On the Postulation of a General Projection of a Curve in $\boldsymbol{P}^N$, $N \geqslant 4$ (*).

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Summary. – *Fix a curve X of genus g and $L \in \mathrm{Pic}^d(X)$. Let $\varphi_L(X)$ be the image of X through the complete linear system $H^0(X, L)$. Here we prove that a general projection of $\varphi_L(X)$ into $\boldsymbol{P}^N$ has maximal rank if either (a) $N \geqslant 4$, $0 \leqslant g \leqslant N-1$, $d \geqslant g+N$, or (b) $d \geqslant d(g, N)$ for suitable $d(g, N)$.*

Introduction.

Let C be a curve in $\boldsymbol{P}^N$. We say that C is of maximal rank if for every $k \geqslant 1$ the natural map of restriction:

$$r_C(k)\colon H^0(\mathcal{O}_{\boldsymbol{P}^N}(k)) \to H^0(\mathcal{O}_C(k)) \quad \text{is injective or surjective}.$$

The maximal rank conjecture (see [EH]) states that a general embedding of a general curve in $\boldsymbol{P}^N$ is of maximal rank (if $N = 3$ this is proved in [BE, 3, 4]). In [Ha] 4.3.4, R. Hartshorne raised the following projection problem:

« Let Z be a projectively normal curve in $\boldsymbol{P}^N$. Take n with $3 \leqslant n < N$ and let $C \leqslant \boldsymbol{P}^n$ be a general projection of Z. Is C of maximal rank? ».

Examples are known where the answer is negative. For example in [GP] it is proved that the general projection into $\boldsymbol{P}^3$ of a canonical curve of genus 5 or 6 is not of maximal rank (see also [Be, 6] for other examples). But it seems reasonable to hope that the projection problem has an affirmative answer if the genus of Z is low or the degree of Z is high. Here we consider only the case of general projections of curves in $\boldsymbol{P}^N$, $N \geqslant 4$ (if $N = 3$ see [BE, 5]). We prove the following results:

Theorem I. – Fix, N, d, g with $N \geqslant 4$, $0 \leqslant g \leqslant N-1$, $d \geqslant g + N$. Fix a curve X of genus g and $L \in \mathrm{Pic}^d(X)$. Let $\varphi_L(X)$ be the image of X through the complete linear system $H^0(X, L)$. Then a general projection of $\varphi_L(X)$ into $\boldsymbol{P}^N$ has maximal rank.

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THEOREM II. – Fix N, g with $N \geqslant 4$, $g \geqslant 0$. There exists an integer $d(g, N)$ such that for every $d \geqslant d(g, N)$, for every smooth curve X of genus g and every L in $\mathrm{Pic}^d(X)$, the general projection of $\varphi_L(X)$ into $\boldsymbol{P}^N$ has maximal rank.

Note that in both theorems $\varphi_L(X)$ is projectively normal ([M, 1]). The restriction $d \geqslant g + N$ in theorem I means that we are working with non special embeddings. Thus for $g \leqslant 3$, theorem I gives an affirmative answer to the projection conjecture if the projection is in $\boldsymbol{P}^N$, $N \geqslant 4$. In particular this proves the maximal rank conjecture in that range.

For the proofs of both theorems we use a result of [BE, 1] which gives examples of reducible curves which are flat limits of projections of $\varphi_L(X)$. By semicontinuity it is sufficient to construct such a reducible curve of maximal rank.

This is done, via many lemmata, by an inductive procedure, the so called « méthode d'Horace » (see [HH, 1, 2], [Hi, 1, 2], [BE, 1, ..., 5]). In particular the proofs of [HH, 1] showed us the right path. In sections $I, \ldots, X$ we give all the details of the proof of theorem I. In sections $A, \ldots, F$ we show how the constructions of the previous sections yield theorem II.

0. – Preliminaries, definitions.

We work over an algebraically closed field of characteristic zero.

0.1 DEFINITION. – If $S \subset \boldsymbol{P}^N$ is a set of distinct points we say that S is in linear general position (l.g.p.) if any $t \leqslant N + 1$ points of S span a linear subspace of dimension $t - 1$.

0.2. DEFINITION. – A tree of degree d in $\boldsymbol{P}^N$ is a connected reduced curve in $\boldsymbol{P}^N$, union of d lines, with arithmetic genus zero and only ordinary double points as singularities.

0.3. DEFINITION. – A bamboo is a tree, T, which looks like a chain: we can order the lines of T, say $L_1, \ldots, L_d$, in such a way that L_i and L_j intersect if and only if $|i - j| \leqslant 1$.

0.4. DEFINITION. – Let X be a curve and T be a bamboo. We say that T is linked to X at the point $p \in X$ if T intersects X only at p and quasi-transversally (i.e. T and X are smooth at p and have distinct tangents at p).

0.5. DEFINTION. – A final line of a tree, T, is a line of T which intersects one (and only one) irreducible component of T. If no confusion can arise we will denote by $(T)_f$ such a line.

0.6. DEFINITION. – Assume $Z = X \cup T$ where T is the union of k distinct bamboos, $T_1, \ldots, T_k$ and X is a curve. A final free line of Z (or of T in Z, or of T) is any final line of a T_i which intersects one (and only one) irreducible component of Z.

0.7. DEFINITION. – Let Y be a subscheme of the scheme Z. We denote by $\mathcal{J}_{Y,Z}$ the ideal sheaf of Y in $\mathcal{O}_Z$. The natural restriction map: $H^0(\mathcal{O}_Z(t)) \to H^0(\mathcal{O}_Y(t))$ is denoted by $r_{Y,Z}(t)$. If there is no danger of confusion we write more simply $\mathcal{J}_Y$, $r_Y(t)$.

0.8. DEFINITION. – Let Y be a subscheme of $\boldsymbol{P} = \boldsymbol{P}^N$ and denote by H a divisor of $\boldsymbol{P}$. The residual scheme of Y with respect to H, $\mathrm{Res}_H(Y)$, has for defining ideal the kernel of $\mathcal{O}_{\boldsymbol{P}} \xrightarrow{f} \mathcal{O}_Y(H)$ where f is the composite of the natural maps:

$$\mathcal{O}_{\boldsymbol{P}} \to \mathcal{O}_{\boldsymbol{P}}(H), \qquad \mathcal{O}_{\boldsymbol{P}}(H) \xrightarrow{\mathrm{res}} \mathcal{O}_X(H).$$

The two main facts used in this paper are «la méthode d'Horace» (see [Hi, 1] especially 2.1, [Hi, 2], [HH, 1, 2], [BE, 2, 3, 4]) and theorem 0 below. As far as we know «la méthode d'Horace» is the only method to construct curves of maximal rank in a systematic way. Using reducible curves and modulo some arguments of general position, one may work by induction. But as we are interested in smooth curves this procedure requires results on smoothability of reducible curves. In our particular case this is achieved by theorem 0. Let X be a smooth curve and $L \in \mathrm{Pic}^d(X)$ be a very ample line bundle. Consider the embedding, φ_L, of X into $\boldsymbol{P}(H^0(L)^\vee) \simeq \boldsymbol{P}^N$ given by the sections of L. Let H be a linear subspace of $\boldsymbol{P}^N$.

0.9. DEFINITION. – With notations as above $\mathrm{Pr}_d(L, H)$ is the closure in Hilb (H) of the set of general projections of $\varphi_L(X)$ into X.

0.10. REMARK. – Clearly $\mathrm{Pr}_d(L, H)$ is irreducible.

THEOREM 0. – Let X be a smooth, connected curve embedded in $\boldsymbol{P}^N$ with $\deg(X) = d$. Let $P_1, \dots, P_k$ be distinct points of X and $a_1, \dots, a_k$ be positive integers. Set $L \simeq \mathcal{O}_X(1)$, $r = \sum_{i=1}^{k} a_i$ and $M = L(\sum a_i P_i)$. Assume $h^0(X, M) = h^0(X, L) + r$. Let T_i, $1 \leqslant i \leqslant k$, be disjoint bamboos with $l(T_i) = a_i$. Assume that T_i is linked to X at P_i.

Then $X \cup T_1 \cup \dots \cup T_k$ is in $\mathrm{Pr}_{d+k}(M, \boldsymbol{P}^N)$.

0.11. REMARKS. – (1) We do not require that X is non degenerate. (2) This theorem was proved (but stated in a weaker form) in [BE, 1] (Prop. II.5). See also [B], § 7.

As said above when using the «méthode d'Horace» one has to solve some problems of general position. For instance when working in $\boldsymbol{P}^3$ the typical situation is like this: Q is a smooth quadric, Y is a reducible curve intersecting Q transversally, $S \subset Q$ is a set of s distinct points; a, b are some integers and we have to show that $Y \cup S$ (or a little deformation of it) satisfies:

$$h^0(J_{Y\cup S, Q}(a, b)) = \mathrm{Max}\left((a+1)(b+1) - s - 2\deg(Y), 0\right).$$

The results needed in this paper (when working in $\boldsymbol{P}^3$) are contained in [BE, 2], § 6.

