On a Characteristic Cauchy Problem (*).

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Sunto. – *Si studia un problema di Cauchy caratteristico (per un operatore di cui quello di Klein-Gordon, in coordinate cono luce, è un modello). Si stabiliscono teoremi di esistenza ed unicità. Si prova che la velocità di propagazione è infinita.*

0. – Introduction.

For several problems in quantum field theories it is useful to consider a reference frame which is, in a certain sense, moving with the speed of the light. Strictly speaking this is impossible: there is not transmission of signals between this system and the laboratory system, because of course the limit of a Lorentz transformation for $v \to c$ does not exist. Nevertheless let us assume, for simplicity, $c = 1$ and consider a reference frame moving in the $\mathscr{x}$ direction with speed $v \simeq 1 = c$; by means of Lorentz transformation, ℓ and $\mathscr{x}$ axes are rotated anticlockwise of almost $\pi/4$, while the transverse coordinates $\mathscr{y}, \mathscr{z}$ are unchanged. This induces us to define, from the ordinary Minkowski coordinates $(\ell, \mathscr{x}, \mathscr{y}, \mathscr{z})$ a new reference frame (t, s, x^α) $(\alpha = 1, 2)$, called «infinite momentum frame», by

$$(0.1)\qquad \begin{cases} t \ = 2^{-\frac{1}{2}}(\ell + \mathscr{x}) \\ s \ = 2^{-\frac{1}{2}}(\ell - \mathscr{x}) \\ x^1 = \mathscr{y} \\ x^2 = \mathscr{z} \,. \end{cases}$$

The formulation of quantum field theories in infinite momentum frame is profitable towards the following subjects: current algebra, quantum field theory and laser beam (see [9], [5] and [7] respectively.) Such a reformulation involves a study of the most important equations (Klein-Gordon, Dirac, etc.) and in this connection the Cauchy problems with data on the $t = 0$ hyperplane naturally arise. Indeed R. A. NEVILLE and F. ROHRLICH in [6] consider the Klein-Gordon equation and

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the characteristic Cauchy problem

$$(0.2)\qquad \begin{cases} (2\partial_{ts}^2 - \Delta_{x^\alpha} + m^2)\, u = 0 \\ u(0, s, x^\alpha) = g(s, x^\alpha)\,. \end{cases}$$

We think that is not devoid of interest to attend to a rigorous study of (0.2). In this paper we will concerned with a more general operator

$$Pu = \Big(\partial_{ts}^2 + \sum_1^n a_{jk}\, \partial_{x_j x_k}^2 + \sum_1^n b_j\, \partial_{x_j} + c\Big) u$$

with coefficients $a_{jk}, b_j, c \in C^\infty([0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n)$ for a suitable $T > 0$; moreover we assume that the coefficients are constant for $\|(s, x)\| \gg 1$ and the matrix $[a_{jk}]$ is selfadjoint definite. We shall show an existence and uniqueness theorem for the characteristic Cauchy problem

$$(0.3)\qquad \begin{cases} Pu = f & \text{in } [0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n \\ u(0, s, x) = g(s, x) & (s, x) \in \boldsymbol{R} \times \boldsymbol{R}^n\,. \end{cases}$$

Since P is characteristic on the set $t = 0$, $x = 0$, the usual Cauchy problem makes no more sense; thus we have only one Cauchy datum. As we shall see in the following this is not yet enough, since it is well known, see e.g. [4] that some additional growth conditions must be imposed on u in order to get a « well-posed problem » from an « ill-posed » one.

Let D be a neighborhood of $(0, s_0, x)$ in $\overline{\boldsymbol{R}}^+ \times [s_0, +\infty)$, ALINHAC [1] has considered the following Goursat problem

$$\begin{cases} (a\partial_{ts}^2 + b)\, u = f(t, s) & (t, s) \in \overline{D} \\ u(t, s_0) = u(0, s) = 0 \end{cases}$$

where $a = a(t, s) \in C^1(\overline{D})$, $b = b(t, s)$ is bounded in $\overline{D}$, and $f \in L^2(\overline{D})$, proving energy estimates for u and the existence of a solution in $L^2(\overline{D})$. Now (0.3) is obviously related to some sort of pseudodifferential Goursat pb. where the line $s = s_0$ goes to $-\infty$. The last circumstance as well as Alinhac's results suggests that some « speed of propagation » along the s-axis should be infinite (see Th. 4.1 below) and reaffirm the need of some growth condition on data at $s = -\infty$.

We would also like mention the paper [8] by UHLMANN: it is concerned with the propagation of singularities for an hyperbolic operator with double involutive characteristics, admitting a C^∞-factorisation and Levi conditions on lower order terms. The parametrix (or rather its construction) though points out the close link between this problem and the classical Goursat pb.; no growth condition has to be imposed however, since Levi conditions are satisfied.

Let us now describe the plan of the paper. We suppose that the matrix $[a_{jk}]$ is negative definite. In § 1 we introduce an auxiliary operator $Q = P - \beta_t - \alpha\partial_s + \alpha\beta$ $(\alpha, \beta \in \boldsymbol{R}, 0 > \alpha > \beta)$, for which we establish an energy estimate. Then, in § 2, by means of this estimate and a functional analysis argument we show an existence and uniqueness theorem for the problem $Qv = h$, $v(0, s, x) = k(s, x)$. In § 3, we define suitable Sobolev space H^r_β with weight (see Definition 3.1) and, thanks to the relation $Q(\exp[\alpha t + \beta s]u) = \exp[\alpha t + \beta s]Pu$ we show the following Theorem (see Th. 3.2): Let $\beta < 0$. Let $f \in L^2([0, T]; H^r_\beta(\boldsymbol{R}\times\boldsymbol{R}^n))$ and $g \in H^r_\beta(\boldsymbol{R}\times\boldsymbol{R}^n)$. There exists a unique $u \in C^0([0, T]; H^{r+1}_\beta(\boldsymbol{R}\times\boldsymbol{R}^n))$ such that (0.3) holds.

Moreover we shall prove that if $f \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r-k}_\beta(\boldsymbol{R}\times\boldsymbol{R}^n))$ then

$$u \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r+1-k}_\beta(\boldsymbol{R}\times\boldsymbol{R}^n)) \cap C^{m+1}([0, T]; H^{r-1-m}_\beta(\boldsymbol{R}\times\boldsymbol{R}^n)) .$$

The next sections are devoted to the study of the range of influence for problem (0.3): by means of a Cauchy problem for ${}^tP$ (see § 5)—as in Holmgren theorem—and an other energy estimate (§ 6) we can prove (§ 7) that the speed of propagation is infinite in the s direction (see Th. 4.1). This result allows us to improve Theorem 3.2: actually we conclude that (see Th. 8.1) if the data are C^∞ functions with a suitable behaviour for $s \to -\infty$, then there is a unique u solution of (0.3) such that u is C^∞ and $\exp[\beta s]u(t, s, x)$ is bounded for $s \to -\infty$.

NOTATIONS. – We shall write ∂_j, ∂^2_{jk} instead of ∂_{x_j}, $\partial^2_{x_jx_k}$; ∇ and ∇_x mean $(\partial_t, \partial_s, \partial_1, \dots, \partial_n)$ and $(\partial_1, \dots, \partial_n)$ respectively.

If $b = (b_1, \dots, b_n)$ and $c = (c_1, \dots, c_n) \in \boldsymbol{C}^n$, then $b\cdot c = \sum b_j c_j$ and $\langle b, c\rangle = \sum b_j \bar{c}_j$: If H is an Hilbert space, then $\langle\ ,\ \rangle_H$ denotes its inner product.

Let $n > 0$, $r \in \boldsymbol{R}$; $H^r(\boldsymbol{R}\times\boldsymbol{R}^n)$ is the Sobolev space $\{u = u(t, s, x) \in \mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R}^n);\ (1 + |\sigma|^2 + |\xi|^2)^{r/2}\hat{u}(\sigma, \xi) \in L^2(\boldsymbol{R}\times\boldsymbol{R}^n)\}$ with norm

$$\|u\|^2_{H^r} = \iint (1 + |\sigma|^2 + |\xi|^2)^r |\hat{u}(\sigma, \xi)|^2 \, d\sigma\, d\xi .$$

We often shall write $H^r(s, x)$, or H^r, instead of $H^r(\boldsymbol{R}\times\boldsymbol{R}^n)$.

Finally $B_m(y; \varrho)$ denotes the open ball in $\boldsymbol{R}^m$ of center $y \in \boldsymbol{R}^m$ and radius ϱ.

1. – An energy estimate.

In this note we will be concerned with an operator P of the form

$$P = \partial^2_{ts} + \sum_{jk=1}^{n} a_{jk}\partial^2_{jk} + \sum_{j=1}^{n} b_j\partial_j + c \tag{1.1}$$

with, for $T > 0$,

(1.2)
(i) $a_{jk}, b_j, c \in C^\infty([0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n)$ and are constant outside of a compact subset of $[0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n$;

(ii) the matrix $A = [a_{jk}]$ is negative definite, then—by (i)—there exist $\gamma, \delta > 0$ such that

$$-\gamma I_n \geqslant A(t, s, x) \geqslant -\delta I_n$$

for every $(t, s, x) \in [0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n$.

