On Conditional Expectation of Random Sets (*).

MICHEL VALADIER (Montpellier, France)

Summary. — *Let Γ be a random set with values in closed convex non empty subsets of the dual of a separable Fréchet space. Then its conditional expectation with respect to a sub-tribe is proved to exist and is related to regular conditional probability.*

Introduction.

This paper intends to give new results in the field of conditional expectation of random sets. The framework is that of SAINT-ÉBE [9]: the random sets are closed convex subsets of the dual space of a separable Fréchet space. The new points are:

1) the « separability » assumption on $(\Omega, \mathcal{B})$ is relaxed (in [9] theorem 6, p. 367 $\mathcal{B}$ is assumed to be the Borel tribe of a Suslin topology) and the expectation we obtain is smaller (in [9] it may happen that Γ is $\mathcal{B}$ -measurable and $E^{\mathcal{B}} \Gamma \supsetneq \Gamma$, in the present paper $E^{\mathcal{B}} \Gamma$ equals Γ);

2) the existence proof uses the notion of essential supremum of a family of multifunctions. This notion has been introduced by the author [10], [11];

3) the conditional expectation is related to regular conditional probability (when it exists). Thus [16], theorem 9.1, p. 27 and [17], theorem A of § 26.1, p. 354 are extended to multifunctions.

These results extend foregoing theorems of the author [12] and CASTAING-VALADIER [3] chapter VIII. Other references are [15], [1], [7], [4], [5], [6].

1. — Prerequisites and notations.

Let E be a separable Fréchet space with a fundamental sequence of seminorms (p_n) . Let V_n be the neighbourhood of 0: $V_n = \{x | p_n(x) \leq 1\}$.

The dual space E' with the weak topology $\sigma(E', E)$ is denoted E'_s . It is the union of the V_n^0 (the polar set of V_n) which are compact metrizable. Hence E'_s is a Lusin space. We denote by $\mathcal{B}(E'_s)$ its Borel tribe.

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Now let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{B}$ a sub tribe of $\mathcal{A}$. We denote $\mathcal{L}_E^1(\Omega, \mathcal{A}, P)$ (shortly $\mathcal{L}_E^1$ or $\mathcal{L}_E^1(\mathcal{A})$ to specify the tribe) the space of « integrable » functions from Ω to E' (see [9] for a detailed study). Namely $u \in \mathcal{L}_E^1$ if

- 1) u is $\mathcal{A} - \mathcal{B}(E'_s)$ measurable (see the appendix for equivalent properties);
- 2) $\exists \alpha \in \mathcal{L}_R^1$ and $\exists n \in N$ such that $\alpha \geq 0$ and $u(\omega) \in \alpha(\omega) V_n^0$ a.e.

We denote $\mathcal{L}_E^\infty(\Omega, \mathcal{A}, P)$ (shortly $\mathcal{L}_E^\infty$ or $\mathcal{L}_E^\infty(\mathcal{A})$) the space of all measurable bounded function $v: \Omega \rightarrow E$. Such a v is limit of a sequence of measurable bounded functions assuming a countable number of values.

The space $\mathcal{L}_E^1$ and $\mathcal{L}_E^\infty$ are in duality and are decomposable ([3] VII.3, p. 197). We shall often use the quotient spaces L_E^1 and L_E^∞ which are in separated duality.

So we have three pairs of spaces in duality: (E, E') , $(L_E^1(\mathcal{A}), L_E^\infty(\mathcal{A}))$ and $(L_E^1(\mathcal{B}), L_E^\infty(\mathcal{B}))$. In any of these pairs the bilinear form will be denoted $\langle \cdot, \cdot \rangle$, the indicator function of a set C will be denoted $\delta(\cdot | C)$ and its support function $\delta^*(\cdot | C)$.

If Γ is a multifunction from Ω to E' we denote by $\mathcal{L}_\Gamma^1(\mathcal{A})$ the set $\{u \in \mathcal{L}_E^1(\mathcal{A}) | u(\omega) \in \Gamma(\omega) \text{ a.e.}\}$ and $L_\Gamma^1(\mathcal{A})$ the corresponding space of equivalence classes. We say that Γ is $\mathcal{A}$ -measurable or is a $\mathcal{A}$ -random set if its graph $G(\Gamma)$ belongs to $\mathcal{A} \otimes \mathcal{B}(E'_s)$. Application of some theorems about measurable multifunctions will give $\hat{\mathcal{A}}$ -random set ($\hat{\mathcal{A}}$ is the completion of $\mathcal{A}$) but the appendix shows the existence of an equivalent (for equality almost everywhere) $\mathcal{A}$ -random set.

