Generalization of a Problem of Evelyn-Linfoot and Page in Additive Number Theory (*).

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Summary. – *Let k and l be integers such that $2 \leqslant k \leqslant l$. Let S_k and S'_l be two subsets of positive integers with $S_k \supseteq Q_k$ and $S'_l \supseteq Q_l$, where Q_k denotes the set of k-free integers. Further suppose that the characteristic functions of S_k and S'_l are multiplicative. Let $T(n)$ denote the number of representations of n in the form $n = a + b$, where $a \in S_k$ and $b \in S'_l$. In this paper we establish an asymptotic formula for $T(n)$, when n is sufficiently large; and deduce several known asymptotic formulae as particular cases.*

1. – Introduction.

Let k and l be two integers such that $2 \leqslant k \leqslant l$. Let S_k and S'_l be two subsets of positive integers with $S_k \supseteq Q_k$ and $S'_l \supseteq Q_l$, where Q_k denotes the set of k-free integers; that is, integers which are not divisible by the k-th power of any prime. Let q_k denote the characteristic function of the set Q_k. Let f_k and f'_l denote the characteristic functions of the sets S_k and S'_l; that is, $f_k(n) = 1$ or 0 according as $n \in S_k$ or $n \notin S_k$: We assume throughout that f_k and f'_l are multiplicative. Define

$$g_k(m) = \sum_{d|m} \mu(d) f_k(m/d) , \tag{1.1}$$

and

$$g'_l(m) = \sum_{d|m} \mu(d) f'_l(m/d) , \tag{1.2}$$

where μ denotes the well-known Möbius function. Then by the converse of Möbius Inversion formula, we have

$$f_k(m) = \sum_{d|m} g_k(d) , \tag{1.3}$$

and

$$f'_l(m) = \sum_{d|m} g'_l(d) . \tag{1.4}$$

Let $T(n) = T(S_k; S'_l, n)$ denote the number of representations of n in the form $n = a + b$, where $a \in S_k$ and $b \in S'_l$. Then by (1.3) and (1.4) we have

$$T(n) = \sum_{n=a+b} f_k(a) f'_l(b) = \sum_{n=d_1\delta_1+d_2\delta_2} g_k(d_1) g'_l(d_2) . \tag{1.5}$$

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In this paper, we prove the following:

THEOREM 1. – For sufficiently large n, we have

$$T(n) = n\cdot s(n)\cdot \prod_p \left\{\sum_{j=0}^{\infty} g_l'(p^j)p^{-j}\right\}\cdot \prod_p \left\{1 + \left(\sum_{j=0}^{\infty} g_l'(p^j)p^{-j}\right)^{-1} \sum_{t=1}^{\infty} g_k(pt)p^{-t}\right\} + O(n^{(k+l-2)/(kl-1)+\varepsilon}), \tag{1.6}$$

for every $\varepsilon > 0$, where the 0-constant depends only on k, l and ε,

$$S(n) = \prod_{p|n}\left\{1 + \left(\sum_{j=0}^{\infty} g_l'(p^j)p^{-j}\right)^{-1} \sum_{t=1}^{\infty} g_k(pt)p^{-t}\right\}^{-1} \times \prod_{p|n}\left\{1 + \left(\sum_{j=0}^{\infty} g_l'(p^j)p^{-j}\right)^{-1}\cdot \sum_{t=1}^{\infty} g_k(p^t)p^{-t} \sum_{j=0}^{\infty} g_l'(p^j)\varepsilon_n((p^j, p^t))p^{-j}\right\}, \tag{1.7}$$

and

$$\varepsilon_n(m, b) = \begin{cases} (m, b), & \text{if } (m, b)|(n, b), \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

THEOREM 2. – Let $S_k = Q_k$ and $T(Q_k; S_l', n) = T_k(S_l'; n)$. Then for sufficiently large n, we have

$$T_k(S_l'; n) = \frac{n}{\zeta(k)} \sum_{\substack{m=1 \\ (m,n)\in Q_k}}^{\infty} \frac{g_l'(m)\, m^{k-1}}{J_k(m)} + F_{k,l}(n), \tag{1.9}$$

where $\zeta(k)$ is the Riemann zeta function defined by $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$,

$$J_k(m) = m^k \prod_{p|m} (1 - p^{-k}), \tag{1.10}$$

the Jordan totient function, and for every $\varepsilon > 0$,

$$F_{k,l}(n) = O(n^{(k+l)/l(k+1)+\varepsilon}) \quad \text{or} \quad O(n^{1/k}), \tag{1.11}$$

according as $l \leqslant k^2$ or $l > k^2$, where the O-estimates depend only on k, l and ε.

THEOREM 3. – When $l \leqslant k^2$, the error term $F_{k,l}(n)$ in formula (1.9) has the better O-estimate given by

$$F_{k,l}(n) = O\left(n^{(k+l)/l(k+1)}(\log n)^{(k^{1/(k+1)}-1)(l-1)/l}\right), \tag{1.12}$$

where the O-constant depends only on k, l and ε.

In section 2, we prepare the necessary background for proving Theorems 1, 2 and 3 and in section 4, we give several applications of Theorems 1, 2, 3 and refer to the results that exist in the literature which are particular cases of these Theorems.

2. – Preliminaries.

In this section, we prove some lemmas which are needed in our present discussion. Throughout the following r denotes a fixed integer $\geqslant 2$.

LEMMA 2.1 (cf. [15], Lemma 2.6). – Let h_r be the multiplicative function defined by

$$h_r(p^\alpha) = \begin{cases} 0\,, & \text{if } 1 \leqslant \alpha \leqslant r-1 \\ 1\,, & \text{if } \alpha \geqslant r\,. \end{cases} \tag{2.1}$$

Then $\sum_{m=1}^{\infty} h_r(m)\, m^{-s}$ converges for any $s > 1/r$.

LEMMA 2.2. – We have

$$h_r(m) = \sum_{d^r\delta = m} h_r^*(\delta)\,, \tag{2.2}$$

where h_r^* is a multiplicative function such that the series

$$\sum_{m=1}^{\infty} h_r^*(m)\, m^{-s} \text{ is convergent for } s > (r+1)^{-1}\,. \tag{2.3}$$

PROOF. – This is implicit in the proof of lemma 2.8 of [15].

LEMMA 2.3 (cf. [15], Lemma 2.8). – For $x \geqslant 1$,

$$\sum_{m \leqslant x} h_r(m) = O(x^{1/r})\,,$$

where the O-constant depends only on r.

