. Let $x \in X$ and $B \in \mathfrak{B}$. Every element in $M_H[C(K, X), C]$ is of the form m_A where $A \in M_H[C(K, X), C]$ with $A(f) = \int f dm_A$. Define $\chi_B \hat{x}$ by

$$\langle \chi_B \hat{x}, m_A \rangle = m_A(B)x.$$

It is easy to see that $\chi_B \hat{x}$ is a linear functional on $M[C(K, X), C]$. Let $p_\alpha(A) \leq 1$. Since

$$\chi_B x = \frac{\|x\|}{\alpha} \chi_B \frac{x}{\|x\|} \alpha,$$

we can assume without loss of generality that $\|x\| \leq \alpha$. Then

$$|\langle \chi_B \hat{x}, m_A \rangle| = |m_A(B)x| \leq \alpha m_A(B) = p_\alpha(A) \leq 1.$$

Thus

$$\hat{\chi}_B x \in M_H[C(K, X), X]'.$$

(It is important to recall that in [3] it has been shown that if A and m_A correspond, then $sv_\alpha m_A = \|A_\alpha\|$.) It is easy to see that $\hat{\cdot}$ is additive on simple functions with disjoint support and hence on $M(X, \mathcal{B})$. Of course if $h \in M(X, \mathcal{B})$, then

$$\langle \hat{\chi}_B h, m_A \rangle = \int_B h dm_A.$$

Now

$$\begin{aligned} \left| \left\langle \sum_i \frac{\|x_i\|}{\alpha} \chi_{C_i} \frac{\alpha x_i}{\|x_i\|}, m_A \right\rangle \right| &= \left| \sum_i \frac{\|x_i\|}{\alpha} m_A(C_i) \left(\frac{\alpha x_i}{\|x_i\|} \right) \right| \\ &\leq \frac{\text{Max } \|x_i\|}{\alpha} sv_\alpha m_A \leq \frac{\|\sum \chi_{C_i} x_i\|}{\alpha} \quad \text{if } sv_\alpha m_A \leq 1. \end{aligned}$$

Thus

$$q_\alpha(\sum \hat{\chi}_{C_i} x_i) \leq \frac{\|\sum \chi_{C_i} x_i\|}{\alpha}.$$

COROLLARY 2. — Let s_n be a sequence of X simple functions where s_n converges pointwise to f . Let $T \in M_H[C(K, X), C]$. Then there exists a disjoint family of sets in $\mathcal{B}$, N and $\{B_k\}$ such that

- (1) $T''[\hat{\chi}_{B_k}(s_n - f)]$ converges to 0 (as $n \rightarrow \infty$) and
- (2) $\int_N s_n dm_T = 0$ for all n .

Proof. — Let N and B_k be as earlier. $\chi_{B_k} s_n$ converges uniformly to $\chi_{B_k} f \in M(X, \mathcal{B})$ (previous corollary). For k fixed,

$$\langle \chi_{B_k}(s_n - f), m_A \rangle = \left| \int_{B_k} (s_n - f) dm_A \right| \leq sv_\delta(m_A)_\alpha$$

where $\|\chi_{B_k} s_n\| \leq \alpha$, $\|\chi_{B_k} f\| \leq \alpha$, $\|\chi_{B_k}(s_n - f)\| \leq \delta$.

Since m_A has the property that $sv_\delta(m_A)_\alpha$ converges to 0 as $\delta \rightarrow 0$ it follows that $\chi_{B_k} s_n$ converges to $\chi_{B_k} f$ in the topology of $M_H[C(K, X), X]'$. Since T is A weakly compact it follows from [4] that T' is $\sigma(F', F)$, $\sigma M_H[C(K, X), C]$, $M_H[C(K, X), C]'$ continuous. Since T' is linear it follows (see [12]) that T'' is weakly continuous and the corollary follows. Again the proof goes through if pointwise convergence is replaced by m_T a.e.

We now introduce some new notation. If $B \in \mathcal{B}$ then $\pi_0(B)$ will denote the collection of all compact subsets of B ordered by $K_1 \leq K_2$ if $K_1 \subset K_2$. $\pi(B)$ will denote the collection of all open sets containing B and ordered by $G_1 \leq G_2$ if $G_2 \subset G_1$.