I. - The first theorem.

The first par of this work is devoted to the proof of:

THEOREM 1. - Fix N, g, d with $N \geqslant 4$, $0 \leqslant g \leqslant N-1$, $d \geqslant g+N$. Let X be a curve of genus g and let L be a degree d line bundle on X. Denote by $\varphi_L(X)$ the image of X by the embedding:

$$\varphi_L\colon X \mapsto \boldsymbol{P}(H^0(X, L)^v)\,, \qquad x \mapsto \{s \in H^0(X, L)\colon s(x) = 0\}\,.$$

Then the general projection of $\varphi_L(X)$ into $\boldsymbol{P}^N$ has maximal rank.

I.1. REMARKS. - (*a*) Note that $h^1(X, L) = 0$ since $\deg(L) \geqslant 2g+1$.
Also if $d = g+N$ then $\varphi_L(X) \subseteq \boldsymbol{P}^N$ is projectively normal according to [M1].

(*b*) As a corollary of theorem 1 and [BE, 2] we obtain that the general curve in $\boldsymbol{P}^n$, $n \geqslant 3$, of genus g, $0 \leqslant g \leqslant 3$ and degree d ($d \geqslant g+n$) is of maximal rank.

(*c*) A similar theorem for projections into $\boldsymbol{P}^3$ is proved in [BE, 5].

(*d*) The conditions $0 \leqslant g \leqslant N-1$, $d \geqslant g+N$ seem not too bad according to the following fact: if X is an hyperelliptic curve of genus 3, none of its embeddings of degree 7 in $\boldsymbol{P}^3$ has maximal rank [BE, 6].

II. - The inductive hypothesis $H_{n,N}(g)$.

Theorem 1 is first reduced to the inductive hypothesis $H_{n,N}(g)$. To state $H_{n,N}(g)$ we need some preliminaries.

II.1. LEMMA. - Let $X \subseteq \boldsymbol{P}^N$ be a smooth, non degenerate curve of genus g, degree $g+N$ with $N \geqslant 4$ and $g \leqslant N-1$. Set $L := \mathcal{O}_X(1)$. Fix a point $p \in X$. Then for a general hyperplane, H, through p, $H \cap X$ consist of $g+N$ distinct points in linear general position if either:

(1) $g \leqslant N-2$ or $g = N-1$ and X is not hyperelliptic, or:

(2) $g = N-1$, X is hyperelliptic and $L(-p)$ is not g times the g_2^1 on X.

PROOF. - For general H through p, $X \cap H$ is reduced. It is enough to show that when (1) or (2) hold, the linear system corresponding to $V := H^0(X, L(-p))$ has no base points and gives a birational map. Indeed in this case, the 3-secant lemma shows that for any $N-2$ points $\{p_i\}$ in $(X \cap H)\setminus\{p\}$, $\{p_i\}$ and p span an hyperplane in H. If the lemma does not hold, by monodromy $(H \cap X)\setminus\{p\}$ is contained in a linear space R. By Bezout's theorem, a pencil of hyperplanes through R shows that $g = 0$. For $g = 0$, the lemma is clear.

Since $\deg\big(L(-p)\big)\geqslant 2g$, $L(-p)$ has no base points and it is not very ample if and only if $g=N-1$ and $L(-p)\simeq\omega_X(p_1+p_2)$ for some $p_i\in X$. Thus when (1) holds, V gives a birational map. If X is hyperelliptic and $g=N-1$, then V gives a birational map or has a rational curve as image. In the second case $L(-p)$ is g times the g_2^1.

II.2. LEMMA. – Let X be a smooth, non degenerate curve of genus g and degree $g+N$ in $\boldsymbol{P}^N$, $N\geqslant 5$, $g\leqslant N-1$. Set $L:=\mathcal{O}_X(1)$. Fix two distinct points, p,q of X. Assume one of the following conditions holds:

(1) $g\leqslant N-3$ or $g=N-2$ and X is not hyperelliptic,

(2) $g=N-2$, X is hyperelliptic and $L(-p-q)$ is not g times the g_2^1 on X,

(3) $g=N-1$ and $L(-p-q)$ is not isomorphic to $\omega_X(x)$ for some $x\in X$.

Then if H is a general hyperplane through p and q, $X\cap H$ consists of $g+N$ distinct points in linear general position.

PROOF. – The conditions give precisely that $L(-p-q)$ has no base points, i.e. the line $[p,q]$ is not a 3-secant to X, and that $X\cap H$ is reduced for general H. As in II.1, (1), (2) or (3) imply that $L(-p-q)$ gives a birational map.

II.3. THE CONDITION (*). – Given a curve $C\subseteq\boldsymbol{P}^N$ we will say that $P\in C$ (resp. $(P,R)\in C\times C$) satisfies ($\circ$) (resp. ($\circ\circ$)) if for a general hyperplane, H, through P (resp. P, R) $C\cap H$ is in linear general position.

A $(1,2)$-index of length k is a couple of integers (x,y) such that: $x+2y=k$.

Given an integer k and a $(1,2)$-index of length k, $\tau=(x,y)$, we will say that a collection of points of C: $(p_1,\dots,p_x;q_1,q_1';\dots;q_z,q_y')$ satisfies condition (*) for τ if any p_i satisfies ($\circ$), any (q_j,q_j') satisfies ($\circ\circ$).

Clearly we will often drop the index τ and just speak of the condition (*). The context will indicate what is meant.

II.4. THE NUMBERS $r(n,N,g)$, $q(n,N,g)$. – For $n\geqslant 1$, $N\geqslant 3$ and $g\geqslant 0$ we define integers $r(n,N,g)$, $q(n,N,g)$ by:

$$n\cdot r(n,N,g)-g+1+q(n,N,g)=\binom{N+n}{N},\qquad 0\leqslant q(n,N,g)\leqslant n-1\,.$$

II.5. THE INDUCTIVE HYPOTHESIS $H_{n,N}(g)$. – For $n\geqslant 1$, $N\geqslant 4$, $0\leqslant g\leqslant N-1$, we make the following statement:

$H_{n,N}(g)$: « There exist an integer k, a $(1,2)$-index of length k, τ, a sequence of integers $(a_1,\dots,a_k)$ such that: if $C\subseteq\boldsymbol{P}^N$ is a smooth connected curve of genus g, degree $g+N$ and if $P_1, \ldots, P_k$ are k distinct points on C satisfying condition (*) for τ, then there exist:

1) a curve, Y, of degree $r(n, N, g)$: $Y = C \cup B_1 \cup \ldots \cup B_k$, where the B_i's are bamboos of length a_i linked at C at the P_i's

2) an index j_0, $1 \leqslant j_0 \leqslant k$, and a line L intersecting the final free line of B_{j_0}

3) a set $S \subset L \setminus (Y \cap L)$ of $q(n, N, g)$ distinct points.

These data satisfy: $h^0(\boldsymbol{P}^N, \mathfrak{I}_{Y \cup S}(n)) = 0$ ».

III. – Reduction of theorem 1 to the inductive hypothesis $H_{n,N}(g)$.

III.1. PROPOSITION. – If $H_{n,N}(g)$ is true for $n \geqslant 1$, $N \geqslant 4$ and $0 \leqslant g \leqslant N-1$ then theorem 1 is true.

PROOF. – Take L in $\mathrm{Pic}^d(X)$ with $d > g+N$. We can write: $r(n, N, g) < d \leqslant r(n-1, N, g)$ for some $n \geqslant 1$. Let k, τ, $(a_1, \ldots, a_k)$ be given by $H_{n,N}(g)$. Choose integers $a_1', \ldots, a_k'$ such that $a_i' \geqslant a_i$, $g + N + \sum_1^k a_j' = d$, $a_{j_0}' > a_{j_0}$. Then take k distinct points on X: $P_1, \ldots, P_k$ and consider $\mathfrak{L} := L\left(-\sum_1^k a_j' P_j\right)$. We have $\deg(\mathfrak{L}) = g + N$ and $\mathfrak{L}$ is very ample. Let $C \subseteq \boldsymbol{P}^N$ be the image of X by the complete linear system $|H^0(X, \mathfrak{L})|$: Set $p_i := \varphi_{\mathfrak{L}}(P_i)$. We may assume that $(p_1, \ldots, p_k)$ satisfies condition (*) for τ (II.1, II.2). Now $H_{n,N}(g)$ applied to C and $(p_1, \ldots, p_k)$ give us a good curve for C of degree $r(n, N, g)$: $Y = C \cup B_1 \cup \ldots \cup B_k$ and a set S, of $q(n, N, g)$ points, such that $h^0(\boldsymbol{P}^N, \mathfrak{I}_{Y \cup S}(n)) = 0$. We may find a good curve for C: $Y' = C \cup B_1' \cup \ldots \cup B_k'$ with the B_i''s of length a_i', linked to C at the p_i's and such that $(Y \cup S) \subset Y'$. A fortiori: $h^0(\boldsymbol{P}^N, \mathfrak{I}_{Y'}(n)) = 0$. Finally since: $\mathcal{O}_C(1)\left(\sum_1^k a_i' p_i\right) \simeq L$ we conclude that Y' is in $\mathrm{Pr}_d(\varphi_L(X), \boldsymbol{P}^N)$ (see § 0, theorem 0).