We use the following symbols:

$\text{CRS}(\mathcal{A}; E')$ denotes the set of closed non empty $\mathcal{A}$ -random subsets of E' ;

$\text{CRS}(\mathcal{A}; V_n^0)$ denotes those which are contained in V_n^0 , and we add one C to mean that convexity is assumed: $\text{CCRS}(\mathcal{A}; E')$ etc.

THEOREM 1. — *Let $(\Gamma_i)_{i \in I}$ be a family in $\text{CRS}(\mathcal{A}; V_n^0)$, then it admits an essential supremum Γ in $\text{CRS}(\mathcal{A}; V_n^0)$: that is*

- 1) $\forall i, \Gamma_i(\omega) \subset \Gamma(\omega)$ a.e.;
- 2) if $\Sigma \in \text{CRS}(\mathcal{A}; V_n^0)$ and $\forall i, \Gamma_i(\omega) \subset \Sigma(\omega)$ a.e., then $\Gamma(\omega) \subset \Sigma(\omega)$ a.e.

Moreover there exists a countable subfamily $(\Gamma_i)_{i \in D}$ such that $\Gamma(\omega) = \text{cl} \bigcup_{i \in D} \Gamma_i(\omega)$ a.e. (where cl abbreviates closure).

For the proof we refer to VALADIER [10, 11] (prop. 1.14).

REMARKS:

1) In [10, 11] we use the measurability property « $\{\omega | \Gamma_i(\omega) \cap \mathcal{U} \neq \emptyset\} \in \mathcal{A}$ for every open set $\mathcal{U}$ ». Here this is verified with $\hat{\mathcal{A}}$: see appendix.

2) In [10, 11] the result is stated with a separable metrizable space in place of V_n^0 . If E is a separable reflexive Banach space, the theorem can be stated for a family in $\text{CRS}(\mathcal{A}; E')$.

THEOREM 2. — *There exists a linear mapping*

$$E^{\mathfrak{B}}: L_{E'}^1(\mathcal{A}) \rightarrow L_{E'}^1(\mathfrak{B}) \quad \text{such that } \forall u \in L_{E'}^1(\mathcal{A}), \forall B \in \mathfrak{B}, \quad \int_B u dP = \int_B E^{\mathfrak{B}}(u) dP.$$

Reference: SAINTE-BEUVE [9], prop. 7, p. 366.

We shall still denote $E^{\mathfrak{B}}(f)$ for a scalar integrable (or quasi-integrable) function f .

LEMMA 1. — *If $u \in L_{E'}^1(\mathcal{A})$ and $v \in L_E^\infty(\mathfrak{B})$ one has $E^{\mathfrak{B}}(\langle v(\cdot), u(\cdot) \rangle)(\omega) = \langle v(\omega), E^{\mathfrak{B}}(u)(\omega) \rangle$ a.e. and $\int_\Omega \langle v, u \rangle dP = \int_\Omega \langle v, E^{\mathfrak{B}}(u) \rangle dP$.*

PROOF. — It suffices to prove the last formula. Indeed if it holds for $v\chi_B$ ($B \in \mathfrak{B}$) then $\int_B \langle v, u \rangle dP = \int_B \langle v, E^{\mathfrak{B}}(u) \rangle dP$ hence $\int E^{\mathfrak{B}}(\langle v, u \rangle) dP = \int \langle v, E^{\mathfrak{B}}(u) \rangle dP$ which is equivalent to $E^{\mathfrak{B}}(\langle v, u \rangle) = \langle v, E^{\mathfrak{B}}(u) \rangle$.

The equality $\int_\Omega \langle v, u \rangle dP = \int_\Omega \langle v, E^{\mathfrak{B}}(u) \rangle dP$ holds if v assumes a countable number of values. By a limiting process the equality is valid for any v .

2. — Preliminary version of the main theorem.