We point out that, if A is positive definite, then using the change of variables $s' = -s$, we have (1.2.ii).

We shall often use the operator

$$Q = P - \beta\partial_t - \alpha\partial_s + \alpha\beta \tag{1.3}$$

where α, β are constants such that $0 > \alpha > \beta$.

Our aim is to study the following characteristic Cauchy problem

$$\begin{cases} Qv = h & \text{in } [0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n \\ v(0, s, x) = k(s, x) & (s, x) \in \boldsymbol{R} \times \boldsymbol{R}^n . \end{cases} \tag{1.4}$$

For this purpose we shall prove, in this section, energy estimates for Q and Q^*.

To begin with, we calculate $2\,\mathrm{Re}\,\langle(\partial_t + \partial_s)v(t), Qv(t)\rangle_{L^2(s,x)}$ for $t \in [0, T]$ and $v \in C_0([0, T]; H^2(s, x)) \cap C^1([0, T]; H^1(s, x))$:

$$2\,\mathrm{Re}\,[(\partial_t + \partial_s)v(t) \cdot \overline{Qv}(t)] =$$

$$(A) \quad = \partial_t v(t) \cdot \partial_{ts}^{2} \bar{v}(t) + \partial_t \bar{v}(t) \cdot \partial_{ts}^2 v(t) +$$

$$(B) \quad + \partial_s v(t) \cdot \partial_{ts}^2 \bar{v}(t) + \partial_s \bar{v}(t) \cdot \partial_{ts}^2 v(t) -$$

$$- 2\beta|\partial_t v(t)|^2 - 2\alpha|\partial_s v(t)|^2 - (\alpha + \beta) 2\,\mathrm{Re}\,[\partial_t v(t) \cdot \partial_s \bar{v}(t)] +$$

$$(C) \quad + \partial_t v(t) \cdot \sum \bar{a}_{jk}(t)\, \partial_{jk}^{2} \bar{v}(t) + \partial_t \bar{v}(t) \cdot \sum a_{jk}(t)\, \partial_{jk}^2 v(t) +$$

$$(D) \quad + \partial_s v(t) \cdot \sum \bar{a}_{jk}(t)\, \partial_{jk} \bar{v}[t) + \partial_s \bar{v}(t) \cdot \sum a_{jk}(t)\, \partial_{jk}^2 v(t) +$$

$$+ 2\,\mathrm{Re}\,[(\partial_t + \partial_s)v(t) \cdot \sum \bar{b}_j(t)\, \partial_j \bar{v}(t)] + \partial_t v(t) \cdot (\alpha\beta + \bar{c}(t))\bar{v}(t) +$$

$$\partial_t \bar{v}(t) \cdot (\alpha\beta + c(t))v(t) +$$

$$(E) \quad + \partial_s v(t) \cdot (\alpha\beta + \bar{c}(t))\bar{v}(t) + \partial_s \bar{v}(t) \cdot (\alpha\beta + c(t))v(t) .$$

In order to integrate on $\boldsymbol{R}_s \times \boldsymbol{R}_x^n$, remark that: (A) Let φ_n be a sequence of test functions converging to $\partial_t v(t)$, then $\iint A\, ds\, dx = \lim_n \iint \partial_s |\varphi_n|^2\, ds\, dx = 0$. (B) From hypotheses on v it follows $B = \partial_t|\partial_s v(t)|^2$, hence $\iint B\,ds\,dx = d_t\|\partial_s v(t)\|^2_{L^2}$. (*C*) Approximating by test functions we can integrate by parts; therefore

$$\iint C\,ds\,dx = -\iint\sum \bar{a}_{jk}(t)\,\partial_t[\partial_j v(t)\cdot\partial_k\bar{v}(t)]\,ds\,dx - 2\,\mathrm{Re}\,\langle\partial_t v(t), \sum\partial_j a_{jk}(t)\cdot\partial_k v(t)\rangle_{L^2} =$$
$$= -\,d_t\iint\sum a_{kj}(t)\,\partial_j v(t)\cdot\partial_k\bar{v}(t)\,ds\,dx + \iint\sum\partial_t a_{kj}(t)\cdot\partial_j v(t)\cdot\partial_k\bar{v}(t)\,ds\,dx -$$
$$- 2\,\mathrm{Re}\,\langle\partial_t v(t), \sum\partial_j a_{jk}(t)\cdot\partial_k v(t)\rangle_{L^2}.$$

(*D*) Formally we make the same calculation, but we remark that $\iint\partial_s[a_{kj}(t)\ \partial_j\ v(t)\cdot\partial_k\bar{v}(t)]\,ds\,dx = 0$ as in (*A*). (*E*) Again as in (*A*): $\alpha\beta\iint\partial_s|v(t)|^2\,ds\,dx = 0$.

Hence

$$\begin{aligned}
&2\,\mathrm{Re}\,\langle(\partial_t+\partial_s)v(t), Qv(t)\rangle_{L^2} =\\
&= d_t\|\partial_s v(t)\|^2_{L^2} - 2\beta\|\partial_t v(t)\|^2_{L^2} - 2\alpha\|\partial_s v(t)\|^2_{L^2} -\\
&-(\alpha+\beta)\,2\,\mathrm{Re}\,\langle\partial_t v(t), \partial_s v(t)\rangle_{L^2} - d_t\langle A(t)\nabla_x v(t), \nabla_x v(t)\rangle_{L^2} +\\
&+\langle[(\partial_t+\partial_s)A(t)]\nabla_x v(t), \nabla_x v(t)\rangle_{L^2} +\\
&+2\,\mathrm{Re}\,\langle(\partial_t+\partial_s)v(t), [-\,{}^t\nabla_x A(t) + b(t)]\cdot\nabla_x v(t) + c(t)v(t)\rangle_{L^2} +\\
&+\alpha\beta d_t\|v(t)\|^2_{L^2}, \qquad \text{where } b = (b_1, \dots, b_n).
\end{aligned}\tag{1.5}$$

Let

$$\tilde{E}(t) = \alpha\beta\|v(t)\|^2_{L^2(s,x)} + \|\partial_s v(t)\|^2_{L^2(s,x)} - \langle A(t)\nabla_x v(t), \nabla_x v(t)\rangle_{L^2(s,x)}.$$

From (1.5) it follows that

$$\begin{aligned}
&\tilde{E}'(t) = 2\,\mathrm{Re}\,\langle(\partial_t+\partial_s)v(t), Qv(t) + [{}^t\nabla_x A(t) - b(t)]\cdot\nabla_x v(t) - c(t)v(t)\rangle_{L^2} +\\
&+(\beta-\alpha)\|\partial_t v(t)\|^2_{L^2} + (\alpha-\beta)\|\partial_s v(t)\|^2_{L^2} +\\
&+(\alpha+\beta)\|(\partial_t+\partial_s)v(t)\|^2_{L^2} - \langle[(\partial_t+\partial_s)A(t)]\nabla_x v(t), \nabla_x v(t)\rangle_{L^2}.
\end{aligned}\tag{1.6}$$

Thus we are led to the following

1.1. DEFINITION. – We choose as energy the function

$$E(t) = \alpha\beta\|v(t)\|^2_{L^2(s,x)} + \|\partial_s v(t)\|^2_{L^2(s,x)} + \gamma\|\nabla_x v(t)\|^2_{L^2(s,x)}.\tag{1.7}$$

1.2. LEMMA. – Let Q be the operator (1.3). There exists $C > 0$ such that for every $t\in[0, T]$, for every $v\in C^0([0, T]; H^2(s; x))\cap C^1([0, T]; H^1(s, x))$:

$$E(t)\leqslant C\Big\{E(0) + \int_0^t[\|Qv(t')\|^2_{L^2(s,x)} + E(t')]\,dt'\Big\}.$$

PROOF. – We can choose $\varepsilon > 0$ such that $\alpha + \beta + \varepsilon^{-1} < 0$. From (1.6) and (1.2.i) it follows that there exist constants $C_j > 0$ $(j = 1, \dots, 4)$, depending only on the coefficients of Q, such that:

$$\begin{aligned}\tilde{E}'(t) \leqslant \varepsilon^{-1} \|(\partial_t + \partial_s) v(t)\|^2_{L^2} + \\ &+ \varepsilon \|Qv(t) + [{}^t\nabla_x A(t) - b(t)] \cdot \nabla_x v(t) - c(t) v(t)\|^2_{L^2} + \\ &+ (\alpha - \beta) \|\partial_s v(t)\|^2_{L^2} + (\alpha + \beta) \|(\partial_t + \partial_s) v(t)\|^2_{L^2} + C_1 \|\nabla_x v(t)\|^2_{L^2} \leqslant \\ &\leqslant (\alpha + \beta + \varepsilon^{-1}) \|(\partial_t + \partial_s) v(t)\|^2_{L^2} + 3\varepsilon \|Qv(t)\|^2_{L^2} + \\ &+ (3\varepsilon C_2 + C_1) \|\nabla_x v(t)\|^2_{L^2} + 3\varepsilon C_3 \|v(t)\|^2_{L^2} + (\alpha - \beta) \|\partial_s v(t)\|^2_{L^2} \leqslant \\ &\leqslant C_4 [\|Qv(t)\|^2_{L^2} + E(t)] \,.\end{aligned}$$

By integration over $[0, t]$ we get:

$$\tilde{E}(t) \leqslant \tilde{E}(0) + C_4 \int_0^t [\|Qv(t')\|^2_{L^2} + E(t')] \, dt' \,.$$

Since $E(t) \leqslant \tilde{E}(t) \leqslant C_5 E(t)$, $(t \in [0, T])$, for a suitable $C_5 > 0$, the lemma is proved. Q.E.D.