Therefore a general element, Z, of $\mathrm{Pr}_d(\varphi_L(X), \boldsymbol{P}^N)$ satisfies: $h^0(\boldsymbol{P}^N, \mathfrak{I}_Z(n)) = 0$. In a similar way, using $H_{n+1,N}(g)$, we prove that a general element, Z', of $\mathrm{Pr}_d(\varphi_L(X), \boldsymbol{P}^N)$ satisfies: $h^1(\boldsymbol{P}^N, \mathfrak{I}_{Z'}(n+1)) = 0$. By irreducibility of $\mathrm{Pr}_d(\varphi_L(X), \boldsymbol{P}^N)$, a general element, Z'', of $\mathrm{Pr}_d(\varphi_L(X), \boldsymbol{P}^N)$ satisfies $h^0(\boldsymbol{P}^N, \mathfrak{I}_{Z''}(n)) = 0$ and $h^1(\boldsymbol{P}^N, \mathfrak{I}_{Z''}(n+1)) = 0$. Hence Z'' is of maximal rank ([M2], p. 99).

IV. – Reduction of $H_{n,N}(g)$ to $S_{n,N}(g)$ and $A_{n,N-1}(s)$.

As we have seen the proof of theorem 1 is reduced to the proof of $H_{n,N}(g)$, $n \geqslant 1$, $N \geqslant 4$, $0 \leqslant g \leqslant N-1$. In this section we perform a further reduction. We introduce two other inductive statements, $S_{n,N}(g)$, $A_{n,N-1}(s)$ and show that: $S_{n-1,N}(g) + A_{n,N-1}(s)$ imply $H_{n,N}(g)$.

The new statements are:

$\underline{S_{n\,N}(g)}$: »Assume $N\geqslant 4$, $n\geqslant 1$, $0\leqslant g\leqslant N-1$ and

$$r(n, N, g) - q(n, N, g) - g - N \geqslant n \tag{+}$$

Let $C \subseteq \boldsymbol{P}^N$ be a smooth connected curve of genus g and degree $g + N$. Let $(P_1, \dots, P_n)$ be n distinct points on C satisfying condition (*). Then there exists a curve X such that:

(*a*) $\deg(X) = r(n, N, g)$

(*b*) $X = Y \amalg D_1 \amalg \dots \amalg D_{q(n,N,g)}$ has $q(n, N, g) + 1$ connected components. The first one, Y, is a good curve for C: $Y = C \cup T_1 \cup \dots \cup T_n$, with the T_i's linked to C at the P_i's. The other connected components, $D_1, \dots, D_{q(n,N,g)}$, are disjoint lines.

(*c*) Finally: $h^0(\boldsymbol{P}^N, \mathfrak{J}_X(n)) = 0$".

IV.1. REMARK. – In view of (*a*), (*b*), the condition (+) is necessary. The following lemma shows that it is almost always satisfied.

IV.2. LEMMA. – Assume: (*a*) $N\geqslant 4$, $n\geqslant 3$, $0\leqslant g\leqslant N-1$ or (*b*) $N\geqslant 5$, $n=2$, $0\leqslant g\leqslant N-1$, or (*c*) $N=4$, $n=2$, $g=0$ or $g=2$.
Then: $r(n, N, g) - q(n, N, g) - g - N \geqslant n$.

PROOF. – Suppose

$$r(n, N, g) - q(n, N, g) - g - N < n \tag{x}$$

From the definition of $r(n, N, g)$, using (x) and $q(n, N, g)\leqslant n-1$, $g\leqslant N-1$, we get:

$$G(N, n) \geqslant 0 \tag{xx}$$

with:

$$G(N, n) = 2n(n-1) + N(2n-1) + 1 - \big((N+n)!/(N!\,n!)\big)\,.$$

Since $G(N+1, n) - G(N, n) = 2n - 1 - \binom{N+n}{n-1}$ we easily see that for $n\geqslant 2$ and $N\geqslant 4$: $G(N+1, n) < G(N, n)$. Furthermore:

$$G(4, n+1) - G(4, n) = 4\Big[n + 2 - \big((n+4)!/(4!(n+1)!)\big)\Big]\,.$$

Thus $G(4, n+1) < G(4, n)$, $n\geqslant 2$. Since $G(4, 3) < 0$ we get: $G(N, n) < 0$, $N\geqslant 4$, $n\geqslant 3$ which is a contradiction with (xx) and proves (*a*).

Part (*b*) is proved using (xx) directly. Finally (*c*) is checked just computing everything.

The second new statement is:

$A_{n,N-1}(s)$: « Assume $N-1 \leqslant s \leqslant 2(N-1)$, $n \geqslant 2$, $N \geqslant 4$. Let r, q', q'', s be integers such that:

$$rn + q' + q'' + s = \binom{N-1+n}{N-1} \text{ with } r \geqslant n\,, \quad 0 \leqslant q'' \leqslant n-1\,, \quad q' \geqslant n\,.$$

Let $P_0, P_1, \ldots, P_s$ be $s+1$ points in $\boldsymbol{P}^{N-1}$ in general linear position. Then there exist:

1) n disjoint bamboos whose union, W, has degree r. Furthermore there is a final line of a bamboo in W containing P_0,

2) a set, R, of $q'-n$ points in general linear position,

3) a set, S, of q'' points contained in a line, D, such that $W \cup D$ is the union of n disjoint bamboos containing P_0 in one of its final lines.

Finally if $X = W \cup R \cup S \cup \{P_1, \ldots, P_s\}$ then: $h^0(\boldsymbol{P}^{n-1}, \mathfrak{J}_X(n)) = 0$ ».

IV.3. REMARK. – Note that $A_{n,N-1}(s)$ is concerned with subschemes of $\boldsymbol{P}^{N-1}$ (and not of $\boldsymbol{P}^N$). We will need $A_{n,N-1}(g+N-1)$, $0 \leqslant g \leqslant N-1$.

So we take: $N-1 \leqslant s \leqslant 2(N-1)$. Later on (VII) we will allow a larger range for s.

IV.4. PROPOSITION. – Assume: (*a*) $N \geqslant 4$, $n \geqslant 4$, $0 \leqslant g \leqslant N-1$; or (*b*) $N \geqslant 5$, $n = 3$, $0 \leqslant g \leqslant N-1$: or (*c*) $N = 4$, $n = 3$, $g = 0$ or $g = 2$.

Then $S_{n-1,N}(g) + A_{n,N-1}(g+N-1)$ imply $H_{n,N}(g)$.

PROOF. – Let $C \subseteq \boldsymbol{P}^N$ be a smooth connected curve of genus g, degree $g+N$ and let $P_1, \ldots, P_{n-1}$, x_0 be n distinct points on C satisfying (*). By $S_{n-1,N}(g)$ (which is well defined according to IV.2) we are given a curve $X = Y \cup D_1 \cup \ldots \cup D_{q(n-1,N,g)}$, of degree $r(n-1, N, g)$, where $Y = C \cup T_1 \cup \ldots \cup T_{n-1}$ is a good curve for C. Furthermore the T_i's are linked to C at the P_i's, the D_j's are disjoint lines and $h^0(\boldsymbol{P}^N, \mathfrak{J}_X(n-1)) = 0$. Consider a general hyperplane H through x_0. We may assume that H intersects X in $r(n-1, N, g)$ distinct points in linear general position. Let $C \cap H = \{x_0, x_1, \ldots, x_{g+N-1}\}$. Now we apply $A_{n,N-1}(g+N-1)$ to $x_0, x_1, \ldots, x_{g+N-1}$ and with:

$$q' - n = r(n-1, N, g) - g - N - n - q(n-1, N, g) + 1\,, \qquad q'' = q(n, N, g)\,,$$
$$r = r(n, N, g) - r(n-1, N, g)\,, \qquad s = g + N - 1\,.$$

Note that, by IV.2, $q' - n \geqslant 0$. Also:

$$rn + q' + q'' + s = n \cdot r(n, N, g) - (n-1)\, r(n-1, N, g) + q(n, N, g) - q(n-1, N, g)$$

which is:

$$\binom{N-n}{N}-\binom{N+n-1}{N}=\binom{N-1+n}{N-1}.$$

The only thing it remains to check is: $r\geqslant n$, this follows from IV.5 below. So we get W, R, S which, together with $x_0, x_1, \ldots, x_{g+N-1}$, satisfy $A_{n,N-1}(g+N-1)$. Set $W:=B_0\cup B_1\cup\ldots\cup B_{n-1}$. Moving the lines in X (but keeping the intersections with C fixed) we may assume that $Y:=X\cup W$ is a good curve for C. More precisely we may assume that: $Y=C\cup\mathcal{C}_0\cup\mathcal{C}_1\cup\ldots\cup\mathcal{C}_{n-1}$ with $\mathcal{C}_i:=T_i\cup\cup B_i\cup D_i$, $1\leqslant i\leqslant q(n-1,N,g)$; $\mathcal{C}_i:=T_i\cup B_i$, $q(n-1,N,g)<i\leqslant n-1$, $\mathcal{C}_0=B_0$.