LEMMA 2. — *Let $\Gamma \in \text{CCRS}(\mathcal{A}; V_n^0)$. There exists $\Sigma \in \text{CCRS}(\mathfrak{B}; V_n^0)$ such that*

1) Σ is the smallest (for inclusion a.e.) of the $\Theta \in \text{CCRS}(\mathfrak{B}; V_n^0)$ such that

$$\forall u \in L_{E'}^1(\mathcal{A}), \quad E^{\mathfrak{B}}(u)(\omega) \in \Theta(\omega) \text{ a.e.}$$

2) Σ is the unique $\Theta \in \text{CCRS}(\mathfrak{B}; V_n^0)$ such that $\forall v \in L_E^\infty(\mathfrak{B})$,

$$\int_\Omega \delta^*(v(\omega)|\Gamma(\omega)) P(d\omega) = \int_\Omega \delta^*(v(\omega)|\Theta(\omega)) P(d\omega).$$

REMARK. — Assertion 1) means that Σ is the essential supremum of the family of multifunction $(\omega \mapsto \{E^{\mathfrak{B}}(u)(\omega)\})$, $u \in L_{E'}^1(\mathcal{A})$.

PROOF. — 1) Let Σ be the essential supremum of the $\omega \mapsto \{E^{\mathfrak{B}}(u)(\omega)\}$. By theorem 1 there exists a sequence (u_n) in $L_{E'}^1(\mathcal{A})$ such that $\Sigma(\omega) = \text{cl} \bigcup_n \{E^{\mathfrak{B}}(u_n)(\omega)\}$.

Consider the countable family $(u'_i)_{i \in \mathbb{N}}$ of barycenters with rational coefficients of all finite subsets of $\{u_n | n \in \mathbb{N}\}$. Each u'_i belongs to $L_{E'}^1(\mathcal{A})$.

So

$$\Sigma(\omega) \ni E^{\mathfrak{B}}(u'_i)(\omega) \quad \text{a.e.}$$

and

$$\Sigma(\omega) = \text{cl} \bigcup_{i \in D} \{E^{\mathfrak{B}}(u'_i)(\omega)\} \quad \text{a.e.}$$

That proves the convexity of $\Sigma(\omega)$.

2) a) We begin to prove that Σ verifies .

$$\delta^*(x|\Sigma(\omega)) = E^{\mathfrak{B}}[\delta^*(x|I(\cdot))](\omega) .$$

One has

$$\begin{aligned} \delta^*(x|\Sigma(\omega)) &= \sup_n \langle x, E^{\mathfrak{B}}(u_n)(\omega) \rangle \\ &= \sup_n E^{\mathfrak{B}}[\langle x, u_n(\cdot) \rangle](\omega) \\ &\leq E^{\mathfrak{B}}[\delta^*(x|I(\cdot))](\omega) . \end{aligned}$$

Conversely, for every n there exists (thanks to [3] III.22) $f_n \in \mathcal{L}_T^1(\mathcal{A})$ such that

$$\langle x, f_n(\omega) \rangle \geq \delta^*(x|I(\omega)) - \frac{1}{n} \quad \text{a.e.}$$

Then for every $B \in \mathfrak{B}$

$$(1) \quad \int_B \langle x, f_n \rangle dP \rightarrow \int_B \delta^*(x|I) dP .$$

As $E^{\mathfrak{B}}(f_n)(\omega) \in \Sigma(\omega)$ a.e. one has

$$\begin{aligned} \int_B \langle x, f_n \rangle dP &= \int_B \langle x, E^{\mathfrak{B}}(f_n) \rangle dP \\ &\leq \int_B \delta^*(x|\Sigma) dP \end{aligned}$$

which combined with (1) gives

$$\int_B \delta^*(x|I) dP \leq \int_B \delta^*(x|\Sigma) dP ,$$

hence

$$E^{\mathfrak{B}}[\delta^*(x|I(\cdot))](\omega) \leq \delta^*(x|\Sigma(\omega)) \quad \text{a.e.}$$

b) We turn now to prove, $\forall v \in L^\infty_{\mathcal{E}}(\mathcal{B})$

$$\int_{\Omega} \delta^*(v|I) dP = \int_{\Omega} \delta^*(v|\Sigma) dP.$$

If $v = \chi_B x$ that follows from a). If v assumes a countable number of values, $v = \sum \chi_{B_n} x_n$ (the B_n disjointed), then for every n ,

$$\int_{B_n} \delta^*(x_n|I) dP = \int_{B_n} \delta^*(x_n|\Sigma) dP.$$

The sum gives the expected equality.

