Propagation of Singularities for Operators with Constant Coefficient Hyperbolic-Elliptic Principal Part (*).

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Summary. – *In this paper we consider partial differential operators of the type* $P(x, D) = P_m(D) + Q(x, D)$, *where the constant coefficient principal part* P_m *is supposed to be hyperbolic-elliptic. We study the propagation of Gevrey singularities for solutions* u *of the equation* $P(x, D)u = f$, *for ultradistributions* f, *finding exactly to which spaces of ultradistributions* u *microlocally belongs. The results are obtained by constructing a fundamental solution for* P *when the lower order part* Q *is with constant coefficients, and a parametrix otherwise.*

0. – Introduction.

The study of operators splitting into hyperbolic or elliptic factors leads to define a subclass of the set of locally hyperbolic polynomials, which has been studied by Fehrman in [2]. The polynomials P of such a subclass, named hyperbolic-elliptic, are characterized algebraically by the following property: P is hyperbolic-elliptic with respect to the direction θ if $P_m(\theta) \neq 0$ and there exist positive numbers c_1, c_2 such that

$$P(\xi - it\theta) \neq 0 \quad \text{when } c_1 < t < c_2|\xi| , \qquad \xi, t \text{ real};$$

here P_m denotes the principal part of P.

For these polynomials the theory parallels, under many aspects, that of hyperbolicity; but while hyperbolic operators have fundamental solutions with support in a convex cone, for the hyperbolic-elliptic ones we can only say that such a cone contains the analytic singularities of their fondamental solutions. This notable difference does not permit to consider the Cauchy problem for these operators, unless to add further hypotheses as it is done by Kumano-go in [9].

In this paper we study operators of the type

$$(*) \qquad P(x, D) = P_m(D) + Q(x, D)$$

$D_j = -i\partial/\partial x_j$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, where P_m is supposed with constant coefficients and Q is the lower order part; P_m is always assumed to be hyperbolic-elliptic. Con-

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sidering the principal part with constant coefficients is owed to the remark that, when this is also the case for Q, it is possible to construct a fundamental solution for $P(D)$ by the fundamental solutions for the iterates of $P_m(D)$, [1], [2], that is

$$E(P) = \sum_{k=0}^{\infty} (-1)^k Q(D)^k E(P_m^{k+1}) .$$

In particular we are interested in the Gevrey singularities for such (ultra) distributions, and the reason of this choice lies in Ivrii's result [6], for which, although the Cauchy problem is not in general C^∞-well posed for operators with hyperbolic principal part, this comes true in some Gevrey classes. A microlocal study of this problem has been carried out by WAKABAYASHI in [13] and [14] for operators with constant coefficient hyperbolic principal part, and the techniques we use here have been inspired by those papers.

In the first chapter we briefly recall the definitions and some results about hyperbolic-elliptic polynomials, following [2]; we introduce then Gevrey classes and ultradistributions, as well as the relative wave front sets. For more informations about this second part we refer to [5], [7], [8], [13], [14].

In the second chapter we consider constant coefficient operators with hyperbolic-elliptic principal part: at first, by means of a precise study of the zeros of the polynomials, we find a result (Theorem 2.1.7) which extends Svensson-Fehrman theorem [12], [2] (see also [10]). Then, through the choice of a suitable integration path, we construct a fundamental solution in a class of ultradistributions depending on the polynomial, improving a result by ZAMPIERI [15] (see also MARI [11]). Furthermore we show that this fundamental solution is analytic outside a convex cone and give outer estimates for the wave front sets of it and of the solutions u of the equation $P(D)u = f$, for ultradistributions f (Theorem 2.3.3).

In the third chapter we apply to variable coefficient operators the techniques before employed, retaining however the hypothesis that the principal part is with constant coefficients; we obtain then results of propagation of singularities, similar to those of the previous chapter, by constructing a left parametrix for the operator $P(x, D)$ (Theorem 3.2.1). On the other hand, by using a right parametrix, we give a result of semiglobal solvability for the equation $P(x, D)u = f$, modulo analytic functions (Theorem 3.3.1), and finally supply some examples, calculating explicitly the degree of regularity of the solutions.

1. – Preliminaries.

1.1. *Hyperbolic-elliptic polynomials.*

Let $P(\xi) = \sum_{|\alpha| \leqslant m} a_\alpha \xi^\alpha$ be a polynomial with complex coefficients and $P_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$ its principal part.

DEFINITION 1.1.1. – A polynomial P will be called hyperbolic-elliptic or a hybrid with respect to the direction θ if $P_m(\theta) \neq 0$ and there exist positive constants c_1 and c_2 such that

$$P(\xi - it\theta) \neq 0 \quad \text{if } c_1 < t < c_2|\xi|, \quad \xi \in \mathbf{R}^n . \tag{1.1.1}$$

The class of all such polynomials P will be denoted by $\mathrm{he}(\theta)$.

For homogeneous elements H in $\mathrm{he}(\theta)$, (1.1.1) can be replaced by

$$H(\xi - it\theta) \neq 0 \quad \text{if } 0 < |t| < c|\xi|, \quad \xi \in \mathbf{R}^n , \tag{1.1.1$'$}$$

for some positive constant c, and we shall write $H \in \mathrm{He}(\theta)$.

DEFINITION 1.1.2. – Let H be a homogeneous polynomial and $\xi \in \mathbf{R}^n$. The localization H_ξ of H at ξ is the first not identically vanishing term in the expansion

$$H(\xi + t\zeta) = t^r H_\xi(\zeta) + O(t^{r+1}) \quad \text{as } t \to 0;$$

r is called the multiplicity of H at ξ.

We are going now to list the main results proved in [2]; hereafter $\mathrm{hyp}(\theta)$ (resp. $\mathrm{Hyp}(\theta)$) denotes the class of all hyperbolic (resp. and homogeneous) polynomials with respect to the direction θ.

THEOREM 1.1.3 ([2]). – (i) $P \in \mathrm{he}(\theta)$ if and only if $P_m \in \mathrm{He}(\theta)$ and P is weaker (1) than P_m; (ii) $\mathrm{he}(\theta) = \mathrm{he}(-\theta)$; (iii) if $P \in \mathrm{he}(\theta)$ then $P_{m\xi} \in \mathrm{Hyp}(\theta)$ for every $\xi \in \mathbf{R}^n \setminus \{0\}$; (iv) if $P \in \mathrm{he}(\theta)$ then $P \in \mathrm{he}(\eta)$ for every direction η in $\Gamma(P_m, \theta) = \bigcap_{0 \neq \xi \in \mathbf{R}^n} \Gamma(P_{m\xi}, \theta)$, where $\Gamma(P_{m\xi}, \theta)$ is the component of θ in the set $\{\zeta;\, P_{m\xi}(\zeta) \neq 0\}$.

The foregoing statements generalize well known results for hyperbolic polynomials, in particular (i) is an extension of the corresponding theorem proved by SVENSSON in [12]. However the open cone $\Gamma(P_m, \theta)$ defined in (iv) is in general smaller than the component of $\{\xi;\, P_m(\xi) \neq 0\}$ containing θ. The dual cone

$$\Gamma(P_m, \theta)^* = \{\eta;\, x \cdot \eta \text{ (2)} \geqslant 0 \text{ for every } x \in \Gamma(P_m, \theta)\}$$

is equal to the convex hull of the wave front surface

$$\bigcup_{0 \neq \xi \in \mathbf{R}^n} \Gamma(P_{m\xi}, \theta)^*$$

and, as for hyperbolic operators, we have:

(1) If $\tilde{P}(\xi) = (\sum_\alpha |P^{(\alpha)}(\xi)|^2)^{\frac{1}{2}}$, $P^{(\alpha)} = \partial^\alpha P$, Q weaker than P means $\tilde{Q}(\xi) \leqslant C\tilde{P}(\xi)$ for some $C > 0$ ([5]).

(2) If $x = (x_1, \dots, x_n)$, $\eta = (\eta^1, \dots, \eta^n)$, $x \cdot \eta = \sum_j x_j \eta^j$.

THEOREM 1.1.4 ([2]). – If $P \in \mathrm{he}(\theta)$ then $P(D)$ has a fundamental solution (in the space of distributions) which is analytic outside the wave front surface.

As above there is a geometrical difference with the hyperbolic case; in fact the boundary of $\Gamma(P_m, \theta)^*$ is not generally contained in the wave front surface.

The last result we state in this section is a converse of Theorem 1.1.4 and really characterizes hybrid operators.

THEOREM 1.1.5 ([2]). – If $P(D)$ has a fundamental solution (in the space of distributions) which is analytic outside a proper closed cone Γ with vertex at the origin, then P is hyperbolic-elliptic with respect to every direction Γ in the open cone $\Gamma^0 = \{\eta;\, x\cdot\eta > 0 \text{ for every } x \in \Gamma\setminus\{0\}\}$.

1.2. *Ultradistributions and generalized wave front sets.*

DEFINITION 1.2.1. – Let K be a compact set in $\boldsymbol{R}^n$. For $1 < \varkappa < \infty$ and $h > 0$ we denote by $\mathcal{E}^{\{\varkappa\},h}(K)$ the space of all functions $f \in C^\infty(K)$ satisfying for every $\alpha \in \boldsymbol{Z}_+^n$

$$\sup_{x\in K} |D^\alpha f(x)| \leqslant C h^{|\alpha|}(\alpha!)^\varkappa$$

for some positive constant C, and by $\mathcal{D}_K^{\{\varkappa\},h}$ the space $\mathcal{E}^{\{\varkappa\},h}(K) \cap C_0^\infty(K)$. $\mathcal{E}^{\{\varkappa\},h}(K)$ and $\mathcal{D}_K^{\{\varkappa\},h}$ are Banach spaces under the norm:

$$\|f\|_{\mathcal{E}^{\{\varkappa\},h}(K)} = \sup_\alpha \sup_{x\in K} \frac{|D^\alpha f(x)|}{h^{|\alpha|}(\alpha!)^\varkappa}\,.$$

For Ω an open set in $\boldsymbol{R}^n$, we put

$$\mathcal{E}^{\{\varkappa\}}(\Omega) = \varprojlim_{K\subset\Omega} \varinjlim_{h\to\infty} \mathcal{E}^{\{\varkappa\},h}(K) \qquad \mathcal{E}^{(\varkappa)}(\Omega) = \varprojlim_{K\subset\Omega} \varprojlim_{h\to 0} \mathcal{E}^{\{\varkappa\},h}(\Omega)$$

$$\mathcal{D}^{\{\varkappa\}}(\Omega) = \varinjlim_{K\subset\Omega} \varinjlim_{h\to\infty} \mathcal{D}_K^{\{\varkappa\},h} \qquad \mathcal{D}^{(\varkappa)}(\Omega) = \varinjlim_{K\subset\Omega} \varprojlim_{h\to\infty} \mathcal{D}_K^{\{\varkappa\},h}\,.$$

The elements of the preceding spaces are called Gevrey functions of order $\varkappa$ on Ω. We shall denote by $*_\varkappa$ either $\{\varkappa\}$ or $(\varkappa)$ when we deal with the two cases together.

The spaces of ultradistributions of order $\varkappa$ on Ω, $\mathcal{D}^{*_\varkappa\prime}(\Omega)$ and $\mathcal{E}^{*_\varkappa\prime}(\Omega)$, are defined as the strong dual spaces of $\mathcal{D}^{*_\varkappa}(\Omega)$ and $\mathcal{E}^{*_\varkappa}(\Omega)$ respectively. $\mathcal{E}^{*_\varkappa\prime}(\Omega)$ may be identified with $\{u \in \mathcal{D}^{*_\varkappa\prime}(\Omega);\ \operatorname{supp} u \text{ is compact}\}$.

$\mathcal{E}^{*_\varkappa}(\Omega)$ and $\mathcal{D}^{*_\varkappa}(\Omega)$ are complete Montel spaces; moreover $\mathcal{E}^{\{\varkappa\}}(\Omega)$ is a Schwartz space, $\mathcal{E}^{(\varkappa)}(\Omega)$ is Fréchet-Schwartz, $\mathcal{D}^{\{\varkappa\}}(\Omega)$ (resp. $\mathcal{D}^{(\varkappa)}(\Omega)$) is an inductive limit (strict) of Fréchet-Schwartz spaces, ([7]). To have uniform notations, for $\mathcal{C}^\infty$ functions and usual distributions we put

$$\mathcal{E}^{\{\infty\}}(\Omega) = \mathcal{E}^{(\infty)}(\Omega) = \mathcal{C}^\infty(\Omega) \qquad \mathcal{D}^{\{\infty\}\prime}(\Omega) = \mathcal{D}^{(\infty)\prime}(\Omega) = \mathcal{D}'(\Omega)$$

but such a writing is formal indeed, since

$$\mathcal{C}^\infty(\Omega) \neq \bigcup_{\varkappa>1} \mathcal{E}^{\{\varkappa\}}(\Omega) = \bigcup_{\varkappa>1} \mathcal{E}^{(\varkappa)}(\Omega) .$$

The Fourier-Laplace transform of $u \in \mathcal{E}^{*\varkappa'}(\Omega)$ is the entire analytic function $\hat{u}(\zeta) = u_x(\exp(-ix\cdot\zeta))$. An analogous of the Paley-Weiner theorem holds for Gevrey functions and ultradistributions; for later use we state it here.

THEOREM 1.2.2 ([7], [8]). – Let K be a compact convex set in $\boldsymbol{R}^n$ and $1 < \varkappa < \infty$.

(i) An entire function ϕ is the Fourier-Laplace transform of a function $\varphi \in \mathcal{D}^{(\varkappa)}(\Omega)$ (resp. $\mathcal{D}^{\{\varkappa\}}(\Omega)$), with support in K, if and only if for every $L > 0$ there is a positive constant C (resp. there are positive constants L and C) such that

$$|\phi(\zeta)| \leqslant C \exp\left(-L|\zeta|^{1/\varkappa} + H_K(\operatorname{Im}\zeta)\right) , \qquad \zeta \in \boldsymbol{C}^n ,$$

where $H_K(\eta) = \sup_{x\in K} x\cdot\eta$ denotes the supporting function of K.

(ii) An entire function U is the Fourier-Laplace transform of an ultradistribution $u \in \mathcal{D}^{(\varkappa)'}(\Omega)$ (resp. $\mathcal{D}^{\{\varkappa\}'}(\Omega)$), with support in K, if and only if there exist positive constants L and C (resp. for every $L > 0$ there exists a positive constant C) such that

$$|U(\zeta)| \leqslant C \exp\left(L|\zeta|^{1/\varkappa} + H_K(\operatorname{Im}\zeta)\right) , \qquad \zeta \in \boldsymbol{C}^n .$$

We can now define the generalized wave front sets for ultradistributions; $\dot{T}^*(\Omega)$ will denote $T^*(\Omega)\setminus\{0\}$.

DEFINITION 1.2.3. – Let Ω be an open set in $\boldsymbol{R}^n$, $\varkappa_1 > 1$, $f \in \mathcal{D}^{(\varkappa_1)'}(\Omega)$, $\varkappa_1 \leqslant \varkappa < \infty$.

(i) We say that f is regular of class $\mathcal{E}^{(\varkappa)}$ (resp. $\mathcal{E}^{\{\varkappa\}}$) in a conic neighborhood of $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ if there exist a neighborhood U of x_0, an open cone $\Gamma \subset \boldsymbol{R}^n\setminus\{0\}$ containing ξ^0, a function $\varphi \in \mathcal{D}^{(\varkappa_1)}(U)$ with $\varphi(x_0) \neq 0$ and for every $L > 0$ a positive constant C (resp. there exist positive constants L and C) satisfying

$$|\widehat{(\varphi f)}(\xi)| \leqslant C \exp\left(-L|\xi|^{1/\varkappa}\right) , \qquad \xi \in \Gamma .$$

We define the wave front set $WF_{(\varkappa)}(f)$ (resp. $WF_{\{\varkappa\}}(f)$) as the complement in $\dot{T}^*(\Omega)$ of the collection of all $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ such that f is regular of class $\mathcal{E}^{(\varkappa)}$ (resp. $\mathcal{E}^{\{\varkappa\}}$) in a conic neighborhood of (x_0, ξ^0).

(ii) We say that f is in $\mathcal{D}^{(\varkappa)'}$ (resp. $\mathcal{D}^{\{\varkappa\}'}$) in a conic neighborhood of $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ if there exist a neighborhood U of x_0, an open cone $\Gamma \subset \boldsymbol{R}^n\setminus\{0\}$ containing ξ^0, a function $\varphi \in \mathcal{D}^{(\varkappa_1)}(U)$ with $\varphi(x_0) \neq 0$ and positive constants L and C (resp. for every $L > 0$ there exists a positive constant C) satisfying

$$|\widehat{(\varphi f)}(\xi)| \leqslant C \exp\left(L|\xi|^{1/\varkappa}\right) , \qquad \xi \in \Gamma .$$

We define the wave front set $WF^{(\varkappa)}(f)$ (resp. $WF^{\{\varkappa\}}(f)$) as the complement in $\dot{T}^*(\Omega)$ of the collection of all $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ such that f is in $\mathcal{D}^{(\varkappa)'}$ (resp. $\mathcal{D}^{\{\varkappa\}'}$) in a conic neighborhood of (x_0, ξ^0).

(iii) Moreover we say that f is regular of class C^∞ (resp. is in $\mathcal{D}'$) in a conic neighborhood of $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ if there exist a neighborhood U of x_0 and an open cone $\Gamma \subset \boldsymbol{R}^n \setminus \{0\}$ containing ξ^0 such that for every $\varphi \in \mathcal{D}^{(\varkappa_1)}(U)$ and for every non negative integer N there is a positive constant C (resp. there are an integer N and a positive constant C) satisfying

$$|(\widehat{\varphi f})(\xi)| \leqslant C(1 + |\xi|)^{-N}, \qquad \xi \in \Gamma .$$

We define the wave front set $WF_{(\infty)}(f) = WF_{\{\infty\}}(f) = WF(f)$ (resp. $WF^{(\infty)}(f) = WF^{\{\infty\}}(f)$) as the complement in $\dot{T}^*(\Omega)$ of the collection of all $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ such that f is regular of class C^∞ (resp. is in $\mathcal{D}'$) in a conic neighborhood of (x_0, ξ^0).