$V(C, G')$ denotes a function continuous on K , 1 on the compact set C , 0 off G where G is open and $C \subset G$ and $0 \leq v(C, G') \leq 1$. We introduce another family of semi-norms on $M_H[C(K, X), C]$ by

$$p_\alpha^\delta(T) = D_\delta T_\alpha \quad \text{for all } \delta > 0 \text{ and } \alpha > 0.$$

THEOREM 1. - Let T_n be a sequence in $M_H[C(K, X), C]$ such that for all $B \in \mathcal{B}$ $T_n''(\chi_B x)$ converges uniformly in x for $\|x\| \leq \alpha$. Assume $\sup_n p_\alpha(T_n) < \infty$ and $\lim_{\delta \rightarrow 0} p_\alpha^\delta(T_n) = 0$ uniformly in n . Then there exists $T \in M_H[C(K, X), C]$ such that

$$\lim_{C \in \pi_0(B)} \lim_{G \in \pi(C)} (T_n - T)[V(C, G')x] = 0$$

uniformly in $\|x\| \leq \alpha$. In addition if $B_i \in \mathcal{B}$ and $B_i \searrow \varnothing$ then

$$\sup_n \sup_{\|x\| \leq \alpha} \lim_{C \in \pi_0(B_i)} \lim_{G \in \pi(C)} T_n[V(C, G')x]$$

converges to 0 as $i \rightarrow \infty$.

PROOF. - Let m_n represent T_n . By [3] the $(m_n)_\alpha$ are countably additive measures. The $(m_n)_\alpha$ may be viewed as vector measures into the completion of $M(X, C)/\ker r_\alpha$ where r_α is a semi norm on $M(X, C)$ defined by

$$r_\alpha(W) = \sup_{\|x\| \leq \alpha} |W(x)|$$

For every $\alpha > 0$ let $w_{n,\alpha}$ be a positive measure such that

$$w_{n,\alpha}(A) \leq |(m_n)_\alpha(A)| \quad \text{where} \quad |(m_n)_\alpha(A)| = r_\alpha m_n(A)$$

and where in addition $(m_n)_\alpha \ll w_{n,\alpha}$ (see [7], p. 321).

Let

$$w_\alpha(A) = \sum_n \frac{w_{\alpha,n}(A)}{2^n [1 + w_{\alpha,n}(K)]}.$$

Now by [3] $(m_n)_\alpha$ are regular measures, thus $w_{\alpha,n}$ are regular and hence w_α . Using the previous notations we now show that $T''[\chi_B x] = m_{T'(B)}x$. Let $a \in C$ then

$$\langle T''[\chi_B x], a \rangle = \langle \chi_B x, T'(a) \rangle = m_{T'(a)}(B)x.$$

Now

$$\begin{aligned} \langle m_{T'(B)}x, a \rangle &= \lim_{C \in \Pi_0(B)} \lim_{G \in \Pi(C)} \langle T[V(C, G')x], a \rangle \\ &= \lim_{C \in \Pi_0(B)} \lim_{G \in \Pi(C)} \langle V(C, G')x, T'(a) \rangle \\ &= \lim_{C \in \Pi_0(B)} \lim_{G \in \Pi(C)} \int v(C, G')x dm_{T'(a)}. \end{aligned}$$

By regularity of $m_{T'(a)}$ this is $m_{T'(a)}(B)x$.

Thus,

$$T''[\hat{\chi}_B x] = m_T(B)x.$$

Since $T''[\hat{\chi}_B x]$ converges uniformly for $\|x\| \leq \alpha$ it follows that $(m_n)_\alpha(B)$ converge in the r_α semi-norms. By the Vitali-Hahn-Saks theorem $(m_n)_\alpha$ are uniformly absolutely continuous with respect to w_α : (Hence $(m_n)_\alpha$ are regular and also uniformly countably additive.) Let $m(B) = \lim m_n(B)$ in the r_α topology. Again by the Nikodym theorem $(m)_\alpha$ is countably additive (and regular). Now by the Nikodym boundedness theorem $r_\alpha[m_n(B)]$ is uniformly bounded in n , say

$$\sup_n r_\alpha[m_n(B)] \leq K_\alpha(B) \quad \text{so} \quad r_\alpha m(B) \leq r_\alpha[m(B) - m_n(B)] + K_\alpha(B)$$

If we pick n so that $r_\alpha[m(B) - m_n(B)] < 1$ then

$$\sup_{\|x\| \leq \alpha} |m(B)x| \leq K_\alpha(B) + 1.$$

Since $sv_\alpha(m_n) = p_\alpha(T_n)$ from the finiteness of $\sup_n sv_\alpha(m_n)$ it follows that since

$$|\sum m(B_i)x_i| \leq |\sum (m(B_i) - m_n(B_i))x_i| + |\sum m_n(B_i)x_i|$$

then $sv_\alpha(m) < \infty$.