Now if $v = \lim v_k$ where v_k assumes a countable number of values and the limit is uniform, one has

$$|\delta^*(v(\omega)|I(\omega)) - \delta^*(v_k(\omega)|I(\omega))| \leq p_n(v(\omega) - v_k(\omega))$$

and the same inequality with Σ in place of I .

Thus

$$\begin{aligned} \int \delta^*(v|I) dP &= \lim \int \delta^*(v_k|I) dP \\ &= \lim \int \delta^*(v_k|\Sigma) dP \\ &= \int \delta^*(v|\Sigma) dP. \end{aligned}$$

c) It remains to prove uniqueness. Let Θ_1 and Θ_2 in $\text{CCRS}(\mathcal{B}; V_n^0)$ such that for every $v \in L^\infty_{\mathcal{E}}(\mathcal{B})$

$$\int \delta^*(v|\Theta_i) dP = \int \delta^*(v|I) dP.$$

By [3] (VII.7, p. 200) the polar of

$$\begin{aligned} v &\mapsto \int \delta^*(v|\Theta_i) dP \quad \text{is} \\ u &\mapsto \int \delta(u|\Theta_i) dP = \delta(u|L^1_{\Theta_i}(\mathcal{B})). \end{aligned}$$

Thus $L^1_{\Theta_1}(\mathcal{B}) = L^1_{\Theta_2}(\mathcal{B})$. This set is non empty (because $= L^1_{\mathcal{E}}(\mathcal{B})$). By standard arguments (see for example [13] Séminaire d'Analyse Convexe, lemma 5) $\Theta_1(\omega) = \Theta_2(\omega)$ a.e.

3. - The main theorem.

In lemma 2 we obtained the conditional expectation Σ of $I \in \text{CCRS}(\mathcal{A}; V_n^0)$. We turn now to $I \in \text{CCRS}(\mathcal{A}; E')$.

THEOREM 3. — Let $\Gamma \in \text{CCRS}(\mathcal{A}; E')$ such that $L_\Gamma^1(\mathcal{A}) \neq \emptyset$. There exists $\Sigma \in \text{CCRS}(\mathcal{B}; E')$ such that:

- 1) Σ is the smallest of the $\Theta \in \text{CRS}(\mathcal{B}; E')$ such that $\forall u \in L_\Gamma^1(\mathcal{A}), E^\mathcal{B}(u)(\omega) \in \Theta(\omega)$ a.e.
- 2) Σ is the unique Θ in $\text{CCRS}(\mathcal{B}; E')$ such that $\forall v \in L_E^\infty(\mathcal{B})$,

$$\int \delta^*(v|\Gamma) dP = \int \delta^*(v|\Theta) dP.$$

3) Σ is the unique $\mathcal{B}$ -measurable multifunction such that $L_\Sigma^1(\mathcal{B}) = \text{cl } E^\mathcal{B}(L_\Gamma^1(\mathcal{A}))$ where cl denotes the closure with respect to $\sigma(L_E^1(\mathcal{B}), L_E^\mathcal{B}(\mathcal{B}))$.

4) if there exists a regular conditional probability $P^\mathcal{B}(\cdot|\cdot)$ on $(\Omega, \mathcal{A})$, then for P -almost every ω , $L_\Sigma^1(\Omega, \mathcal{A}, P^\mathcal{B}(\cdot|\omega))$ is $\neq \emptyset$ and

$$\Sigma(\omega) = \text{cl} \left\{ \int u(\omega') P^\mathcal{B}(d\omega'|\omega) \mid u \in L_\Gamma^1(\Omega, \mathcal{A}, P^\mathcal{B}(\cdot|\omega)) \right\}.$$

REMARK. — A regular conditional probability is a function $P^\mathcal{B}(\cdot|\cdot): \mathcal{A} \times \Omega \rightarrow [0, 1]$ such that

- 1) $\forall \omega, P^\mathcal{B}(\cdot|\omega)$ is a probability over $\mathcal{A}$,
- 2) $\forall A \in \mathcal{A}, P^\mathcal{B}(A|\cdot)$ is $\mathcal{B}$ -measurable,
- (2) 3) $\forall A \in \mathcal{A}, \forall B \in \mathcal{B}, P(A \cap B) = \int_B P^\mathcal{B}(A|\omega) P(d\omega)$.