(iv) Finally we define the analytic wave front set of f. Let $U_1 \subset U_2$ be neighborhoods of $x_0 \in \boldsymbol{R}^n$. Then there exist a sequence $\{\chi_N\} \subset \mathcal{D}^{(\varkappa_1)}(U_2)$ and a positive constant C such that for every $h > 0$ there is a positive number C_h satisfying

$$|D^{\alpha+\beta}\chi_N(x)| \leqslant C_h (CN)^{|\alpha|} h^{|\beta|} (\beta!)^{\varkappa_1}, \qquad |\alpha| \leqslant N$$

and $\chi_N \equiv 1$ in U_1 for $N = 1, 2, \ldots$.

The analytic wave front set $WF_A(f)$ of f is defined as the complement in $\dot{T}^*(\Omega)$ of the collection of all $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ such that for some U_1, U_2, $\{\chi_N\}$ as above, there are a conic neighborhood $\Gamma \subset \boldsymbol{R}^n \setminus \{0\}$ of ξ^0 and a positive constant C satisfying

$$|(\widehat{\chi_N f})(\xi)| \leqslant C(CN)^N (1 + |\xi|)^{-N}, \qquad \xi \in \Gamma, \qquad N = 1, 2, \ldots .$$

For the properties of $WF_{*\varkappa}$ and WF_A we refer to [5] and [14], for $WF^{*\varkappa}$ see [14].

To state the main result we need about propagation of singularities for continuous linear maps $K\colon \mathcal{E}^{*\varkappa'} \to \mathcal{D}^{*\varkappa'}$, we now briefly recall some definitions; details and complete statements may be found in [5], [8], [14].

Definition 1.2.4. – Let Ω_1, Ω_2 be open sets in $\boldsymbol{R}^n$.

(i) Let $\phi\colon \Omega_1 \to \Omega_2$ be a $\mathcal{E}^{*\varkappa}$-mapping. We denote by $\phi^*\colon \mathcal{E}^{*\varkappa}(\Omega_2) \to \mathcal{E}^{*\varkappa}(\Omega_1)$ the pull-back, defined by $(\phi^* u)(x) = u(\phi(x))$.

(ii) Let $\psi\colon \Omega_1 \to \Omega_2$ be a real analytic proper mapping. We denote by $\psi_* = {}^t(\psi^*)\colon \mathcal{E}^{*\varkappa'}(\Omega_1) \to \mathcal{E}^{*\varkappa'}(\Omega_2)$ the push-forward. ψ_* may be extended to an operator from $\mathcal{D}^{*\varkappa'}(\Omega_1)$ to $\mathcal{D}^{*\varkappa'}(\Omega_2)$.

For every continuous linear map $K\colon \mathcal{D}^{*\varkappa}(\Omega_2) \to \mathcal{D}^{*\varkappa'}(\Omega_1)$ there exists a kernel $K(x, y) \in \mathcal{D}^{*\varkappa'}(\Omega_1 \times \Omega_2)$ such that for every $u \in \mathcal{D}^{*\varkappa}(\Omega_2)$

$$\langle Ku, \varphi \rangle = \langle K(x, y), \varphi(x)\, u(y) \rangle, \qquad \varphi \in \mathcal{D}^{*\varkappa}(\Omega_1)$$

(see [8]). This result leads us to write formally for $u \in \mathcal{E}^{*\varkappa'}(\Omega_2)$

$$Ku = \Pi_{1*}\circ \Delta^*(K(x, y)\otimes u(y)), \quad Ku \in \mathcal{D}^{*\varkappa'}(\Omega_1) \tag{1.2.1}$$

where $\Pi_1: \Omega_1\times\Omega_2 \to \Omega_1$ is the projection $\Pi_1(x, y) = x$ and $\Delta: \Omega_1\times\Omega_2\to\Omega_1\times\Omega_2\times\Omega_2$ is the map $\Delta(x, y) = (x, y, y)$. Under suitable hypotheses Ku is well defined by (1.2.1) and we have outer estimates of the wave front sets $WF_{*\varkappa}(Ku)$ and $WF^{*\varkappa}(Ku)$:

THEOREM 1.2.5. – Let $1<\varkappa_1\leqslant\varkappa<\infty$, $u\in\mathcal{E}^{*\varkappa_1'}(\Omega_2)$, $K(x, y)\in\mathcal{D}^{*\varkappa_1'}(\Omega_1\times\Omega_2)$. Put $WF'_{*\varkappa_1\Omega_2}(K) = \{(y, \eta)\in \dot{T}^*(\Omega_2);\ (x, y; 0, -\eta)\in WF_{*\varkappa_1}(K(x, y)) \text{ for some } x\in\Omega_1\}$ and for $W\subset T^*(\Omega_1\times\Omega_2)$

$$W' = \{((x, \xi), (y, \eta))\in T^*(\Omega_1)\times T^*(\Omega_2);\ (x, y; \xi, -\eta)\in W\}.$$

If $WF'_{*\varkappa_1\Omega_2}(K)\cap WF_{*\varkappa_1}(u) = \phi$, then Ku is well defined by (1.2.1) and we obtain the following estimates:

$$\begin{aligned} WF_{*\varkappa}(Ku)\subset \bigcap_{\varkappa_1\leqslant s\leqslant\varkappa} \{&WF^{*'}_{s}(K)\circ WF_{*\varkappa_1}(u)\cup WF'_{*\varkappa_1}(K)\circ WF^{*s}(u)\cup\\ &\cup WF'_{*s}(K)\circ WF_{*s}(u)\cup WF'_{*\varkappa_1}(K)\circ \mathrm{supp}_0^{*s}(u)\cup WF'_{*s}(K)\circ\mathrm{supp}_0(u)\}\\ WF^{*\varkappa}(Ku)\subset \bigcap_{\varkappa_1\leqslant s\leqslant\varkappa} \{&WF^{*'}_{s}(K)\circ WF_{*\varkappa_1}(u)\cup WF^{*\varkappa'}(K)\circ WF_{*s}(u)\cup WF'_{*\varkappa_1}(K)\circ WF^{*s}(u)\cup\\ &\cup WF'_{*s}(K)\circ WF^{*s}(u)\cup WF^{*\varkappa'}(K)\circ\mathrm{supp}_0(u)\cup WF'_{*\varkappa_1}(K)\circ\mathrm{supp}_0^{*s}(u)\cup\\ &\cup WF'_{*s}(K)\circ\mathrm{supp}_0^{*s}(u)\} \end{aligned}$$

where $\mathrm{supp}_0(u) = \{(x, 0)\in T^*(\Omega_2);\ x\in\mathrm{supp}\, u\}$ and $\mathrm{supp}_0^{*s}(\phi) = \{(x, 0)\in T^*(\Omega_2);\ x\in\mathrm{sing\,supp}^{*s}(\phi)\}$.

2. – Operators with constant coefficients.

In this chapter we shall deal with differential operators $P(D)$ with principal part $P_m(D)\in \mathrm{He}(\theta)$ and general lower order terms. Our goal is to construct, for such an operator $P(D)$, a fundamental solution E with analytic singular support contained in the wave front surface; so, in view of Theorems 1.1.3 and 1.1.5, we have to consider the more general spaces than $\mathcal{D}'$ introduced in section 1.2. Precisely, refering to Definition 2.1.2 for $\varrho(P)$ and $\varrho(\xi)$, we shall find a solution E, with the required property, in $\mathcal{D}^{(1/\varrho(P))'}$; E is microlocally in $\mathcal{D}'$ at every $(x_0, \xi^0)\in\dot{T}^*(\boldsymbol{R}^n)$ with $\varrho(\xi^0) = 0$, (see Theorems 2.3.1 and 2.3.2).

During the whole of this chapter, $P(\xi) = P_m(\xi) + \sum_{j=0}^{m-1} Q_j(\xi)$ denotes a polynomial with hyperbolic-elliptic principal part P_m with respect to the direction θ and lower order homogeneous terms Q_j, $\deg Q_j = j$, $j = 0, \dots, m-1$.

2.1. *Polynomial with hyperbolic-elliptic principal part.*

LEMMA 2.1.1. – Let $P_m \in \mathrm{He}(\theta)$, $\xi^0 \in \boldsymbol{R}^n \setminus \{0\}$ and put $a(\xi^0) = a = \deg P_{m\xi^0}$. Then there exist a conic neighborhood Γ of ξ^0 and positive constants c_1, c_2 such that

$$P(\xi - is\theta) \neq 0 \quad \text{if } c_1|\xi|^{(a-1)/a} < s < c_2|\xi|, \quad \xi \in \Gamma .$$

Moreover if $\mu = \max_{0 \neq \xi} a(\xi)$ denotes the highest multiplicity of the real roots ξ in the equation $P_m(\xi) = 0$, $\xi \neq 0$, we have

$$P(\xi - is\theta) \neq 0 \quad \text{if } C_1|\xi|^{(\mu-1)/\mu} < s < C_2|\xi|, \quad \xi \in \boldsymbol{R}^n$$

for some positive constants C_1 and C_2.

PROOF. – $P_m \in \mathrm{He}(\eta)$ for every η in the open cone $\Gamma(P_m, \theta)$, so we have

$$P_m(\xi + \zeta - i\theta) = P_m(\xi + \mathrm{Re}\,\zeta - i(\theta - \mathrm{Im}\,\zeta)) \neq 0$$

for sufficiently large real ξ, say $|\xi| \geqslant 1/c_2$, and small $\zeta \in \boldsymbol{C}^n$. Thus from Lemma 11.1.4 in [5] it follows that

$$\tilde{P}_m(\xi - i\theta) \leqslant C|P_m(\xi - i\theta)| \quad \text{if } |\xi| \geqslant 1/c_2 . \tag{2.1.1}$$

On the other hand from $\sum_{|\alpha|=a} |P^{(\alpha)}(\xi^0)|^2 \neq 0$ by homogeneity we obtain

$$C|\xi|^{m-a} \leqslant \tilde{P}_m(\xi) \tag{2.1.2}$$

for ξ in suitable conic neighborhood Γ of ξ^0. The estimates (2.1.1)-(2.1.2) yield

$$\frac{|Q_j(\xi - is\theta)|}{|P_m(\xi - is\theta)|} \leqslant Cs^{-a}|\xi|^{a+j-m} \quad \text{if } 0 < s \leqslant c_2|\xi|, \ \xi \in \Gamma , \tag{2.1.3}$$

and then

$$|P(\xi - is\theta)| \geqslant |P_m(\xi - is\theta)| \big(1 - C \sum_j s^{-a}|\xi|^{a+j-m}\big) > 0$$

for $c_1|\xi|^{(a-1)/a} < s \leqslant c_2|\xi|$ and every ξ in Γ if c_1 is sufficiently large. The second assertion in the lemma may be proved using the first one by an easy covering argument.

DEFINITION 2.1.2. – Let P_m be in $\mathrm{He}(\theta)$, Γ a cone in $\mathrm{R}^n \setminus \{0\}$, $\xi^0 \in \boldsymbol{R}^n \setminus \{0\{$. We define

$$\varrho(P; \Gamma) = \varrho(\Gamma) = \inf\{\varrho \geqslant 0;\ \text{there exist positive constant } \gamma_1 \text{ and } \gamma_2 \text{ such that } P(\xi - is\theta) \neq 0 \text{ if } \xi \in \Gamma \text{ and } \gamma_1|\xi|^\varrho < s < \gamma_2|\xi|\}$$

$$\varrho(P; \xi^0) = \varrho(\xi^0) = \inf\{\varrho(\Gamma);\ \Gamma \text{ conic neighborhood of } \xi^0\}$$

$$\varrho(P) = \sup\{\varrho(\xi);\ \xi \in \boldsymbol{R}^n \setminus \{0\}\} .$$

By Lemma 2.1.1 we have $\varrho(\xi)\leqslant(a-1)/a$ and $\varrho(P)\leqslant(\mu-1)/\mu$. If $P\in \mathrm{he}(\theta)$ then $\varrho(\xi)=0$ for every $\xi\in \boldsymbol{R}^n\setminus\{0\}$; conversely, if $\varrho(P)=0$, from (iii) in the following lemma, we obtain $P\in \mathrm{he}(\theta)$ by a covering of the unit sphere.

LEMMA 2.1.3. – Let $\xi^0\in \boldsymbol{R}^n\setminus\{0\}$. Then the following properties hold:

(i) $\varrho(\xi^0)=\varrho(\lambda\xi^0)$ if $\lambda>0$;

(ii) $\varrho(\Gamma_1)\leqslant\varrho(\Gamma_2)$ if $\Gamma_1\subset\Gamma_2$ are conic neighborhood of ξ^0;

(iii) there exists a conic neighborhood Γ_0 of ξ^0 such that $\varrho(\xi^0)=\varrho(\Gamma_0)$;

(iv) there exists a conic neighborhood Γ_0 of ξ^0 such that $\varrho(\xi)\leqslant\varrho(\xi^0)$ for every ξ in Γ_0;

(v) $\varrho(\xi^0)$ is a rational number;

(vi) if $\varrho(\xi^0)>0$, then for every conic neighborhood Γ of ξ^0 there are Puiseux series $s(r)\in\boldsymbol{R}$, $\xi(r)\in\Gamma$, converging for large positive r, with $|\xi(r)|=r$, $s(r)=cr^{\varrho(\xi^0)}(1+o(1))$ as $r\to\infty$, $c>0$, and $P(\xi(r)-is(r)\theta)=0$;

(vii) when $P_m\in \mathrm{Hyp}(\theta)$ we have $\varrho(\xi^0)=\delta(\xi^0)$ with $\delta(\xi^0)$ defined in [14];

(viii) the set $\{\varrho(\xi);\ \xi\in\boldsymbol{R}^n\setminus\{0\}\}$ is a finite set.

PROOF. – The assertions (i)-(ii) are trivial. Let $E_\varepsilon\subset\boldsymbol{R}^{2+n}$ be the semi-algebraic set defined by the system

$$\begin{cases} P(\xi-is\theta)=0\,, & s\in\boldsymbol{R}\\ |\xi|^2=r^2\,, & r>0\\ |\xi-r\xi^0|^2\leqslant\varepsilon^2r^2\,, & \varepsilon>0\\ s^a\leqslant c_1^a r^{a-1} & \end{cases}\tag{2.1.4}$$

i.e. $E_\varepsilon=\{(r,s,\xi)\in\boldsymbol{R}^{2+n};\ (r,s,\xi)\text{ satisfies (2.1.4)}\}$, where we may assume that $|\xi^0|=1$ and a, c_1 the same constants of Lemma 2.1.1.

From Tarski-Seidenberg lemma (3), repeating the same arguments as in the proofs of Lemma 1.1.2 in [14] and Theorem A.2.5 in [5], we can find a positive integer l, $r_1>0$, $\varepsilon_0>0$ and analytic functions $\varphi_i(r,\varepsilon)$, $i=1,\dots,\nu$, in

$$D=\{(r,\varepsilon)\in\boldsymbol{R}^2;\ r>r_1,\ r^{-1/l}\leqslant\varepsilon\leqslant\varepsilon_0\}$$

such that:

a) $-\infty=\varphi_0<\varphi_1(r,\varepsilon)<\varphi_2(r,\varepsilon)<\dots<\varphi_\nu(r,\varepsilon)$, $(r,\varepsilon)\in D$;

(3) See [4], [5].

b) $\varphi_i(r, \varepsilon) = c_i(\varepsilon)^{\varrho_i}(1 + o(1))$ as $r \to \infty$, with rational ϱ_i, where ϱ_i and $\operatorname{sgn} c_i(\varepsilon)$ are independent of ε for $(r, \varepsilon) \in D$, $i = 1, \ldots, \nu$;

c) $F_\varepsilon = \{(s, r) \in \boldsymbol{R}^2;\ (s, r, \xi) \in E_\varepsilon,\ (r, \varepsilon) \in D\}$ is the union of the curves $s = \varphi_i(r, \varepsilon)$ and of the strips bounded by them.

Define $\varphi(r, \varepsilon) = \sup\{s;\ (r, s, \xi) \in E_\varepsilon\}$ and $\Gamma_0 = \{\xi;\ .\xi - |\xi|\xi^0. < \varepsilon_0|\xi|\}$; in consequence of *c)* there exists $i_0 \in \{0, 1, \ldots, \nu\}$ such that

$$\varphi(r, \varepsilon) = \varphi_{i_0}(r, \varepsilon) \quad \text{in } D$$

with the convention $i_0 = 0$ if $F_\varepsilon = \phi$. Therefore

$$\varrho(\Gamma) = \begin{cases} \max(\varrho_{i_0}, 0) & \text{if } i_0 \neq 0 \text{ and } c_{i_0}(\varepsilon) > 0 \\ 0 & \text{otherwise} \end{cases}$$

for every conic neighborhood Γ of ξ^0 contained in Γ_0. This proves (iii) and (v). The assertion (iv) easily follows from (iii), while (vi) holds by virtue of Theorem A.2.8 in [5]. If $P_m \in \mathrm{Hyp}(\theta)$ the last inequality in (2.1.4) follows from the first two equalities of the system, hence we have (vii). Finally, (viii) may be proved in the same way as Lemma 1.1.3 in [14].

Definition 2.1.4. - Let Γ be a cone in $\boldsymbol{R}^n \setminus \{0\}$, $\xi^0 \in \boldsymbol{R}^n \setminus \{0\}$ and $R > 0$ a constant such that $P_m(\xi - i\theta) \neq 0$ for $|\xi| > R$, (see (1.1.1)$'$). For $j = 0, \ldots, m-1$, we define

$$n_j(P; \Gamma) = n_j(\Gamma) = \inf\left\{k;\ \frac{|Q_j(\xi - i\theta)|}{|P_m(\xi - i\theta)|} \leqslant C|\xi|^k \text{ for every } \xi \in \Gamma,\ |\xi| > R\right\}$$

$$n_j(P; \xi^0) = n_j(\xi^0) = \inf\{n_j(\Gamma);\ \Gamma \text{ conic neighborhood of } \xi^0\}$$

$$n_j^+ = \max(n_j, 0)\,.$$

By (2.1.1) and Lemma 10.4.2 in [5] we have $n_j^+(\xi) = 0$ for every $\xi \in \boldsymbol{R}^n \setminus \{0\}$ if and only if Q_j is weaker than P_m, that is $P \in \mathrm{he}(\theta)$ if and only if $n_j^+(\xi) = 0$ for every $\xi \in \boldsymbol{R}^n \setminus \{0\}$ and $j = 0, \ldots, m-1$.