We may also assume $R=(Y\setminus W)\cap H$. Under these conditions we claim that: $h^0(P^N,\mathfrak{J}_{Y\cup S}(n))=0$. Indeed if $f\in H^0(\boldsymbol{P}^N,\mathfrak{J}_{Y\cup S}(n))$ then $f|H$ vanishes on $W\cup R\cup\cup S\cup\{x_1,\ldots,x_{g+N-1}\}$. By $A_{n,N-1}(g+N-1)$, $f|H\equiv 0$. So we get a form f' of degree $n-1$ vanishing on $\operatorname{Res}_H(Y\cup S)=X$. By $S_{n-1,N}(g)$, $f'\equiv 0$. Hence $f\equiv 0$ as wanted. Setting $a_i:=$ length $(\mathcal{C}_{i-1})$, $1\leqslant i\leqslant n$, this proves $H_{n,N}(g)$.

IV.4.1. REMARK. - If $N=4$ in the proof above we need $A_{n,3}(g+3)$, $n\geqslant 4$, $0\leqslant g\leqslant 3$ with $q'=r(n-1,4,g)-g-3-q(n-1,4,g)$. We note that $q'\geqslant 2n-3$ (see IV.4.2 below). This remark will be used in X where we will prove $A_{n,3}(g+3)$ with the extra condition $q'\geqslant 2n-3$ (see the statement $A'_{n,3}(g+3)$ of X). A similar remark applies for proposition VI.1.

IV.4.2. Let q' be as in IV.5, then $q'\geqslant 2n-3$

PROOF. - If $n\geqslant 5$ this will follow from $r:=r(n-1,4,g)\geqslant 3n+1$. Indeed by definition: $(n-1)r+1-g+q=\binom{n+3}{3}$ with $0\leqslant q\leqslant n-2$.

If $r\leqslant 3n$ we get: $n^4+6n^3-61n^2+54n+24\leqslant 0$ which is false if $n\geqslant 5$. The case $n=4$ is checked by computing everything.

IV.5. LEMMA. - For $N\geqslant 4$, $n\geqslant 3$, $0\leqslant g\leqslant N-1$ we have:

$$r(n,N,g)-r(n-1,N,g)\geqslant n\,.$$

PROOF. - Assume

$$r(n,N,g)\leqslant r(n-1,N,g)+n-1\,. \tag{x}$$

From the definitions of $r(n,N,g)$, $r(n-1,N,g)$ we get:

$$n\cdot r(n,N,g)-(n-1)\,r(n-1,N,g)+q(n,N,g)-q(n-1,N,g)=\binom{N-1+n}{N-1}.$$

Using (x), $q(n, N, g) \leqslant n-1$ and $q(n-1, N, g) \geqslant$ we get:

$$r(n-1, N, g) + n^2 - 1 \geqslant \binom{N-1+n}{N-1}. \tag{xx}$$

On the other hand: $r(n-1, N, g) \leqslant \left[N-2+\binom{N-n-1}{N}\right]/(n-1)$.

Combining with (xx): $F(N, n) \geqslant 0$ where

$$F(N, n) := N-2+(n-1)(n^2-1)+\left[(N+n-1)!(n-N(n-1))/n!N!\right].$$

It is easy to check that for $N \geqslant 4$ and $n \geqslant 3$ $F(N+1, n) < F(N, n)$. Since for $n \geqslant 3$: $F(4, n) < 0$ we get a contradiction and the lemma is proved.

IV.6. CHECK-POINT. – In order to get theorem 1 it remains to prove:

$S_{n,N}(g)$ for: $n \geqslant 3$, $N \geqslant 4$, $0 \leqslant g \leqslant N-1$

$n = 2$, $N \geqslant 5$, $0 \leqslant g \leqslant N-1$

$n = 2$, $N = 4$, $g = 0$ and $g = 2$

$A_{n,N-1}(g+N-1)$ for: $n \geqslant 3$, $N \geqslant 5$, $0 \leqslant g \leqslant N-1$;

$n \geqslant 4$, $N = 4$, $0 \leqslant g \leqslant 3$ (but with $q' \geqslant 2n-3$, see IV.4.1)

$n = 3$, $N = 4$, $g = 0$ and $g = 2$

$H_{n,N}(g)$ for: $1 \leqslant n \leqslant 2$, $N \geqslant 4$, $0 \leqslant g \leqslant N-1$

$n = 3$, $N = 4$, $g = 1$ and $g = 3$.

V. – Proof of some initial cases.

First we need some lemmas:

V.1. LEMMA. – Let N, g be integers such that: $N \geqslant 5$, $2 \leqslant g \leqslant N-1$. Denote by $P, \bar{P}, P_1, \ldots, P_{g-2}$, g points of $\boldsymbol{P}^{N-2}$ in linear general position. Let τ_1, τ_2 be integers satisfying: $\tau_1 + \tau_2 + g = N-1$, $\tau_i \geqslant 0$, $1 \leqslant i \leqslant 2$. Then there exist two disjoint bamboos, T_1, T_2, such that:

(a) $\deg(T_i) = \tau_i$

(b) if $\tau_1 > 0$ (resp. $\tau_2 > 0$) then $P \in (T_1)_f$ (resp. $\bar{P} \in (T_2)_f$).

(c) $h^0(\boldsymbol{P}^{N-2}, \mathfrak{I}_{T_1 \cup T_2 \cup \{P, \bar{P}, P_1, \ldots, P_{g-2}\}}(1)) = 0$, $0 \leqslant i \leqslant 1$.

PROOF. – Since $\tau_1+\tau_2+g=N-1$ it is enough to show

$$h^0(\mathfrak{I}_{T_1\cup T_2\cup\{P,\bar{P},P_1,\dots,P_{g-2}\}}(1))=0\,.$$

The proof is by induction on N. The initial case $N=5$ is easy. If $g\leqslant N-2$ we may assume $\tau_1>0$. Take an hyperplane H of $\boldsymbol{P}^{N-2}$ containing $P,\overline{P},P_1,..,P_{g-2}$. By induction we get two bamboos in H: T_1',T_2' with $\deg(T_1')=\tau_1-1$, $\deg(T_2)=\tau_2$. To conclude, just add a line $L\not\subset H$ to T_1'.

If $g=N-1$ then $\tau_1=\tau_2=0$ and the lemma is clear.

V.2. LEMMA. – Let N,g be integers such that: $N\geqslant 5$, $2\leqslant g\leqslant N-1$ and $g\equiv\binom{N}{N-2}$ (mod 2). Let $P,\overline{P},P_1,\dots,P_{g-2}$ be g points of $\boldsymbol{P}^{N-2}$ in linear general position. Then there exist two disjoint bamboos T_1,T_2 such that: $P\in(T_1)_f$, $\overline{P}\in(T_2)_f$ and

$$h^i(\boldsymbol{P}^{N-2},\mathfrak{I}_{T_1\cup T_2\cup\{P_1,\dots,P_{g-2}\}}(2))=0\,,\qquad 0\leqslant i\leqslant 1\,.$$

PROOF. – Induction on N. The initial case $N=5$ follows easily with $(t_1,t_2)=(1,3)$ if $g=2$, $(t_1,t_2)=(1,2)$ if $g=4$ $(t_i=\deg(T_i))$. Assume N is even. Let k be defined by: $2k+1=N-1-2g$.