Now we are able to obtain an energy estimate for Q, which will be used in § 2 in order to prove a uniqueness theorem.

1.3. THEOREM. – Let Q be the operator (1.3). For every $r \in \boldsymbol{R}$ there exists $C_r > 0$ such that for every $t \in [0, T]$ and for every $v \in C_0([0, T]; H^{r+2}(s, x)) \cap C^1([0, T]; H^{r+1}(s, x))$:

$$(1.8)_r \quad \begin{aligned}&\alpha\beta \|v(t)\|^2_{H^r} + \|\partial_s v(t)\|^2_{H^r} + \gamma \|\nabla_x v(t)\|^2_{H^r} \leqslant \\ &\leqslant C_r [\alpha\beta \|v(0)\|^2_{H^r} + \|\partial_s v(0)\|^2_{H^r} + \gamma \|\nabla_x v(0)\|^2_{H^r} + \|Qv\|^2_{L^2([0,T]; H^r)}] \,.\end{aligned}$$

PROOF. – Let $v \in C^0([0, T]; H^{r+2}) \cap C^1([0, T]; H^{r+1})$. Denote by Λ_r the p.d.o. with symbol $(1 + |\sigma|^2 + |\xi|^2)^{r/2}$. Define

$$E_r(t) = \alpha\beta \|v(t)\|^2_{H^r} + \|\partial_s v(t)\|^2_{H^r} + \gamma \|\nabla_x v(t)\|^2_{H^r} .$$

From Lemma 1.2 it follows:

$$E_r(t) \leqslant C \left\{ E_r(0) + \int_0^t [\|Q\Lambda_r v(t')\|^2_{L^2} + E_r(t')] \, dt' \right\} .$$

In order to estimate $\|Q\Lambda_r v(t')\|_{L^2}$, write: $Q\Lambda_r v(t) = \Lambda_r Q v(t) + (S\Lambda_r - \Lambda_r S) v(t)$, where $S = \sum a_{jk} \partial^2_{jk} + \sum b_j \partial_j + c$. Thus $S\Lambda_r - \Lambda_r S$ is a p.d.o. of order $r + 2 - 1$ and its symbol, for every $t \in [0, T]$, does not depend on (s, x) outside of a compact subset of $\boldsymbol{R}\times\boldsymbol{R}^n$; then (see IV.11.1 (m) in [2]) $S(t)\Lambda_r - \Lambda_r S(t)\colon H^{r+1}(s, x) \to H^0(s, x)$ is continuous; i.e. $K(t) = \sup_{u\in H^{r+1},\|u\|=1} \|[S(t)\Lambda_r - \Lambda_r S(t)]u\|_{H^0} < \infty$. By the Banach-Steinhaus theorem: $K = \sup_t K(t) < \infty$. Hence

$$\|Q\Lambda_r v(t)\|_{L^2} \leqslant \|Qv(t)\|_{H^r} + K\|v(t)\|_{H^{r+1}} \leqslant \|Qv(t)\|_{H^r} + K_1 E_r(t)^{\frac{1}{2}}.$$

Now:

$$E_r(t) \leqslant K_2\Big\{E_r(0) + \int_0^t [\|Qv(t')\|^2_{H^r} + E_r(t')]\,dt'\Big\}$$

and the theorem follows from Lemma VI.4.4 in [2]. Q.E.D.

We need an energy estimate for $Q^* = \partial^2_{ts} + \beta\partial_t + \alpha\partial_s + \sum \partial^2_{jk}(\bar{a}_{jk}\cdot) - \sum \partial_j(\bar{b}_j\cdot) + \alpha\beta + \bar{c}$ too. Such an estimate will be used in § 2 in order to prove an existence theorem.

1.4. THEOREM. – Let Q^* be as above. For every $r \in \boldsymbol{R}$, there exists $K_r > 0$ such that for every $t \in [0, T]$ and for every $v \in C^0([0, T]; H^{r+2}(s, x)) \cap C^1([0, T]; H^{r-1}(s, x))$:

$$(1.9)_r \quad \alpha\beta\|v(t)\|^2_{H^r} + \|\partial_s v(t)\|^2_{H^r} + \gamma\|\nabla_x v(t)\|^2_{H^r} \leqslant$$
$$\leqslant K_r[\alpha\beta\|v(T)\|^2_{H^r} + \|\partial_s v(T)\|^2_{H^r} + \gamma\|\nabla_x v(T)\|^2_{H^r} + \|Q^* v\|^2_{L^2([0,T];H^r)}].$$

PROOF. – Let $v \in C^0([0, T] + H^2(s, x)) \cap C^1([0, T]; H^1(s, x))$. By obvious modification in (1.6) we get:

$$\begin{aligned}\tilde{E}'(t) = {}& 2\mathrm{Re}\,\langle(\partial_t + \partial_s)v(t), Q^*v(t) + [-\,{}^t\nabla_x \bar{A}(t) + \bar{b}(t)]\cdot\nabla_x v(t) + \\ &+ [-\textstyle\sum \partial^2_{jk}a_{jk}(t) + \sum \partial_j\bar{b}_j(t) - \bar{c}(t)]v(t)\rangle_{L^2} + (\alpha - \beta)\|\partial_t v(t)\|^2_{L^2} + \\ &+ (\beta - \alpha)\|\partial_s v(t)\|^2_{L^2} - (\alpha + \beta)\|(\partial_t + \partial_s)v(t)\|^2_{L^2} - \\ &- \langle[(\partial_t + \partial_s)\bar{A}(t)]\nabla_x v(t), \nabla_x v(t)\rangle_{L^2}.\end{aligned}$$

Arguing as in Lemma 1.2:

$$\begin{aligned}\tilde{E}'(t) \geqslant {}& -(\alpha + \beta + \varepsilon^{-1})\|(\partial_t + \partial_s)v(t)\|^2_{L^2} - 3\varepsilon\|Q^*v(t)\|^2_{L^2} - (3\varepsilon C_2 + C_1)\|\nabla_x v(t)\|^2_{L^2} - \\ &- 3\varepsilon C_3\|v(t)\|^2_{L^2} - (\alpha - \beta)\|\partial_s v(t)\|^2_{L^2} \geqslant - C_4[\|Q^*v(t)\|^2_{L^2} + E(t)].\end{aligned}$$

By integration over $[t, T]$ we get

$$E(t) \leqslant C\Big\{E(T) + \int_t^T [\|Q^*v(t')\|^2_{L^2} + E(t')]\,dt'\Big\}.$$

As in the proof of Theorem 1.3 it follows that there exists $K_r > 0$ such that for every $v \in C^0([0, T]; H^{r+2}(s, x)) \cap C^1([0, T]; H^{r+1}(s, x))$

$$E_r(t) \leqslant K_r\left\{E_r(T) + \int_t^T [\|Q^* v(t')\|_{H^r}^2 + E_r(t')]\, dt'\right\}.$$

To finish our argument it is enough to put $Y(t) = E_r(T - t)$ and $\varphi(t) = \|Q^* v(T - t)\|_{H^r}^2$; so we can apply Lemma VI.4.4 in [2]. Q.E.D.

2. – Existence and uniqueness of solution.

By an argument of functional analysis and Theorems 1.3 and 1.4 we shall prove a theorem of existence and uniqueness for the Cauchy problem (1.4).

We begin with

2.1. PROPOSITION. – Let Q be the operator (1.3) and let $r \in \mathbf{R}$. If $h \in L^1([0, T]; H^r(s, x))$ and $k \in H^r(s, x)$, then there exists $v \in C^0([0, T]; H^{r-2}(s, x))$ such that (1.4) holds.