Lemma 2.1.5. - If $\xi^0 \in \boldsymbol{R}^n \setminus \{0\}$, then for every $j = 0, \ldots, m-1$ we have

(i) $n_j(\Gamma_1) \leqslant n_j(\Gamma_2)$ if $\Gamma_1 \subset \Gamma_2$ are conic neighborhoods of ξ^0;

(ii) $n_j(\xi^0) = n_j(\lambda\xi^0)$ if $\lambda \neq 0$;

(iii) there exists a conic neighborhood Γ_0 of ξ^0 such that $n_j(\xi^0) = n_j(\Gamma_0)$;

(iv) there exists a conic neighborhood Γ_0 of ξ such that $n_j(\xi) \leqslant n_j(\xi^0)$ for every ξ in Γ_0;

(v) for every conic neighborhood Γ of ξ^0 there are Puiseux series $s_j(r) \in \boldsymbol{R}_+$, $\xi^j(r) \in \Gamma$, converging for large positive r, with $|\xi^j(r)| = r$,

$$s_j(r) = cr^{n_j(\xi_0)}(1 + o(1)) \quad \text{as } r \to \infty \text{ and } \frac{|Q_j(\xi^j(r) - i\theta)|}{|P_m(\xi^j(r) - i\theta)|} = s_j(r);$$

(vi) $j - m + \deg P_{m\xi_0} - \deg Q_{j\xi_0} \leqslant n_j(\xi^0) \leqslant j - m + \deg P_{m\xi_0}$;

(vii) $n_j(\xi^0) = j - m + \deg P_{m\xi_0}$ if $Q_j(\xi^0) \neq 0$;

(viii) $n_j(\xi^0) = j - m$ if $P_m(\xi^0) \neq 0$.

PROOF. - The statements (i)-(ii) are obvious. Let $E_\varepsilon \subset \boldsymbol{R}^{2+n}$ be the semi-algebraic set defined by the system

$$\begin{cases} s|P_m(\xi - i\theta)|^2 - |Q_j(\xi - i\theta)|^2 = 0\,, & s \in \boldsymbol{R} \\ |\xi|^2 = r^2\,, & r > 0 \\ |\xi - r\xi^0|^2 \leqslant \varepsilon^2 r^2\,. & \end{cases}$$

Refering to the arguments and the notations we used in the proof of Lemma 2.1.3, we can find $\varepsilon_0 > 0$ such that

$$\varphi(r, \varepsilon) = \sup\{s;\, (r, s, \xi) \in E_\varepsilon\} = c(\varepsilon) r^\varrho (1 + o(1)) \quad \text{as } r \to \infty$$

for $(r, \varepsilon) \in D$, with ϱ and $\operatorname{sgn} c(\varepsilon)$ independent of ε. Hence if

$$\Gamma_0 = \{\xi;\, |\xi - |\xi|\xi^0| \leqslant \varepsilon_0|\xi|\}$$

we have

$$\varrho/2 = n_j(\Gamma_0) = n_j(\xi^0)$$

proving thus (iii). The assertion (iv) is a trivial consequence of (iii) and, as in Lemma 2.1.3, (v) follows by Theorem A.2.8 in [5]. We have already proved the second inequality in (vi) by (2.1.3); for what concerns the first one, choose α such that $Q_j^{(\alpha)}(\xi^0) \neq 0$, $|\alpha| = \deg Q_{j\xi_0}$; then we have

$$|Q_j^{(\alpha)}(\xi) \leqslant |\tilde{Q}_j(\xi) \leqslant C|\xi|^{n_j(\xi^0)}|P_m(\xi - i\theta)| \leqslant C|\xi|^{n_j(\xi^0)} \sum_{k=0}^{m} |\langle \partial, \theta \rangle^k P_m(\xi)|$$

for every $\xi \in \Gamma_0$. Replacing ξ by $t\xi^0$ and letting $t \to \infty$ we get

$$j - \deg Q_{j\xi_0} \leqslant n_j(\xi^0) + m - \deg P_{m\xi_0}\,.$$

Finally, (vii)-(viii) easily follows from (vi).

Lemma 1.2.4 in [14] holds also in the hyperbolic-elliptic case; the following remark is the only variant in its laborious proof.

Lemma 2.1.6. – Let ξ^0 be in $\mathbf{R}^n\setminus\{0\}$, l a positive integer and $\eta(s)$ the Puiseux series, converging for small s,

$$\eta(s) = s^{-l}\left(\xi^0 + \sum_{j=1}^{\infty} s^{j/l}\xi^j\right), \qquad \xi^j \in \mathbf{R}^n$$

Moreover let $\tau_k(s)$ be the roots of $P_m(s\eta(s) + \tau\theta) = 0$, $\tau_k(s) = a_k s^{\mu_k}(1 + o(1))$ as $s \to 0$, $a_k \neq 0$, with the convention $\mu_k = \infty$ if $\tau_k \equiv 0$, $k = 1, \ldots, m$. Then we have

$$P_m(\eta(s) + s^{-\sigma}(\tau - i)\theta) = s^b(c_\sigma(\tau) + o(1)) \quad \text{as } s \to 0 \tag{2.1.5}$$

where $b = m + \sum_{k=1}^{m} \min(\mu_k, 1-\sigma)$ and $c_\sigma(\tau)$ has no real roots τ for all $0 \leqslant \sigma < 1$.

Proof. – The only non-trivial part of the lemma concerns the polynomial $c_\sigma(\tau)$. If $\tau_k(s)$ is not real for real s, then $|\tau_k(s)| > c|s\eta(s)|$, $c > 0$ (see (2.3) in [2]), so $\mu_k \leqslant 0 < 1 - \sigma$. Since

$$c_\sigma(\tau) = C(\tau - i)^{m_1} \prod_{k=1}^{m_2} (\tau - i - a_k)$$

where $\{1, \ldots, m_2\} = \{k;\ \mu_k = 1 - \sigma\}$, we have $c_\sigma(\tau) \neq 0$ for real τ, and the lemma is proved.

We can now state the following result, parallel to Theorem 1.2.5 in [14].

Theorem 2.1.7. – Let $P_m \in \mathrm{He}(\theta)$ and $\xi^0 \in \mathbf{R}^n\setminus\{0\}$. Then we have

$$\max\left(\varrho(\xi^0), \varrho(-\xi^0)\right) = \max_{0 \leqslant j \leqslant m-1} n_j^+(\xi^0)/\left(m - j + n_j^+(\xi^0)\right). \tag{2.1.6}$$

In particular

$$\varrho(P) = \max_{|\xi|=1} \max_{0 \leqslant j \leqslant m-1} n_j^+(\xi)/\left(m - j + n_j^+(\xi)\right).$$

Proof. – Put $\delta = \max_{0 \leqslant j \leqslant m-1} n_j^+(\xi^0)/(m - j + n_j^+(\xi^0))$ and let Γ_0 be a conic neighborhood of ξ^0 such that $n_j(\xi^0) = n_j(\Gamma_0)$ for every $j = 0, \ldots, m-1$. Then we obtain

$$|P(\xi - is\theta)| \geqslant |P_m(\xi - is\theta)| \left(1 - C\sum_j s^{j-m-n_j^+(\xi^0)} |\xi|^{n_j^+(\xi^0)}\right) > 0$$

for every $\xi \in \pm\Gamma_0$, if $\gamma_1|\xi|^\delta < s < \gamma_2|\xi|$, with γ_1 and γ_2 suitable positive constants. Hence $\max\left(\varrho(\xi^0), \varrho(-\xi^0)\right) \leqslant \delta$.

To prove the reverse inequality, we may assume $\delta > 0$ and proceed in the same way as in [14], considering the improvement we have given by the assertion (v) of

Lemma 2.1.5. Thus for every conic neighborhood Γ of ξ^0 we can find Puiseux series $s(t)\in \boldsymbol{R}$, $\xi(t)\in \pm\Gamma$, converging for small positive t, such that $s(t)=c|\xi(t)|^{\delta}(1+o(1))$ as $t\to 0^+$, $c>0$, and $P(\xi(t)-is(t)\theta)=0$. This proves $\max(\varrho(\xi^0),\varrho(-\xi^0))\geqslant\delta$ and then the theorem.

REMARK 2.1.8. - The above theorem is an extension of Fehrman's result we reported in Theorem 1.1.3 (i). In fact it follows from (2.1.6) that

$$\varrho(P)\leqslant\varrho<1 \text{ if and only if } \frac{|Q_j(\xi-i\theta)|}{|P_m(\xi-i\theta)|}\leqslant C|\xi|^{(m-j)\varrho/(1-\varrho)} \tag{2.1.7}$$

for every ξ with $|\xi|>R$, $j=0,\dots,m-1$.

2.2. *Construction of the fundamental solution*

To have simpler notations we shall write $\varrho=\varrho(P)$ and $\varkappa_0=1/\varrho$.

LEMMA 2.2.1. - Let P_m be in $\mathrm{He}(\theta)$ and M a compact set in $\Gamma(P_m,\theta)$. Then there exist positive constants c_1, γ_1 and γ_2 such that

$$|P(\xi-is\eta)|\geqslant c_1|s|^m \quad \text{if } \gamma_1|\xi|^{\varrho}<|s|<\gamma_2(|\xi|, \quad \eta\in M\,.$$

PROOF. - $P_m\in\mathrm{He}(\theta)$ implies the existence of a positive constant γ_2 satisfying

$$\tilde{P}_m(\xi-i\eta)\leqslant C|P_m(\xi-i\eta)| \quad \text{if } |\xi|\geqslant 1/\gamma_2\,, \quad \eta\in M \tag{2.2.1}$$

(see (2.1.1)). Moreover, by Lemma 10.4.2 in [5] and (2.1.7) we obtain

$$|Q_j(\xi-i\eta)|\leqslant\tilde{Q}_j(\xi-i\eta)\leqslant C|\xi|^{(m-j)\varrho/(1-\varrho)}\tilde{P}_m(\xi-i\eta)\,, \quad \text{if } |\xi|\geqslant 1/\gamma_2,\ \eta\in M\,. \tag{2.2.2}$$

Hence, from (2.2.1)-(2.2.2) we can find positive constants c_1 and γ_1 such that

$$|P(\xi-is\eta)|\geqslant|P_m(\xi-is\eta)|\Big(1-C\sum_j|s|^{(j-m)/(1-\varrho)}|\xi|^{(m-j)\varrho/(1-\varrho)}\Big)\geqslant c_1|s|^m$$

if $\gamma_1|\xi|^{\varrho}<|s|<\gamma_2|\xi|$, for every η in M. The lemma is proved.

DEFINITION 2.2.2. - A map $t\to\Omega_t$, from a topological space T to open sets in $\boldsymbol{R}^n$ will be called inner continuous if, for every $t_0\in T$, any compact part of Ω_t is contained in Ω_{t_0} when t is close enough to t_0.

DEFINITION 2.2.3. - A map $t\to M_t$, from a topological space T to compact sets in $\boldsymbol{R}^n$ will be called outer continuous if, for every $t_0\in T$, any compact neighborhood of M_{t_0} contains M_t when t is close enough to t_0.

The family $\Gamma(P_{m\xi},\theta)$, $\xi\in\boldsymbol{R}^n\setminus\{0\}$, is inner continuous, ([2], [3]).

LEMMA 2.2.4. – Let P_m be in $\mathrm{He}(\theta)$, M a compact set in $\Gamma(P_{m\xi^0},\theta)$, $\xi^0\in \boldsymbol{R}^n\setminus\{0\}$. Then there exist a conic neighborhood Γ of ξ^0 and positive constants γ_1, γ_2 and R such that

$$P(\xi - is\theta - it\eta) \neq 0 \quad \text{if } \gamma_1|\xi|^\varrho < s < \gamma_2|\xi|, \quad 0\leqslant t\leqslant \gamma_1|\xi|, \quad \xi\in\Gamma, \ |\xi|\geqslant R, \ \eta\in M .$$

PROOF. – We may assume that $|\xi^0| = 1$ without loss of generality. Define

$$f(s, t, \tau, \xi; \eta) = P_m(\xi^0 + \xi + s\theta + t\eta) + \sum_j \tau^{m-j} Q_j(\xi^0 + \xi + s\theta + t\eta)$$

for $(s, t, \tau, \xi)\in \boldsymbol{C}^{3+n}$ and $\eta\in M$. By «Main Lemma» in [3] and Theorem 2.10 in [2] we have

$$f(s, t, 0, \xi; \eta) \neq 0 \tag{2.2.3}$$

$$\text{if } \operatorname{Im} s \operatorname{Im} t\geqslant 0, \ \operatorname{Im}(s+t)\neq 0, \ \xi \text{ real}, \ \xi, \ s, \ t \text{ small enough};$$

on the other hand, from (2.1.7) we easily obtain

$$f(s, 0, \tau, \xi; \eta) \neq 0 \tag{2.2.4}$$

$$\text{if } |\operatorname{Im} s| > \gamma_1|\tau|^{1-\varrho}, \ \xi \text{ real}, \ \xi, \ s \text{ sufficiently small}$$

for some positive constant γ_1. Let us now prove

$$f(s, t, \tau, \xi; \eta) \neq 0 \tag{2.2.5}$$

$$\text{if } \operatorname{Im} s \operatorname{Im} t\geqslant 0, \ |\operatorname{Im} s| > \gamma_1|\tau|^{1-\varrho}, \ \xi \text{ real}, \ \xi, \ s, \ t, \ \tau \text{ small enough} .$$

Since $f(0, t, 0, 0; \eta) = P_m(\xi^0 + t\eta) = t^r P_{m\xi_0}(\eta) + O(t^{r+1})$ as $t\to 0$, we can factorize f in the following way:

$$f(s, t, \tau, \xi; \eta) = P_{m\xi^0}(\eta) \prod_{i=1}^{r} \big(t - \lambda_i(s, \tau, \xi; \eta)\big) F(s, t, \tau, \xi; \eta)$$

where $F(-;\eta)$ and $\lambda_i(-;\eta)$ are analytic functions for small s, t, τ, ξ with $F(0, 0, 0, 0; \eta) = 1$, $\lambda_i(0, 0, 0; \eta) = 0$, $i = 1, \dots, r$. From (2.2.3) we get

$$\operatorname{Im}\lambda_i(s, \tau, \xi; \eta) < 0 \quad \text{if } \operatorname{Im} s > 0, \ \xi \text{ real}, \ \xi, \ s \text{ small enough}$$

for $i = 1, \dots, r$, while (2.2.4) yields

$$\operatorname{Im}\lambda_i(s, \tau, \xi; \eta) \neq 0 \quad \text{if } \operatorname{Im} s > \gamma_1|\tau|^{1-\varrho}, \ \xi \text{ real}, \ \xi, \ s, \ \tau \text{ sufficiently small}$$

for $i = 1, \ldots, r$; hence by continuity

$$f(s, t, \tau, \xi; \eta) \neq 0 \quad \text{if } \operatorname{Im} t \geqslant 0, \ \operatorname{Im} s > \gamma_1 |\tau|^{1-\varrho}$$

under the above conditions on ξ, s, t, τ. With an application of the same argument to the reverse inequalities $\operatorname{Im} s < -\gamma_1 |\xi|^{1-\varrho}$ and $\operatorname{Im} t \leqslant 0$, the proof of (2.2.5) may be completed.

Choose a positive number ε small enough and define $K = \{\zeta \in \boldsymbol{R}^n;\ |\zeta| = 1$ and $|\zeta - \xi^0| \leqslant \varepsilon\}$ in such a way to have from (2.2.5) that there exists a positive constants γ_2 satisfying

$$P\big(\tau^{-1}(\zeta - is\theta - it\eta)\big) \neq 0 \quad \text{if } s > \gamma_1 |\tau|^{1-\varrho},\ s,\ t,\ |\tau| < \gamma_2,\ t \geqslant 0,\ \eta \in M$$

for every ζ in K. If $\Gamma = \{\xi;\ \xi = \lambda\zeta,\ \lambda > 0,\ \zeta \in K\}$, then

$$P(\xi - is\theta - it\eta) = P\Big(|\xi|\big(\xi/|\xi| - is\theta/|\xi| - it\eta/|\xi|\big)\Big) \neq 0$$

if $\gamma_1 |\xi|^{\varrho} < s < \gamma_2 |\xi|$ and $0 \leqslant t \leqslant \gamma_2 |\xi|$, for every $\xi \in \Gamma$ with $|\xi| > 1/\gamma_2$ and every $\eta \in M$. This ends the proof.

REMARK 2.2.5. – Let $\xi \to M_\xi$ be an outer continuous map, where M_ξ are compact sets in $\Gamma(P_{m\xi}, \theta)$ satisfying $M_{t\xi} = M_\xi$ wher $t > 0$. Then, by the inner continuity of the cones $\Gamma(P_{m\xi}, \theta)$, the property (2.2.5) and a finite covering of the unit sphere, we can prove the existence of positive constants C_1, C_2 and R such that

$$P(\xi - is\theta - it\eta) \neq 0 \quad \text{if } C_1 |\xi|^{\varrho} < s < C_2 |\xi|, \quad 0 \leqslant t \leqslant C_2 |\xi|, \quad \eta \in M_\xi$$

for every $\xi \in \boldsymbol{R}^n$ with $|\xi| > R$, and

$$P(\xi - is\eta) \neq 0 \quad \text{if } C_1 |\xi|^{\varrho} < s < C_2 |\xi|, \quad \eta \in M_\xi .$$

We construct now a fundamental solution for $P(D)$ assuming, as usual, that the principal part $P_m \in \operatorname{He}(\theta)$. From the definition of $\varrho = \varrho(P)$ there exist positive constants c_1 and c_2 satisfying

$$P\Big(\xi - i\gamma(1 + |\xi|)^{\varrho}\theta\Big) \neq 0 \quad \text{if } c_1 < \gamma < c_2 |\xi|^{1-\varrho};$$

thus we can define, with a fixed $\gamma > c_1$,

$$E(P, \theta, \gamma, C; x) = (2\pi)^{-n} \int_{\Lambda} \frac{\exp(ix \cdot \zeta)}{P(\zeta)}\, d\zeta \tag{2.2.6}$$

where $\Lambda = \{\zeta = \xi - i\gamma(1 + |\xi|)^{\varrho}\theta;\ \xi \in \boldsymbol{R}^n,\ |\xi| \geqslant C > (\gamma/c_2)^{1/(1-\varrho)}\}$; when $P_m \in \operatorname{Hyp}(\theta)$ we can choose $C = 0$.