Note that $g+2k+1\equiv\binom{N-1}{N-3}$ (mod 2). Also if $g\leqslant N-3$ we have: $2\leqslant g+2k+1\leqslant N-2$. Let $\mathcal{H}$ be an hyperplane of $\boldsymbol{P}^{N-2}$ containing none of the points $P,\overline{P},P_1,\dots,P_{g-2}$. By induction there exist in $\mathcal{H}$ two disjoint bamboos T_1',T_2' such that their union with $g+(2k+1)-2$ general points is not contained in a quadric. Set $t_i':=\deg(T_i')$. Out of $\mathcal{H}$ take two bamboos, $\overline{T}_i$, of degree t_i-t_i' with: $t_1+t_2=g+(2k+1)+t_1'+t_2'$; $t_i\geqslant t_i'-1$. We may assume that the folowing conditions hold: $T_i:=\overline{T}_i\cup T_i'$ is a bamboo of length t_i, $P\in(T_1)_f$, $\overline{P}\in(T_2)_f$. If $S:=(T_1\cap T_2)\cap(\mathcal{H}\setminus(T_1'\cup T_2'))$ then $h^i(\mathcal{H},\mathfrak{I}_{T_1'\cup T_2'\cup S}(2))=0$, $0\leqslant i\leqslant 1$ (note that $\#(S)=g+(2k+1)-2$). Now if $f\in H^0(\boldsymbol{P}^{N-2},\mathfrak{I}_{T_1\cup T_2\cup\{P_1,\dots,P_{g-2}\}}(2))$ then $f|\mathcal{H}=0$.

Hence $f=hh'$ where h is an equation of $\mathcal{H}$ and where h' is a linear form vanishing on

$$R:=\operatorname{Res}_{\mathcal{H}}(T_1\cup T_2\cup\{P_1,\dots,P_{g-2}\})=\overline{T}_1\cup\overline{T}_2\cup\{P_1,\dots,P_{g-2}\}\,.$$

Since $h^0(\boldsymbol{P}^{N-2},\mathcal{O}_R(1))=N-1$, we may assume (V.1) that $h'\equiv 0$ and therefore $f=0$. If $g=N-2$ the same proof works with $2k+1=-N+5$; if $g=N-1$ with $2k+1=-N+3$. Finally observe that, by construction: $h_0(\mathcal{O}_X(2))=\binom{N}{N-2}$, $X:=T_1\cup T_2\cup\{P_1,\dots,P_{g-2}\}$. Therefore $h^0(\mathfrak{I}_X(2))=0\Leftrightarrow h^1(\mathfrak{I}_X(2))=0$.

The case N odd is similar.

V.3. LEMMA. - Let N, g be integers such that: $N \geqslant 5$, $2 \leqslant g \leqslant N-1$ and $g \not\equiv \binom{N}{N-2}$ (mod 2). Let $P, \bar{P}, p_1, \ldots, p_{g-2}$ (resp. $Q, P, \bar{P}, P_1, \ldots, P_{g-2}$) be g (resp. $g+1$) points of $\boldsymbol{P}^{N-2}$ in linear general position. Then there exist three disjoint bamboos, T_1, T_2, L such that:

(*a*) L is a line

(*b*) $P \in (T_1)_f$, $\bar{P} \in (T_2)_f$ (resp. $Q \in L$)

(*c*) $h^i(\boldsymbol{P}^{N-2}, \mathfrak{J}_{T_1 \cup T_2 \cup L \cup \{P_1, \ldots, P_{g-2}\}}(2)) = 0$, $0 \leqslant i \leqslant 1$.

As in the previous lemma the proof is by induction on N. The initial case $N = 5$, $g = 3$ is the well known fact that three skew lines in $\boldsymbol{P}^3$ are contained in a unique quadric surface.

V.4. LEMMA. - Let P be a point of $\boldsymbol{P}^{N-2}$, $N \geqslant 5$, [resp. P, Q be two distinct points of $\boldsymbol{P}^{N-2}$].

Define τ by: $2\tau + 1 = \binom{N}{N-2}$ $\left[\text{resp. if } \binom{N}{N-2} \text{ is even: } 2\tau + 2 := \binom{N}{N-2}\right]$.
Then there exists a bamboo, T, of degree τ such that: $P \in (T)_f$ and $h^0(\boldsymbol{P}^{N-2}, \mathfrak{J}_T(2)) = 0$ [resp. $P \in (T)_f$, $Q \notin T$ and $h_0(\boldsymbol{P}^{N-2}, \mathfrak{J}_{T \cup \{Q\}}(2)) = 0$].

PROOF. - Note that $\binom{N}{N-2} = \binom{N-2}{N-4} + 2N - 3$. The proof is a double induction on N, according to the parity of N. The initial cases $N = 5$, $N = 6$ are easy. Assume $\binom{N}{N-2}$ is odd. Let H be an hyperplane of $\boldsymbol{P}^{N-2}$ and $\mathcal{K} \subset H$ a $(N-4)$-linear subspace such that $P \in \mathcal{K}$.

By induction there exist T' of degree $\tau - (N-1)$, contained in $\mathcal{K}$ and a point $Q \in \mathcal{K}$ such that: $P \in (T')_f$ and $h^0(\mathcal{K}, \mathfrak{J}_{T' \cup Q}(2)) = 0$. Let L be a line in H, not contained in $\mathcal{K}$, and such that $L \cup T'$ is a bamboo. Finally let $\mathcal{C}$ be a bamboo of degree $N-2$ such that $X := \mathcal{C} \cup L \cup T'$ is a bamboo, $Q \in (X)_f$ and $\mathcal{C}$ intersects $H \setminus (L \cup \mathcal{K})$ in $N-4$ general points $x_1, \ldots, x_{N-4}$. If $\varphi \in H^0(\mathfrak{J}_X(2))$ set $\tilde{\varphi} = \varphi | H$. Then $\tilde{\varphi} | \mathcal{K} = 0$, thus $\tilde{\varphi} = \mathcal{K} \cdot l$ where l is a linear form vanishing on $Z = L \cup \{x_1, \ldots, x_{N-4}\}$. Since $h^0(\mathcal{O}_Z(1)) = N-2$ we may assume $l \equiv 0$. Thus $\varphi = H \cdot l'$ with l' vanishing on $\mathcal{C}$. Again we may assume $l' = 0$ hence $\varphi = 0$.

If $\binom{N}{N-2}$ is even the proof is similar: by induction one gets T' in $\mathcal{K}$ of degree $\tau - (N-2)$. Then just add a point Q in H and a suitable $\mathcal{C}$ of degree $N-2$.

V.5. PROOF OF $H_{1,N}(g)$, $N \geqslant 4$, $0 \leqslant g \leqslant N-1$. - This follows directly from a theorem of Mumford [M1].

V.6. PROOF OF $H_{2,N}(g)$, $N \geqslant 5$, $2 \leqslant g \leqslant N-1$. - We distinguish two cases according to the value of $q(2, N, g)$.

V.6.1. Assume $q(2, N, g) = 0$.

We have $2\cdot r(2, N, g) - g + 1 = \binom{N+2}{2}$. Since $\binom{N+2}{2} = \binom{N}{N-2} + 2N + 1$ we have: $g \equiv \binom{N}{N-2} \pmod 2$. Let $C \subseteq \boldsymbol{P}^N$ be a smooth curve of degree $g + N$. Let H be a general hyperplane and set: $C \cap H = \{P_1, \ldots, P_{g+N}\}$. Take $\mathcal{H}$, an hyperplane of H, containing $P_1, \ldots, P_g$ but not $P_{g+1}, \ldots, P_{g+N}$. Now by V.2 there exist T_1, T_2, two bamboos in $\mathcal{H}$ such that:

$$P_{g-1} \in (T_1)_f\,, \quad P_g \in (T_2)_f \quad \text{and } h^i(\mathcal{H}, \mathfrak{J}_{T_1 \cup T_2 \cup \{P_1, \ldots, P_{g-2}\}}(2)) = 0\,, \quad 0 \leqslant i \leqslant 1\,.$$

Set $X = C \cup T_1 \cup T_2$. Since $\{P_{g+1}, \ldots, P_{g+N}\}$ is not contained in a hyperplane of H and since $C \subseteq \boldsymbol{P}^N$ is non degenerate, we get: $h^0(\boldsymbol{P}^N, \mathfrak{J}_X(2)) = 0$. Furthermore $p_a(X) = g$ and $h^0(\mathcal{O}_X(2)) = \binom{N+2}{N}$, thus $\deg(X) = r(2, N, g)$ and we are done.

V.6.2. Assume $q(2, N, g) = 1$.

This time we have $g \not\equiv \binom{N}{N-2} \pmod 2$. With the same notations as above we take for $\mathcal{H}$ an hyperplane of H containing $P_1, \ldots, P_{g+1}$ if $g < N - 1$ (resp. $P_1, \ldots, P_{N-1}$ if $g = N - 1$) but no other P_i.