Finally,

$$\begin{aligned} |\sum m(B_i)x_i - \sum m(B_i)x'_i| &\leq |\sum m(B_i)x_i - \sum m_n(B_i)x_i| + \\ &\quad + |\sum m_n(B_i)x_i - \sum m_n(B_i)x'_i| + |\sum m_n(B_i)x'_i - \sum m(B_i)x'_i| \end{aligned}$$

which implies $\lim_{\delta \rightarrow 0} sv_\delta(m)_\alpha = 0$, since the middle term is $p_\alpha^\delta(T_n)$ and is assumed to converge to 0 uniformly in n . By [3] it follows that there exists $T \in M_H[C(K, X), C]$ for which $m_T = m$. Also

$$\lim_{C \in \Pi_0(B)} \lim_{G \in \Pi(C)} (T_n - T)[V(C, G')x] = |(m_n(B) - m(B))x|$$

and this converges to 0 in the r_α semi-norms i.e. uniformly as $\|x\| \leq \alpha$. Finally let $B_i \downarrow \varphi$, then $r_\alpha m_n(B_i)$ converges to 0 uniformly in n , by uniform countable additivity. Thus $\sup_n \sup_{\|x\| \leq \alpha} |m_n(B_i)x|$ converges to 0.

Let $(S, \mathcal{B}, \tau)$ be a finite measure space. Let v be a map from S into $M(X, C)$ and let ϱ be a lifting of $L^\infty(\tau)$, see [6]. v is called a K -kernel if

- (1) $v(\cdot, x): t \rightarrow v(t)(x)$ is in $L^\infty(\tau)$ for each x ;
- (2) $v(\cdot, x) = \varrho v(\cdot, x)$ for each x ;

- (3) $t \rightarrow \sup_{\|x\| \leq \alpha} |v(t, x)|$ is bounded for each $\alpha > 0$;
 (4) $\sup_{\|x\| \leq \alpha} |v(t)(x)|$ and $D_\delta v(\cdot)_\alpha: t \rightarrow D_\delta v(t)_\alpha$ are $\mathcal{B}$ measurable and
 $\lim_{\delta \rightarrow 0} D_\delta v(t)_\alpha = 0$, τ a.e.

THEOREM 2. - Let T_n be a sequence in $M_H[C(K, X), C]$ satisfying the properties of the previous theorem. Then there exists finite positive measures μ_n and μ such that

$$T_n(f) = \int u_n(t, f(t)) d\mu_n(t)$$

$$T(f) = \int u(t, f(t)) d\mu(t)$$

(T is as in the previous theorem) where u_n and u are K -kernels. Moreover

$$\int_B u_n(t, x) d\mu_n(t) \quad \text{converges to} \quad \int_B u(t, x) d\mu(t)$$

uniformly for $\|x\| \leq \alpha$.

PROOF. - Let m_n and m represent T_n and T . It was shown that $m_n(B)$ converges to $m(B)$ in the r_α semi-norms. Since the variation dominates the semi-variation and is dominated by four times the semi-variation (and the same is true for the δ -variation) it follows that

$$\sup v[(m_n)_k] < \infty \quad \text{and} \quad \lim_{\delta \rightarrow 0} v_\delta(m_n)_k = 0$$

where v and v_δ denote the variation and the δ variation.

Let

$$\mu_n(B) = \sum_k \frac{1}{2^k} \frac{v(m_n)_k(B)}{[1 + v(m_n)_k]}.$$

Then $(m_n)_\alpha$ is countably additive, $v(m_n)_\alpha < \infty$, $\lim_{\delta \rightarrow 0} v_\delta(m_n) = 0$ uniformly in n and

$$v(m_n)_\alpha(B) \leq L_\alpha^n \mu_n(B)$$

where L_α^n is some constant. By [3] there exists K -kernels u_n such that

$$T_n(f) = \int u_n(t, f(t)) d\mu_n(t)$$

and similarly

$$T(f) = \int u(t, f(t)) d\mu(t).$$

Since

$$m_n(B)x = \int_B u_n(t, x) d\mu_n(t) \quad \text{and} \quad m(B)x = \int_B u(t, x) d\mu(t)$$

the theorem follows.

The last theorem states that one can pick $\mu_n = \mu$ provided the double adjoints of T_n are dominated by probability functions.

THEOREM 3. - Let T_n be a sequence in $M_H[C(K, X), C]$ such that for every $B \in \mathfrak{B}$, $T_n''(\chi_B x)$ converges uniformly for $\|x\| < \alpha$. Assume that $\lim_{\delta \rightarrow 0} p_\alpha^\delta(T_n) = 0$ uniformly in n and that for every positive integer k there exists probabilities p_k such that

$$\sup_{\|x\| \leq k} |T_n''(\chi_B x)| \leq L'_k p_k(B) \quad \text{for } B \in \mathfrak{B}.$$

Then the conclusion of the previous theorem hold with $\mu_n = \mu$.

PROOF. - Let

$$\mu(B) = \sum_k \frac{p_k(B)}{[1 + p_k(K)]^{2^k}}.$$

From the hypothesis it follows that $v(m_n)_k(B) \leq L'_k p_k(B) \leq L_k \mu(B)$.

The proof then proceeds as in Theorem 2.

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