LEMMA 2.4. – Let m be a fixed positive integer. Then

$$\sum_{n \leqslant x} h_r(mn) = O\big((xm)^{1/r}\big)\,,$$

where the O-constant depends only on r.

PROOF. – We have

$$\sum_{n \leqslant x} h_r(mn) = \sum_{\substack{t \leqslant xm \\ m|t}} h_r(t) \leqslant \sum_{t \leqslant xm} h_r(t) = O\big((xm)^{1/r}\big)\,,$$

by Lemma 2.3.

Hence Lemma 2.4 follows.

LEMMA 2.5. – For each fixed m,

$$\sum_{n>x} \frac{h_r(mn)}{n} = O(x^{-1+1/r}\cdot m^{1/r}) ,$$

where the O-constant depends only on r.

PROOF. – Follows from Lemma 2.4 and partial summation.

LEMMA 2.6. – For each fixed m,

$$\sum_{n=1}^{\infty} h_r(mn)\, n^{-1} = O(m^{\varepsilon}) , \qquad \text{for every } \varepsilon > 0,$$

where the O-constant depends only on ε.

PROOF. – We have

$$\sum_{n\leqslant x} h_r(mn)\, n^{-1} = m \sum_{\substack{t\leqslant xm \\ m|t}} h_r(t)\, t^{-1} \leqslant m \sum_{t\leqslant xm} h_r(t)\, t^{-1} .$$

Since the series $\sum_{t=1}^{\infty} h_r(t)\, t^{-1}$, converges by Lemma 2.1, it follows that the series $\sum_{n=1}^{\infty} h_r(mn)\, n^{-1}$ also converges.

We have

$$\sum_{n=1}^{\infty} h_r(mn)\, n^{-1} = \Bigl(\sum_{\substack{n_1=1 \\ (n_1,m)=1}}^{\infty} h_r(n_1)\, n_1^{-1}\Bigr) \prod_{p^{\alpha}\|m} \Bigl\{h_r(p^{\alpha}) + \frac{h_r(p^{\alpha+1})}{p} + \dots\Bigr\} =$$

$$= \prod_{p\nmid m} \Bigl\{1 + \frac{1}{p^r} + \frac{1}{p^{r+1}} + \dots\Bigr\} \prod_{p^{\alpha}\|m} \Bigl\{h_r(p^{\alpha}) + \frac{h_r(p^{\alpha+1})}{p} + \dots\Bigr\} <$$

$$< \prod_{p} \Bigl\{1 + \frac{1}{p^r} + \frac{1}{p^{r+1}} + \dots\Bigr\} \prod_{p|m} \Bigl\{1 + \frac{1}{p} + \frac{1}{p^2} + \dots\Bigr\} \leqslant \zeta(2)\cdot 2^{\omega(m)} = O(m^{\varepsilon}) ,$$

where $\omega(m)$ is the number of distinct prime factors of m.

Hence lemma 2.6 follows.

LEMMA 2.7. – If g_k, g_l' and h_r are the multiplicative functions defined in (1.1), (1.2) and (2.1), then

$$|g_k(m)| \leqslant h_k(m) \tag{2.4}$$

and

$$|g_l'(m)| \leqslant h_l(m) , \tag{2.5}$$

for all m.

PROOF. – The proof of (2.4) is given in Lemma 2.14 of [15]. (2.5) similarly follows.

LEMMA 2.8 (cf. [9], Lemma 2.1, p. 132). – If a, b, n are positive integers and $R(a, b, n)$ denotes the numbers of solutions in positive integers x, y of the Diophantine equation

$$ax + by = n\,,$$

then

$$R(a, b, n) = \begin{cases} \dfrac{n(a, b)}{ab} + 2\theta_1, & \text{if } a + b \leqslant n,\ (a, b)|n, \\ 0\,, & \text{otherwise,} \end{cases}$$

where $\theta_1 = \theta_1(a, b, n)$ is a number absolutely less than 1.

For positive integers a and H, let $Q_k(x; a, H)$ denote the number of positive integers $n \leqslant x$ with $n \equiv a \pmod H$. As usual, H is called K-full if $p^k|H$ for any prime $p|H$.

LEMMA 2.9. – If $(a, H) \in Q_k$ and H is k-full, then

$$Q_k(x; a, H) = \frac{xH^{k-1}}{\zeta(k)\, J_k(H)} + E_k(x; a, H)\,, \tag{2.6}$$

where

$$E_k(x; a, H) = O\big(k^{\omega(H)}(x^{1/k} H^{-1/k^2} + H^{1/k})\big)\,, \tag{2.7}$$

the O-estimates being uniform in x, a and H; $J_k(H)$ being given by (1.10).

REMARK 2.1. – K. PRACHAR (cf. [14], p. 175) proved (2.6) under the assumption $(a, H) = 1$, rather than the assumption $(a, H) \in Q_k$ and H is k-full. However, the proof in either case is almost identical (cf. [14], § 3). Infact, C. POMERENCE and D. SURYANARAYANA [13] recently made small improvements in the O-estimates of (2.7) even under less stringent conditions on a and H.

LEMMA 2.10 (cf. [13], corollary). – We have

$$E_k(x; a, H) = \begin{cases} O\big((xk^{\omega(H)})^{1/k+1}\big)\,, & \text{if } x \leqslant H^k \\ O\big(x^{1/k} H^{-1/(k+3/2)}\big)\,, & \text{if } x > H^k\,, \end{cases} \tag{2.8}$$

where the implied constants in the O-terms are uniform in x, a and H.

REMARK 2.2. – It is clear that $Q_k(x; a, H) = 0$, if $(a, H) \notin Q_k$. Further, if $Q_k^*(x; a, H)$ denotes the number of positive integers strictly less than x with $n \equiv a \pmod H$, then $Q_k(x; a, H)$ and $Q_k^*(x; a, H)$ differ by atmost 1 so that

$$Q_k^*(x; a, H) = Q_k(x; a, H) + O(1).$$

3. – Main results.

In this section we give the proofs of Theorems 1, 2 and 3 stated in the introduction.