From V.3 there exist P_1, T_2, L in $\mathcal{H}$ such that: $P_1 \in (T_1)_f$, $P_2 \in (T_2)_f$, $P_3 \in L$ and, if $g < N - 1$: $h^0(\mathcal{H}, \mathfrak{J}_{T_1 \cup T_2 \cup L \cup \{P_1, \ldots, P_{g-2}\}}(2)) = 0$.

If $g < N - 1$ we take one further point, Q, in H such that $Q, P_{g+2}, \ldots, P_{g+N}$ span H. Then $X := C \cup T_1 \cup T_2 \cup L \cup Q$ works. If $g = N - 1$ we take Q in $\mathcal{H}$ such that $h^0(\mathcal{H}, \mathfrak{J}_{T_1 \cup T_2 \cup L \cup \{Q, P_1, \ldots, P_{N-1}\}}(2)) = 0$. Again $X := C \cup T_1 \cup T_2 \cup L \cup Q$ works.

V.7. PROOF OF $H_{2,N}(1)$, $N \geqslant 5$. – Assume $q(2, N, 1) = 0$. For C a smooth elliptic curve of degree $N + 1$ let $\{P_1, \ldots, P_{N+1}\} = C \cap H$, H a general hyperplane. Let $\mathcal{H} \subset H$ be an hyperplane through P_{N+1} not containing P_i, $1 \leqslant u \leqslant N$. By V.4 there exists a bamboo, T, in $\mathcal{H}$ such that: $P_{N+1} \in (T)_f$ and $h^0(\mathcal{H}, \mathfrak{J}_T(2)) = 0$. Then $X := C \cup T$ works. If $q(2, N, 1) = 1$ the same proof (using V.4 again) works adding one further point, Q, in $\mathcal{H}$.

V.8. PROOF OF $H_{2,N}(0)$, $N \geqslant 5$. – Assume $q(2, N, 0) = 0$. Then $2r(2, N, 0) + 1 = \binom{N}{N-2} + 2N + 1$.

So $\binom{N}{N-2} \equiv 0 \pmod 2$. Let $C \subseteq \boldsymbol{P}^N$ be a rational normal curve. Let H be a general hyperplane and $C \cap H = \{P_1, \ldots, P_N\}$. In H take an hyperplane, $\mathcal{H}$, through P_1 and not containing any further P_i. Denote by L a line (in H) through P_2. Set $L \cap \mathcal{H} := Q$ (we may assume $Q \neq P_1$). By V.4 there exists a bamboo, T, of length τ $\left(\text{where } 2\tau + 2 = \binom{N}{N-2}\right)$ such that: $P_1 \in (T)_f$ and $h^0(\mathcal{H}, \mathfrak{J}_{T \cup Q}(2)) = 0$. Since

$R = L \cup \{P_3, \ldots, P_N\}$ has $h^0(\mathcal{O}_R(1)) = N$, we may assume $h^i(H, \mathcal{I}_R(1)) = 0$, $0 \leqslant i \leqslant 1$. So $X := C \cup L \cup T$ works.

If $q(2, N, 0) = 1$, the proof is similar.

V.9. PROOF OF $H_{2,4}(g)$, $0 \leqslant g \leqslant 3$. – These cases are easy, suppose for example $g = 3$. We have $r(2, 4, 3) = 8$, $q(2, 4, 3) = 1$. Let $C \subseteq \boldsymbol{P}^4$ be a smooth curve of genus 3, degree 7. Let H be a general hyperplane and $C \cap H = \{P_1, \ldots, P_7\}$. Let $\mathcal{H} \subset H$ be a plane containing P_1, P_2, P_3. Let L be a general line in $\mathcal{H}$ through P_1. Also let Q be a point of $\mathcal{H}$ not collinear with P_2, P_3 and not lying on L. Then $X := C \cup L \cup \{Q\}$ works. The other cases are similar.

V.10. PROOF OF $S_{2,N}(g)$ FOR $N \geqslant 5$, $2 \leqslant g \leqslant N - 1$. – It is identical to the proof of $H_{2,N}(g)$, $N \geqslant 5$, $2 \leqslant g \leqslant N - 1$ (see V.6) and therefore is omitted.

VII. PROOF OF $S_{2,N}(g)$ FOR $N \geqslant 5$, $0 \leqslant g \leqslant 1$. – As usual C denotes a smooth curve of genus g, degree $g + N$ in $\boldsymbol{P}^N$ and H a general hyperplane.

(*a*) $g = 0$, $q(2, N, 0) = 0$: let $\{P_1, \ldots, P_N\} = C \cap H$. Take $\mathcal{H}$ an hyperplane in H containing P_1, P_2 and no further P_i. In $\mathcal{H}$ consider two disjoint bamboos T'_i such that: $P_1 \in (T'_i)_f$, $1 \leqslant i \leqslant 2$, and $h^j(\mathcal{H}, \mathcal{I}_{T'_1 \cup T'_2}(2)) = 0$, $0 \leqslant j \leqslant 1$ (see V.2). Then let D be a line in H, not contained in $\mathcal{H}$, and intersecting the other final line of T'_1: Finally set: $T_1 = T'_1 \cup D$, $T_2 = T'_2$. We easily get that $X := C \cup T_1 \cup T_2$ satisfies $S_{2,N}(0)$.

(*b*) $g = 1$, $q(2, N, 1) = 0$: as above but this time $\mathcal{H}$ contains three points of $C \cap H$.

(*c*) $g = 0$, $q(2, N, 0) = 1$: as in (*a*) but in $\mathcal{H}$ we take two bamboos T'_i and a line L such that: $P_i \in (T'_i)_f$, $1 \leqslant i \leqslant 2$, $h^j(\mathcal{H}, \mathcal{I}_{T'_1 T \cup'_2 \cup L}(2)) = 0$, $0 \leqslant j \leqslant 1$ (see V.3).

(*d*) $g = 1$, $q(2, N, 1) = 1$: as in (*c*) but with $\mathcal{H}$ containing three points of $C \cap H$.

V.12. PROOF OF $S_{2,4}(g)$, $0 \leqslant g \leqslant 3$. – If $g = 1$ or 3 then $S_{2,4}(g)$ is meaningless. Finally the cases $N = 4$, $g = 0$ or 2 are easy.

VI. – Reduction of $S_{n,N}(g)$ to $A_{n,N}(s)$.

VI.1. PROPOSITION. – For $N \geqslant 4$ and $n \geqslant 3$:

$$S_{n-1,N}(g) \text{ and } A_{n,N-1}(g + N - 1) \text{ imply } S_{n,N}(g)$$

PROOF. – Let $C \subseteq \boldsymbol{P}^N$ be a smooth connected curve of genus g and degree $g + N$ and let $P_1, \ldots, P_n$ be n points of C satisfying (*) (see II.3). By $S_{n-1,N}(g)$ for $P_2, \ldots, P_n$ we get a curve X of degree $r(n-1, N, g)$ of the form: $X = Y \cup D_1 \cup \ldots \cup D_{q(n-1,N,g)}$. Let H be a general hyperplane through P_1. In H we consider a curve Z of degree

$r(n, N, g) - r(n-1, N, g)$ which is the disjoint union of n bamboos (see IV.2) and such that $X \cup Z$ is a curve of type $(q(n, N, g), g)$ (i.e. the configuration required by $S_{n,N}(g)$). Furthermore we may assume that exactly one of the bamboos of Z intersects C (at P_1). All this is possible since Y has at least $n-1$ free lines. Indeed if $q(n-1, N, g) > q(n, N, g)$, all the bamboos of Z intersect X at free lines except the one intersecting C at P_1), $q(n-1, N, g) - q(n, N, g)$ of them intersect two bamboos at free lines, at most one of these two bamboos intersects C. If $q(n-1, N, g) \leqslant q(n, N, g)$, exactly $q(n, N, g) - q(n-1, N, g)$ of the bamboos of Z do not intersect X, one intersects C at P_1 and $n - q(n, N, g) + q(n-1, N, g) - 1$ of them intersect X at free lines.

Now $Q' = (X \setminus C) \cap (H \setminus Z)$ is a set of:

$$q' - n = r(n-1, N, g) - q(n-1, N, g) + q(n, N, g) + 1 - g - N - n \text{ points}.$$

By semi-continuity we may assume that: $C \cap H$, Z and Q' satisfy $A_{n,N-1}(g + N - 1)$. Note that: $n \cdot \deg(Z) + g + N - 1 + q' = \binom{n+N-1}{N-1}$.

Since $\operatorname{Res}_H(X \cup Z) = X$, by $S_{n-1,N}(g)$ we get: $h^0(\boldsymbol{P}^N, \mathfrak{J}_{X\cup Z}(n)) = 0$ and $X \cup Z$ satisfies $S_{n,N}(g)$.