Formula (2) extends in the following way: for every $\varphi: \Omega \times \Omega \rightarrow [0, \infty]$ which is $\mathcal{B} \otimes \mathcal{A}$ -measurable

$$(3) \quad \int_\Omega \varphi(\omega, \omega) P(d\omega) = \int_\Omega \left[\int_\Omega \varphi(\omega, \omega') P^\mathcal{B}(d\omega'|\omega) \right] P(d\omega).$$

Let Q be the probability on $(\Omega \times \Omega, \mathcal{B} \otimes \mathcal{A})$ which is the image of P by $\omega \mapsto (\omega, \omega)$. Then $P^\mathcal{B}(\cdot|\cdot)$ is the disintegration of Q with respect to the projection on $(\Omega, \mathcal{B})$. Such a disintegration exists when $\mathcal{A}$ is the Borel tribe of a Suslin topology (Jirina's theorem). Formula (3) holds also for $\varphi \in \mathcal{L}^1(\Omega \times \Omega, \mathcal{B} \otimes \mathcal{A}, Q)$.

PROOF. — A) We suppose first that $0 \in \Gamma(\omega)$ a.e. We define $\Gamma_n \in \text{CCRS}(\mathcal{A}; V_n^0)$ by $\Gamma_n(\omega) = \Gamma(\omega) \cap V_n^0$. Let Σ_n be the conditional expectation of Γ_n given by lemma 2. It is clear that $\Sigma_n(\omega) \subset \Sigma_{n+1}(\omega)$ a.e.

We put $\Sigma(\omega) = \text{cl} \left(\bigcup_n \Sigma_n(\omega) \right)$. It is a closed convex set containing 0. Thanks to the formula

$$\delta(x|\Sigma(\omega)) = \left[\sup_n \delta^*(\cdot|\Sigma_n(\omega)) \right]^*(x)$$

and the result of [3] (ch. VII) one has

$$G(\Sigma) \in \hat{\mathfrak{B}} \otimes \mathfrak{B}(E'_s).$$

Using the appendix we may suppose that Σ is $\mathfrak{B}$ -measurable.

B) PROOF OF 2).

a) First we suppose $0 \in \Gamma(\omega)$ a.e.

By Fatou lemma the formula valid for Γ_n and Σ_n remains valid for Γ and

$$\Sigma: \quad \int \delta^*(v|\Gamma) dP = \int \delta^*(v|\Sigma) dP.$$

b) Now we relax the hypothesis $0 \in \Gamma(\omega)$. Let $u_0 \in \mathfrak{L}_T^1(\mathcal{A})$. Put $\Gamma'(\omega) = \Gamma(\omega) - u_0(\omega)$. The bijection $(\omega, x') \mapsto (\omega, x' - u_0(\omega))$ (which maps $\Omega \times E'$ into itself) transforms $G(\Gamma)$ into $G(\Gamma')$, so $G(\Gamma') \in \mathcal{A} \otimes \mathfrak{B}(E'_s)$. Let Σ' be the conditional expectation of Γ' and $\Sigma(\omega) = \Sigma'(\omega) + E^{\mathfrak{B}}(u_0)(\omega)$. The foregoing argument applies to Σ' and Σ : Σ is measurable. At last, if $v \in L_E^\infty(\mathfrak{B})$,

$$\begin{aligned} \delta^*(v|\Gamma) &= \delta^*(v|\Gamma') + \langle v, u_0 \rangle, \\ \delta^*(v|\Sigma) &= \delta^*(v|\Sigma') + \langle v, E^{\mathfrak{B}}(u_0) \rangle. \end{aligned}$$

Then, thanks to a) and lemma 1), $\int \delta^*(v|\Gamma) dP = \int \delta^*(v|\Sigma) dP$. The uniqueness is proved as in lemma 2.

C) PROOF OF 1).

First we prove $u \in L_T^1(\mathcal{A}) \Rightarrow E^{\mathfrak{B}}(u) \in L_E^1(\mathfrak{B})$.

One has for $v \in L_E^\infty(\mathfrak{B})$,

$$\begin{aligned} \langle v, E^{\mathfrak{B}}(u) \rangle &= \int \langle v, u \rangle dP \quad (\text{lemma 1}) \\ &\leq \int \delta^*(v|\Gamma) dP \\ &= \int \delta^*(v|\Sigma) dP. \end{aligned}$$

Then, with the polar functionals, for every $u' \in L_E^1(\mathfrak{B})$

$$\delta(u'|\{E^{\mathfrak{B}}(u)\}) \geq \int \delta(u'|\Sigma) dP.$$

Taking $u' = E^{\mathfrak{B}}(u)$ that proves $E^{\mathfrak{B}}(u)(\omega) \in \Sigma(\omega)$ a.e.