PROOF OF THEOREM 1. – We give the proof of this theorem by combining the methods due to CARLITZ [2] and PAGE [12]. Let $1 < t_i = t_i(n) < n$, $i = 1, 2$, be two functions of n. By (1.5), we have

$$T(n) = \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_1\leqslant t_1, d_2>t_2}} g_k(d_1)\, g_l'(d_2) + \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_1\leqslant t_1, d_2\leqslant t_2}} g_k(d_1)\, g_l'(d_2) + \sum_{\substack{d_1>t_1\\ n=d_1\delta_1+d_2\delta_2}} g_k(d_1)\, g_l'(d_2)\,.$$

Let

$$T'(n) = \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_1\leqslant t_1, d_2\leqslant t_2}} g_k(d_1)\, g_l'(d_2)\,, \tag{3.1}$$

so that by (2.4) and (2.5), we have

$$|T(n) - T'(n)| < \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_2>t_2}} h_k(d_1)\, h_l(d_2) + \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_1>t_1}} h_k(d_1)\, h_l(d_2) = T_2 + T_1\,, \quad \text{say.} \tag{3.2}$$

By Lemma 2.5 ($m = 1$, $r = l$) we have

$$T_2 \leqslant \sum_{\substack{d_2\delta_2<n\\ d_2>t_2}} h_l(d_2) \sum_{d_1|n-d_2\delta_2} 1 = \sum_{\substack{d_2\delta_2<n\\ d_2>t_2}} h_l(d_2)\, \tau(n - d_2\,\delta_2) = O\Big(n^{\varepsilon} \sum_{\substack{d_2<n\\ d_2>t_2}} h_l(d_2) \sum_{\delta_2<n/d_2} 1\Big) =$$

$$= O\Big(n^{1+\varepsilon} \sum_{d_2>t_2} h_l(d_2)\, d_2^{-1}\Big) = O(n^{1+\varepsilon}\, t_2^{-1+1/l})\,,$$

where in the above, we used that $\tau(n) = O(n^{\varepsilon})$ for every $\varepsilon > 0$, $\tau(n)$ being the number of divisors of n.

Similarly $T_1 = O(n^{1+\varepsilon} t_1^{-1+1/k})$. Hence we have

$$T(n) = T'(n) + O(n^{1+\varepsilon} t_2^{-1+1/l}) + O(n^{1+\varepsilon} t_1^{-1+1/k})\,. \tag{3.3}$$

From (3.1), we have

$$T'(n) \leqslant \sum_{m|n} \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_1\leqslant t_1, d_2\leqslant t_2\\ (d_1,d_2)=m}} g_k(d_1)\, g_l'(d_2) = \sum_{m|n} T^*(m)\,, \quad \text{say.}$$

Now, by lemma 2.8,

$$(3.4)\qquad T^*(m) = \sum_{\substack{n=d_1\delta_1+d_2\delta_2\\ d_1\leqslant t_1, d_2\leqslant t_2\\ (d_1,d_2)=m}} g_k(d_1)\,g_l'(d_2) = \sum_{\substack{n/m=a_1\delta_1+a_2\delta_2\\ (a_1,a_2)=1\\ a_1\leqslant t_1/m,\, a_2\leqslant t_2/m}} g_k(a_1 m)\,g_l'(a_2 m) =$$

$$= \sum_{\substack{a_1\leqslant t_1/m\\ a_2\leqslant t_2/m\\ (a_1,a_2)=1}} g_k(a_1 m)\,g_l'(a_2 m)\left\{\frac{n}{m a_1 a_2} + O(1)\right\} =$$

$$= nm^{-1}\sum_{\substack{a_1\leqslant t_1/m\\ a_2\leqslant t_2/m\\ (a_1,a_2)=1}} g_k(a_1 m)\,g_l'(a_2 m)/a_1 a_2 + O\Big(\sum_{\substack{a_1\leqslant t_1/m\\ a_2\leqslant t_2/m}} h_k(a_1 m)\,h_l(a_2 m)\Big).$$

By lemma 2.3, the O-term in (3.4) is $O(t_1^{1/k}t_2^{1/l})$, so that

$$(3.5)\qquad T^*(m) = nm^{-1}\sum_{\substack{a_1\leqslant t_1/m\\ a_2< t_2/m\\ (a_1,a_2)=1}} g_k(a_1 m)\,g_l'(a_2 m)/a_1 a_2 + O(t_1^{1/k}t_2^{1/l}) =$$

$$= nm^{-1}\sum_{a_1\leqslant t_1/m} g_k(a_1 m)\,a_1^{-1}\Big\{\sum_{\substack{a_2=1\\ (a_2,a_1)=1}}^{\infty} g_l'(a_2 m)\,a_2^{-1} + O\Big(\sum_{a_2>t_2/m} h_l(a_2 m)\,a_2^{-1}\Big)\Big\} + O(t_1^{1/k}t_2^{1/l}) =$$

$$= nm^{-1}\sum_{a_1\leqslant t_1/m} g_k(a_1 m)\,A_l'(a_1; m)\,a_1^{-1} + O\Big(t_2^{-1+1/l}\,n\sum_{a_1\leqslant t_1/m} h_k(a_1 m)/a_1\Big) + O(t_1^{1/k}t_2^{1/l}).$$

by Lemma 2.5, where

$$(3.6)\qquad A_l'(a_1; m) = \sum_{\substack{a_2=1\\ (a_2,a_1)=1}}^{\infty} g_l'(a_2 m)\,a_1^{-1}.$$

Now, by Lemma 2.6,

$$(3.7)\qquad \sum_{a_1\leqslant t_1/m} h_k(a_1 m)\,a_1^{-1} \leqslant \sum_{a_1=1}^{\infty} h_k(a_1 m)\,a_1^{-1} = O(m^{\varepsilon}).$$

Further, we have

$$T_3 = \sum_{a_1\leqslant t_1/m} g_k(a_1 m)\,A_l'(a_1; m)\,a_1^{-1} = \sum_{a_1=1}^{\infty} g_k(a_1 m)\,A_l'(a_1; m)\,a_1^{-1} + $$

$$+ O\Big(\sum_{a_1>t_1/m} h_k(a_1 m)\,A_l'(a_1; m)\,a_1^{-1}\Big).$$

Since by Lemma 2.6, $A_l'(a_1; m) \leqslant \sum_{a_2=1}^{\infty} h_l(a_2 m)\,a_2^{-1} = O(m^{\varepsilon})$, it follows from Lemma 2.5 $(r = k)$ that

$$(3.8)\qquad T_3 = \sum_{a_1=1}^{\infty} g_k(a_1 m)\,A_l'(a_1; m)\,a_1^{-1} + O(t_1^{-1+1/k}m^{1+\varepsilon}).$$