PROOF. – Let $E = \{\varphi \in C^\infty([0, T]; H^{+\infty}(s, x));\ \varphi(T) = 0\}$. We are going to define an antilinear functional $l: Q^* E \to \mathbf{C}$ and we shall show that we can continue it to a continuous functional on $L^2([0, T]; H^{-r}(s, x))$. Let

$$(f, g) = \iint \hat{f}(\sigma, \xi)\, \overline{\hat{g}(\sigma, \xi)}\, đ\sigma\, đ\xi \qquad \text{for } f \in H^r(s, x)\,, \qquad g \in H^{-r}(s, x)\,;$$

define, for $\varphi \in E$,

$$l(Q^* \varphi) = \int_0^T (h(t), \varphi(t))\, dt + (k, -\partial_s \varphi(0) - \beta \varphi(0))\,. \tag{2.1}$$

From $\varphi(T) = 0$, it follows that $\partial_s \varphi(T) = 0$ and $\nabla_x \varphi(T) = 0$. Thus, by $(1.9)_{-r}$, l is well defined. By $(1.9)_{-r}$ we obtain also:

$$|l(Q^* \varphi)| \leqslant \|h\|_{L^2([0,T]; H^r)} \left[\int_0^T \|\varphi(t)\|_{H^{-r}}^2 dt\right]^{\frac{1}{2}} + \\ + 2^{\frac{1}{2}} \|k\|_{H^r} [\|\partial_s \varphi(0)\|_{H^{-r}}^2 + \beta^2 \|\varphi(0)\|_{H^{-r}}^2]^{\frac{1}{2}} \leqslant C \|Q^* \varphi\|_{L^2([0,T]; H^{-r})}$$

for a suitable $C > 0$, for every $\varphi \in E$. By an application of the Hahn-Banach theorem, there exists $w \in L^2([0, T]; H^{-r}(s, x))$ such that $\langle w, Q^* \varphi \rangle_{L^2([0,T]; H^{-r})} = l(Q^* \varphi)$, $(\varphi \in E)$.

Let $v(t) = \Lambda_{-2r} w(t)$, then $v \in L^2([0, T]; H^r)$ and

$$\int_0^T (v(t), Q^* \varphi(t))\, dt = l(Q^* \varphi) \qquad (\varphi \in E)\,.$$

Let $\{\psi_j\}$ be a C^∞ partition of unity of $\boldsymbol{R}\times\boldsymbol{R}^n$. We write

$$\big(v(t), Q^*\varphi(t)\big) = \sum_j \big(v(t), Q^*\psi_j\varphi(t)\big);$$

thus by an integration by parts with respect to s and x_j, we get

$$(2.2)\qquad l(Q^*\varphi) = \int_0^T \Big\{-\big([\partial_s - \beta]v(t), \partial_t\varphi(t)\big) + \\ + \big([-\alpha\partial_s + \sum a_{jk}(t)\partial^2_{jk} + \sum b\,(t)\partial_j + \alpha\beta + c(t)]v(t), \varphi(t)\big)\Big\}\,dt\,.$$

If, in particular, φ is a test function on $(0, T)\times\boldsymbol{R}\times\boldsymbol{R}^n$, with an integration by parts, with respect to t, we have

$$l(Q^*\varphi) = \int_0^T \big(Qv(t), \varphi(t)\big)\,dt\,.$$

Now, from (2.1), it follows

$$\int_0^T \big(Qv(t), \varphi(t)\big)\,dt = \int_0^T \big(h(t), \varphi(t)\big)\,dt$$

for every test function φ of $(0, T)\times\boldsymbol{R}\times\boldsymbol{R}^n$. Hence

$$Qv = h \quad \text{ in } \mathcal{D}'\big((0, T)\times\boldsymbol{R}\times\boldsymbol{R}^n\big)\,.$$

Then

$$\partial_t(\partial_s - \beta v) = \alpha\partial_s v - \sum a_{jk}\partial^2_{jk}v - \sum b_j\partial_j v - (\alpha\beta + c)v + h \in L^2([0, T]; H^{r-2});$$

and, due to the Sobolev theorem

$$(2.3)\qquad \partial_s v - \beta v \in C^0\big([0, T]; H^{r-2}\big)\,.$$

In general, if $\varphi \in E$, from an integration by parts with respect to t in (2.2) it follows

$$l(Q^*\varphi) = \int_0^T \big(Qv(t), \varphi(t)\big)\,dt + \big(\partial_s v(0) - \beta v(0), \varphi(0)\big);$$

i.e. $\big(k, -\partial_s\varphi(0) - \beta\varphi(0)\big) = \big(\partial_s v(0) - \beta v(0), \varphi(0)\big)$; it is enough to say that $k = v(0)$.

To finish our argument we must show $v \in C^0([0, T]; H^{r-2})$. Let $t, t' \in [0, T]$; since

$$\langle\partial_s v(t) - \partial_s v(t'),\ v(t) - v(t')\rangle_{H^{r-2}} = i\iint \sigma\big(1 + |\sigma|^2 + |\xi|^2\big)^{r-2}\,|\widehat{v(t) - v(t')}|^2\,d\sigma\,d\xi$$

is pure imaginary, then

$$\|\partial_s v(t) - \beta v(t) - [\partial_s v(t') - \beta v(t')]\|^2_{H^{r-2}} = \|\partial_s v(t) - \partial_s v(t')\|^2_{H^{r-2}} + \beta^2 \|v(t) - v(t')\|^2_{H^{r-2}} .$$

From (2.3) it follows $\lim_{t'\to t} \|v(t) - v(t')\|^2_{H^{r-2}} = 0$. Q.E.D.

Now we show the uniqueness:

2.2. PROPOSITION. – With the same hypotheses of Proposition 2.1 the solution v is unique in $C^0([0, T]; H^{r-2}(s, x))$.

PROOF. – Let $w \in C^0([0, T]; H^{r-2})$ such that $Qw = 0$ and $w(0) = 0$. Then

$$\partial_t(\partial_s w - \beta w) = \alpha \partial_s w - \sum a_{jk} \partial^2_{jk} w - \sum b_j \partial_j w - (\alpha\beta + c) w \in C^0([0, T]; H^{r-4}) .$$

Hence by the above argument:

$$\partial_t \beta w \in C^0([0, T]; H^{r-4});$$

i.e. $w \in C^1([0, T]; H^{r-4})$. Thus we can apply $(1.8)_{r-6}$ to see $w = 0$. Q.E.D.

2.3. REMARK. – If the data are smooth, i.e. $h \in C^\infty([0, T]; H^{+\infty}(s, x))$ and $k \in H^{+\infty}(s, x)$, then there exists an unique $v \in C^0([0, T]; H^{+\infty}(s, x))$ such that $Qv = h$ and $v(0) = k$. Moreover, since $\partial_t(\partial_s - \beta) v \in C^0([0, T]; H^{+\infty})$, it follows $v \in C^1([0, T]; H^{+\infty})$. Thus with a step by a step argument $v \in C^\infty([0, T]; H^{+\infty})$.

Using this remark we shall improve Propositions 2.1 and 2.2:

2.4 THEOREM. – Let Q be the operator (1.3). If $h \in L^2([0, T]; H^r)$ and $k \in H^{r+1}$, then there exists a unique $v \in C^0([0, T]; H^{r+1})$ such that (1.4) holds. Moreover v satisfied $(1.8)_r$.

PROOF. – Let (h_n) be a sequence in $C_0^\infty([0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n)$ converging to h in $L^2([0, T]; H^r)$ and let (k_n) be a sequence of test functions in $\boldsymbol{R} \times \boldsymbol{R}^n$ converging to k in $H^{r+1}(s, x)$. By previous Remark 2.3, for every $n \in \boldsymbol{N}$ there exists a unique $v_n \in C^\infty([0, T]; H^{+\infty})$ such that $Qv_n = h_n$ and $v_n(0) = k_n$. To see that the sequences $v_n, \partial_s v_n, \partial_j v_n$ $(j = 1, \ldots, n)$ are Cauchy sequence in $C^0([0, T]; H^r)$ it is enough to apply $(1.8)_r$ to $v_n - v_m$; thus there exists v such that v_n converges to v in $C^0([0, T]; H^{r+1})$. Then $v(0) = k$. Moreover

$$\partial_t(\partial_s - \beta) v_n = (\alpha \partial_s - \sum a_{jk} \partial^2_{jk} - \sum b_j \partial_j - \alpha\beta - c) v_n + h_n$$

is a Cauchy sequence in $L^2([0, T]; H^{r-1})$; hence $Qv = h$.

Since $v_n \to v$, we see that v satisfied $(1.8)_r$. Q.E.D.

Moreover we shall show that there is a relation between the regularity of the data and the regularity of the solution.

2.5. COROLLARY. – With the same hyptheses of Theorem 2.4, for every $m \in \boldsymbol{N}$:

(A_m) If $h \in \bigcap_{k=0}^{m} H^k([0, T]; H^{r-k}(s, x))$ then

$$v \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r-k+1}(s, x)) \cap H^{m+1}([0, T]; H^{r-m-1}(s, x));$$

(B_m) If $h \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r-k}(s, x))$ then

$$v \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r-k+1}(s, x)) \cap C^{m+1}([0, T]; H^{r-m-1}(s, x)) .$$

PROOF. – We shall write $H^k(q)$, $C^k(q)$ instead of $H^k([0, T]; H^q)$, $C^k([0, T]; H^q)$ respectively.