VI.2. REMARKS. - (1) As in IV.4 we notice that in the proof above we need $A_{n,3}(g+3)$ with $q' = r(n-1, 4, g) - q(n-1, 4, g) + q(n, 4, g) + 1 - g - 4$ and that $q' \geqslant 2n - 3$ (see IV.4.2).

(2) To get theorem 1 it remains to prove:

$$A_{n\,N-1}(g + N - 1)\,, \quad N \geqslant 5\,, \quad n \geqslant 3\,, \quad 0 \leqslant g \leqslant N-1$$

$$A_{n,3}(g+3)\,, \quad n \geqslant 4\,, \quad 0 \leqslant g \leqslant 3 \ (\text{with } q' \geqslant 2n-3)$$

$$A_{3,3}(g+3)\,, \quad g = 0 \text{ or } 2; \qquad H_{3,4}(g)\,, \quad g = 1 \text{ or } 3\,.$$

VII. - The statements $\bar{A}_{n,N}(s)$ and $P_{n,N}(s)$.

In order to prove $A_{n,N}(s)$ we first define a more general form of $A_{n,N}(s)$ (denoted by $\bar{A}_{n,N}(s)$) and we also introduce a new inductive statement, $P_{n,N}(s)$. Then we show that $P_{2,N}(s)$, $N \geqslant 4$, and $\bar{A}_{n,3}(0)$, $n \geqslant 3$, yield $\bar{A}_{n,N}(s)$, $n \geqslant 3$, $N \geqslant 4$. Finally it is obvious that $\bar{A}_{n,N}(s)$ implies $A_{n,N}(s)$, $n \geqslant 3$, $N \geqslant 4$.

The proof of $P_{2,N}(s)$, $N \geqslant 4$, is fairly easy (see VIII). Instead the proof of $\bar{A}_{n,3}(0)$, $n \geqslant 3$, is quite tricky and therefore is postponed to the next section (see IX: initial cases in $\boldsymbol{P}^3$).

The more general form of $A_{n,N}(s)$ we will consider is:

$\underline{\bar{A}_{n,N}(s)}$: $N \geqslant 3$, $n \geqslant 2$, $s \geqslant 0$:

Let $P_0, P_1, \ldots, P_s$ be $s+1$ points in $\boldsymbol{P}^N$ in general linear position. Let r, q', q'' be integers such that:

(a) $nr + q' + q'' + s = \binom{N+n}{N}$

(b) $r > 0,\ 0 \leqslant q'' \leqslant n-1$.

(c) If $N \geqslant 4$: $q' \geqslant \min(n, r)$ and if $N = 3$: $q' \geqslant \operatorname{Max}[\min(n, r), 2n-3]$.

Then there exists a subscheme X of $\boldsymbol{P}^N$ such that: $h^0(\boldsymbol{P}^N, \mathfrak{J}_X(n)) = 0$ and X is the disjoint union of the following subschemes:

(1) $P_i,\ 0 \leqslant i \leqslant s$

(2) $q' - \min(n, r)$ points of $\boldsymbol{P}^N$

(3) the union, W, of $\min(r, n)$ disjoint bamboos with $\deg(W) = r$, P_0 being contained in a final line of W

(4) q'' collinear points on a line D such that $W \cup D$ is the union of $\min(r, n)$ disjoint bamboos and contains P_0 in a final line.

To define $P_{2,N}(s)$ we need some preliminaries.

VII.1. DEFINITION. – We define numbers $a(n, N, s)$, $b(n, N, s)$ by:

$$n \cdot a(n, N, s) + s + b(n, N, s) = \binom{N+n}{N},$$

$$n \leqslant b(n, N, s) \leqslant 2n - 1\,, \qquad n \geqslant 2\,, \qquad N \geqslant 3,\ 0 \leqslant s \leqslant 2N\,.$$

VII. LEMMA. – (1) Assume $N \geqslant 5$, $n \geqslant 2$ or $N = 4$, $n \geqslant 3$ or $N = 3$, $n \geqslant 6$. Then for $0 \leqslant s \leqslant 2N$, $a(n, N, s)$ and $b(n, N, s)$ are well defined and satisfy: $a(n, N, s) \geqslant b(n, N, s)$.

(2) The same conclusion holds if $N = 4$, $n = 2$, $0 \leqslant s \leqslant 7$ or $N = 3$ and: $n = 5$, $0 \leqslant s \leqslant 6$; $n = 4$, $0 \leqslant s \leqslant 6$, $s \neq 4$; $n = 3$, $0 \leqslant s \leqslant 5$, $s \neq 3$; $n = 2$, $0 \leqslant s \leqslant 4$, $s \neq 3$.

PROOF. – Dividing $\binom{N+n}{N} - n - s$ by n we get $a(n, N, s)$ and $\bar{q}$, $0 \leqslant \bar{q} \leqslant n-1$. Then we put $b(n, N, s) = n + \bar{q}$. In order to do this we need:

$$(*) \qquad \binom{N+n}{N} - n - s \geqslant n\,.$$

Furthermore we want:

$$(**) \qquad a(n, N, s) \geqslant b(n, N, s)\,.$$

Claim: if

$$(+) \qquad 0 \leqslant s \leqslant \binom{N+n}{N} - (n+1)(2n-1);$$

then (*) and (**) hold.

Proof of the claim. First $\binom{N+n}{N} - (n+1)(2n+1) \leqslant \binom{N+n}{N} - 2n$ so $(+)$ implies (*).

Now suppose $a(n, N, s) < b(n, N, s)$. Then $\binom{N+n}{N} - s < (n+1)b(n, N, s) \leqslant$ $\leqslant (n+1)(2n-1)$ in contradiction with $(+)$.

So if we want $s \leqslant 2N$ it is enough to check: $F(N, n) \leqslant 0$ where

$$F(N, n) = 2N + (n+1)(2n-1) - \binom{N+n}{N}.$$

We first have:

$$F(N+1, n) - F(N, n) = 2 - \binom{N+n}{n-1}.$$

Hence $F(N+1, n) < F(N, n)$, $N \geqslant 3$, $n \geqslant 2$. Then one easily checks that: $F(3, n) < 0$, $n \geqslant 6$; $F(4, n) < 0$, $n \geqslant 3$ and $F(N, 2) < 0$, $N \geqslant 5$. This proves (1).

Part (2) is checked by direct computations.

VII.3. DEFINITION. – For $N \geqslant 3$, $n \geqslant 2$, we define

$$S(N, n) = \{s \in \boldsymbol{N}\colon 0 \leqslant s \leqslant 2N \text{ and } a(n, N, s) \geqslant b(n, N, s)\}\,.$$

Then we set:

$\underline{P_{n,N}(s)}$: For $N \geqslant 4$, $n \geqslant 2$ and $s \in S(N, n)$.

Let $P_0, P_1, \ldots, P_s$ be $s+1$ points of $\boldsymbol{P}^N$ in general linear position. Then there exists (Y, W) such that:

(1) W is the union of $b(n, N, s)$ disjoint bamboos of $\boldsymbol{P}^N$. Furthermore P_0 is contained in a final line of W and $\deg(W) = a(n, N, s)$.

(2) $Y = W \cup \{P_1, \ldots, P_s\}$ and $h^0(\boldsymbol{P}^N, \mathfrak{I}_Y(n)) = 0$.

VIII. – Proof of $P_{2,N}(s)$.

VIII..1 PROPOSITION. – For $N \geqslant 4$ and $s \in S(N, 2)$, $P_{2,N}(s)$ is true.

PROOF. – By induction on N. Assume $N \geqslant 5$ and $P_{2,N-1}(s)$ true. First assume $s \geqslant N+1$. Take an hyperplane H containing P_i for $i = 0$, $i > N+1$. In H take W', of degree $a(2, N-1, s')$ which satisfies $P_{2,N-1}(s')$ for P_i, $i = 0$, $i > N+1$ $(s' = s - N - 1)$.

We have:

$$2\big(a(2, N, s) - a(2, N-1, s-N-1)\big) = -\,b(2, N, s) + b(2, N-1, s-N-1)\,.$$

So $a(2, N, s) = a(2, N-1, s-N-1)$, $b(2, N, s) = b(2, N-1, -N+s-1)$ and we are done because any linear form of $\boldsymbol{P}^N$ vanishing on $P_1, \ldots, P_{N+1}$ is identically zero. If $s \leqslant N-1$ we take for H an hyperplane containing no P_i. Outside of H we consider a bamboo, T, of degree $N-s$, containing P_0 and such that T and the P_i's span $\boldsymbol{P}^N$. In H we have $N-s$ points $Q_0, Q_1, \ldots, Q_{N-s-1}$ of $T \cap H$. We take W in H satisfying $P_{2,N-1}(N-s-1)$ for the Q_i's. We conclude as usual (note that $b(2, N, s) = b(2, N-1, N-s-1)$). Finally if $s = N$ we take for H an hyperplane containing P_s and no other P_i. Then we repeat the construction done for $s = N-1$.