Collecting (3.5)-(3.8), we have

$$T^*(m) = nm^{-1}\sum_{a_1=1}^{\infty} g_k(a_1 m)\, A_l'(a_1;\, m)\, a_1^{-1} + \\ + O(t_1^{-1+1/k} m^{1+\varepsilon}) + O(t_2^{-1+1/l} nm^{\varepsilon}) + O(t_1^{1/k} t_2^{1/l}),$$

so that

$$T'(n) = \sum_{m|n} T^*(m) = n\sum_{m|n} m^{-1}\sum_{a_1=1}^{\infty} g_k(a_1 m)\, A_l'(a_1; m)\, a_1^{-1} + \\ + O\Big(t_1^{-1+1/k}\sum_{m|n} m^{1+\varepsilon}\Big) + O\Big(nt_2^{-1+1/l}\sum_{m|n} m^{\varepsilon}\Big) + O(t_1^{1/k} t_2^{1/l}\tau(n)).$$

It is clear that $\sum_{m|n} m^{1+\varepsilon} \leqslant n^{1+\varepsilon}\cdot\tau(n) = O(n^{1+2\varepsilon})$, and $\sum_{m|\varepsilon} m^{\varepsilon} \leqslant n^{\varepsilon}\tau(n) = O(n^{2\varepsilon})$, for every $\varepsilon > 0$. Hence we have

$$T'(n) = n\sum_{m|n} H(m)\, m^{-1} + O(n^{1+2\varepsilon} t_1^{-1+1/k}) + O(n^{1+2\varepsilon} t_2^{-1+1/l}) + O(t_1^{1/k} t_2^{1/l} n^{\varepsilon}),$$

where $H(m) = \sum_{a_1=1}^{\infty} g_k(a_1 m)\, A_l'(a_1; m)\, a_1^{-1}$. Now by (3.3), it follows that

$$T(n) = n\sum_{m|n} H(m)\, m^{-1} + O(n^{1+2\varepsilon} t_1^{-1+1/k}) + O(n^{1+2\varepsilon} t_2^{-1+1/l}) + O(n^{\varepsilon} t_1^{1/k} t_2^{1/l}).$$

Taking $t_1 = n^{k(l-1)/(kl-1)}$ and $t_2 = n^{l(k-1)/(kl-1)}$, we see that each of the above O-terms reduce to $O(n^{(k+l-2)/(kl-1)+2\varepsilon})$. Now, it can be shown using the Euler- Infinite Product theorem that

$$\sum_{m|n} H(m)\, m^{-1} = \sum_{\substack{a_1, a_2 = 1 \\ (a_1, a_2)|n}}^{\infty} g_k(a_1)\, g_l'(a_2)(a_1, a_2)(a_1 a_2)^{-1} = \\ = S(n)\cdot\prod_p \Big\{\sum_{j=0}^{\infty} g_l'(p^j)\, p^{-j}\Big\}\prod_p \Big\{1 + \Big(\sum_{j=0}^{\infty} g_l'(p^j)\, p^{-j}\Big)^{-1}\sum_{t=1}^{\infty} g_k(pt)\, p^{-t}\Big\},$$

where $S(n)$ is given by (1.7).

Hence Theorem 1 follows.

REMARK 3.1. – We can further simplify the main term in (1.6) so that $T(n)$ is also given by

$$(3.9)\qquad T(n) = n\prod_p \Big\{\sum_{j=0}^{\infty} g_l'(p^j)\, p^{-j} + \sum_{t=1}^{\infty} g_k(pt)\, p^{-t}\Big\}\prod_{p|n}\Big\{\sum_{j=0}^{\infty} g_l'(p^j)\, p^{-j} + \sum_{t=1}^{\infty} g_k(pt)\, p^{-t}\Big\}^{-1}\times \\ \times\prod_{p|n}\Big\{\sum_{j=0}^{\infty} g_l'(p^j)\, p^{-j} + \sum_{t=1}^{\infty} g_k(p^t)\, p^{-t}\sum_{j=0}^{\infty} g_l'(p^j)\, p^{-j}\varepsilon_n((p^j, p^t))\Big\} + O(n^{(k+l-2)/(kl-1)+\varepsilon}).$$

PROOF OF THEOREM 2. – By (1.3), Remark 2.2 and Lemma 2.3, we have

$$(3.10)\qquad T(S_l'; n) = \sum_{n=a+b} f_l'(a)\, q_k(b) = \sum_{n=d\delta+b} g_l'(d)\, q_k(b) =$$
$$= \sum_{d<n} g_l'(d)\, Q_k^*(n; n, d) = \sum_{d<n} g_l'(d)\, Q_k(n; n, d) + O\Big(\sum_{d<n} h_l(d)\Big) =$$
$$= \sum_{d<n} g_l'(d)\, Q_k(n; n, d) + O(n^{1/l})\,.$$

Let $1 < t = t(n) < n$. Then by Remark 2.2,

$$(3.11)\qquad \sum_{d<n} g_l'(d)\, Q_k(n; n, d) = \sum_{d<n} g_l'(d)\, q_k\big((d, n)\big) Q_k(n; n, d) =$$
$$= \sum_{d\leqslant t} g_l'(d)\, q_k\big((d, n)\big) Q_k(n; n, d) + \sum_{t<d<n} g_l'(d)\, q_k\big((d, n)\big) Q_k(n; n, d) = T_4 + T_5\,, \quad \text{say.}$$

Since $Q_k(n; n, d) \leqslant 2nd^{-1}$, it follows from Lemma 2.5 $(r = l,\ m = 1)$ that

$$(3.12)\qquad |T_5| \leqslant 2n \sum_{d>t} h_l(d)\, d^{-1} = O(nt^{-1+1/l})\,.$$