First we prove (A_0): let $h \in H^0(r)$; by Theorem 2.4, $v \in C^0(r+1)$; then $\partial_t(\partial_s - \beta)v \in H^0(r-1)$, and, by the same argument used before in the proof of 2.1, we get $\partial_t v \in H^0(r-1)$, i.e. $v \in H^1(r-1)$. Let (A_m) holds. Let $h \in \bigcap_{k=0}^{m+1} H^k(r-k)$. From $Qv = h$ we obtain

$$Q\partial_t^{m+1}v = Q_{m+1}v + \partial_t^{m+1}h$$

where Q_{m+1} is a differential operator of order m in t and order 2 in (s, x). Since, by (A_m), $v \in C^m(r+1-m)$, then

$$Q\partial_t^{m+1}v \in C^0(r-1-m) \cap H^0(r-1-m) .$$

Finally, by (A_0), $\partial_t^{m+1}v \in C^0(r-m) \cap H^1(r-2-m)$. This proves (A_{m+1}). A similar argument proves (B_m). Q.E.D.

3. – Conclusions about the operator P.

Consider the operator P in (1.1). We shall show that the characteristic Cauchy problem (0.3) is well posed if the data f and g belong to suitable spaces defined as follows:

3.1. DEFINITION. – $r, \beta \in \boldsymbol{R}$. Define

$$H_\beta^r(\boldsymbol{R}\times\boldsymbol{R}^n) = \{\varphi \in \mathcal{D}'(\boldsymbol{R}\times\boldsymbol{R}^n);\ \exp[\beta s]\varphi(s, x) \in H^r(s, x)\} ,$$

and $\|\varphi\|_{H_\beta^r} = \|\exp[\beta s]\varphi\|_{H^r}$.

3.2. THEOREM. – Let P be the operator (1.1). Let $\beta < 0$, $f \in L^2([0, T]; H^r_\beta(s, x))$ and $g \in H^{r+1}_\beta(s, x)$. Then there exists a unique $u \in C^0([0, T]; H^{r+1}_\beta(s, x))$ such that

$$\begin{cases} Pu = f & \text{in } [0, T]\times \mathbf{R}\times\mathbf{R}^n \\ u(0, s, x) = g(s, x) & (s, x) \in \mathbf{R}\times\mathbf{R}^n . \end{cases}$$

Moreover if $f \in \bigcap_{k=0}^{m} H^k([0, T]; H^{r-k}_\beta(s, x))$, then

$$u \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r+1-k}_\beta(s, x)) \cap H^{m+1}([0, T]; H^{r-1-m}_\beta(s, x));$$

if $f \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r-k}_\beta(s, x))$, then

$$u \in \bigcap_{k=0}^{m} C^k([0, T]; H^{r+1-k}_\beta(s, x)) \cap C^{m+1}([0, T]; H^{r-1-m}_\beta(s, x)) .$$

PROOF. – Since $\beta < 0$, we can choose α such that $0 > \alpha > \beta$. Let $h = \exp[\alpha t + \beta s]f$ and $k = \exp[\beta s]g$; by Theorem 2.4 there exists a unique $v \in C^0([0, T]; H^{r+1})$ such that $Qv = h$ and $v(0) = k$.

To finish it is enough to put $u = \exp[-\alpha t - \beta s]v$, and remark that

$$Q(\exp[at + \beta s]u) = \exp[\alpha t + \beta s]Pu . \qquad \text{Q.E.D.}$$

4. – Range of influence: the statement.

We will study the range of influence of the operator P (1.1), with reference to the Cauchy problem (0.3).

Let $(t_0, s_0, x_0) \in [0, T]\times\mathbf{R}\times\mathbf{R}^n$ and consider the cone

$$\mathcal{C} = \{(t, s, x) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^n;\ 4\delta(t_0 - t)(s_0 - s) - \|x - x_0\|^2 > 0 \text{ and } 0 \leqslant t < t_0\} \tag{4.1}$$

with δ is as in (1.2.ii).

Now we state that the speed of propagation is infinite; in fact we shall prove in section 7 the following

4.1. THEOREM. – Let P be the operator (1.1). If $u = u(t, s, x)$ satisfies the following conditions

(4.2)
(i) $u \in C^2(\mathcal{C})$;
(ii) there exists $\beta < 0$ such that $\exp[\beta s]u(t, s, x)$ is bounded in $\mathcal{C}$;
(iii) $\begin{cases} Pu = 0 & \text{in } \mathcal{C} \\ u(0, s, x) = 0 \text{ for every } (s, x) \text{ such that } (0, s, x) \in \mathcal{C} \end{cases}$

then $u(t, s, x) = 0$ in $\mathcal{C}$.

5. - An auxiliary Cauchy problem.

5.1. DEFINITION. - Let $(t_1, s_1, x_1) \in C$, and $\alpha_1 = 4\delta(t_0 - t_1)(s_0 - s_1) - \|x_1 - x_0\|^2 > 0$. Define

$$\mathcal{T}_1 = \{(t, s, x);\ 4\delta(t_0 - t)(s_0 - s) - \|x - x_0\|^2 \geqslant \alpha_1 \text{ and } t < t_0\}.$$

In this section we prove the existence of a solution of the equation: ${}^tPw = 0$ in $\mathcal{T}_1$, when the Cauchy data are assigned on the hyperboloid $\partial\mathcal{T}_1$. To this end we define coordinates (p, q, x) by means of

$$\left\{\begin{aligned} p &= t + s \\ q &= t - s \\ x_j &= x_j \qquad (j = 1, \ldots, n)\,. \end{aligned}\right. \tag{5.1}$$

Let (p_0, q_0, x_0) be the coordinates, in the frame (5.1), of the point (t_0, s_0, x_0). We introduce also coordinates $(\tilde{p}, \tilde{q}, \tilde{x})$ by means of

$$\left\{\begin{aligned} \tilde{p} &= p_0 - p - \theta \\ \tilde{q} &= q - q_0 \\ \tilde{x}_j &= x_j - x_{0j} \qquad (j = 1, \ldots, n) \end{aligned}\right. \tag{5.2}$$

where

$$\theta = \theta(q, x) = [\alpha_1\delta^{-1} + (q - q_0)^2 + \|x - x_0\|^2\delta^{-1}]^{\frac{1}{2}} = (\alpha_1\delta^{-1} + \tilde{q}^2 + \|\tilde{x}\|^2\delta^{-1})^{\frac{1}{2}}\,.$$

It is straightforward to check that $(t, s, x) \to (\tilde{p}, \tilde{q}, \tilde{x})$ is a C^∞ one-to-one transformation of $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}^n$ onto itself. Moreover if $\tilde{\mathcal{T}}_1$ corresponds to $\mathcal{T}_1$ by means of above change of coordinates, then $\tilde{\mathcal{T}}_1 = \{\tilde{p} \geqslant 0\}$.

5.2. LEMMA. - Let $\delta_1 > \delta$. There exist $a'_{jk}, b'_j, c' \in C^\infty((-\infty, T]\times\boldsymbol{R}\times\boldsymbol{R}^n)$ such that they extend the coefficients $a_{jk}, b_j, c \in C^\infty([0, T]\times\boldsymbol{R}\times\boldsymbol{R}^n)$ of the operator P (1.1) and

$$0 > A'(t, s, x) = [a'_{jk}(t, s, x)] \geqslant -\delta_1 I_n$$

for every $(t, s, x) \in (-\infty, T]\times\boldsymbol{R}\times\boldsymbol{R}^n$.

5.3. LEMMA. - Let $\delta_1 > \delta$. Let $R = \partial^2_{ts} + \sum_{jk=1}^{n} b_{jk}\partial^2_{jk} + \sum_{j=1}^{n} c_j\partial_j + r$ be an operator with coefficients b_{jk}, c_j, r belonging to $C^\infty(\mathcal{T}_1)$, and let $B = [b_{jk}]$ be selfadjoint definite such that $0 > B \geqslant -\delta_1 I_n$ in $\mathcal{T}_1$. Denote by $\tilde{R}$ the operator that corresponds to R

in the $(\tilde{p}, \tilde{q}, \tilde{x})$ coordinates (5.2). Then

$$(5.3) \qquad \tilde{R} = \tilde{\varphi}(\tilde{p}, \tilde{q}, \tilde{x})\partial^2_{\tilde{p}} + \tilde{R}_1(\tilde{p}, \tilde{q}, \tilde{x}; \partial_{\tilde{q}}, \partial_{\tilde{x}_1}, \dots, \partial_{\tilde{x}_n})\partial_{\tilde{p}} + \tilde{R}_2(\tilde{p}, \tilde{q}, \tilde{x}; \partial_{\tilde{q}}, \partial_{\tilde{x}_1}, \dots, \partial_{\tilde{x}_n})$$

where: $\tilde{R}_j$ $(j = 1, 2)$ is a linear differential operator of order j, with coefficients belonging to $C^\infty\big((\boldsymbol{R}^+ \cup \{0\}) \times \boldsymbol{R} \times \boldsymbol{R}^n\big)$; $\tilde{\varphi}$ is a smooth function such that $\tilde{\varphi} > 0$ if $\|\tilde{x}\| < [\delta\alpha_1(\delta_1 - \delta)^{-1}]^{\frac{1}{2}}$; $\tilde{R}$ is strictly hyperbolic in the direction $d\tilde{p}$, on the domain

$$\big(\boldsymbol{R}^+ \cup \{0\}\big) \times \boldsymbol{R} \times B_n\big(0; [\delta\alpha_1(\delta_1 - \delta)^{-1}]^{\frac{1}{2}}\big)\,.$$