It remains to prove that if $\Theta(\omega) \ni E^{\mathfrak{B}}(u)(\omega)$ a.e. for every $u \in L_T^1(\mathcal{A})$, then $\Theta(\omega) \supset \Sigma(\omega)$ a.e. Let $u_0 \in \mathfrak{L}_T^1(\mathcal{A})$. One has $\Theta(\omega) - E^{\mathfrak{B}}(u_0)(\omega) \ni E^{\mathfrak{B}}(u)(\omega) - E^{\mathfrak{B}}(u_0)(\omega)$. Hence $\Theta(\omega) - E^{\mathfrak{B}}(u_0)(\omega)$ contains all the $E^{\mathfrak{B}}(u')$ where $u' \in L_{T_n}^1(\mathcal{A})$, so it contains $\Sigma'_n(\omega)$.

Hence $\Theta(\omega) - E^{\mathfrak{B}}(u_0)(\omega) \supset \Sigma'(\omega) = \Sigma(\omega) - E^{\mathfrak{B}}(u_0)(\omega)$.

D) PROOF OF 3)

By 1) we have $L^1_{\Sigma}(\mathcal{B}) \supset E^{\mathcal{B}}(L^1_R(\mathcal{A}))$. Moreover $L^1_{\Sigma}(\mathcal{B})$ is $\sigma(L^1_{\Sigma}(\mathcal{B}), L^{\infty}_{\Sigma}(\mathcal{B}))$ closed because its indicator function is the polar of the functional

$$v \mapsto \int \delta^*(v|\Sigma) dP.$$

It remains to prove

$$\delta^*(v|L^1_{\Sigma}(\mathcal{B})) = \delta^*(v|E^{\mathcal{B}}(L^1_R(\mathcal{A}))).$$

One has

$$\begin{aligned} \delta^*(v|E^{\mathcal{B}}(L^1_R(\mathcal{A}))) &= \sup \{ \langle v, E^{\mathcal{B}}(u) \rangle | u \in L^1_R(\mathcal{A}) \} \\ &= \sup \{ \langle v, u \rangle | u \in L^1_R(\mathcal{A}) \} \quad (\text{lemma 1}) \\ &= \delta^*(v|L^1_R(\mathcal{A})) \\ &= \int \delta^*(v|I) dP \\ &= \int \delta^*(v|\Sigma) dP \\ &= \delta^*(v|L^1_{\Sigma}(\mathcal{B})). \end{aligned}$$

At last uniqueness has still be noticed.

E) PROOF OF 4)

Let $u_0 \in \mathcal{L}^1_{\Sigma}(\mathcal{A})$. There exist $\alpha \in \mathcal{L}^1_R$ and $n \in \mathbb{N}$ such that $\alpha \geq 0$ and $u_0(\omega) \in \alpha(\omega) V_n^0$ a.e. Applying formula (3) to $\varphi(\omega, \omega') = \alpha(\omega')$, one obtains

$$\int_{\Omega} \alpha dP = \int_{\Omega} \left[\int_{\Omega} \alpha(\omega') P^{\mathcal{B}}(d\omega'|\omega) \right] P(d\omega).$$

Hence for P -almost every ω (that is except if $\omega \in N$, N negligible) α is $P^{\mathcal{B}}(\cdot|\omega)$ -integrable and so $u_0 \in \mathcal{L}^1_{\Sigma}(\Omega, \mathcal{A}, P^{\mathcal{B}}(\cdot|\omega))$ (remark: if $u_0(\omega) \in I'(\omega)$ if $\omega \notin N'$ where $P(N') = 0$, then for P -almost every ω , N' is $P^{\mathcal{B}}(\cdot|\omega)$ negligible).

Put for $\omega \notin N$

$$\Theta(\omega) = \text{cl} \left\{ \int u(\omega') P^{\mathcal{B}}(d\omega'|\omega) | u \in \mathcal{L}^1_R(\Omega, \mathcal{A}, P^{\mathcal{B}}(\cdot|\omega)) \right\}.$$

It is a closed convex non empty subset of E' .