THEOREM 2.2.6. – Assume $P_m \in \mathrm{He}(\theta)$. Then (2.2.6) defines an ultradistribution $E \in \mathfrak{D}^{(\varkappa_0)'}(\boldsymbol{R}^n)$, $\varkappa_0 = 1/\varrho$, by the correct interpretation

$$\langle E, \varphi\rangle = (2\pi)^{-n}\int_{\Lambda} \frac{\hat{\varphi}(-\zeta)}{P(\zeta)}\, d\zeta\,, \qquad \varphi \in \mathfrak{D}^{(\varkappa_0)}(\boldsymbol{R}^n)\,. \tag{2.2.7}$$

Modulo entire analytic functions, E is a fundamental solution of $P(D)$ independent of γ and C as long as these constants satisfy the condition $c_1 < \gamma < c_2|\xi|^{1-\varrho}$ when $|\xi| \geqslant C$.

PROOF. – Putting $M = \{\theta\}$ in Lemma 2.2.1 we have $|P(\zeta)| \geqslant C_1 > 0$ for every ζ in Λ; moreover, by Theorem 1.2.2, if $\varphi \in \mathfrak{D}^{(\varkappa_0)}(\boldsymbol{R}^n)$ for every $L > 0$ there is a positve constant C_L such that

$$|\hat{\varphi}(-\zeta)| \leqslant C_L \exp\left(-L|\zeta|^{1/\varkappa_0} + A|\mathrm{Im}\,\zeta|\right)\,, \qquad \zeta \in \boldsymbol{C}^n$$

with some $A > 0$ depending on φ. Therefore, choosing L sufficiently large, we can easily prove the convergence of the integral in (2.2.7).

If $\varphi_j \to 0$ as $j \to \infty$ in $\mathfrak{D}^{(\varkappa_0)}(\boldsymbol{R}^n)$, then there exist $A > 0$ and for every $L > 0$ positive constants $C_L^j \to 0$ as $j \to \infty$, satisfying

$$|\hat{\varphi}_j(-\zeta)| \leqslant C_L^j \exp\left(-L|\zeta|^{1/\varkappa_0} + A|\mathrm{Im}\,\zeta|\right)\,, \qquad \zeta \in \boldsymbol{C}^n;$$

hence $\langle E, \varphi_j\rangle \to 0$, which proves $E \in \mathfrak{D}^{(\varkappa_0)'}(\boldsymbol{R}^n)$.

Obviously $P(D)E = \delta + h$ where δ is the Dirac measure and h the entire function

$$h(x) = -(2\pi)^{-n}\int_{\Lambda_0} \exp(ix\cdot\zeta)\, d\zeta\,, \qquad \Lambda_0 = \{\zeta = \zeta - i\gamma(1 + |\xi|)^{\varrho}\theta;\ |\xi| \leqslant C\}$$

To show that $E(\gamma, C) - E(\gamma', C')$ is an entire analytic function first note that if $C' < C$, the ultradistribution defined by

$$\langle E(\gamma, C') - E(\gamma, C), \varphi\rangle = (2\pi)^{-n}\int_{\Lambda_1} \frac{\hat{\varphi}(-\zeta)}{P(\zeta)}\, d\zeta\,, \qquad \varphi \in \mathfrak{D}^{(\varkappa_0)}(\boldsymbol{R}^n)\,,$$

where $\Lambda_1 = \{\zeta = \xi - i\gamma(1 + |\xi|)^{\varrho}\theta;\ C' < |\xi| < C\}$, has this property; then it suffices to prove that $E(\gamma, C) - E(\gamma', C)$ has no analytic singularities when C is chosen as large as the condition $c_1 < s\gamma + (1 - s)\gamma' < c_2|\xi|^{1-\varrho}$ is satisfied for all $0 \leqslant s \leqslant 1$ and ξ with $|\xi| \geqslant C$. Define $\zeta_s = \xi - i(s\gamma + (1 - s)\gamma')(1 + |\xi|)^{\varrho}\theta$, $|\xi| \geqslant C$, $0 \leqslant s \leqslant 1$; by Stokes' theorem we have

$$\langle E(\gamma, C) - E(\gamma', C), \varphi\rangle = (2\pi)^{-n}\int_{\substack{|\xi| = C\\ 0 \leqslant s \leqslant 1}} \frac{\hat{\varphi}(-\zeta_s)}{P(\zeta_s)}\, d\zeta_s\,, \qquad \varphi \in \mathfrak{D}^{(\varkappa_0)}(\boldsymbol{R}^n)$$

which defines an entire function. The proof is complete.

REMARK 2.2.7. – Since the principal part P_m is in $\mathrm{He}(\theta)$, it is locally hyperbolic, (see [2], [3]). By choosing then, as it is always possible, an entire function f such that $P(D)f = h$, $E - f$ becomes a fundamental solution for $P(D)$.

In [15] is proved the existence of a fundamental solution for $P(D)$ in the space $\mathfrak{D}^{(\sigma)'}$, where $\sigma = p/(p-1)$ with p equal to the number of the real roots t of $P_m(\xi + t\theta)$, when ξ is not parallel to θ. We improve then this result, since σ is generally smaller than $\mu/(\mu - 1) \leqslant \varkappa_0$ with μ the same constant of Lemma 2.1.1.

In [11] is considered a class of operators $P(D)$ satisfying

$$P(\xi - is\theta) \neq 0 \quad \text{if } A|\xi|^\beta < s < B|\xi|^\alpha, \quad \theta \text{ in a open cone } \Gamma$$

where $\beta \in [0, 1[$, $\alpha \in]\beta, 1[$; for an operator $P(D)$ in this class is proved the existence of a fundamental solution $E \in \mathfrak{D}^{(1/\beta)'}$ with E regular of class $\mathcal{E}^{\{1/\alpha\}}$ outside the dual cone Γ^*. Theorems 2.2.6 and 2.3.1 show that this result holds for $\alpha = 1$ too; furthermore our techniques may be adapted to the general case.

2.3. *Propagation of singularities.*

THEOREM 2.3.1. – Let $(x_0, \xi^0) \in \dot{T}^*(\boldsymbol{R}^n)$ with $x_0 \notin \Gamma(P_{m\xi^0}, \theta)^*$. Then we have $(x_0, \xi^0) \notin WF_A(E)$, that is

$$WF_A(E) \subset \bigcup_{0 \neq \xi} (P_{m\xi}, \theta)^* \times \{\xi\};$$

in particular $\operatorname{sing\,supp}_A(E)$ is contained in the wave front surface.

PROOF. – From the definition of the dual cone we can find $\eta^0 \in \Gamma(P_{m\xi^0}, \theta)$ and a neighborhood U of x_0 such that $x \cdot \eta^0 < 0$ for every $x \in U$. Let K be a compact set in U and $\{\chi_N\}$ a sequence in $\mathfrak{D}^{(\varkappa_0)}(U)$ satisfying

$$|D^{\alpha+\beta}\chi_N(x)| \leqslant C_h (CN)^{N_h|\beta|}(\beta!)^{\varkappa_0} u, \qquad |\alpha| \leqslant N = 1, 2, \ldots \tag{2.3.1}$$

for every $h > 0$, with $C_h > 0$ depending on h, $\operatorname{supp}\chi_N \subset K$. By the property (2.3.1), for every $B > 0$ there is $C_B > 0$ to get

$$|\hat{\chi}_N(\zeta)| \leqslant C_B (CN)^N (1 + |\zeta|)^{-N} \exp\left(-B|\zeta|^{1/\varkappa_0} + H_K(\operatorname{Im}\zeta)\right), \qquad \zeta \in \boldsymbol{C}^n. \tag{2.3.2}$$

Write $\chi_{N,\xi}(x) = \exp(-ix\cdot\xi)\,\chi_N(x)$, and consider

$$\widehat{(\chi_N E)}(\xi) = \langle E(x), \chi_{N,\xi}(x)\rangle = (2\pi)^{-n}\int_A \frac{\hat{\chi}_N(\xi - \zeta)}{P(\zeta)}\,d\zeta, \qquad \xi\zeta = w - i\gamma(1 + |w|)^\varrho\theta.$$

We split the above integral into $I^0 + I^\infty$, where in I^0 the integration is performed in $\{\zeta = w - i\gamma(1 + |w|)^\varrho\theta;\ |w - \xi| < \varepsilon|\xi|,\ \varepsilon > 0\}$ and in I^∞ consequently. Then we have, from (2.3.2),

$$|I^\infty| \leqslant C_B(CN)^N(1+\varepsilon|\xi|)^{-N}\int \exp\left[-B|\xi - w|^{1/\varkappa_0} + A\gamma(1+|w|)^\varrho\right] dw \leqslant$$
$$\leqslant C_B(CN)^N(1+|\xi|)^{-N}$$

where $A > 0$ depends on K and B is taken larger than $A\gamma(\varepsilon/2)^{-1/\varkappa_0}$.

Let us now estimate I^0. Choose a conic neighborhood Γ of ξ^0 for which Lemma 2.2.4 holds, and let $\Gamma_1 \subset \Gamma$ to satisfy

$$w \in \Gamma \quad \text{if } \xi \in \Gamma_1 \quad \text{and} \quad |w - \xi| < \varepsilon|\xi| .$$

Then we have, by Tarsky-Seidenberg lemma and Lemma 2.2.4,

$$\left|P\left(w - i\gamma(1+|w|)^\varrho\theta - it|w|\eta^0\right)\right| \geqslant C(1+|w|)^a, \quad 0 \leqslant t \leqslant t_0$$

for some positive constants C and t_0, $a \in \boldsymbol{Q}$, if $\xi \in \Gamma_1$ is large enough and $|w - \xi| < \varepsilon|\xi|$. Thus, by Stokes' theorem, we can write $I^0 = I^0(B) + I^0(t_0)$, where the integration is performed in $\{\zeta_t = w - i\gamma(1+|w|)^\varrho\theta - it|w|\eta^0;\ |w-\xi| = \varepsilon|\xi|,\ 0 \leqslant t \leqslant t_0\}$ and in $\{\zeta_{t_0} = w - i\gamma(1+|w|)^\varrho\theta - it_0|w|\eta^0;\ |w - \xi| \leqslant \varepsilon|\xi|\}$ respectively. We can estimate $I^0(B)$ in the same way as I^∞. For what concerns $I^0(t_0)$,note that there exists a positive constant c such that

$$|I^0(t_0)| \leqslant C_B(CN)^N(1+|\xi|)^{-N}\int \exp\left[-B|w-\xi|^{1/\varkappa_0} + A\gamma(1+|w|)^\varrho - ct_0|w|\right](1+|w|)^a\, dw$$
$$\leqslant C_B(CN)^N(1+|\xi|)^{-N}, \quad \xi \in \Gamma_1, \quad \xi \text{ sufficiently large} .$$

The proof si complete.

Theorem 2.3.2. – Let (x^0, ξ^0) be in $T^*(\boldsymbol{R}^n)$. Then $(x_0, \xi^0) \notin WF^{(1/\varrho(\xi^0))}(E)$, intending $(x_0, \xi^0) \notin WF^{(\infty)}(E)$ if $\varrho(\xi^0) = 0$.

Proof. – Let φ be in $\mathcal{D}^{(\varkappa_0)}$ with $\varphi(x_0) \neq 0$ and choose a conic neighborhood Γ of ξ^0 satisfying $\varrho(\Gamma) = \varrho(\xi^0)$. Refering to the arguments and the notations of the preceding proof, we have only to give a new estimate of I^0. First assume $\varrho(\xi^0) > 0$ and put $\zeta_t = w - i\left[\gamma(1+|w|)^\varrho + t\gamma\left((1+|w|)^{\varrho(\xi^0)} - (1+|w|)^\varrho\right)\right]\theta$ for $0 \leqslant t \leqslant 1$. Lemma 2.2.1, the definition of $\varrho(\xi^0)$ and Tarsky-Seidenberg lemma imply

$$|P(\zeta_t)| \geqslant C(1+|w|)^a \quad \text{if } \xi \in \Gamma_1, \quad |w-\xi| < \varepsilon|\xi|, \quad \xi \text{ large enough}$$

for all $0 \leqslant t \leqslant 1$ and some $C > 0$, $a \in \boldsymbol{Q}$. Again, by Stokes' theorem, we can write $I^0 = I^0(B) + I^0(1)$ where $I^0(B)$ may be estimated in the same way as I^∞, while for every $L>0$ there is a positive constant C_L to get

$$|I^0(1)|\leqslant C_L\int\limits_{|\xi-w|<\varepsilon|\xi|}\exp\left[-L|\xi-w|^{1/\varkappa_0}+A\gamma(1+|w|)^{\varrho(\xi^0)}\right]dw$$
$$\leqslant C\exp\left(C|\xi|^{\varrho(\xi^0)}\right),\quad \xi\in\Gamma_1,\quad \xi \text{ sufficiently large}$$

if L is suitably chosen. Finally, if $\varrho(\xi^0)=0$, we easily obtain

$$|\widehat{(\varphi E)}(\xi)|\leqslant C(1+|\xi|)^M,\quad \xi\in\Gamma_1$$

for some constants $C>0$ and M, ending thus the proof.

We are going now to give outer estimates for the wave front sets of $E*f$. When $P_m\in\mathrm{Hyp}(\theta)$ we may prove, by Lemma 2.2.1, that $\operatorname{supp}E(\gamma,0)$ is contained in the closed cone $\Gamma(P_m,\theta)^*$; of course this property does not hold for general P_m in $\mathrm{He}(\theta)$, therefore we can only consider convolution with ultradistributions f which have compact support.

Theorem 2.3.3. – Let $1<\varkappa_1\leqslant\varkappa_0=1/\varrho(P)$, $\varkappa_1\leqslant\varkappa\leqslant\infty$; then we have

$$WF_{(\varkappa)}(E*f)\subset\{(x+y,\xi)\in\dot T^*(\boldsymbol{R}^n);\ x\in\Gamma(P_{m\xi},\theta)^*,\ (y,\xi)\in WF_{(1/\varrho(\xi))}(f)$$
$$\text{if } 1/\varrho(\xi)\leqslant\varkappa,\ (y,\xi)\in WF_{(\varkappa)}(f)\text{ if }1/\varrho(\xi)>\varkappa\},\qquad f\in\mathcal{E}^{(\varkappa_1)'};$$

$$WF_{\{\varkappa\}}(E*f)\subset\{(x+y,\xi)\in\dot T^*(\boldsymbol{R}^n);\ x\in\Gamma(P_{m\xi},\theta)^*,\ (y,\xi)\in WF_{(1/\varrho(\xi))}(f)$$
$$\text{if } 1/\varrho(\xi)\leqslant\varkappa,\ (y,\xi)\in WF_{\{\varkappa\}}(f)\text{ if }1/\varrho(\xi)>\varkappa\},\qquad f\in\mathcal{E}^{\{\varkappa_1\}'};$$

$$WF^{(\varkappa)}(E*f)\subset\{(x+y,\xi)\in\dot T^*(\boldsymbol{R}^n);\ x\in\Gamma(P_{m\xi},\theta)^*,\ (y,\xi)\in WF_{(1/\varrho(\xi))}(f)$$
$$\text{if } 1/\varrho(\xi)<\varkappa,\ (y,\xi)\in WF^{(\varkappa)}(f)\text{ if }1/\varrho(\xi)\geqslant\varkappa\},\qquad f\in\mathcal{E}^{(\varkappa_1)'};$$

$$WF^{\{\varkappa\}}(E*f)\subset\{(x+y,\xi)\in\dot T^*(\boldsymbol{R}^n);\ x\in\Gamma(P_{m\xi},\theta)^*,\ (y,\xi)\in WF_{(1/\varrho(\xi))}(f)$$
$$\text{if } \varkappa_1<1/\varrho(\xi)\leqslant\varkappa,\ (y,\xi)\in WF^{\{\varkappa\}}(f)\text{ if }1/\varrho(\xi)>\varkappa,$$
$$(y,\xi)\in WF_{\{1/\varrho(\xi)\}}(f)\text{ if }1/\varrho(\xi)=\varkappa_1\},\qquad f\in\mathcal{E}^{\{\varkappa_1\}'}.$$

Proof. – The result follows from a simple application of Theorems 1.2.5, 2.3.1, 2.3.2.

Corollary 2.3.4. – Under the hypotheses of Theorem 2.3.3 we have:

(i) assume that the fiber $WF_{*\varkappa}(f)|_\xi=\phi$ for $\varkappa<1/\varrho(\xi)$ and that $WF_{(1/\varrho(\xi))}(f)|_\xi=\phi$ for $\varkappa\geqslant1/\varrho(\xi)$, $\xi\in\boldsymbol{R}^n\setminus\{0\}$; then $WF_{*\varkappa}(E*f)=\phi$, that is, $E*f\in\mathcal{E}^{*\varkappa}$;

(ii) assume that $WF^{(\varkappa)}(f)|_\xi=\phi$ (resp. $WF^{\{\varkappa\}}(f)|_\xi=\phi$) for $\varkappa\leqslant1/\varrho(\xi)$ (resp. $\varkappa<1/\varrho(\xi)$) and that $WF_{(1/\varrho(\xi))}(f)|_\xi=\phi$ if $\varkappa>1/\varrho(\xi)$ (resp. $\varkappa\geqslant1/\varrho(\xi)$); than $WF^{(\varkappa)}(E*f)=\phi$ (resp. $WF^{\{\varkappa\}}(E*f)=\phi$), that is, $E*f\in\mathcal{D}^{(\varkappa)'}$ (resp. $E*f\in\mathcal{D}^{\{\varkappa\}'}$).