From (2.5) and (2.1) $(r = l)$ it follows that $g_l'(p^\alpha) = 0$ for any prime p and $1 \leqslant \alpha \leqslant l-1$. Hence we can assume that d in the sum T_4 is l-full and hence k-full since $l \geqslant k$. Hence by Lemma 2.9, we have

$$(3.13)\qquad T_4 = \sum_{d\leqslant t} g_l'(d)\, q_k\big((d, n)\big)\left\{\frac{nd^{k-1}}{\zeta(k)\, J_k(d)} + O(n^{1/k} k^{\omega(d)} d^{-1/k^2}) + O(k^{\omega(d)} d^{1/k})\right\} =$$
$$= \frac{n}{\zeta(k)} \sum_{d\leqslant t} \frac{g_l'(d)\, q_k\big((d, n)\big)\, d^{k-1}}{J_k(d)} + O\Big(n^{1/k} \sum_{d\leqslant t} h_l(d)\, k^{\omega(d)} d^{-1/k^2}\Big) +$$
$$+ O\Big(\sum_{d\leqslant t} h_l(d)\, k^{\omega(d)} d^{1/k}\Big) = T_{11} + T_{12} + T_{13}, \quad \text{say.}$$

The series $\sum_{d=1}^{\infty} g_l'(d)\, q_k\big((d, n)\big)\, d^{k-1}/J_k(d)$ converges absolutely by (2.5), the fact that $d^k/J_k(d)$ is bounded for all d and Lemma 2.1. Further by (2.5) and Lemma 2.5 $(m = 1,\ r = l)$,

$$\sum_{d>t} g_l'(d)\, q_k\big((d, n)\big)\, d^{k-1}/J_k(d) = O\Big(\sum_{d>t} h_l(d)\, d^{-1}\Big) = O(t^{-1+1/l})\,,$$

so that

$$(3.14)\qquad T_{11} = n\big(\zeta(k)\big)^{-1} \sum_{d=1}^{\infty} g_l'(d)\, q_k\big((d, n)\big)\, d^{k-1}/J_k(d) + O(nt^{-1+1/l})\,.$$

By Lemmas 2.1 and 2.3, and the fact that $k^{\omega(d)} = O(d^\varepsilon)$ for every $\varepsilon > 0$ and partial summation, we have

$$(3.15)\qquad T_{12} = \begin{cases} O(n^{1/k}), & \text{if } l > k^2 \\ O(n^{1/k} t^{1/l - 1/k^2 + \varepsilon}), & \text{if } l \leqslant k^2 . \end{cases}$$

Also, by Lemma 2.3,

$$(3.16)\qquad T_{13} = O\big(t^{1/k+\varepsilon} \sum_{d \leqslant t} h_l(d)\big) = O(t^{1/l + 1/k + \varepsilon}) .$$

Collecting now the results (3.10)-(3.16) we obtain,

$$T(S_l'; n) = n(\zeta(k))^{-1} \sum_{d=1}^{\infty} g_l'(d)\, q_k((d, n))\, d^{k-1}/J_k(d) + F_{k,l}(n) ,$$

where

$$F_{k,l}(n) = O(n^{1/l}) + O(nt^{-1+1/l}) + O(t^{1/l+1/k+\varepsilon}) + \begin{cases} O(n^{1/k}), & \text{if } l > k^2 \\ O(n^{1/k} t^{1/l - 1/k^2 + \varepsilon}), & \text{if } l \leqslant k^2 . \end{cases}$$

Taking $t = n^{k/(k+1)}$ we obtain Theorem 2.

PROOF OF THEOREM 3. – From (3.11), we have

$$\begin{aligned} (3.17)\qquad T_4 &= \sum_{d<t} g_l'(d)\, q_k((d, n))\, Q_k(n; n, d) = \\ &= n(\zeta(k))^{-1} \sum_{d \leqslant t} g_l'(d)\, d^{k-1}\, q_k((d, n))/J_k(d) + \sum_{d < n^{1/k}} g_l'(d)\, q_k((d, n))\, E_k(n; n, d) + \\ &\qquad + \sum_{n^{1/k} \leqslant d \leqslant t} g_l'(d)\, q_k((d, n))\, E_k(n; n, d) = \\ &= T_{11} + T_{12}' + T_{13}' \end{aligned}$$

where $E_k(x; a, H)$ is the error term in the formula (2.6).

By (2.5), Lemma 2.10, Lemmas 2.1 and 2.3, and partial summation,

$$\begin{aligned} (3.18)\qquad T_{12}' &= O\Big(n^{1/k} \sum_{d < n^{1/k}} h_l(d)\, d^{-1/(k+3/2)}\Big) = \begin{cases} O(n^{1/k}), & \text{if } l \geqslant k+2 \\ O(n^{1/k + 1/kl - 1/k(k+3/2)}), & \text{if } l < k+2. \end{cases} \\ &= O(n^{(k+l)/l(k+1)}) . \end{aligned}$$

Again by (2.5) and lemma 2.10, we have

$$\begin{aligned} T_{13}' &= O\Big(\sum_{d<t} h_l(d)(nk^{\omega(d)})^{1/(k+1)}\Big) \\ &= O\Big(n^{1/(k+1)} \sum_{d<t} h_l(d)(k^{\omega(d)})^{1/(k+1)}\Big) . \end{aligned}$$

Now by (2.2) and (2.3),

$$\sum_{m\leqslant t} h_l(m)(k^{\omega(m)})^{1/(k+1)} \leqslant \sum_{d^l\delta\leqslant t} h_l^*(\delta)(k^{\omega(\delta)})^{1/(k+1)}(k^{\omega(d)})^{1/(k+1)}$$
$$= \sum_{\delta\leqslant t} h_l^*(\delta)(k^{\omega(\delta)})^{1/(k+1)} \sum_{d\leqslant (t/\delta)^{1/l}} (\tau(d))^{\log_2 k^{1/(k+1)}}$$
$$= O\Big(t^{1/l}(\log n)^{(k^{1/(k+1)}-1)} \sum_{\delta\leqslant t} h_l^*(\delta)\,\delta^{-1/l}(k^{\omega(\delta)})^{1/(k+1)}\Big)$$
$$= O\big(t^{1/l}(\log n)^{(k^{1/(k+1)}-1)}\big)\,,$$

where we use a formula of Ramanujan (see WILSON [19], eq. (2.39)). Hence we have

$$T'_{13} = O\big(n^{1/(k+1)}t^{1/l}(\log n)^{(k^{1/(k+1)}-1)}\big)\,. \tag{3.19}$$

Collecting now the results (3.10)-(3.12), (3.14) and (3.16)-(3.18), we obtain

$$F_{k,l}(n) = O(nt^{-1+1/l}) + O(n^{(k+l)/l(k+1)}) + O(n^{1/(k+1)}t^{1/l}(\log n)^{(k^{1/(k+1)}-1)})\,.$$

Taking $t = n^{k/(k+1)}(\log n)^{-(k^{1/(k+1)}-1)}$, we obtain Theorem 3.