For $x \in E$

$$\begin{aligned} \delta^*(x|\Theta(\omega)) &= \sup \left\{ \langle u(\omega'), x \rangle P^{\mathcal{B}}(d\omega'|\omega) | u \in \mathcal{L}^1_R(\Omega, \mathcal{A}, P^{\mathcal{B}}(\cdot|\omega)) \right\} \\ &= \int \delta^*(x|I(\omega')) P^{\mathcal{B}}(d\omega'|\omega). \end{aligned}$$

So $(\omega, x) \mapsto \delta^*(x|\Theta(\omega))$ is an integrand over $(\Omega, \mathcal{B}) \times E$.

For $v \in L^\infty_{\mathcal{B}}(\mathcal{B})$, one has

$$\begin{aligned} \int_{\Omega} \delta^*(v(\omega)|\Theta(\omega)) P(d\omega) &= \int_{\Omega} \left[\int_{\Omega} \delta^*(v(\omega)|\Gamma(\omega')) P^{\mathcal{B}}(d\omega'|\omega) \right] P(d\omega) \\ &= \int_{\Omega} \delta^*(v(\omega)|\Gamma(\omega)) P(d\omega) \end{aligned}$$

(formula (3) applied to $\varphi(\omega, \omega') = \delta^*(v(\omega)|\Gamma(\omega'))$).

By 2) $\Theta = \Sigma$ a.e.

NOTATION. — We shall denote $E^{\mathcal{B}}(\Gamma)$ in place of Σ .

COROLLARY:

- 1) $\Gamma_1 \subset \Gamma_2$ implies $E^{\mathcal{B}}(\Gamma_1) \subset E^{\mathcal{B}}(\Gamma_2)$.
- 2) if Γ is $\mathcal{B}$ -measurable, $E^{\mathcal{B}}(\Gamma) = \Gamma$ a.e.
- 3) if $\mathcal{C}$ is a sub-tribe of $\mathcal{B}$, $E^{\mathcal{C}}(\Gamma) = E^{\mathcal{C}}(E^{\mathcal{A}}(\Gamma))$.
- 4) if $\Sigma = E^{\mathcal{B}}(\Gamma)$, for every $B \in \mathcal{B}$:

$$\text{cl} \left\{ \int_B u dP | u \in L^1_{\Gamma}(\mathcal{A}) \right\} = \text{cl} \left\{ \int_B u dP | u \in L^1_{\Sigma}(\mathcal{B}) \right\} = \text{cl} \left\{ \int_B u dP | u \in L^1_{\Sigma}(\mathcal{A}) \right\}.$$

PROOF:

- 1) follows from 1) of the theorem;
- 2) follows from 2) of the theorem;
- 3) follows easily from 2) of the theorem;
- 4) follows from the calculus of the support functions.

4. — Further remarks.

- 1) Part 2) of theorem 3 implies

$$\forall x \in E \quad \forall B \in \mathcal{B}, \quad \int_B \delta^*(x|\Gamma(\omega)) P(d\omega) = \int_B \delta^*(x|\Sigma(\omega)) P(d\omega),$$

hence $\forall x \in E$, $\delta^*(x|\Sigma(\omega)) = E^{\mathcal{B}}[\delta^*(x|\Gamma(\cdot))](\omega)$ a.e.

But this condition does not characterize Σ . It gives uniqueness if $\Sigma(\omega)$ is supposed to be a closed convex weakly locally compact set which contains no line (see [3], III.35, p. 83).

We give a counter example: let $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B} = \mathcal{B}([0, 1])$, P the Lebesgue measure, $E = \mathbf{R}^2$, $\Sigma(\omega) = \{(x, y) \in \mathbf{R}^2 | y = \omega x\}$, $\Sigma'(\omega) = \mathbf{R}^2$.

It is easy to see that

$$\int_B \delta^*(x | \Sigma(\omega)) P(d\omega) = \int_B \delta^*(x | \Sigma'(\omega)) P(d\omega)$$

(the integrals are $+\infty$ as soon as $x \neq 0$ and $P(B) > 0$).