Here $WF^{(\infty)}(E*f)=\phi$ if $WF^{(\infty)}(f)|_\xi=\phi$ for $\varrho(\xi)=0$ and $WF_{(1/\varrho(\xi))}(f)|_\xi=\phi$ for $\varrho(\xi)>0$, $\xi\in\boldsymbol{R}^n\setminus\{0\}$.

3. – Operators with constant coefficient principal part.

3.1. *Construction of a parametrix.*

In this section we shall construct a left parametrix for differential operators $P(x, D) = P_m(D) + Q(x, D)$, where P_m is in $\mathrm{He}(\theta)$ and the coefficients of the lower order part Q are in $\mathcal{E}^{\{\varkappa_1\}}$. We begin by introducing some notations which extend in an obvious way those of the foregoing chapter.

Let $\xi^0 \in \boldsymbol{R}^n \setminus \{0\}$; we define

$$n_j(\xi^0) = \sup_{x \in \boldsymbol{R}^n} n_j^+(P(x, \cdot); \xi^0)$$

and remark, although this will not be used in the following, that employing Tarsky-Seidenberg lemma one may prove that such a supremum is really attained. Furthermore set

$$\begin{aligned} n_j(P) &= \max_{|\xi|=1} n_j(\xi) \\ \varkappa(\xi^0) &= \min_{0 \leqslant j \leqslant m-1} \left(m - j + n_j(\xi^0)\right)/n_j(\xi^0) \\ \varkappa_0 &= \min_{|\xi|=1} \varkappa(\xi) \end{aligned}$$

with $\varkappa(\xi^0) = +\infty$ if $n_j(\xi^0) = 0$ $j = 0, \ldots, m-1$.

If this is the case, i.e. $\varkappa(\xi) = +\infty$ for every ξ in $\boldsymbol{R}^n \setminus \{0\}$, then the operator $P(x, D)$ is hyperbolic-elliptic (with respect to the direction θ) for every fixed x; moreover, it is of constant strength, [5].

We denote, as in the section 2.1, with $a(\xi^0)$ the degree of the localization $P_{m\xi^0}$, and $\mu = \max_\xi a(\xi)$; then from (2.1.3) it follows that $\varkappa(\xi^0)$ depends only on the terms of degree strictly greater than $m - a(\xi^0)$ and, in view of the fact that $\varkappa_0$ will give the class of ultradistributions of the parametrix, we can say that such a class is stable under perturbations of operators of degree less or equal than $m - \mu$. Again, from (2.1.3) and the definition of $\varkappa(\xi^0)$, it follows

$$\varkappa(\xi^0) \geqslant a(\xi^0)/\left(a(\xi^0) - 1\right) \qquad \varkappa_0 \geqslant \mu/(\mu - 1);$$

moreover if one knows $l = \deg Q$ these estimates may be improved:

$$\varkappa(\xi^0) \geqslant a(\xi^0)/\left(a(\xi^0) + l - m\right) \qquad \varkappa_0 \geqslant \mu/(\mu + l - m)$$

(see e.g. [10], Theorem 19).

We are going now to write the lower order part $Q(x, D)$ in such a way to make easier the use of Fourier transform. For this, set

$$V_j = \{p(\xi) \in \boldsymbol{C}[\xi];\ p(\xi) \text{ is homogeneous of degree } j \text{ and}$$
$$n_j^+(P_m + p;\, \eta) \leqslant n_j(\eta) \text{ for every } \eta \text{ in } \boldsymbol{R}^n \setminus \{0\}\}\,.$$

V_j is a finite dimensional vector space; remark that if $\varkappa_0 = +\infty$ then the hypothesis $n_j^+(P_m + p;\, \eta) \leqslant n_j(\eta)$, for every η in $\boldsymbol{R}^n \setminus \{0\}$, is equivalent to the request p weaker than P_m. Clearly $Q_j(x, \cdot) \in V_j$ for every x. Let $\{p^j\}$ be a base of the finite dimensional vector space $V(P) = V_0 + \ldots + V_{m-1}$; we may suppose it as set up by homogeneous elements.

Let $\varkappa_1$ be a number > 1 and suppose the coefficients of $Q(x, D)$ are in $\mathcal{E}^{\{\varkappa_1\}}$; then we have:

LEMMA 3.1.1. – Under the preceding hypothesis and notations me may write

$$Q(x, \xi) = \sum_j q_j(x)\, p^j(\xi) \qquad q_j \in \mathcal{E}^{\{\varkappa_1\}}\,.$$

Moreover, if the coefficients of $Q(x, D)$ have compact supports, the same is true for $\{q_j\}$.

In the following we shall always assume

$$1 < \varkappa_1 < \varkappa_0 \quad \text{ if } \varkappa_0 < +\infty \text{ or } 1 < \varkappa_1 \leqslant +\infty \text{ if } \varkappa_0 = +\infty$$

and work out the construction of the parametrix for the case $\{q_j\} \subset \mathcal{D}^{\{\varkappa_1\}}$: in the next section the hypothesis of compactness of the supports will be easily removed.

We denote $H_\varepsilon = \{x;\ x\cdot\theta > -\varepsilon\}$, $\varepsilon > 0$. Let $v \in \mathcal{E}^{\{\varkappa_1\}'}$, $\operatorname{supp} v \subset H_\varepsilon$, $\varphi \in \mathcal{D}^{\{\varkappa_1\}}$, and define

$$\langle E_l(v), \varphi\rangle = (2\pi)^{-n} \int_\Lambda \hat{\varphi}(-\zeta)\, \widehat{E_l(v)}(\zeta)\, d\zeta \qquad l = 0, 1, 2, \ldots$$

where

$$\widehat{E_0(v)}(\zeta) = \frac{1}{P_m(\zeta)}\, \hat{v}(\zeta)$$

$$\widehat{E_l(v)}(\zeta) = \sum_{j_1, \ldots, j_l} (-1)^l (2\pi)^{-ln} \frac{1}{P_m(\zeta)} \int_{\Lambda_l(\zeta)} d\zeta^1 \ldots d\zeta^l\, \hat{q}_{j_1}(\zeta^1) \frac{p^{j_1}}{P_m}(\zeta - \zeta^1) \ldots$$
$$\ldots \hat{q}_{j_l}(\zeta^l) \frac{p^{j_l}}{P_m}(\zeta - \zeta^1 - \ldots - \zeta^l)\, \hat{v}(\zeta - \zeta^1 - \ldots - \zeta^l)$$

$\zeta \in \Lambda$, $l = 1, 2, \ldots$, with

$$\Lambda = \{\zeta = \xi - i\gamma(1 + |\xi|)^{1/\varkappa_0}\theta;\ \xi \in \boldsymbol{R}^n,\ |\xi| > C_0\gamma(1 + |\xi|)^{1/\varkappa_0}\} \qquad \gamma > 1\,,$$

$$\Lambda_l(\zeta) = \Big\{(\zeta^1, \ldots, \zeta^l);\ \zeta^j = \eta^j + ia(1 + |\eta^j|)^{1/\varkappa_1}\theta,\ |\xi - \eta^1 - \ldots - \eta^j| \geqslant \geqslant C_0\Big[\gamma(1 + |\xi|)^{1/\varkappa_0} + a\sum_{h=1}^{j}(1 + |\eta^h|)^{1/\varkappa_1}\Big], \quad j = 1, \ldots, l\Big\}$$

where $\zeta = \xi - i\gamma(1 + |\xi|)^{1/\varkappa_0}\theta$, $a > 0$, $l = 1, 2, \ldots$, denoting by C_0 a constant such that $P_m(\xi - it\theta) \neq 0$ if $0 < t < C_0^{-1}|\xi|$.

The iteration of the functionals $\{E_l\}$ will yield another sequence $\{R_l\}$ such that

$$E_l(P_m u) = -E_{l-1}(Qu) + R_l(u) \qquad l = 1, 2, \ldots\,, \tag{3.1.1}$$

defined by

$$\langle R_l(v), \varphi\rangle = (2\pi)^{-n}\int_\Lambda \widehat{R_l(v)}(\zeta)\,\hat\varphi(-\zeta)\,d\zeta \qquad l = 1, 2, \ldots\,,$$

where

$$\widehat{R_l(v)}(\zeta) = \sum_{j_1, \ldots, j_l}(-1)^{l+1}(2\pi)^{-ln}\frac{1}{P_m(\zeta)}\int_{\Lambda_l^c(\zeta)} \zeta^1 \ldots d\zeta^l\,\hat q_{j_1}(\zeta^1)\frac{p^j}{P_m}(\zeta - \zeta^1) \ldots \ldots \hat q_{j_l}(\zeta^l)\,p^{j_l}(\zeta - \zeta^1 - \ldots - \zeta^l)\,\hat v(\zeta - \zeta^1 - \ldots - \zeta^l)$$

for $\zeta = \xi - i\gamma(1 + |\xi|)^{1/\varkappa_0}\theta \in \Lambda$ and

$$\Lambda_l^c(\zeta) = \Big\{(\zeta^1, \ldots, \zeta^l);\ (\zeta^1, \ldots, \zeta^{l-1}) \in \Lambda_{l-1}(\zeta),\ \zeta^l = \eta^l + ia(1 + |\eta^l|)^{1/\varkappa_1}\theta\,, |\xi - \eta^1 - \ldots - \eta^l| \leqslant C_0\Big[\gamma(1 + |\xi|)^{1/\varkappa_0} + a\sum_{h=1}^{l}(1 + |\eta^h|)^{1/\varkappa_1}\Big]\Big\}.$$

We shall now prove that

$$J_k^j = \left|\frac{p^j}{P_m}(\zeta - \zeta^1 - \ldots - \zeta^k)\right| \leqslant Ca^{-\lambda}\gamma^{-\sigma} \tag{3.1.2}$$

with $\gamma > 1$ and for some positive constants λ, σ, where $\zeta \in \Lambda$, $(\zeta^1, \ldots, \zeta^k) \in \Lambda_k(\zeta)$ and $1 < \varkappa_1 < \varkappa_0 < +\infty$.

In fact

$$J_k^j \leqslant C\Big\{\gamma(1 + |\xi|)^{1/\varkappa_0} + a\sum_{h=1}^{k}(1 + |\eta^h|)^{1/\varkappa_1}\Big\}^{\nu - m} \cdot \cdot\Big\{\Big[\gamma(1 + |\xi|)^{1/\varkappa_0} + a\sum_{h=1}^{k}(1 + |\eta^h|)^{1/\varkappa_1}\Big]^{-1}|\xi - \eta^1 - \ldots - \eta^k|\Big\}^{n_\nu(P)},$$

denoting $\nu = \deg p^j$. Since $n_\nu(P) - [m - \nu + n_\nu(P)]\varkappa_0^{-1} \leqslant 0$, we get

$$J_k^j \leqslant C\gamma^{\nu - m} \quad \text{if } |\eta^1| + \ldots + |\eta^k| \leqslant |\xi| .$$

Furthermore

$$J_k^j \leqslant C\left\{\gamma^{-\sigma}\left[a^{\nu-m+\sigma-n_\nu(P)} \sum_{h=1}^{k} (1 + |\eta^h|)^{1/\varkappa_1}\right]^{\nu-m+\sigma+(\varkappa_1-1)n_\nu(P)}\right\} \leqslant Ca^{-\lambda}\gamma^{-\sigma} \quad \text{if } |\eta^1| + \ldots + |\eta^h| > |\xi| ,$$

putting $\sigma = \min_{0 \leqslant j \leqslant m-1} (m - j - n_j(P)(\varkappa_1 - 1))$, $\lambda = \min_\nu (m - \nu - \sigma + n_\nu(P))$; since $\varkappa_0 n_\nu(P) - m + \nu - n_\nu(P) \leqslant 0$ and $\varkappa_1 < \varkappa_0$ it follows that σ and λ are > 0.

We can now state the main result of this section.

THEOREM 3.1.2. – Let $1 < \varkappa_1 < \varkappa_0 \leqslant +\infty$, $\{q_j\} \subset \mathfrak{D}^{\{\varkappa_1\}}$. Then the series $\sum_{l=0}^{\infty} \widehat{E_l(v)}$, $\sum_{l=1}^{\infty} \widehat{R_l(v)}$ converge in Λ to $\widehat{E(v)}$, $\widehat{R(v)}$ respectively defining then two ultradistributions $E(v)$, $R(v)$ by

$$\langle E(v), \varphi\rangle = (2\pi)^{-n}\int_\Lambda \hat{\varphi}(-\zeta)\,\widehat{E(v)}(\zeta)\,d\zeta , \qquad \langle R(v), \varphi\rangle = (2\pi)^{-n}\int_\Lambda \hat{\varphi}(-\zeta)\,\widehat{R(v)}(\zeta)\,d\zeta$$

for $v \in \mathcal{E}^{\{\varkappa_1\}'}$, $\operatorname{supp} v \in H_\varepsilon$, $\varphi \in \mathfrak{D}^{\{\varkappa_1\}}$.

Furthermore we have

$$E(Pv) = v + h(v) + R(v) \tag{3.1.3}$$

where $h(v)(x) = -(2\pi)^{-n}\int_{\Lambda^0} \exp(ix\cdot\zeta)\,\hat{v}(\zeta)\,d\zeta$ is an entire function,

$$\Lambda^0 = \{\zeta = \xi - i\gamma(1 + |\xi|)^{1/\varkappa_0}\theta;\ \xi \in \mathbf{R}^n,\ |\xi| \leqslant C_0\gamma(1 + |\xi|)^{1/\varkappa_0}\} .$$

More precisely we get the following estimates [4], for $v \in \mathcal{E}^{\{\varkappa_1\}'}$, $\operatorname{supp} v \subset H_\varepsilon$:

$$|\widehat{E(v)}(\zeta)| \leqslant \frac{C_L}{|P_m(\zeta)|} \exp[L|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1 + |\xi|)^{1/\varkappa_0}] \quad \text{for every } L > 0 \tag{3.1.4}$$

$$|\widehat{R(v)}(\zeta)| \leqslant \frac{C}{|P_m(\zeta)|} \exp[-M|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1 + |\xi|)^{1/\varkappa_0}] \quad \text{for some } M > 0 \tag{3.1.5}$$

which hold true for $\zeta \in \Lambda$ and γ greater than some constant γ_0 depending on P and a.

The rest R is regularizing, i.e. $R(v) \in \mathcal{E}^{\{\varkappa_1\}}$.

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[4] $1/\varkappa_0 = 0$ if $\varkappa_0 = +\infty$.

PROOF. – We begin with proving (3.1.4). By Theorem 1.2.2 there exist positive constants C_1, M, B and for every L a constant C_L such that

$$|q_j(\zeta)|\leqslant C\exp\left[-M|\zeta|^{1/\varkappa_1}+B|\operatorname{Im}\zeta|\right] \tag{3.1.6}$$

$$|\hat{v}(\zeta)|\leqslant C_L\exp\left[L|\zeta|^{1/\varkappa_1}+H_K(\operatorname{Im}\zeta)\right] \tag{3.1.7}$$

for every ζ in $\boldsymbol{C}^n$, $\operatorname{supp} v\subset K$. From (3.1.2), for $\zeta\subset\Lambda$, we get

$$|\widehat{E_l(v)}(\zeta)|\leqslant C(C_1Ca^{-\lambda}\gamma^{-\sigma})^l C_L\frac{1}{|P_m(\zeta)|}\int d\eta^1\ldots\int d\eta^l\exp\Big[-M\sum_{j=1}^{l}|\eta^j|^{1/\varkappa_1}+ + aB\sum_{j=1}^{l}(1+|\eta^j|)^{1/\varkappa_1}\Big]\exp\Big[L|\xi-\eta^1-\ldots-\eta^l|^{1/\varkappa_1}+\varepsilon\Big(\gamma(1+|\xi|)^{1/\varkappa_0}+ + a\sum_{j=1}^{l}(1+|\eta^j|)^{1/\varkappa_1}\Big)\Big].$$

Choosing now $L<M/4$ and fixing once for all $0<a<M/4(B+\varepsilon)$,

$$|\widehat{E_l(v)}(\zeta)|\leqslant C(C_1C\gamma^{-\sigma})^l\frac{1}{|P_m(\zeta)|}C_L\exp\left[L|\xi|^{1/\varkappa_1}+\varepsilon\gamma(1+|\xi|)^{1/\varkappa_0}\right]$$

$l=1,2,\ldots,\zeta\in\Lambda$, and this permit us to get the convergence of the series for γ large enough. Then $E(v)$ satisfies the following estimate, for every positive L:

$$|\widehat{E(v)}(\zeta)|\leqslant\frac{C_L}{|P_m(\zeta)|}\exp\left[L|\xi|^{1/\varkappa_1}+\varepsilon\gamma(1+|\xi|)^{1/\varkappa_0}\right]\qquad\zeta\in\Lambda\,.$$

The convergence of the integral defining $\langle E(v),\varphi\rangle$ is therefore obvious, since if $\varphi\in\mathfrak{D}^{\{\varkappa_1\}}$

$$|\langle E(v),\varphi\rangle|\leqslant C_2C_L\int_{\Lambda}\exp\left[-D_1|\xi|^{1/\varkappa_1}+D_2\gamma(1+|\xi|)^{1/\varkappa_0}+L|\xi|^{1/\varkappa_1}+\varepsilon\gamma(1+|\xi|)^{1/\varkappa_0}\right]d\zeta$$

for every L and constants C_2, D_1, D_2 depending on φ; it is then sufficient to take $L<D_1$. The continuity of the functional $E(v)$ may be seen as in Theorem 2.2.6.