PROOF. – Let $\tilde{r}_2$ and r_2 be the principal symbols of the operators $\tilde{R}$ and R respectively, then they are connected by

$$\tilde{r}_2(\tilde{p}, \tilde{q}, \tilde{x}; \pi, \chi, \zeta) = r_2\left(t, s, x; \frac{{}^t\partial(p, q, x)}{\partial(t, s, x)}\begin{pmatrix}\pi\\ \chi\\ \zeta\end{pmatrix}\right).$$

Therefore, it follows

$$\tilde{r}_2(\tilde{p}, \tilde{q}, \tilde{x}; \pi, \chi, \zeta) = $$
$$= \pi^2\tilde{\varphi}(\tilde{p}, \tilde{q}, \tilde{x}) + \pi[2\tilde{q}\theta^{-1}\chi - \delta^{-1}\theta^{-1}(\langle B\zeta, \tilde{x}\rangle + \langle B\tilde{x}, \zeta\rangle)] + \langle B\zeta, \zeta\rangle - \chi^2$$

where

$$(5.4) \qquad \tilde{\varphi}(\tilde{p}, \tilde{q}, \tilde{x}) = \frac{\delta^2\theta^2 - \delta^2\tilde{q}^2 + \langle B\tilde{x}, \tilde{x}\rangle}{\delta^2\theta^2} \geqslant \frac{\delta\alpha_1 + (\delta - \delta_1)\|\tilde{x}\|^2}{\delta^2\theta^2}\,.$$

Hence $\tilde{\varphi} > 0$, for $\|x\|^2 < \delta\alpha_1(\delta_1 - \delta)^{-1}$.

Since $\chi^2 - \langle B\zeta, \zeta\rangle > 0$ we can see that $\tilde{R}$ is strictly hyperbolic in the direction $d\tilde{p}$ on the domain $\{\tilde{\varphi} > 0\}$. Q.E.D.

5.4. DEFINITION. – Let $s_2 < s_0$. Define

$$\Omega_{s_2} = \{(t, s, x) \in \mathcal{T}_1;\ s \geqslant s_2\}\,.$$

5.5. REMARK. – Let

$$t_{s_2} = t_0 - \frac{\alpha_1}{4\delta(s_0 - s_2)}$$

$$r_{s_2}(t) = r(t) = [4\delta(t_0 - t)(s_0 - s_2) - \alpha_1]^{\frac{1}{2}}$$

$$s_{s_2}(t, x) = s(t, x) = s_0 - \frac{\alpha_1 + \|x - x_0\|^2}{4\delta(t_0 - t)}\,.$$

Then $(t, s, x) \in \Omega_{s_2}$ if only if

$$\begin{cases} t \in [0, t_{s_2}] \\ x \in B_n(x_0; r(t)) \\ s \in [s_2, s(t, x)]; \end{cases}$$

therefore $\partial\Omega_{s_2} = A_{s_2} \cup B_{s_2} \cup C_{s_2}$, with

$$A_{s_2} = \}(t, s_2, x);\ t \in [0, t_{s_2}],\ x \in B_n(x_0; r(t))\}$$
$$B_{s_2} = \{(0, s, x);\ x \in B_n(x_0; r(0)),\ s \in [s_2, s(0, x)]\}$$
$$C_{s_2} = \{(t, s(t, x), x);\ t \in [0, t_{s_2}],\ x \in B_n(x_0; r(t))\}\ .$$

Finally we can prove the following

5.6. THEOREM. – Let P be the operator (1.1) and $\varphi \in C_0^\infty(\partial\mathcal{T}_1)$. For every $s_2 < s_0$ there exists $w \in C^\infty(\Omega_{s_2})$ such that

$$\text{(5.5)} \qquad \begin{cases} {}^tPw = 0 & \text{in } \Omega_{s_2} \\ w \quad = 0 & \text{in } C_{s_2} \\ \partial_n w = \varphi & \text{in } C_{s_2} \end{cases}$$

where n is the unit normal vector, directed outside Ω_{s_2}:

PROOF. – Let $s_2 < s_0$: Since $r_2(t) \leqslant [4\delta t_0(s_0 - s_2) - \alpha]^{\frac{1}{2}}$ and $s_2(t, x) \leqslant s_0$, Ω_{s_2} is bounded. Let $\tilde{\Omega}_{s_2}$ be the domain which corresponds to Ω_{s_2} in the $(\tilde{p}, \tilde{q}, \tilde{x})$ coordinates; then $\tilde{\Omega}_{s_2}$ is bounded, hence there exists δ_1, $\delta_1 > \delta$, such that

$$B = B_{2+n}(0; \tfrac{1}{2}[\delta\alpha_1(\delta_1 - \delta)^{-1}]^{\frac{1}{2}}) \supset \tilde{\Omega}_{s_2}\,.$$

From Lemma 5.2 it follows that there exists an extension P', of P, to $(-\infty, T]\times \times\boldsymbol{R}\times\boldsymbol{R}^n \supset \mathcal{T}_1$, such that $0 > A' \geqslant -\delta_1 I_n$: By Lemma 5.3 the operator ${}^t\tilde{P}'$ is of the form (5.3), strictly hyperbolic in the direction $d\tilde{p}$, on $2B$. Keeping into account Lemma VI.4.12 in [2] there exists an operator $\tilde{L}$ with coefficients belonging to $C^\infty\big((\boldsymbol{R}^+ \cup \{0\})\times\boldsymbol{R}\times\boldsymbol{R}^n\big)$, constant out of $2B$, of the same form as ${}^t\tilde{P}'$ and such that $\tilde{L} = {}^t\tilde{P}'$ in B, strictly hyperbolic in the direction $d\tilde{p}$, on $(\boldsymbol{R}^+ \cup \{0\})\times\boldsymbol{R}\times\boldsymbol{R}^n$.

Let $\tilde{\chi} = -\varphi\|\nabla\tilde{p}\|^{-1}$, $U > \frac{1}{2}[\delta\alpha_1(\delta_1 - \delta)^{-1}]^{\frac{1}{2}}$. There exists $\tilde{w} \in C^\infty([0, U]\times\boldsymbol{R}\times\boldsymbol{R}^n)$ such that

$$\begin{cases} \tilde{L}w = 0 & \text{in } [0, U]\times\boldsymbol{R}\times\boldsymbol{R}^n \\ \tilde{w}(0, \tilde{q}, \tilde{x}) = 0 & \text{in } \boldsymbol{R}\times\boldsymbol{R}^n \\ \partial_{\tilde{p}}\tilde{w}(0, \tilde{q}, \tilde{x}) = \tilde{\chi}(\tilde{q}, \tilde{x}) & \text{in } \boldsymbol{R}\times\boldsymbol{R}^n\,. \end{cases}$$

Put $w(t, s, x) = \tilde{w}(\tilde{p}, \tilde{q}, \tilde{x})$. Since $\tilde{\mathcal{T}}_1 = \{\tilde{p} \geq 0\}$ and $C_{s_2} \subset \partial \mathcal{T}_1$, thus $n = -\nabla\tilde{p}\|\nabla\tilde{p}\|^{-1}$ and, in C_{s_2},

$$\nabla w = \frac{{}^t\partial(\tilde{p}, \tilde{q}, \tilde{x})}{\partial(t, s, x)} \tilde{\nabla}\tilde{w} = (\partial_{\tilde{p}}\tilde{w})\nabla\tilde{p} \qquad \text{holds};$$

therefore $\partial_n w = -\nabla w \cdot \nabla\tilde{p}\|\nabla\tilde{p}\|^{-1} = -\tilde{\chi}\|\nabla\tilde{p}\| = \varphi$. To finish it is enough to remark that $\tilde{L} = {}^t\tilde{P}' = {}^t\tilde{P}$ in $\tilde{\Omega}_{s_2} \subset B \subset [0, U] \times \boldsymbol{R} \times \boldsymbol{R}^n$. Q.E.D.

6. – A Stokes-energy inequality.

To prove Theorem 4.1 we need an estimate of a solution of (5.5).