4. - Applications.

Throughout the following r_1 and r_2 denote integers with $2\leqslant r_1\leqslant r_2$ and k_1 and k_2 denote integers with $k_1 > r_1$ and $k_2 > r_2$. A positive integer m is called Semi-k-free [18] if in the canonical factorization of m, no exponent is equal to k. A positive integer m is called unitarily k-free [3] if no exponent in the canonical factorization of m is a multiple of k and m is called a k-skew integer of rank t, where t is any positive integer, if in the canonical factorization of m, no exponent is equal to jk, for $1\leqslant j\leqslant t$. For convenience, we shall call these integers (k-skew integers of rank t) as unitarily k—t-integers. Clearly the unitarily $k-1$-integers are the semi-k-free integers and the unitarily k—∞-integers are the unitarily k-free integers. Let r be an integer with $2\leqslant r < k$. A positive integer m is called a (k, r)-free integer if no exponent in the canonical factorization of m belongs to the interval $[r, k-1]$. This definition includes as special cases the r-free integers ($k = \infty$) and the Semi-r-free integers ($k = r+1$). A positive integer m is called a (k, r)-integer if m can be represented as $m = m_1^k m_2$ where m_1 is a positive integer and m_2 is an r-free integer. These integers include as a special case the r-free integers ($k = \infty$). The concept of a (k, r)-integer has been introduced independently (without using this name) by E. COHEN [5] and M. V. SUBBA RAO - V. C. HARRIS [16]. Let S_t^* denote the set of all integers m in whose canonical factorization each exponent is just 1 or t. We also include the integer 1 in the set S_t^*. These integers have been introduced by E. COHEN (cf. [4], p. 78).

Let $Q^*_{k,t}$, $Q_{(k,r)}$ and $Q_{k,r}$ respectively denote the set of all unitarily $k-t$, (k, r)-free and (k, r)-integers. Let $q^*_{k,t}$, $q_{(k,r)}$ and $q_{k,r}$ denote the characteristic functions of the sets $Q^*_{k,t}$, $Q_{(k,r)}$ and $Q_{k,r}$: First we have

COROLLARY 4.1. - Let $T_{(k_1,r_1)(k_2,r_2)}(n)$ denote the number of ordered pairs $(a, b) \in \in Q_{k_1,r_1} \times Q_{k_2,r_2}$ with $n = a + b$. Then for sufficiently large n, we have

$$(4.1) \qquad T_{(k_1,r_1)(k_2,r_2)}(n) = n \frac{\zeta(k_2)}{\zeta(r_2)} \prod_p \left(1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \cdot \frac{p^{-k_1}-p^{-r_1}}{1-p^{-r_1}}\right) A(n) + O(n^{(r_1+r_2-2)/(r_1 r_2 - 1)+\varepsilon}),$$

for every $\varepsilon > 0$, where the O-constant is independent of k_1, k_2 and n:

$$(4.2) \qquad A(n) = \prod_{p|n} \left\{1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \left(\sum_{\mu=1}^{\infty} p^{-\mu k_1} \sum_{m=0}^{\infty} \left[\frac{\varepsilon_n((p^{mk_2}, p^{\mu k_1}))}{p^{mk_2}} - \frac{\varepsilon_n((p^{r_2+mk_2}, p^{\mu k_1}))}{p^{r_2+mk_2}} \right] - \right.\right.$$
$$\left.\left. - \sum_{\mu=0}^{\infty} p^{-(r_1+\mu k_1)} \sum_{m=0}^{\infty} \left[\frac{\varepsilon_n((p^{mk_2}, p^{r_1+\mu k_1}))}{p^{mk_2}} - \frac{\varepsilon_n((p^{r_2+mk_2}, p^{r_1+\mu k_1}))}{p^{r_2+mk_2}} \right] \right) \right\} \times$$
$$\times \left(1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \cdot \frac{p^{-k_1}-p^{-r_1}}{1-p^{-k_1}}\right)^{-1}.$$

PROOF. - It has been shown by M. V. Subba Rao and V. C. Harris (cf. [16], Theorem 3) that

$$q_{k,r}(m) = \sum_{d|m} \lambda_{k,r}(d),$$

where $\lambda_{k,r}$ is the multiplicative function defined by

$$(4.3) \qquad \lambda_{k,r}(p^\alpha) = \begin{cases} 1, & \text{if } \alpha \equiv 0 \pmod k \\ -1, & \alpha \equiv r \pmod k \\ 0, & \text{otherwise.} \end{cases}$$

Now we take $S_{r_1} = Q_{k_1,r_1}$, $S'_{r_2} = Q_{k_2,r_2}$, $k = r_1$ and $l = r_2$ in Theorem 1. Noting that in this case $g_k = g_{r_1} = \lambda_{k_1,r_1}$, and $g'_l = g'_{r_2} = \lambda_{k_2,r_2}$, we obtain corollary 4.1 from (4.3) and Theorem 1.

REMARK 4.1. - It has been established by E. BRINITZER (cf. [1], Satz 2) that

$$(4.4) \qquad T_{(k_1,r_1)(k_2,r_2)}(n) = n \frac{\zeta(k_2)}{\zeta(r_2)} \prod_p \left(1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \cdot \frac{p^{-k_1}-p^{-r_1}}{1-p^{-k_1}}\right) B(n) + O(n^{1/r_1+1/r_2-1/r_1 r_2} \zeta(k_1/r_1)\, \zeta(k_2/r_2)),$$

where

$$(4.5)\qquad B(n)=\prod_{p\mid n}\left(1+\frac{1-p^{-k_2}}{1-p^{-r_2}}\left\{\sum_{\mu=1}^{\infty}p^{-\mu k_1}\sum_{m=0}^{\infty}\left[\frac{(p^{mk_2},p^{\mu k_1})}{p^{mk_2}}-\frac{(p^{r_2+mk_2},p^{\mu k_1})}{p^{r_2+mk_2}}\right]-\right.\right.$$
$$\left.\left.-\sum_{\mu=0}^{\infty}p^{-(r_1+\mu k_1)}\sum_{m=0}^{\infty}\left[\frac{(p^{mk_2},p^{r_1+\mu k_1})}{p^{mk_2}}-\frac{(p^{r_2+mk_2},p^{r_1+\mu k_1})}{p^{r_2+mk_2}}\right]\right\}\right)\times\left(1+\frac{1-p^{-k_2}}{1-p^{-r_2}}\cdot\frac{p^{-k_1}-p^{r_1}}{1-p^{-k_1}}\right)^{-1}.$$