The proof of (3.1.5) is rather similar to that of (3.1.4); in fact we have, with the same constants above used,

$$|\widehat{R_l(v)}(\zeta)|\leqslant CC_1^l(C\gamma^{-\sigma})^{l-1}C_L\frac{1}{|P_m(\zeta)|}\int_{\Lambda_l^\varrho(\zeta)}d\zeta^1\ldots d\zeta^l\cdot \cdot\exp\Big[-M\sum_{j=1}^{l}|\eta^j|^{1/\varkappa_1}+aB\sum_{j=1}^{l}(1+|\eta^j|)^{1/\varkappa_1}\Big]\cdot \cdot\exp\Big[L|\xi-\eta^1-\ldots-\eta^l|^{1/\varkappa_1}+\varepsilon\Big(\gamma(1+|\xi|)^{1/\varkappa_0}+a\sum_{j=1}^{l}(1+|\eta^j|)^{1/\varkappa_1}\Big)\Big]|p^j(\zeta-\zeta^1-\ldots-\zeta^l)|$$

for $\zeta\in\Lambda$.

From the definition of the domain of integration it results

$$|\eta^l|^{1/\varkappa_1} \geqslant |\xi|^{1/\varkappa_1} - C_0^{1/\varkappa_1}\Big[\gamma(1+|\xi|)^{1/\varkappa_0} + a\sum_{j=1}^{l}(1+|\eta^j|)^{1/\varkappa_1}\Big]^{1/\varkappa_1} - \sum_{j=1}^{l-1}|\eta^j|^{1/\varkappa_1}$$

and from here we infer, with easy calculations,

$$|\widehat{R_l(v)}(\zeta)| \leqslant C(C_1 C\gamma^{-\sigma})^l C_L \frac{1}{|P_m(\zeta)|}\exp\left[-M|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1+|\xi|)^{1/\varkappa_0}\right],$$

$\zeta \in \Lambda$, for a new positive constant M, where we have chosen L sufficiently small. Therefore, for every γ large enough to assure the convergence of the series, we get (3.1.5).

The functional $R(v)$ is well-defined and continuous arguing as above, and then results an ultradistribution of class $\mathfrak{D}^{\{\varkappa_1\}'}$.

To prove the regularity of the rest R, let φ be in $\mathfrak{D}^{\{\varkappa_1\}}$, ϱ in $\boldsymbol{R}^n$; so

$$|\widehat{\varphi R(v)}(\varrho)| \leqslant C\int_\Lambda \frac{1}{|P_m(\zeta)|}\exp\left[-M|\xi|^{1/\varkappa_1} - A|\xi-\varrho|^{1/\varkappa_1} + B\gamma(1+|\xi|)^{1/\varkappa_0}\right] d\zeta$$

for some positive constants M, A, B. This gives, putting for instance $D = \min(M, A)$

$$|\widehat{\varphi R(v)}(\varrho)| \leqslant C\exp\left[-D|\varrho|^{1/\varkappa_1}/2\right],$$

and so we have $WF_{\{\varkappa_1\}}(R(v)) = \phi$.

Since the equality (3.1.3) is an obvious consequence of (3.1.1), the theorem is completely proved.

REMARK 3.1.3. – When $\varkappa_1 < \varkappa_0 = +\infty$ the proof of Theorem 3.1.3 could be made simpler, since then it is not necessary to consider complex variables ζ^j, $j = 1, \ldots, l$: actually one can take $a = 0$ and use the estimate

$$\left|\frac{p^j}{P_m}(\xi - i\gamma\theta)\right| \leqslant C\gamma^{-1} \qquad 1 < \gamma < C_0^{-1}|\xi| \tag{3.1.8}$$

instead of (3.1.2). This follows from what above observed about the spaces V_j if $\varkappa_0 = +\infty$.

The case $\varkappa_1 = \varkappa_0 = +\infty$ is not treated in the theorem; if this happens we get the following estimates of polynomial kind:

$$|\widehat{E(v)}(\xi - i\gamma\theta)| \leqslant \frac{C}{|P_m(\xi - i\gamma\theta)|}(1+|\xi|)^N \qquad \text{for some } N > 0 \tag{3.1.4$'$}$$

$$|\widehat{R(v)}(\xi - i\gamma\theta)| \leqslant \frac{C_N}{|P_m(\xi - i\gamma\theta)|}(1+|\xi|)^{-N} \qquad \text{for every } N > 0 \tag{3.1.5$'$}$$

with ξ real sufficiently large.

Since (3.1.4)′ is quite clear we trace only a sketch of the proof of (3.1.5)′, which is slightly different from the preceding one. In the case we are, the elements of the sequence defining the rest take the form

$$\widehat{R_l(v)}(\xi - i\gamma\theta) = \sum_{j_1,\dots,j_l} (-1)^{l+1}(2\pi)^{-ln} \frac{1}{|P_m(\xi - i\gamma\theta)|} \int\limits_{|\xi-\xi^1|>C_0\gamma} d\xi^1 \hat{q}_{j_1}(\xi) \frac{p^{j_1}}{P_m}(\xi - \xi^1 - i\gamma\theta) \dots$$
$$\dots \int\limits_{|\xi-\xi^1-\dots-\xi^l|<C_0\gamma} d\xi^l \hat{q}_{j_l}(\xi^l) p^{j_l}(\xi - \xi^1 - \dots - \xi^l - i\gamma\theta)\hat{v}(\xi - \xi^1 - \dots - \xi^l - i\gamma\theta)$$

for $v \in \mathcal{E}'$, $|\xi| > C_0\gamma$. We split the integral into the sum $I_1 + I_2$, where in I_1 suppose $|\xi^j| < \varepsilon|\xi|/(l-1)$, for every $j = 1, \dots, l-1$, and I_2 consequently. From Paley-Wiener theorem and the fact that in I_2 there exists at least an h, $1 \leqslant h \leqslant l-1$, such that $|\xi^h| > \varepsilon|\xi|/(l-1)$ we obtain

$$|I_2| \leqslant C_N(C\gamma^{-1})^l \frac{1}{|P_m(\xi - i\gamma\theta)|}(1 + |\xi|)^{-N}$$

for $|\xi| > C_0\gamma$. A similar estimate holds for I_1, since in this case $|\xi^l| > (1/2 - \varepsilon)|\xi|$ for $|\xi|$ large enough. So we have (3.1.5)′ summing up with respect to l.

Finally the regularity of the rest, in this case of class C^∞, is clearly a consequence of (3.1.5)′.

We want now to show that the functional E has a kernel $E(x, y) \in \mathcal{D}^{\{\varkappa_1\}'}(\boldsymbol{R}^n \times H_\varepsilon)$, [8].

THEOREM 3.1.4. – Let $1 < \varkappa_1 < \varkappa_0 < +\infty$ or $1 < \varkappa_1 \leqslant +\infty$ if $\varkappa_0 = +\infty$. Then E is a continuous functional from $\mathcal{D}^{\{\varkappa_1\}}(H_\varepsilon)$ to $\mathcal{D}^{\{\varkappa_1\}'}(\boldsymbol{R}^n)$ with range contained in $\mathcal{E}^{\{\varkappa_1\}}(\boldsymbol{R}^n)$.

PROOF. – Consider for the moment the case $1 < \varkappa_1 < \varkappa_0 < +\infty$ and take $v \in \mathcal{D}^{\{\varkappa_1\}}(H_\varepsilon)$, $\operatorname{supp} v \subset K$, with K compact subset of H_ε; then there exist positive constants L, C so that

$$|\hat{v}(\zeta)| \leqslant C \exp[-L|\zeta|^{1/\varkappa_1} + H_K(\operatorname{Im}\zeta)] \qquad \zeta \in \boldsymbol{C}^n, \tag{3.1.9}$$

If we take ζ in Λ then

$$|\widehat{E_l(v)}(\zeta)| \leqslant C(C_1 C a^{-\lambda}\gamma^{-\sigma})^l \frac{1}{|P_m(\zeta)|} \int d\eta^1 \dots \int d\eta^l \cdot$$
$$\cdot \exp\Big[-M\sum_{j=1}^{l} |\eta^j|^{1/\varkappa_1} + aB\sum_{j=1}^{l}(1 + |\eta^j|)^{1/\varkappa_1}\Big] \cdot$$
$$\cdot \exp\Big[-L|\xi - \eta^1 - \dots - \eta^l|^{1/\varkappa_1} + \varepsilon\Big(\gamma(1 + |\xi|)^{1/\varkappa_0} + a\sum_{j=1}^{l}(1 + |\eta^j|)^{1/\varkappa_1}\Big)\Big];$$

since $|\xi - \eta^1 - \ldots - \eta^l|^{1/\varkappa_1} \geqslant |\xi|^{1/\varkappa_1} - |\eta^1|^{1/\varkappa_1} - \ldots |\eta^l|^{1/\varkappa_1}$, choosing eventually a smaller L in (3.1.9) to get $L < M/4$, we have

$$|\widehat{E_l(v)}(\zeta)| \leqslant C(C_1 C a^{-\lambda} \gamma^{-\sigma})^l \frac{1}{|P_m(\zeta)|} \exp\left[-L|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1 + |\xi|)^{1/\varkappa_0}\right]$$

for ζ in Λ, and, if γ is sufficiently large,

$$|\widehat{E(v)}(\zeta)| \leqslant \frac{C}{|P_m(\zeta)|} \exp\left[-L|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1 + |\xi|)^{1/\varkappa_0}\right].$$

By following the lines of the proof of Theorem 3.1.2 when was showed the regularity of the rest, one immediately sees that $E(v) \in \mathcal{E}^{\{\varkappa_1\}}(\boldsymbol{R}^n)$ if $v \in \mathcal{D}^{\{\varkappa_1\}}(H_\varepsilon)$.

The continuity is obtained in quite a standard way: let $\{v_j\} \subset \mathcal{D}^{\{\varkappa_1\}}(H_\varepsilon)$ be a sequence converging to 0 in the topology of this space; then there exist a constant L, a sequence $\{C_j\}$, $C_j \to 0$, a compact subset K of H_ε such that

$$|\hat{v}_j(\zeta)| \leqslant C_j \exp\left[-L|\zeta|^{1/\varkappa_1} + H_K(\operatorname{Im}\zeta)\right] \qquad \zeta \in \boldsymbol{C}^n, \quad j = 1, 2, \ldots.$$

Just as above one can easily obtain

$$|\widehat{E(v_j)}(\zeta)| \leqslant C C_j \frac{1}{|P_m(\zeta)|} \exp\left[-L'|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1 + |\xi|)^{1/\varkappa_0}\right] \qquad \xi \in \Lambda$$

for a suitable constant L'. If now $\varphi \in \mathcal{D}^{\{\varkappa_1\}}$ we get

$$|\langle E(v_j), \varphi\rangle| \leqslant C C_j \int_\Lambda \exp\left[-L'|\xi|^{1/\varkappa_1} + \varepsilon\gamma(1 + |\xi|)^{1/\varkappa_0} - D|\xi|^{1/\varkappa_1} + F\gamma(1 + |\xi|)^{1/\varkappa_0}\right] d\zeta$$

for positive constants D and F; then $\langle E(v_j), \varphi\rangle \to 0$.

We leave to the reader the case $\varkappa_1 < \varkappa = +\infty$; finally, if $\varkappa_1 = \varkappa_0 = +\infty$, in the formula defining $\widehat{E_l(v)}$ we perform the integration at first where $|\eta^j| \leqslant \varepsilon|\xi|/l$, for every $j = 1, \ldots, l$, so $|\xi - \eta^1 - \ldots - \eta^l| \geqslant (1 - \varepsilon)|\xi|$, and then in the remaining region, where there will be an h, $1 \leqslant h \leqslant l$, such that $|\eta^h| > \varepsilon|\xi|/l$ holds true; Paley-Wiener theorem and (3.1.8) allow us to reach the thesis. Also in this case continuity is as easy to prove as in the former ones.

3.2. *Propagation of singularities.*

The aim of this section is to extend the results on propagation of singularities achieved in the previous chapter to operators with variable coefficients. The methods of proof will be then similar, although we need a more precise study for the zeros of the principal part of the polynomial (see Lemma 3.2.4) and it will be requested a careful analysis of the sequence defining the parametrix we set up. The following theorem will be the main result:

THEOREM 3.2.1. – Let $P(x, D) = P_m(D) + Q(x, D)$, P_m in $\mathrm{He}(\theta)$, $Q(x, D)$ with coefficients $\{q_j\}$ of class $\mathcal{E}^{\{\varkappa_1\}}$. Let further u be a solution of the equation $P(x, D)u = f$, with u and f in $\mathcal{D}^{\{\varkappa_1\}'}$ with support contained in $\bar{H}_0$ (5); $1 < \varkappa_1 < \varkappa_0$ if $\varkappa_0 < +\infty$, $1 < \varkappa_1 \leqslant +\infty$. if $\varkappa_0 = +\infty$ Then we have

$$WF_{\{\varkappa\}}(u) \subset \{(x + y, \xi) \in \dot{T}^*(\boldsymbol{R}^n);\ x \in \Gamma(P_{m\xi}, \theta)^*,\ (y, \xi) \in WF_{(\varkappa(\xi))}(f)$$
$$\text{if } \varkappa(\xi) \leqslant \varkappa, \text{ or } (y, \xi) \in WF_{\{\varkappa\}}(f) \text{ if } \varkappa(\xi) > \varkappa\}\,.$$

$$WF^{\{\varkappa\}}(u) \subset \{(x + y, \xi) \in \dot{T}^*(\boldsymbol{R}^n);\ x \in \Gamma(P_{m\xi}, \theta)^*,\ (y, \xi) \in WF_{(\varkappa(\xi))}(f)$$
$$\text{if } \varkappa(\xi) \leqslant \varkappa, \text{ or } (y, \xi) \in WF^{\{\varkappa\}}(f) \text{ if } \varkappa(\xi) > \varkappa\}\,.$$

It will be not restrictive to establish this theorem under the hypotheses of compactness of the supports of the coefficients $\{q_j\}$ and of the solution u (and then of f). In fact let $\Phi \in \mathcal{D}^{\{\varkappa_1\}}$, $\Phi \equiv 1$ in a neighborhood of $\operatorname{supp} u$; then

$$[P_m(D) + \Phi Q(x, D)]\, u = f$$

and now the coefficients have compact supports. For what concerns the solution, let us suppose $u \in \mathcal{D}^{\{\varkappa_1\}'}$ and

$$P(x, D)\, u = f\,, \qquad \operatorname{supp} u \subset H$$

where $f \in \mathcal{D}^{\{\varkappa_1\}'}$ and the coefficients $q_j \in \mathcal{E}^{\{\varkappa_1\}}$. Set

$$T_h = [h\theta - \Gamma(P_m, \theta)^*] \cap H_0 \qquad h = 1, 2, \ldots$$

and let $\{\varphi_h\} \subset \mathcal{D}^{\{\varkappa_1\}}$ be a family of functions identically equal to one in a neighborhood of T_h, $\operatorname{supp} \varphi_h \cap H_0 \subset T_{h+1}$. Now $\{\varphi_h u\} \subset \mathcal{E}^{\{\varkappa_1\}'}$ and

$$P(x, D)(\varphi_h u) = \varphi_h f + f_h \qquad h = 1, 2, \ldots$$

with $f_h \in \mathcal{E}^{\{\varkappa_1\}'}$, $\operatorname{supp} f_h \subset T_{h+1} \setminus T_h$. Cutting the coefficients as above we are then under the hypotheses to apply Theorem 3.2.1 in the case when all supports are compact, from which, setting

$$W_h = \{(z, \eta) \in \dot{T}^*(\boldsymbol{R}^n);\ z \in T_h\}$$

---

(5) $\bar{H}_0 = \{x \in \boldsymbol{R}^n;\ x \cdot \theta \geqslant 0\}$.

we deduce

$$WF(u) = \bigcup_h \left(W_h \cap WF_{\{\varkappa\}}(\varphi_h u)\right) \subset$$
$$\subset \{(x+y,\xi) \in \dot{T}^*(\mathbf{R}^n);\ (x,\xi) \in \Gamma(P_{m\xi},\theta)^*,\ (y,\xi) \in WF_{(\varkappa(\xi))}(f) \text{ if } \varkappa(\xi) \leqslant \varkappa,\ (y,\xi) \in WF_{\{\varkappa\}}(f) \text{ if } \varkappa(\xi) > \varkappa\}\,.$$

From what precedes we can remark that, as regards the propagation of singularities of the solution u, it is sufficient to assume $\operatorname{sing\,supp}_{\{\varkappa_1\}} u \subset \bar{H}_0$ instead of $\operatorname{supp} u \subset \bar{H}_0$.

The proof of the theorem, where we shall consider H_ε in place of $\bar{H}_0$, will be achieved by studying the wave front set of the kernel $E(x, y)$; we shall found that such a kernel is in $\mathcal{D}^{(\varkappa_0)'}(\mathbf{R}^n \times H_\varepsilon)$,

$$\operatorname{sing\,supp}_{\{\varkappa_1\}} E(x,y) \subset \{(x,y) \in \mathbf{R}^n \times H_\varepsilon;\ x - y \in \Gamma(P_m,\theta)^*\}$$

(see Theorem 3.2.5). This assumed, the restriction to the case in which only the supports of the coefficients q_j are compact may be seen also in the following way. Let $\chi \in \mathcal{E}^{(\varkappa_0)}(\mathbf{R}^{2n})$, $\chi \equiv 1$ in a neighborhood of $\operatorname{sing\,supp}_{\{\varkappa_1\}} E(x,y)$,

$$\operatorname{supp} \chi \subset \operatorname{sing\,supp}_{\{\varkappa_1\}} E(x,y) + \{|(x,y)| \leqslant r\}\,,$$

r small, and split

$$E = \chi E + (1-\chi) E\,.$$

The second term yields a regularizing operator, while for the first the projection Π_1: $\operatorname{supp} \chi E(x,y) \to \mathbf{R}^n$, $\Pi_1(x,y) = x$, is proper. This permits to extend the functional associated to $\chi E(x,y)$ to the space $\mathcal{D}^{\{\varkappa_1\}'}(H_\varepsilon)$.

As already said, in the proof of Theorem 3.2.1 will be needed some informations on the zeros of the principal part P_m, which we are going now to establish.