6.1. DEFINITION. – Let P be the operator (1.1) and $\alpha, \beta \in \boldsymbol{R}$ such that $0 > \alpha > \beta$. If w is a solution of ${}^tPw = 0$ in an open subset of $[0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n$, define

$$v(t, s, x) = \exp[-\alpha t - \beta s] w(t, s, x), \tag{6.1}$$

and

$$E(t, s, x) = \alpha\beta|v|^2 + |\partial_s v|^2 + \gamma\|\nabla_x v\|^2;$$
$$\tilde{E}(t, s, x) = \alpha\beta|v|^2 + |\partial_s v|^2 - \langle A\nabla_x v, \nabla_x v\rangle;$$
$$F(t, s, x) = \alpha\beta|v|^2 + |\partial_t v|^2 - \langle A\nabla_x v, \nabla_x v\rangle;$$
$$G(t, s, x) = 2\,\mathrm{Re}\,[(\partial_t + \partial_s)\bar{v} \cdot A\nabla_x v].$$

6.2. LEMMA. – There exists a constant $K > 0$ (depending only on P, α, β) such that, for every C^∞ solution w of ${}^tPw = 0$ in a neighbourhood of (t, s, x) in $[0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n$,

$$\partial_t \tilde{E}(t, s, x) + \partial_s F(t, s, x) + \nabla_x \cdot G(t, s, x) \geq -KE(t, s, x). \tag{6.2}$$

PROOF. – Let Q be the operator (1.3). Then by (6.1) ${}^tQv = \exp[-\alpha t - \beta s]\,{}^tPw = 0$, thus $0 = 2\,\mathrm{Re}\,[(\partial_t + \partial_s)\bar{v} \cdot {}^tQv]$. Since

$${}^tQv = \partial^2_{ts} v + \beta\partial_t v + \alpha\partial_s v + \sum a_{jk}\partial^2_{jk} v + \sum l_j \partial_j v + \alpha\beta v + rv,$$

where $l_j = \sum \partial_k(a_{jk} + a_{kj}) - b_j$, $(j = 1, \dots, n)$; and $r = \sum \partial^2_{jk} a_{jk} - \sum \partial_j b_j + c$, by a straightforward calculation we have

$$\begin{aligned} 0 = 2\,\mathrm{Re}\,[(\partial_t + \partial_s)\bar{v} \cdot {}^tQv] &= \partial_s|\partial_t v|^2 + \partial_t|\partial_s v|^2 + 2\beta|\partial_t v|^2 + 2\,\mathrm{Re}\,(\alpha + \beta)\,\partial_t v \cdot \partial_s \bar{v} + \\ &+ 2\alpha|\partial_s v|^2 + \nabla_x 2\,\mathrm{Re}\,[\partial_t\bar{v} \cdot A\nabla_x v] - \partial_t\langle A\nabla_x v, \nabla_x v\rangle + \langle(\partial_t A)\nabla_x v, \nabla_x v\rangle - \\ &- 2\,\mathrm{Re}\,[\partial_t\bar{v} \cdot ({}^t\nabla_x A)\nabla_x v] + \nabla_x 2\,\mathrm{Re}\,[\partial_s\bar{v} \cdot A\nabla_x v] - \partial_s\langle A\nabla_x v, \nabla_x v\rangle + \langle(\partial_s A)\nabla_x v, \nabla_x v\rangle - \\ &- 2\,\mathrm{Re}\,[\partial_s\bar{v} \cdot ({}^t\nabla_x A)\nabla_x v] + 2\,\mathrm{Re}\,[(\partial_t + \partial_s)\bar{v} \cdot (l \cdot \nabla_x v + rv)] + \alpha\beta\partial_t|v|^2 + \alpha\beta\partial_s|v|^2, \end{aligned}$$

where $l = (l_1, \dots, l_n)$.

It follows:

$$\partial_t \tilde{E} + \partial_s F + \nabla_x \cdot G = -(\alpha+\beta)|(\partial_t+\partial_s)v|^2 + (\alpha-\beta)|\partial_t v|^2 + (\beta-\alpha)|\partial_s v|^2 + \\ + 2\,\mathrm{Re}\,\{(\partial_t+\partial_s)\bar{v}\cdot[({}^t\nabla_x A - l)\cdot\nabla_x v - rv]\} - \langle[(\partial_t+\partial_s)A]\nabla_x v, \nabla_x v\rangle .$$

Since $0>\alpha>\beta$, there exists $\varepsilon>0$ such that $\varepsilon^{-1}+\alpha+\beta<0$, therefore

$$\nabla(\tilde{E}, F, G) \geqslant -(\varepsilon^{-1}+\alpha+\beta)|(\partial_t+\partial_s)v|^2 - (\alpha-\beta)|\partial_s v|^2 - \\ - \varepsilon|({}^t\nabla_x A - l)\nabla_x v - rv|^2 - \|(\partial_t+\partial_s)A\|\,\|\nabla_x v\|^2 \geqslant -KE$$

for a suitable $K\gg$, because A and l are constant for $\|(s,x)\|\gg$. Q.E.D.

Now we can prove the required inequality:

6.3. THEOREM. – Let $\varphi\in C_0^\infty(\partial\mathcal{T}_1)$. There exists a constant $M>0$ (depending only on P, α, β and φ) such that for every $s_2<s_0$, for every solution $w\in C^\infty(\Omega_{s_2})$ of (5.5):

$$\iint_{A_{s_2}} F(t,s,x)\,dt\,dx \leqslant M .$$

PROOF. – Let $s_2<s_0$. Put $\Omega_{s_2}(t)=\{(t',s,x)\in\Omega_{s_2};\ t'\geqslant t\}$, $t\in[0,t_{s_2}]$. It follows that $\partial\Omega_{s_2}(t)=A_{s_2}(t)\cup B_{s_2}(t)\cup C_{s_2}(t)$ with

$$A_{s_2}(t)=\{(t',s_2,x);\ t'\in[t,t_{s_2}],\ x\in B_n(x_0;r(t'))\},$$
$$B_{s_2}(t)=\{(t,s,x);\ x\in B_n(x_0;r(t)),\ s\in[s_2,s(t,x)]\},$$
$$C_{s_2}(t)=\{(t',s(t',x)x);\ t'\in[t,t_{s_2}],\ x\in B_n(x_0,r(t'))\}.$$

Applying Stokes theorem to (6.2):

$$-\iint_{B_{s_2}(t)} \tilde{E}(t,s,x)\,ds\,dx - \iint_{A_{s_2}(t)} F(t',s_2,x)\,dt'\,dx + \iint_{C_{s_2}(t)} (\tilde{E},F,G)\cdot n\,dS \geqslant \\ \geqslant -K\iiint_{\Omega_{s_2}(t)} E(t',s,x)\,dt'\,ds\,dx .$$

But, for (1.2.ii), $E\leqslant\tilde{E}$, thus

$$\iint_{B_{s_2}(t)} E\,ds\,dx \leqslant \iint_{C_{s_2}(t)} (\tilde{E},F,G)\cdot n\,dS - \iint_{A_{s_2}(t)} F\,dt'\,dx + K\iiint_{\Omega_{s_2}(t)} E\,dt'\,ds\,dx . \tag{6.3}$$

Now we are going to calculate $(\tilde{E},F,G)\cdot n$ in C_{s_2}. From $w=0$, $\partial_n w=\varphi$ in C_{s_2}, it follows $\nabla w=\varphi n$; therefore, in C_{s_2}, $v=0$ and $\nabla v=\exp[-\alpha t-\beta s]n$. Hence

$$(\tilde{E},F,G) = |\exp[-\alpha t-\beta s]\varphi|^2\big(|n_s|^2-\langle An_x,n_x\rangle,\ |n_t|^2-\langle An_x,n_x\rangle,\ (n_t+n_s)2An_x\big)$$

in C_{s_2}, where $n = (n_t, n_s, n_x)$. Since $n = -\nabla\tilde{p}\|\nabla\tilde{p}\|^{-1}$, hence

$$(\tilde{E}, F, G)\cdot n = -|\exp[-\alpha t - \beta s]\varphi|^2(n_s^2 n_t + n_t^2 n_s + (n_t + n_s)\langle A n_x, n_x\rangle) =$$
$$= |\exp[-\alpha t - \beta s]\varphi|^2 2\|\nabla\tilde{p}\|^{-3}(1 - \tilde{q}\theta^{-2} + \delta^{-2}\theta^{-2}\langle A\tilde{x}, \tilde{x}\rangle)\,.$$

By comparison with (5.4) we see that $(\tilde{E}, F, G)\cdot n = h \in C_0^\infty(\partial\mathfrak{T}_1)$, with $h \geqslant 0$, $\operatorname{supp} h = \operatorname{supp}\varphi$. Therefore there exists a constant $K_1 = K_1(P, \alpha, \beta, \varphi) > 0$ such that

$$\iint_{C_{s_2}} (\tilde{E}, F, G)\cdot n\, dS \leqslant \iint_{\operatorname{supp} h} h\, dS \leqslant K_1: \tag{6.4}$$

Put $y(t) = \iint_{B_s(t)} E(t, s, x)\, ds\, dx$ and $g(t) = \iint_{B_n(x_0; r(t))} F(t, s_2, x)\, dx$, for $t \in [0, t_{s_2}]$. Then $\int_t^{t_{s_2}} y(t')\, dt' = \iiint_{\Omega_{s_2}(t)} E\, dt'\, ds\, dx$ and $\int_t^{t_{s_2}} g(t')\, dt' = \iint_{A_{s_2}} F\, dt'\, dx$. Therefore, from (6.3) and (6.4), it follows $y(t) \leqslant K_1 - \int_t^{t_{s_2}} g(t')\, dt' + K\int_t^{t_{s_2}} y(t')\, dt'$. Arguing as in VI.4.4 [2] we get $y(t) \leqslant \exp[K(t_{s_2} - t)]K_1 - \int_t^{t_{s_2}} \exp[K(t' - t)]g(t')\, dt'$. Since $E, F \geqslant 0$, hence $y, g \geqslant 0$; therefore

$$\iint_{A_{s_2}} F\, dt\, dx = \int_0^{t_{s_2}} g(t')\, dt' \leqslant \int_0^{t_{s_2}} \exp[Kt']g(t')\, dt' + y(0) \leqslant \exp[Kt_{s_2}]K_1 \leqslant \exp[Kt_0]K_1\,. \qquad \text{Q.E.D.}$$

7. – Proof of Theorem 4.1.