2) We give now an example which shows that the conditional expectation of [9] is bigger than ours. Let $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B} = \mathcal{B}([0, 1])$, P the Lebesgue measure, $E = \mathbf{R}$, K a Cantor like set with $P(K) > 0$,

$$\begin{aligned} \Gamma(\omega) &= [0, \infty] & \text{if } \omega \notin K \\ &= \{0\} & \text{if } \omega \in K. \end{aligned}$$

Then, as it is said in OLECH [8] (p. 622) (see also [13]) the $\sigma(L^1, \mathbb{C})$ closure of L^1_Γ is L^1_Σ with $\Sigma(\omega) = [0, \infty[$. So the Sainte-Beuve's conditional expectation of Γ is Σ .

3) It is easy to see that the conditional expectation of HIAI and UMEGAKI [6] can be defined as the essential supremum of $\{E^{\mathcal{B}}(u) | u \in L^1_\Gamma(\mathcal{A})\}$ (see [14] for more details).

4) In [2] (th. 3, p. 9-10) Castaing obtains a weakly compact valued conditional expectation. Namely let F be a separable Banach space, F'_s its strong dual which is assumed separable, Γ a multifunction with closed convex weakly compact values in F . Moreover $\exists \alpha \in \mathbb{C}^1_{\mathbf{R}}$ such that $\Gamma(\omega) \subset \alpha(\omega)B$ where B denotes the unit ball of F .

We can apply theorem 3 with $E = F'$, $E' = F''$. It gives $\Sigma(\omega) = E^{\mathcal{B}}(\Gamma)(\omega) \subset F''$ with $\Sigma(\omega) \subset E^{\mathcal{B}}(\alpha)B''$ where B'' denotes the unit ball of E'' .

As L^1_Γ is $\sigma(L^1_E, L^\infty_E)$ compact ([2] theorem 1) L^1_Σ is equal to $E^{\mathcal{B}}(L^1_\Gamma)$ hence contained in $L^1_{F'}$. The inclusion $L^1_\Sigma \subset L^1_{F'}$ implies $\Sigma(\omega) \subset F$ a.e.

5. - Appendix.

1) Let $u: \Omega \rightarrow E'_s$. There are many equivalent measurability properties. It is well known that « u is $\mathcal{A} - \mathcal{B}(E'_s)$ measurable» is equivalent to « u is scalarly measurable», that is $\forall x \in E$, $\omega \mapsto \langle u(\omega), x \rangle$ is measurable. If u is obtained by application of the von Neumann's selection theorem ([3], III.22) it is $\hat{\mathcal{A}} - \mathcal{B}(E'_s)$ measurable. But there exists a Polish topology $\mathcal{T}$ on E' finer than $\sigma(E', E)$ whose Borel tribe is still $\mathcal{B}(E'_s)$. Hence there exists $u': \Omega \rightarrow E'_s$ which is $\mathcal{A} - \mathcal{B}(E'_s)$ measurable and equal a.e. to u .

2) Let $\Gamma \in \text{CRS}(\hat{\mathcal{A}}; V^n_a)$, then for every open set U $\{\omega | \Gamma(\omega) \cap U \neq \emptyset\}$ belongs to $\hat{\mathcal{A}}$. It follows from [3], III.2 that Γ is measurable with respect to $\hat{\mathcal{A}}$ and the

Borel tribe of the Hausdorff metric on the set of closed subsets of V_n^0 . Hence there exists I' which is $\mathcal{A}$ -measurable and equal a.e. to I .

3) Let $I \in \text{CCRS}(\hat{\mathcal{A}}; E')$. Then $(\omega, x') \mapsto \delta(x'|I(\omega))$ is a convex normal integrand on $(\Omega, \hat{\mathcal{A}}) \times E'_s$. So there exists a sequence (v_n, r_n) of $\hat{\mathcal{A}}$ -measurable selection of $\omega \mapsto \text{epi } \delta^*(\cdot|I(\omega))$ such that

$$\text{epi } \delta^*(\cdot|I(\omega)) = \text{cl} \{ (v_n(\omega), r_n(\omega)) | n \in \mathbb{N} \}.$$

As (v_n, r_n) takes its values in $E \times R$ there exists $\mathcal{A}$ -measurable functions $(\bar{v}_n, \bar{r}_n)$ such that $(\bar{v}_n, \bar{r}_n) = (v_n, r_n)$ a.e.

Then $I'(\omega) = \cap \{ x' | \langle x', \bar{v}_n(\omega) \rangle \leq \bar{r}_n(\omega) \}$ has its graph in $\mathcal{A} \otimes \mathcal{B}(E'_s)$ and is equal to I a.e.

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