LEMMA 3.2.2. – Let $\xi^0 \in \mathbf{R}^n \setminus \{0\}$, $\eta^0 \in \Gamma(P_{m\xi^0},\theta)$. Then there exist positive constants δ, t_0, R and a conic neighborhood Γ of ξ^0 such that

$$P_m(\xi - i\theta - it|\xi|\eta) \neq 0 \qquad (3.2.1)$$

if $0 \leqslant t \leqslant t_0$, $|\eta - \eta^0| < \delta$, $\xi \in \Gamma$, $|\xi| \geqslant R$.

PROOF. – Assume $|\xi^0| = 1$; since the polynomial $P_m(\xi^0 + \xi)$ defines, as a function of ξ, a locally hyperbolic function, the « Main Lemma » in [3] yields

$$P_m(\xi^0 + \xi - is\theta - it\eta) \neq 0 \qquad (3.2.2)$$

for every η in a fixed compact subset M of $\Gamma(P_{m\xi^0}, \theta)$, $\xi \in \boldsymbol{R}^n$, $|\xi|$, t, s all bounded from above by a constant t_0, $t \geqslant 0$, $s > 0$. By homogeneity then there exists a conic neighborhood Γ of ξ^0 such that

$$P_m(\xi - i\theta - it|\xi|\eta) \neq 0$$

if $\xi \in \Gamma$, $|\xi| \geqslant 1/t_0$, $0 \leqslant t \leqslant t_0$, $\eta \in M$, which proves (3.2.1).

LEMMA 3.2.3. – Let $\xi^0 \in \boldsymbol{R}^n \setminus \{0\}$, $\eta^0 \in \Gamma(P_{m\xi^0}, \theta)$. Then there exists positive constants t_0, R and a conic neighborhood Γ of ξ^0 so that

$$\tilde{P}_m(\xi - i\theta - it|\xi|\eta^0) \leqslant C|P_m(\xi - i\theta - it|\xi|\eta^0)|. \tag{3.2.3}$$

$$\tilde{P}_m(\xi/t|\xi| - i\theta/t|\xi| - i\eta^0) \leqslant C|P_m(\xi/t|\xi| - i\theta/t|\xi| - i\eta^0)|, \tag{3.2.4}$$

if $t|\xi| > 0$, for some constant C. Here and in the following we denote with the same letters constants or neighborhoods maybe different.

PROOF. – Let ϱ be a positive number such that $\theta + \varrho\eta^0 \in \Gamma(P_m, \theta)$. If $t|\xi| \leqslant \varrho$ then $\theta + t|\xi|\eta^0$ varies in a compact part of $\Gamma(P_m, \theta)$; and since P_m is hyperbolic-elliptic with respect to every direction of such a cone we get

$$P_m\big(\xi + \operatorname{Re} z - i(\theta + t|\xi|\eta^0 - \operatorname{Im} z)\big) \neq 0$$

for $z \in \boldsymbol{C}^n$ sufficiently small, ξ in $\boldsymbol{R}^n$ large enough. This proves (3.2.3) when $t|\xi| \leqslant \varrho$, in view of Lemma 11.1.4 in [5].

If now Γ, t_0, δ have the same meaning as in Lemma 3.2.2, we choose a cone $\Gamma_1 \subset \Gamma$ in a way that $\xi + |\xi|v \in \Gamma$ if $\xi \in \Gamma_1$, $0 \leqslant |v| \leqslant t_0\delta$. Then (3.2.1) gives

$$P_m\left(\xi + t|\xi| \operatorname{Re} z - i\big(\theta + (t|\xi|\eta^0 - \operatorname{Im} z)\big)\right) \neq 0$$

if $\xi \in \Gamma_1$, $|\xi| \geqslant R/(1 - t_0\delta)$, $0 \leqslant t \leqslant t_0(1 - t_0\delta)$. Again by Lemma 11.1.4 in [5] we have

$$|P_m^{(\alpha)}(\xi - it|\xi|\eta^0 - i\theta)| \leqslant [C/(t|\xi|\delta)]^{|\alpha|}|P_m(\xi - i\theta - it|\xi|\eta^0)|$$

under the above conditions and $t|\xi| > 0$, concluding the proof of (3.2.3). Multiplying this last inequality by $(t|\xi|)^{|\alpha|-m}$ and summing up with respect to α we reach (3.2.4).

With the help of the results just obtained one can deduce the following lemma, the proof of which, quite analogous to that of Lemma 2.4.2 in [14], is skipped away.

LEMMA 3.2.4. – Let $\xi^0 \in \boldsymbol{R}^n \setminus \{0\}$, $\eta^0 \in \Gamma(P_{m\xi^0}, \theta)$, $\nu = \deg p^j$. Then there exist positive constants t_0, R, C and a conic neighborhood Γ of ξ^0 such that

$$\left|\frac{p^j}{P_m}\big(\xi - i\gamma(1 + |\xi|)^{1/\varkappa_0}\theta - it|\xi|\eta^0\big)\right| \leqslant C\gamma^{\nu - m}$$

for $\xi \in \Gamma$, $|\xi| > R\gamma(1 + |\xi|)^{1/\varkappa_0}$, $\gamma \geqslant 1$, $0 \leqslant t \leqslant t_0$.

THEOREM 3.2.5. – $WF_{\{\varkappa_1\}}(E(x,y)) \subset \{(x,y;\xi,-\xi)\in \dot{T}^*(\boldsymbol{R}^n);\ x-y\in\Gamma(P_{m\xi},\theta)^*\}$.

PROOF. – We shall achieve the proof of this theorem under the hypothesis $1<\varkappa_1<\varkappa_0<+\infty$; the remaining cases may be treated with variants analogous to those pointed out in the preceding section.

We begin by proving that if $(x_0,y_0;\xi^0,\eta^1)\in\dot{T}^*(\boldsymbol{R}^{2n})$ and $\xi^0\neq-\eta^1$, then $(x_0,y_0;\xi^0,\eta^1)\notin WF_{\{\varkappa_1\}}(E(x,y))$. In fact let $\tilde{\Gamma}$ be a conic neighborhood of (ξ^0,η^1) in $\boldsymbol{R}^{2n}\setminus\{0\}$ such that

$$|\xi+\eta|\geqslant c(|\xi|+|\eta|)\qquad(\xi,\eta)\in\tilde{\Gamma}$$

for some positive constant c. Let $\varphi\in\mathfrak{D}^{\{\varkappa_1\}}$, $\psi\in\mathfrak{D}^{\{\varkappa_1\}}(H_\varepsilon)$, $\varphi(x_0)\neq0$, $\psi(y_0)\neq0$ and $(\xi,\eta)\in\tilde{\Gamma}$; from (3.1.4) and Theorem 1.2.2 we get then

$$|\widehat{\varphi\psi E(x,y)}(\xi,\eta)|\leqslant$$
$$\leqslant C\int_A \exp\left[-D|\xi-w|^{1/\varkappa_1}+D_1\gamma(1+|w|)^{1/\varkappa_0}-F|w+\eta|^{1/\varkappa_1}+\varepsilon\gamma(1+|w|)^{1/\varkappa_0}\right]d\zeta$$

with $\zeta=w-i\gamma(1+|w|)^{1/\varkappa_0}\theta$, for suitable positive constants C, D, D_1, F. We may assume $D=F$, and written $D=A+B$ we obtain

$$D[|\xi-w|^{1/\varkappa_1}+|w+\eta|^{1/\varkappa_1}]\geqslant A|\xi+\eta|^{1/\varkappa_1}+B[|\xi-w|^{1/\varkappa_1}+|w+\eta|^{1/\varkappa_1}];$$

then

$$|\widehat{\varphi\psi E(x,y)}(\xi,\eta)|\leqslant$$
$$\leqslant C\exp\left[(B-AC_1)(|\xi|^{1/\varkappa_1}+|\eta|^{1/\varkappa_1})\right]\int_A\exp\left[-2B|w|^{1/\varkappa_1}+(D_1+\varepsilon)\gamma(1+|w|)^{1/\varkappa_0}\right]d\zeta\,.$$

Choosing $B<DC_1/(1+C_1)$ we conclude this first part of the proof: the fibers in the wave front set are similar to those of the kernel of a convolution operator.

The statement on the base of the wave front set is essentially inspired to [1], as we have already emphasized. Let $\xi^0\in\boldsymbol{R}^n\setminus\{0\}$, $(x_0,y_0)\in\boldsymbol{R}^n\times H_\varepsilon$ such that $x_0-y_0\notin\Gamma(P_{m\xi^0},\theta)^*$; then there will exist $\eta^0\in\Gamma(P_{m\xi^0},\theta)$ and open neighborhoods U, V in $\boldsymbol{R}^n$ of x_0, y_0 respectively ($V\subset H_\varepsilon$) such that

$$(x-y)\cdot\eta^0<0\qquad x\in U,\quad y\in V\,.\tag{3.2.5}$$

Let again φ, ψ be two functions of class $\mathfrak{D}^{\{\varkappa_1\}}$, but with supports contained in U, V respectively, $\varphi(x_0)\neq0$, $\psi(y_0)\neq0$. Put $\psi_\eta(y)=\exp[-iy\cdot\eta]\psi(y)$; then

$$\widehat{\varphi\psi E(x,y)}(\xi,\eta)=(2\pi)^{-n}\int_A\hat{\varphi}(\xi-\zeta)\,\widehat{E\psi_\eta}(\zeta)\,d\zeta=I\,.$$

Split now $I = I_1 + I_2$, where in I_1 we perform the integration in the region $\{\zeta = w - i\gamma(1+|w|)^{1/\varkappa_0}\theta \in \Lambda;\ |w-\xi| > \delta|\xi|\}$, $\delta > 0$ small enough, and in I_2 consequently. An estimate of I_1 is achieved at once: in fact there will exist positive constants $D = A + B$, D_1 depending on φ such that

$$|I_1| \leqslant C \exp[-A(\delta|\xi|)^{1/\varkappa_1} + B|\xi|^{1/\varkappa_1}]\int_\Lambda \exp[-B|w|^{1/\varkappa_1} + \gamma(D_1+\varepsilon)(1+|w|)^{1/\varkappa_0}]\,dw$$

with (ξ, η) in a suitable conic neighborhood $\bar{\Gamma}$ of $(\xi^0, -\xi^0)$ such that $|(\xi,\eta)| \leqslant c'|\xi|$ if $(\xi,\eta) \in \bar{\Gamma}$, $c' > 0$. Choosing now $B < A\delta^{1/\varkappa_1}$ we get

$$|I_1| \leqslant C \exp[-B_1|\xi|^{1/\varkappa_1}]\,.$$

As to I_2, remark that if $|\xi|$ is sufficiently large the region of integration becomes $\{w - i\gamma(1+|w|)^{1/\varkappa_0}\theta;\ |w-\xi| < \delta|\xi|\} \subset \Lambda$; split again $I_2 = I_2^\infty + I_2^0$, where explicitly

$$\begin{aligned} I_2^\infty = \sum_{l=0}^{\infty} \sum_{j_1,\dots,j_l} (-1)^l (2\pi)^{-(l+1)n} \int_{|w-\xi|<\delta|\xi|} d\zeta \frac{\hat{\varphi}(\xi-\zeta)}{P_m(\zeta)} \times \\ \times \int_{\Lambda_l^\infty(\zeta)} d\zeta^1 \dots d\zeta^l \hat{q}_j(\zeta^1) \frac{p^j}{P_m}(\zeta-\zeta^1) \dots \hat{q}_{j_l}(\zeta^l) \times \\ \times \frac{p^{j_l}}{P_m}(\zeta-\zeta^1-\dots-\zeta^l)\hat{\psi}(\zeta+\eta-\zeta^1-\dots-\zeta^l) = (2\pi)^{-n} \int_{|w-\xi|<\delta|\xi|} \hat{\varphi}(\xi-\zeta)\widehat{E^\infty\psi_\eta}(\zeta)\,d\zeta \end{aligned}$$

with

$$\Lambda_l^\infty(\zeta) = \Big\{(\zeta^1, \dots, \zeta^l) \in \Lambda_l(\zeta);\ \zeta^j = w^j + ia(1+|w^j|)^{1/\varkappa_1}\theta,\ \sum_{j=1}^{l}(1+|w^j|)^{1/\varkappa_1} \geqslant (\delta|\xi|)^{1/\varkappa_1}\Big\}$$

$l = 1, 2, \dots,$ and

$$I_2^0 = (2\pi)^{-n} \int_{|w-\xi|<\delta|\xi|} \hat{\varphi}(\xi-\zeta)\widehat{E^0\psi_\eta}(\zeta)\,d\zeta$$

consequently. If M is the constant in (3.1.6) then

$$|\widehat{E^\infty\psi_\eta}(\zeta)| \leqslant C \exp[-M(\delta|\xi|)^{1/\varkappa_1}/4 + \varepsilon\gamma(1+|w|)^{1/\varkappa_0}]$$

which yields, for constants D, D_1 depending on φ

$$|I_2^\infty| \leqslant C \exp[(-M\delta^{1/\varkappa_1}/4 + D)|\xi|^{1/\varkappa_1}]\int \exp[-D|w|^{1/\varkappa_1} + (D_1+\varepsilon)\gamma(1+|w|)^{1/\varkappa_0}]\,dw;$$

if we suppose now $D < M\delta^{1/\varkappa_1}/4$, we can estimate I_2^∞ as I_1.

We pass then to I_2^0. If $|\xi| > C = C(\gamma, \delta)$, the region of integration of the set of variables $\zeta^1, \ldots, \zeta^l$ in the l-th term of the series defining $\widehat{E^0\psi_\eta}(\zeta)$ is actually given only by the inequality $\sum_{j=1}^{l} (1 + |w^j|)^{1/\varkappa_1} \leqslant (\delta|\xi|)^{1/\varkappa_1}$, since

$$\gamma\Big(1 + |w|^{1/\varkappa_0} + a \sum_{h=1}^{j} (1 + |w^h|)^{1/\varkappa_1}\Big) \leqslant \gamma(1 + (1+\delta)|\xi|)^{1/\varkappa_0} + a(\delta|\xi|)^{1/\varkappa_1} \leqslant$$
$$\leqslant C_0^{-1}(1 - 2\delta)|\xi| \leqslant C_0^{-1}\Big|w - \sum_{h=1}^{j} w^h\Big|\,.$$

Our aim is now to define $\widehat{E^0\psi_\eta}$ on the manifold

$$\{\zeta_t = w - i\gamma(1 + |w|)^{1/\varkappa_0}\theta - it|w|\eta^0;\ 0 \leqslant t \leqslant t_0,\ |w - \xi| < \delta|\xi|\};$$

to control the quotients p^j/P_m we shall make use of Lemma 3.2.4. Therefore let Γ be as in such lemma, $\Gamma_1 \subset \Gamma_2 \subset \Gamma$ conic neighborhoods of ξ^0 with the following properties:

(3.2.6) if $|v| \leqslant \delta|\xi|$, $\xi \in \Gamma_1$ then $\xi + v \in \Gamma_2$;

(3.2.7) if $|v| \leqslant \delta|w|/(1-\delta)$, $w \in \Gamma_2$ then $w + v \in \Gamma$.

This settled, it is easy to see that

$$(3.2.8)\qquad \left|\frac{p^j}{P_m}(\zeta_t - \zeta^1 - \ldots - \zeta^k)\right| =$$
$$= \left|\frac{p^j}{P_m}\Big(w - \sum_{h=1}^{k} w^h - i\Big[\gamma(1 + |w|)^{1/\varkappa_0} + a\sum_{h=1}^{k} (1 + |w^h|)^{1/\varkappa_1}\Big]\theta - it|w|\eta^0\Big)\right| \leqslant C\gamma^{-1}$$

for $0 \leqslant t \leqslant t_0'$, $\xi \in \Gamma_1$, $|\xi| > C_1(\gamma, \xi^0)$, ζ^j in the above specified region, $C = C(\xi^0, \eta^0)$; for simplicity of notations in the following we shall write t_0 instead of t_0'. From (3.2.1) and (3.2.8) we deduce

$$(3.2.9)\qquad |\hat{\varphi}(\xi - \zeta_t)\widehat{E^0\psi_\eta}(\zeta_t)| \leqslant C\exp\left[-D|\xi - \zeta_t|^{1/\varkappa_1} + \gamma(D_1 + \varepsilon)(1 + |w|)^{1/\varkappa_0}\right]$$

of $0 \leqslant t \leqslant t_0$, $\xi \in \Gamma_1$ and $\gamma > C_2(\xi^0, \eta^0)$; this last condition is needed to get the convergence of the series defining $\widehat{E^0\psi_\eta}(\zeta_t)$. By Stokes' theorem we can write $I_2^0 = I_2^0(t_0) + I_2^0(B)$, meaning

$$I_2^0(t_0) = (2\pi)^{-n} \int\limits_{|w-\xi|<\delta|\xi|} \hat{\varphi}(\xi - \zeta_{t_0})\widehat{E^0\psi_\eta}(\zeta_{t_0})\, d\zeta_{t_0}$$

$$I_2^0(B) = (2\pi)^{-n} \int\limits_{\substack{|w-\xi|=\delta|\xi| \\ 0\leqslant t\leqslant t_0}} \hat{\varphi}(\xi - \zeta_t)\widehat{E^0\psi_\eta}(\zeta_t)\, d\zeta_t\,.$$

From (3.2.9) then, for some constant $D' > 0$,

$$|I_2^0| \leqslant C \exp[-D'|\xi|^{1/\varkappa_1}] \qquad \xi \in \Gamma_1$$

since this is obvious as far as it concerns $I_2^0(B)$, while in the region of integration of $I_2^0(t_0)$ it results $|\xi - \zeta_{t_0}| \geqslant \delta'|\xi|$, δ' a suitable positive constant.

The proof is not yet complete, because the inequality (3.2.9) has been reached only for values of γ sufficiently large; therefore we must check that such a change of the path of integration Λ does not compromise the estimates given for I.