In this section we show

THEOREM. – Let u be such that (4.2) holds. Then $u = 0$ in $\mathcal{C}$.

PROOF. – Let $(t_1, s_1, x_1) \in \mathcal{C}$, $(t_1 > 0)$ and $\varphi \in C_0^\infty(\partial\mathfrak{T}_1)$ with $\varphi \geqslant 0$, $\varphi = 1$ near the point (t_1, s_1, x_1). Let $s_2 < s_0$ and $w \in C^\infty(\Omega_{s_2})$ be a solution of (5.5). By a straightforward calculation we get $0 = wPu - u\,{}^tPw = \nabla\cdot H$, where

$$H = (w\partial_s u, -u\partial_t w, wA\nabla_x u - uw\,{}^t\nabla_x A - u({}^t\nabla_x w)A + uwb)$$

with $b = (b_1, \dots, b_n)$. Therefore, by Stockes theorem we get

$$0 = \iiint_{\Omega_{s_2}} (wPu - u\,{}^tPw)\, dt\, ds\, dx = \iint_{A_{s_2}} u\partial_t w\, dt\, dx - \iint_{B_{s_2}} w\partial_s u\, ds\, dx + \iint_{C_{s_2}} H\cdot n\, dS\,. \tag{7.1}$$

Let us calculate $H\cdot n$ in C_{s_2}. From $w = 0$ in C_{s_2}, it follows $\nabla w = \varphi n$ in C_{s_2}; thus it becomes $H\cdot n = (0, -u\varphi n_t, -u\varphi\,{}^t n_x A)\cdot n = -u\varphi(n_t n_s + \langle A n_x, n_x\rangle)$, and (by comparison with (5.4)) $H\cdot n = -u\varphi\psi$ for a suitable function $\psi > 0$. But $\partial_s u = 0$ in B_{s_2}, then, from (7.1), we obtain

$$\left|\iint_{C_{s_2}} u\varphi\psi\, dS\right| \leqslant \iint_{A_{s_2}} |u\partial_t w|\, dt\, dx\,. \tag{7.2}$$

By hypothesis (4.2.ii) there exist constants $\beta_1 < 0$ and $C_1 > 0$ such that $|\exp[\beta_1 s]u(t, s, x)| \leqslant C_1$, $(t, s, x) \in \mathcal{C}$. Let us choose $\beta \in (\beta_1, 0)$, then

$$\iint_{A_{s_2}} |u\partial_t w| dt\, dx \leqslant C_1 \exp[(\beta - \beta_1)s_2](\operatorname{mis} A_{s_2})^{\frac{1}{2}} \left(\iint_{A_{s_2}} |\exp[-\beta s]\partial_t w|^2\right)^{\frac{1}{2}}. \tag{7.3}$$

Let us take $\alpha \in (\beta, 0)$; from (6.1) it follows that $\exp[-\alpha t - \beta s]\partial_t w = \partial_t v + \alpha v$; hence, for every $t \geqslant 0$, $|\exp[-\beta s]\partial_t w|^2 \leqslant 2(\alpha^2|v|^2 + |\partial_t v|^2) \leqslant 2F(t, s, x)$. Applying Theorem 6.3 we get

$$\left(\iint_{A_{s_2}} |\exp[-\beta s]\partial_s w|^2\, dt\, dx\right)^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}} M^{\frac{1}{2}}\,. \tag{7.4}$$

Finally there exists a constant $C_2 > 0$ such that $\operatorname{mis} A_{s_2} \leqslant C_2 |s_s|^{n/2}$ for every $s_2 \ll$. Hence from (7.2), (7.3) and (7.4) it follows

$$\left|\iint_{C_{s_2}} u\varphi\psi\, dS\right| \leqslant C_1 C_2^{\frac{1}{2}} M^{\frac{1}{2}} \exp[(\beta - \beta_1)s_2]|s_2|^{n/2} \xrightarrow[s \to -\infty]{} 0\,.$$

We conclude that $u = 0$ in $\operatorname{supp}\varphi$ and, in particular, $u(t_1, s_1, x_1) = 0$. Q.E.D.

8. – Conclusion.

By Theorem 4.1 we can improve Theorem 3.2. Namely we show

8.1. THEOREM. – Let $\varphi = \varphi(s) \in C^\infty(\boldsymbol{R})$ satisfying the following conditions:

(i) there exists $\beta < 0$ such that $\varphi(s) = \exp[\beta s]$ for $s \ll 0$;

(ii) $\varphi(s) = 0$ for $s \gg 0$.

If $f \in C^\infty([0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n)$ and $g \in C^\infty(\boldsymbol{R} \times \boldsymbol{R}^n)$ are such that $\varphi(s) f(t, s, x)$ (for every $t \in [0, T]$), and $\varphi(s) g(s, x)$ belong to $H^{+\infty}(\boldsymbol{R} \times \boldsymbol{R}^n)$, then there exists a unique solution u of

$$\begin{cases} Pu = f & \text{in } [0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n \\ u(0, s, x) = g(s, x) & (s, x) \in \boldsymbol{R} \times \boldsymbol{R}^n \end{cases} \tag{0.3}$$

such that $u \in C^\infty([0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n)$ and $\varphi(s) u(t, s, x)$ is bounded in $[0, T] \times \boldsymbol{R} \times \boldsymbol{R}^n$.

Actually it is of importance the behaviour of f and g in the $s<0$ half-space only.

PROOF. – Let $\mathcal{Q}_n=\{(s,x)\in \boldsymbol{R}\times\boldsymbol{R}^n;\ (s-n)^2-\|x\|^2\geqslant 1$ and $s\leqslant n-1\}$, $n\in \boldsymbol{N}$.

Since the distance between $\partial\mathcal{Q}_n$ and $\partial\mathcal{Q}_{n+1}$ is greater than a positive constant, we can find $\chi_n\in C^\infty(\boldsymbol{R}\times\boldsymbol{R}^n)$ such that $\chi_n=1$ in $\mathcal{Q}_n$, $\chi_n=0$ in $(\boldsymbol{R}\times\boldsymbol{R}^n)-\mathcal{Q}_{n+1}$, and $\partial^\alpha\chi_n$ is bounded for every $\alpha\in\boldsymbol{N}^{n+1}$.

Define

$$f_n(t,s,x)=\chi_n(s,x)f(t,s,x)\qquad\text{and } g_n(s,x)=\chi_n(s,x)g(s,x)\,.$$

Then $f\in C^\infty([0,T];H_\beta^{+\infty}(\boldsymbol{R}\times\boldsymbol{R}^n)$ and $g_n\in H_\beta^{+\infty}(\boldsymbol{R}\times\boldsymbol{R}^n)$, therefore—by Theorem 3.2—there exists a unique $u_n\in C^\infty([0,T];H_\beta^{+\infty}(\boldsymbol{R}\times\boldsymbol{R}^n))$ such that

$$\begin{cases} Pu_n=f_n & \text{in } [0,T]\times\boldsymbol{R}\times\boldsymbol{R}^n\\ u_n(0)=g_n & \text{in } \boldsymbol{R}\times\boldsymbol{R}^n\,.\end{cases}$$

Let $\mathcal{C}_{s_0}$ be the cone defined by the point $(T,s_0,0)$ through (4.1). If K is a compact subset of $[0,T]\times\boldsymbol{R}\times\boldsymbol{R}^n$ surely there exist $s_0\in\boldsymbol{R}$ and $n_0\in\boldsymbol{N}$ such that $K\subset\overline{\mathcal{C}}_{s_0}\subset\mathcal{Q}_n$, for every $n\geqslant n_0$. Since $(u_n-u_m)(0)=0$ in $\mathcal{Q}_{n\wedge m}$ and $Pu_n-Pu_m=0$ in $[0,T]\times\mathcal{Q}_{n\wedge m}$, by Theorem 4.1, we have $u_n-u_m=0$ in $\overline{\mathcal{C}}_{s_0}$.

Therefore the sequence u_n converges in $C^\infty([0,T]\times\boldsymbol{R}\times\boldsymbol{R}^n)$ to a C^∞ function u which satisfy (0.3). Moreover from $u_n\in C([0,T];H_\beta^{+\infty}(\boldsymbol{R}\times\boldsymbol{R}^n))$ and $u_n=u$ in $\overline{\mathcal{C}}_{s_0}$ (for $n\gg$), it follows that $\varphi(s)u(t,s,x)$ is bounded in $\overline{\mathcal{C}}_{s_0}$.

Uniqueness follows from Theorem 4.1. Q.E.D.

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