We remark here that the main term in (4.4) with $B(n)$ given by (4.5) is incorrect. We get the correct result only if we replace each bracket $(\, , \,)$ by $\varepsilon_n((\, , \,))$ in the definition of $B(n)$ given in (4.5) and then it is clear that the main terms in (4.1) and (4.4) coincide. Further, the error term we obtain from (4.1) is better than the one obtained by E. BRINITZER (cf. [1], Satz 2). It may be mentioned here that M. V. SUBBA RAO and Y. K. FENG [17] established an asymptotic formula for $T_{(k_1,r_1)(k_2,r_2)}(n)$ in the particular case when $(k_i, r_i) = s_i > 1$ for $i = 1, 2$ with a O-estimate for the error term weaker than that of E. BRINITZER [1].

COROLLARY 4.2. - Let $T'_{(k_1,r_1)(k_2,r_2)}(n)$ denote the number of representations of n as the sum of a (k_1, r_1)-free and (k_2, r_2)-free integers. Then we have

$$(4.6)\qquad T'_{(k_1,r_1)(k_2,r_2)}(n)=n\cdot C(n)\prod_{p}(1-p^{-r_1}-p^{-r_2}+p^{-k_1}+p^{-k_2})+O(n^{(r_1+r_2-2)/(r_1r_2-1)+\varepsilon}),$$

for every $\varepsilon > 0$, where the O-constant is independent of k_1, k_2 and n; $C(n)$ being given by

$$(4.7)\qquad C(n)=\prod_{p^{r_1}\mid n}(1-p^{-r_1}-p^{-r_2}+p^{-k_1}+p^{-k_2})^{-1}\times$$
$$\times(1-p^{-r_2}+p^{-k_2}-\varepsilon_n((p^{k_2},p^{r_1}))p^{-(k_2+r_1)}+\varepsilon_n((p^{k_2},p^{k_1}))p^{-(k_1+k_2)}).$$

PROOF. - First we note that for each prime p,

$$(4.8)\qquad f_{(k,r)}(p^{\alpha})=q_{(k,r)}(p^{\alpha})-q_{(k,r)}(p^{\alpha-1})=\begin{cases}-1, & \text{if } \alpha=r\\ 1, & \text{if } \alpha=k\\ 0, & \text{otherwise}\end{cases}$$

Now, we take $k = r_1$, $l = r_2$, $S_k = Q_{(k_1,r_1)}$ and $S'_l = Q_{(k_2,r_2)}$. Taking $k = r_1$ and $l = r_2$ in (1.1) and (1.2), from (4.8) it is clear that $g_k = g_{r_1} = f_{(k_1,r_1)}$ and $g'_l = g'_{r_2} = f_{(k_2,r_2)}$. Now, corollary 4.2 can be deduced from (3.9) with the aid of (4.8).

REMARK 4.2. - Formula (4.6) has been established by G. E. HARDY [10] with error term $O(n^{1-\delta})$ where

$$\delta=\mathrm{Min}\,\{1-1/r_2-\varepsilon(r_2-1),\,1-1/r_1-1/r_2+\varepsilon\}\,,\qquad\text{and}\qquad 0<\varepsilon<1/r_2\,.$$

It is clear that $1-\delta \geqslant 1/r_1 + 1/r_2 - 1/r_1 r_2$ with equality if and only if $\varepsilon = 1/r_1 r_2$. Thus the best O-estimate of the error term that can be obtained from G. E. Hardy's result is $O(n^{1/r_1+1/r_2-1/r_1r_2})$, which is clearly weaker than the one obtained by us in (4.6).

Letting k_1, $k_2 \to \infty$ in Corollaries 4.1 or 4.2, we obtain the important

COROLLARY 4.3. – Let $T_{r_1,r_2}(n)$ denote the number of ordered pairs $(a, b) \in Q_{r_1} \times Q_{r_2}$ with $a+b=n$. Then for sufficiently large n,

$$(4.9) \qquad T_{r_1,r_2}(n) = n \cdot \prod_p (1 - p^{-r_1} - p^{-r_2}) \prod_{p^{r_1}|n} (1 + (p^{r_2} - p^{r_2-r_1} + 1)^{-1}) + \\ + O(n^{(r_1+r_2-2)/(r_1r_2-1)+\varepsilon}),$$

or every $\varepsilon > 0$.

REMARK 4.3. – Formula (4.9) is due to A. PAGE [12]. E. COHEN and R. L. ROBINSON (cf. [7], Corollary 2, p. 291) established the asymptotic formula (4.9) without O-term. In case $r_1 = r_2$, (4.9) reduces to a formula established originally by C. J. A. EVELYN and E. H. LINFOOT [9].

COROLLARY 4.4. – Let $E_{r_1,r_2}(n)$ denote the error term in the formula (4.9). Then we have the following better O-estimates than in (4.9):

$$E_{r_1,r_2}(n) = O\left(n^{(r_1+r_2)/r_2(r_1+1)} (\log n)^{(r_1^{1/(r_1+1)}-1)(r_2-1)/r_2}\right) \qquad \text{or} \ O(n^{1/r_1}),$$

according as $r_2 \leqslant r_1^2$ or $r_2 > r_1^2$.

PROOF. – Taking $l = r_2$ and $S_l' = Q_{r_2}$ in Theorems 2 and 3, we obtain Corollary 4.4.

REMARK 4.4. – The result in corollary 4.4 has been recently established by C. POMERENCE and D. SURYANARAYANA [13]. Taking $r_1 = r_2 = 2$ in corollary 4.4, we obtain that $E_{2,2}(n) = E(n) = O\left(n^{2/3} (\log n)^{(2^{1/3}-1)/2}\right)$ which is an improvement over the result $E(n) = O(n^{2/3} \log^2 n)$, established by E. COHEN [6].

REMARK 4.5. – Taking $k_1 = r_1 + 1$ and $k_2 = r_2 + 1$ in (4.6) we obtain an asymptotic formula for $T^{s^*}_{r_1,r_2}(n)$, where $T^{s^*}_{r_1,r_2}(n)$ is the number of representations of n as the sum of a Semi-r_1-free and a Semi-r_2-free integer.