Let then $1 < \gamma_0 < \gamma_1$ and under the condition $|w - \xi| < \delta|\xi|$ consider the path $\zeta_s = w - i(\gamma_0 + s(\gamma_1 - \gamma_0))(1 + |w|)^{1/\varkappa_0}\theta$, $0 \leqslant s \leqslant 1$. Arguing as we did in the first part of the proof of (3.1.2) we get

$$\left|\frac{p^j}{P_m}(\zeta_s - \zeta^1 - \ldots - \zeta^k)\right| \leqslant C\gamma_0^{-1}$$

if $\gamma_1(1 + |w|)^{1/\varkappa_0} < C_0^{-1}|w|$, i.e. for sufficiently large values of $|\xi|$ (remember that $|w - \xi| < \delta|\xi|$). For such values of $|\xi|$

$$\left|\int\limits_{|w-\xi|<\delta|\xi|} \hat{\varphi}(\xi - \zeta_0)\widehat{E^0\psi_\eta}(\zeta_0)\,d\zeta_0 - \int\limits_{|w-\xi|<\delta|\xi|} \hat{\varphi}(\xi - \zeta_1)\widehat{E^0\psi_\eta}(\zeta_1)\,d\zeta_1\right| \leqslant$$

$$\leqslant \int\limits_{\substack{|w-\xi|=\delta|\xi|\\ 0\leqslant s\leqslant 1}} |\hat{\varphi}(\xi - \zeta_s)\widehat{E^0\psi_\eta}(\zeta_s)|\,|d\zeta_s| \leqslant$$

$$\leqslant C \int\limits_{\substack{|w-\xi|=\delta|\xi|\\ 0\leqslant s\leqslant 1}} \exp[-D|\xi - w|^{1/\varkappa_1} + \gamma_1(D_1 + \varepsilon)(1 + |w|)^{1/\varkappa_0}]\,|d\zeta_s| \leqslant C\exp[-D'|\xi|^{1/\varkappa_1}]\,.$$

This proves that the choice $\gamma > C_2(\xi^0, \eta^0)$ in (3.2.9) is not restrictive and moreover we have

$$E(x, y; \gamma_1) - E(x, y; \gamma_0) \in \mathcal{E}^{\{\varkappa_1\}}(\boldsymbol{R}^n \times H_\varepsilon)$$

taking into account the former estimates. The theorem is now completely proved.

With the help of the proof of Theorem 3.2.5 we can now prove another result of regularity.

Theorem 3.2.6. – Under the hypotheses of Theorem 3.2.1,

$$(x_0, y_0; \xi^0, -\xi^0) \notin WF^{(\varkappa(\xi^0))}(E(x, y)) \quad \text{for} \quad (x_0, y_0) \in \boldsymbol{R}^{2n}, \quad \xi^0 \in \boldsymbol{R}^n \setminus \{0\}\,.$$

Proof. – Suppose $1 < \varkappa_1 < \varkappa_0 < +\infty$, since if $\varkappa_0 = +\infty$ there is nothing to show. Let U, V be neighborhoods in $\boldsymbol{R}^n$ of x_0, y_0 respectively, $V \subset H_\varepsilon$, and

$\varphi \in \mathcal{D}^{\{\varkappa_1\}}(U)$, $\psi \in \mathcal{D}^{\{\varkappa_1\}}(V)$, $\varphi(x_0) \neq 0$, $\psi(y_0) \neq 0$. Looking back to the proof of the preceding theorem one sees that the condition about the base of the wave front set has been obtained by the suitable choice of two neighborhoods of x_0, y_0, for which held $x_0 - y_0 \notin \Gamma(P_{m\xi^0}, \theta)^*$, but this was needed only when we took I_2^0 into account. Then it is now sufficient to give an estimate of this term only, without any hypothesis on the base, since all the other integrals decrease exponentially with $|\xi|^{1/\varkappa_1}$.

Consider the path

$$\zeta_t = w - i\gamma\Big[(1+|w|)^{1/\varkappa_0} + t\big((1+|w|)^{1/\varkappa(\xi^0)} - (1+|w|)^{1/\varkappa_0}\big)\Big]\theta \qquad 0 \leqslant t \leqslant 1$$

and let Γ be a conic neighborhood of ξ^0 such that $n_j(\xi^0) = n_j(\Gamma)$ for every j (see Lemma 2.1.5 (iii)); choose then two cones $\Gamma_1 \subset \Gamma_2 \subset \Gamma$ as in the proof of Theorem 3.2.5. If $\nu = \deg p^j$ then

$$\left|\frac{p^j}{P_m}(\zeta_t - \zeta^1 - \ldots - \zeta^k)\right| \leqslant$$
$$\leqslant C(\xi^0)\left[\gamma(1+|w|)^{1/\varkappa(\xi^0)}\right]^{\nu-m}\left[\left|w - \sum_{h=1}^{k} w^h\right|(1+|w|)^{-1/\varkappa(\xi^0)}\right]^{n_\nu(\xi^0)} \leqslant C/\gamma$$

since

$$\sum_{h=1}^{k} |w^h| \leqslant \delta|w|(1-\delta) \qquad \text{and then} \qquad w - \sum_{h=1}^{k} w^h \in \Gamma \quad \text{if } \xi \in \Gamma_1,$$

$(\nu - m)/\varkappa(\xi^0) + n_\nu(\xi^0) - n_\nu(\xi^0)/\varkappa(\xi^0) \leqslant 0$ from the definition of $\varkappa(\xi^0)$, and

$$\left|\operatorname{Im}\zeta_t + a\sum_{h=1}^{k}(1+|w^h|)^{1/\varkappa_1}\theta\right| \leqslant C\left|w - \sum_{h=1}^{k} w^h\right| \quad \text{if } \xi \in \Gamma_1,$$

$|\xi|$ large enough.

We can therefore give meaning to $\widehat{E^0\psi_\eta}(\zeta_t)$, obtaining

$$|\widehat{E^0\psi_\eta}(\zeta_t)| \leqslant C\exp\left[\varepsilon|\operatorname{Im}\zeta_t| - F|w+\eta|^{1/\varkappa_1}\right] \tag{3.2.10}$$

with $0 \leqslant t \leqslant 1$ and F a positive constant depending on ψ. Once again by Stokes' theorem

$$I_2^0 = (2\pi)^{-n}\int\limits_{\substack{|w-\xi|=\delta|\xi| \\ 0\leqslant t\leqslant 1}} \hat\varphi(\xi - \zeta_t)\,\widehat{E^0\psi_\eta}(\zeta_t)\,d\zeta_t + (2\pi)^{-n}\int\limits_{|w-\xi|<\delta|\xi|} \hat\varphi(\xi-\zeta_1)\,\widehat{E^0\psi_\eta}(\zeta_1)\,d\zeta_1\,.$$

About the first integral we may proceed as we have done other times, and then get the exponential decrease with $|\xi|^{1/\varkappa_1}$; on the other hand, if $\varkappa(\xi^0) < +\infty$ the second one is estimated by

$$C \int\limits_{|w-\xi|<\delta|\xi|} \exp\left[-D|\xi-w|^{1/\varkappa_1}+(D_1+\varepsilon)\gamma(1+|w|)^{1/\varkappa(\xi^0)}-F|w+\eta|^{1/\varkappa_1}\right]dw$$
$$\leqslant C\exp\left[(D_1+\varepsilon)\gamma|\xi|^{1/\varkappa(\xi^0)}\right]\int\limits_{|\xi-w|<\delta|\xi|}\exp\left[-D|\xi-w|^{1/\varkappa_1}+(D_1+\varepsilon)\gamma(1+|\xi-w|)^{1/\varkappa(\xi^0)}\right]dw$$
$$\leqslant C\exp\left[D'|\xi|^{1/\varkappa(\xi^0)}\right]$$

and the thesis in this case is reached; if $\varkappa(\xi^0)+=\infty$ from Tarsky-Seidenberg lemma

$$|P_m(w-i\gamma\theta)|^{-1}\leqslant C(1+|w|)^b$$

for some constants C and b, $w\in\Gamma_2$, and then

$$|I_2^0|\leqslant C(1+|\xi|)^b\,.$$

The proof is now complete.

Corollary. - The kernel $E(x, y)$ of the parametrix E is an ultradistribution of class $\mathcal{D}^{(\varkappa_0)'}(\boldsymbol{R}^n\times H_\varepsilon)$.

The proof of Theorem 3.2.1 is now gained making use of Theorems 3.2.5 and 3.2.6 and recalling the rules of composition of wave front sets (Theorem 1.2.5).

3.3. *Semiglobal solvability. Examples.*

The parametrix constructed and studied in the previous sections is also useful to obtain results of semiglobal solvability for data with compact support, modulo analytic functions.

Let $P(x, D)=P_m(D)+\sum_j q_j(x)p^j(D)$ be the operator till now considered, $\{q_j\}\in\mathcal{E}^{\{\varkappa_1\}}$; the transposed operator of $P(x, D)$ is

$${}^tP(x, D)=(-1)^m P_m(D)+\sum_{j,\alpha}(-1)^{\nu_j}1/\alpha!\,D^\alpha q_j(x)\,p^{j(\alpha)}(D)$$

where $\nu_j=\deg p^j$. For the operator ${}^tP$ it results $\varkappa({}^tP,\xi)=\varkappa(P,\xi)$ for every ξ in $\boldsymbol{R}^n\setminus\{0\}$ (see [14], Lemma 2.1.4); in view of the problems are we dealing with, we shall assume that the coefficients have compact supports, and write again for simplicity of notations

$${}^tP(x, D)=(-1)^m P_m(D)+\sum_j q_j(x)\,p^j(D)$$

where $q_j\in\mathcal{D}^{\{\varkappa_1\}}$, $p^j\in V(P)$.

We shall denote by $E({}^tP)$ the left parametrix of the operator ${}^tP$, by $E({}^tP; x, y) \in \mathcal{D}^{(\varkappa_0)'}(\mathbf{R}^n \times H_\varepsilon)$ its kernel and by $R({}^tP)$ the rest. The transposed functional ${}^tE({}^tP)$, which we indicate here with E^*, is well defined by

$$\langle E^* v, \varphi\rangle = \langle v, E({}^tP)\varphi\rangle = (2\pi)^{-n}\int_A \hat{v}(-\zeta)\,\widehat{E({}^tP)\varphi}(\zeta)\,d\zeta$$

for $\varphi \in \mathcal{D}^{(\varkappa_0)}(H_\varepsilon)$, $v \in \mathcal{E}^{(\varkappa_0)'}(\mathbf{R}^n)$, has $E^*(x, y) = {}^tE({}^tP; x, y) \in \mathcal{D}^{(\varkappa_0)'}(H_\varepsilon \times \mathbf{R}^n)$ as kernel and rest $R^* = {}^tR({}^tP)$; furthermore

$$PE^* = I + R^*$$

where I is the identity operator. So E^* is a right parametrix.

THEOREM 3.3.1. – For the wave front set of the kernel $E^*(x, y)$ of the right parametrix E^* holds the result of Theorem 3.2.5; instead

$$WF_A(R^* v) = \phi \qquad v \in \mathcal{E}^{(\varkappa_0)'}(\mathbf{R}^n)\,.$$

PROOF. – The first statement is obvious in view of the preceding remarks. As regards the regularity of the rest, suppose at first $1 < \varkappa_1 < \varkappa_0 < +\infty$; let $U \in H_\varepsilon$ be a neighboroohd of x_0, $\{\chi_N\}$ a sequence in $\mathcal{D}^{(\varkappa_0)}(U)$, $\operatorname{supp}\chi_N \subset K$, K compact subset of U, such that for every $h > 0$ there exists a constant C_h satisfying

$$|D^{\alpha+\beta}\chi_N(x)| \leqslant C_h (CN)^{|\alpha|} h^{|\beta|} (\beta!)^{\varkappa_0} \qquad |\alpha| \leqslant N, \quad x \in K\,.$$

Setting as usual $\chi_{N,\xi}(\zeta) = \exp[-i\zeta\cdot\xi]\,\chi_N(\zeta)$ we have

$$\widehat{\chi_N R^*(v)}(\xi) = (2\pi)^{-n}\int_A \hat{v}(-\zeta)\,\widehat{R({}^tP)\,\chi_{N,\xi}}(\zeta)\,d\zeta \tag{3.3.2}$$

where $\zeta = w - i\gamma(1 + |w|)^{1/\varkappa_0}\theta$, and, as in the previous sections,

$$\widehat{R_l({}^tP)\,\chi_{N,\xi}}(\zeta) = \frac{1}{P_m(\zeta)} \sum_{j_1,\dots,j_l} (-1)^{l(m+1)+1} (2\pi)^{-ln} \times$$

$$\times \int_{A_l^Q(\zeta)} d\zeta^1 \dots d\zeta^l\, \hat{q}_{j_1}(\zeta^1)\,\frac{p^{j_1}}{P_m}(\zeta - \zeta^1) \dots \hat{q}_{j_l}(\zeta^l)\,p^{j_l}(\zeta - \zeta^1 - \dots - \zeta^l)\,\widehat{\chi}_N(\xi + \zeta - \zeta^1 - \dots - \zeta^l)$$

for $\zeta \in A$, $\zeta^j = w^j + ia(1 + |w^j|)^{1/\varkappa_1}\theta$. From (3.3.1) it easily follows that for every $h > 0$ there exists a constant C_h such that

$$|\widehat{\chi_N}(z)| \leqslant C_h (CN)^N (1 + |z|)^{-N} \exp[-h|z|^{1/\varkappa_0} + H_K(\operatorname{Im} z)]$$

for $z \in \boldsymbol{C}^n$, $N = 0, 1, 2, \ldots$, and then, recalling the geometry of the region of integration, splitting it into the subsets $|w + \xi| > \delta|w|$ and $|w + \xi| \leqslant \sigma|w|$ and proceeding as above (see for instance the proof of Theorem 3.1.2) we get

$$|\widehat{\chi_N R^*(v)}(\xi)| \leqslant C(CN)^N (1 + |\xi|)^{-N}$$

and then the thesis if $\varkappa_0 < +\infty$. The case $\varkappa_0 = +\infty$ is left to the reader.

The equation $P(x, D)u = f$ is therefore solvable for data f of class $\mathcal{E}^{\{\varkappa_1\}'}(\boldsymbol{R}^n)$, modulo analytic functions; the propagation of singularities of the solution u is clearly given by Theorem 3.2.1.

4. – Some examples.

Example 3.3.2. – Let $P_m(\xi) = \zeta_1^m - \xi_2^m - \ldots - \xi_n^m$, m positive and even; in any way we choose a polynomial Q of degree less than m, $P = P_m + Q$ is hyperbolic-elliptic with respect to the direction $\theta = (1, 0, \ldots, 0)$ by Theorem 10.4.10 in [5] and 1.1.3 (i) Then $\varkappa_0 = +\infty$.

Example 3.3.3. – Let $P = P_4 + Q_3 + Q$, where $P_4(\xi) = \xi_1^4 + \xi_1^2(\xi_2^2 + \xi_3^2) - \xi_3^4$, $Q_3(\xi) = \xi_2^3$, $\deg Q \leqslant 2$; P_m is hyperbolic-elliptic with respect to the direction $\theta = (1, 0, 0)$. If ξ is not parallel to the vector $\xi^0 = (0, 1, 0)$ one has

$$\sum_{j=1}^{3} |\partial P_4/\partial \xi_j(\xi)| \neq 0$$

so $\varkappa(\xi) = +\infty$; on the other hand, since $Q_3(\xi^0) \neq 0$, Lemma 2.1.5 (vii) yields $n_3(\xi^0) = 1$, and threfore $\varkappa(\xi^0) = 2$. In this case then $\varkappa_0$ assumes the least value allowed to a polynomial with principal part with at most double characteristics.

Example 3.3.4. – Let $P = P_5 + Q_4 + Q_3 + Q$, denoting $P_5(\xi) = \xi_1^2(\xi_1^3 + \xi_2^3 + \xi_3^3)$, $Q_4(\xi) = \xi_1^2(\xi_2^2 + \xi_3^2)$, $Q_3(\xi) = \xi_2 \xi_3^2$, and Q a polynoamil with $\deg Q \leqslant 2$; $\theta = (1, 1, 1)$ is a direction of hyperbolic-ellipticity for P_m. Since Q_4 is weaker than P_5, it follows that $n_4(\xi) = 0$ for every $\xi \in \boldsymbol{R}^n \setminus \{0\}$ and so $\varkappa(\xi) = +\infty$ if ξ is not parallel to $\xi^0 = (0, 1, -1)$. For this vector $Q_3(\xi^0) \neq 0$ and then $n_3(\xi^0) = 1$, which implies $\varkappa(\xi^0) = 3$.

Therefore $\varkappa_0 = 3$, while the least value for a polynomial with principal part P_5 is $3/2$.

Example 3.3.5. – Let $P(\xi) = \xi_1^m + \sum_{j=0}^{m-1} \xi_1^j P^j(\xi')$ with $\xi = (\xi_1, \xi')$, $\deg(P^j) = m_j$, $\max_{0 \leqslant j \leqslant m-1} m_j/(m - j) = p < 1$; P_m is hyperbolic with respect to the direction $\theta = (1, 0, \ldots, 0)$. For every $\xi \in \boldsymbol{R}^n \setminus \{0\}$ with $\xi_1 \neq 0$ it results $\varkappa(\xi) = +\infty$. The homogeneous term of degree ν is

$$Q_\nu(\xi) = \sum_{\substack{0 \leqslant j \leqslant \nu \\ j + m_j \geqslant \nu}} \xi^j P^j_{\nu-j}(\xi')$$

where $P^j_{\nu-j}$ stands for the homogeneous term of degree $\nu - j$ in P^j. For the indexes j in the above sum the inequality $j \geqslant (\nu - pm)/(1-p)$ is satisfied, from which $n_\nu(\xi) \leqslant p(m-\nu)/(1-p)$ and $\varkappa(\xi) \geqslant 1/p$ for every $\xi \in \boldsymbol{R}^n \setminus \{0\}$; on the other side this lower bound is really reached when $\xi = (0, \xi')$ and $\nu = m_j + j$, $m_j = p(m-j)$. In conclusion, $\varkappa_0 = 1/p$.

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