COROLLARY 4.5. – Let $T^*_{(k_1,t_1)(k_2,t_2)}(n)$ denote the number of ordered pairs $(a, b) \in Q^*_{k_1,t_1} \times Q^*_{k_2,t_2}$ with $n = a + b$. Then for sufficiently large n we have

$$(4.10) \qquad T^*_{(k_1,t_1)(k_2,t_2)}(n) = n \cdot \prod_p \left(1 - \frac{(p-1)}{p}\left(\frac{p^{k_1t_1}-1}{p^{k_1t_1}(p^{k_1}-1)} + \frac{p^{k_2t_2}-1}{p^{k_2t_2}(p^{k_2}-1)}\right)\right) \times \\ \times \prod_{p|n} \left(1 - \frac{(p-1)}{p}\left(\frac{p^{k_1t_1}-1}{p^{k_1t_1}(p^{k_1}-1)} + \frac{p^{k_2t_2}-1}{p^{k_2t_2}(p^{k_2}-1)}\right)\right)^{-1} \times H'(n) + O(n^{(k_1+k_2-2)/(k_1k_2-1)+\varepsilon}),$$

for every $\varepsilon > 0$, where the O-constant is independent of t_1, t_2 and n; $H'(n)$ being given by

$$H'(n) = \prod_{p|n} \left\{1 - \frac{(p-1)}{p}\left(\frac{p^{k_1t_1}-1}{p^{k_1t_1}(p^{k_1}-1)} + \frac{p^{k_2t_2}-1}{p^{k_2t_2}(p^{k_2}-1)}\right) - \right.$$
$$- \sum_{t=1}^{t_2} p^{-tk_2}\left[\sum_{j=1}^{t_1}\left(\frac{\varepsilon_n\big((p^{tk_2}, p^{jk_1+1})\big)}{p^{jk_1+1}} - \frac{\varepsilon_n\big((p^{tk_2}, p^{jk_1})\big)}{p^{jk_1}}\right)\right] -$$
$$\left. - \sum_{t=1}^{t_2} p^{-(tk_2+1)}\left[\sum_{j=1}^{t_1}\left(\frac{\varepsilon_n\big((p^{tk_2+1}, p^{jk_1+1})\big)}{p^{jk_1+1}} - \frac{\varepsilon_n\big((p^{tk_2+1}, p^{jk_1})\big)}{p^{jk_1}}\right)\right]\right\}. \tag{4.11}$$

PROOF. – It can be shown that

$$f^*_{k,t}(p^\alpha) = q^*_{k,t}(p^\alpha) - q^*_{k,t}(p^{\alpha-1}) = \begin{cases} -1, & \text{if } \alpha = jk,\ 1 \leqslant j \leqslant t \\ 1, & \text{if } \alpha = jk+1,\ 1 \leqslant j \leqslant t \\ 0, & \text{otherwise.} \end{cases} \tag{4.12}$$

Now we take $k = k_1$, $l = k_2$, $S_k = Q^*_{k_1,t_1}$ and $S'_l = Q^*_{k_2,t_2}$. Then it is clear that $g_{k_1} = f^*_{k_1,t_1}$ and $g_{k_2} = f^*_{k_2,t_2}$.

Now, Corollary 4.5 can be deduced from (3.9) with the aid of (4.12).

REMARK 4.6. – Letting t_1, $t_2 \to \infty$ in (4.10) we obtain an asymptotic formula for $T^*_{k_1,k_2}(n)$, where $T^*_{k_1,k_2}(n)$ denotes the number of representations of n as the sum of a unitarily k_1-free integer and a unitarily k_2-free integer.

COROLLARY 4.6. – Let $T^*_{(r_2,t_2)r_1}(n)$ denote the number of ordered pairs $(a, b) \in Q^*_{r_2,t_2} \times Q_{r_1}$ with $n = a + b$. Then for sufficiently large n we have

$$T^*_{(r_2,t_2)r_1}(n) = n \cdot \prod_p \left\{1 - \frac{1}{p^{r_1}} - \frac{1}{p^{r_2t_2+1}}\left(\frac{p^{r_2t_2}-1}{p^{r_2}-1}\right)\right\} \times$$
$$\times \prod_{p^{r_1}|n}\left(1 - \frac{p^{r_1-1-r_2t_2}(p-1)(p^{r_2t_2}-1)}{(p^{r_1}-1)(p^{r_2}-1)}\right)^{-1} + E^{(t_2)}_{r_1,r_2}(n)\,, \tag{4.13}$$

where

$$E^{(t_2)}_{r_1,r_2}(n) = O\big(n^{(r_1+r_2)/r_2(r_1+1)}(\log n)^{(r_1^{1/(r_1+1)}-1)(r_2-1)/r_2}\big) \quad \text{or} \quad O(n^{1/r_2})\,,$$

according as $r_2 \leqslant r_1^2$ or $r_2 > r_1^2$; the O-estimates being independent of t_2 and n.

PROOF. – Taking $k = r_1$, $l = r_2$, $S'_l = Q^*_{r_2,t_2}$ in Theorems 2 and 3, and using (4.12) ($k = r_2$, $t = t_2$) we obtain Corollary 4.6.

Let $t \geqslant 3$ and S^*_t denote the set introduced in the beginning of section 4. Now, we have

COROLLARY 4.7. – Let $T^*_t(n)$ denote the number of ordered pairs $(a, b) \in S^*_t \times Q_2$

with $n = a + b$. Then for sufficiently large n,

$$(4.14)\qquad T_t^*(n) = n \prod_p (1 - 2p^{-2} + p^{-t} - p^{-(t+1)}) \cdot \prod_{p^2 | n} \left(1 + \frac{p^2}{p^2-1}(p^{-t} - p^{-(t+1)} - p^{-2})\right)^{-1} + O\left(n^{2/3}(\log n)^{(2^{1/3}-1)/2}\right),$$

where the O-estimate is independent of t and n.

PROOF. – Since $Q_2 \subseteq S_t^*$, taking $k = l = 2$ and $S_2 = S_t^*$ in Theorem 3, we obtain (4.14).

Added in proof. – We note that Theorem 1 of this paper enables us to obtain an asymptotic formula for $T_0(n)$, where $T_0(n)$ denotes the number of representations of n in the form $n = a + b$, where a is a K_1-void integer and b is a K_2-void integer. For the definition of K-void integers we refer to G. J. RIEGER (J. reine angew. Math., **262/263** (1973), pp. 189-193).

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