The starting case $N = 4$ can be checked in the same way, reducing to elementary assertions in $\boldsymbol{P}^3$.

IX. - Reduction of $\bar{A}_{n,N}(s)$ and $P_{n,N}(s)$ to the initial cases in $\boldsymbol{P}^3$.

The next two lemmas prove part of the $\bar{A}_{n,N}(s)$.

IX.1. LEMMA. - $P_{n-1,N}(s)$ and $\bar{A}_{n,N-1}(0)$ imply $P_{n,N}(s)$, $n \geqslant 3$, $N \geqslant 4$, $0 \leqslant s \leqslant 2N$.

IX.2. LEMMA. - $P_{n-1,N}(s)$ and $\bar{A}_{n,N-1}(0)$ imply $\bar{A}_{n,N}(s)$, $n \geqslant 3$, $N \geqslant 4$, $0 \leqslant s \leqslant 2N$.

PROOF OF IX.1. - Let $P_0, P_1, \ldots, P_s$ be $s+1$ points in general linear position in $\boldsymbol{P}^N$. By $P_{n-1,N}(s)$ we are given (Y, W) where W has degree $a(n-1, N, s)$ and is the union of $b(n-1, N, s)$ bamboos (for this we need $s \in S(N, n-1)$ so the case $N = 4$, $n = 3$, $s = 8$ has to be handled separately). Let H be a general hyperplane. In H consider the union, T, of n disjoint bamboos with: $\deg(T) = r = a(n, N, s) - a(n-1, N, s)$. By IX.3: $r \geqslant n$. Furthermore we require that $T \cup W$ is the union of $b(n, N, s)$ disjoint bamboos. This is possible because:

$$|b(n, N, s) - b(n-1, N, s)| \leqslant n \qquad \text{and } b(n-1, N, s) \geqslant n-1\,.$$

By $\bar{A}_{n,N-1}(0)$ we may assume that the union of T and $d := \binom{N-1+n}{N-1} - rn - n$ points of H in general position is not contained in a degree n hypersurface of H. Now $W \cap (H \setminus T)$ consists precisely in

$$d = a(n-1, N, s) - b(n-1, N, s) + b(n, N, s) - n$$

points that we may assume in general position.

By semi-continuity we may suppose that $T \cup (W \cap H)$ satisfies $\bar{A}_{n,N-1}(0)$. If $f \in H^0(\boldsymbol{P}^N, \mathfrak{J}_{Y\cup T}(n))$ then $f|H \equiv 0$ by $\bar{A}_{n,N-1}(0)$. Thus f is divided by the equation z of H. Since f/z vanishes on Y, by $P_{n-1,N}(s)$, we get $f \equiv 0$ as wanted. In a similar way we prove $P_{2,4}(7) + A_{3,3}(0)$ implies $P_{3,4}(8)$ taking for H an hyperplane through P_8 (and with $P_i \notin H$, $0 \leqslant i \leqslant 7$).

IX.3. LEMMA. – For $N \geqslant 4$, $n \geqslant 3$, $s \in S(N, n) \cap S(N, n-1)$ we have: $a(n, N, s) - a(n-1, N, s) \geqslant n$. Also: $a(3, 4, 8) - a(2, 4, 7) \geqslant 3$.

PROOF. If $a(n, N, s) \leqslant a(n-1, N, s) + n - 1$ then from the definition we get:

$$\binom{N+n-1}{N-1} \leqslant a(n-1, N, s) + n(n-1) + b(n, N, s) - b(n-1, N, s) .$$

Since:

$$a(n-1, N, s) < \binom{N+n-1}{N} / (n-1)$$

and $b(n, N, s) - b(n-1, N, s) \leqslant n$, we get $F(N, n) < 0$, where

$$F(N, n) = [(N+n-1)!(Nn - N - n)]/(N!n!) - n^2(n-1) .$$

It is easy to check that:

$$F(N+1, n) > F(N, n) , \quad N \geqslant 4 , \quad n \geqslant 3 \quad \text{and } F(4, n) > 0 , \quad n \geqslant 3 .$$

This proves the first part of the lemma. The last statement is checked directly.

PROOF OF IX.2. – Let $P_0, P_1, \ldots, P_s$ be the data of $\bar{A}_{n,N}(s)$. Put

$$x := a(n-1, N, s) + |n - b(n-1, N, s)| \quad \text{and } r' = r - a(n-1, N, s) .$$

First assume: $r' > 0$, $r \geqslant x$ and $s \in S(N, n-1)$. Let (Y, W) be given by $P_{n-1,N}(s)$ for the P_i's. Take a general hyperplane H and consider in H the union, T, of $k := \min(n, r - a(n-1, N, s))$ disjoint bamboos, with $\deg(T) = r'$ and such that $W \cup T$ is the union of n disjoint bamboos. This is possible by the assumption $r \geqslant x$. Then in H add $q' - \min(n, r)$ general points and q'' points on a line D such that $W \cup T \cup D$ satisfies (4) of $\bar{A}_{n,N}(s)$. Furthermore in H there are the $a(n-1, N, s) - k + n - b(n-1, N, s)$ points of $W \cap (H \setminus T)$, we may assume these points in general position. So we have $\tilde{q}' - k$ points in general position in H with $\tilde{q}' = q' + a(n-1, N, s) - b(n-1, N, s) + n - \min(n, r)$. We want to apply $\bar{A}_{n,N-1}(0)$ to the union of these $\tilde{q}' - k$ points, the q'' points on D and T. For this we have to check: $nr' + \tilde{q}' + q'' \geq \binom{N-1+n}{N-1}$. From the definitions we get: $nr' + q' + q'' - b(n-1, N, s) = \binom{N-1+n}{N-1}$, since $a(n-1, N, s) + n - \min(n, r) \geq 0$, we are done.

The same proof works if $r' = 0$, $r \geq x$, adding only points in H. If $s \notin S(N, n-1)$ then $N = 4$, $n = 3$, $s = 8$: just consider an hyperplane H containing one of the P_i's.

Now assume $r < x$. Arguing as above we prove $\bar{A}_{n,N}(s)$ for: $r = x$, q'', $q' - n(x-r)$. We get: $\deg(W) = x$, union of n disjoint bamboos, P_0 in a final line of W; a set, Q', of $q' - n(x-r+1)$ points of $\boldsymbol{P}^N$; a set, Q'', of q'' points on a suitable line such that: $X := W \cup \{P_i\} \cup Q' \cup Q''$ satisfies: $h^0(\boldsymbol{P}^N, \mathfrak{I}_X(n)) = 0$.

Note that:

$$nx + q'' + q' - n(x-r) + s = nr + q' + q'' + s = \binom{N+n}{N}.$$

Note also that: $\min(n, x) = n$(IX.3) and: $q' \geq n(x-r+1)$ (IX.4). Now we take $W' \subset W$, $\deg(W') = r$, W' the union of $\min(r, n)$ bamboos. We may assume that $X' = X \setminus (W \setminus W')$ satisfies the conditions of $\bar{A}_{n,N}(s)$. The natural map: $r_{X'}(n)$: $H^0(\mathcal{O}_{\boldsymbol{P}^N}(n)) \to H^0(\mathcal{O}_{X'}(n))$ is surjective, hence we just have to add $n(x-r)$ suitable points to X' to prove $\bar{A}_{n,N}(s)$ for r, q', q''.

IX.4. LEMMA. - With the notations of the proof of IX.2 if $r < x$ then $q' \geq n(x-r+1)$.

PROOF. - We have $x \leq a(n-1, N, s) + n - 3$ hence:

$$x - r + 1 \leq a(n-1, N, s) - r + n - 2\,. \tag{*}$$

From the definitions we get:

$$n(r - a(n-1, N, s)) + a(n-1, N, s) + q'' + q' - b(n-1, N, s) = \binom{N+n-1}{N-1}.$$

If $q' < n(x-r+1)$, combining with (*): $a(n-1, N, s) + n(n-2) > \binom{N+n-1}{N-1}$. Since $\binom{N+n-1}{N-1}/(n-1) > a(n-1, N, s)$ we get $F(N, n) > 0$ with:

$$F(N, n) = (N+n-1)!(n - Nn + N)/(n!N!) + (n-1)(n-2)h\,.$$

But $F(N+1, n) < F(N, n)$, $N \geq 4$, $n \geq 2$ and $F(4, n) < 0$, $n \geq 2$, which proves the lemma.

X. – Initial cases in P^3 and end of the proof of theorem 1.

In this section we will prove $\bar{A}_{n,3}(0)$, $n\geq 3$. According to IX this will prove $\bar{A}_{n,N}(s)$, $n\geq 3$, $N\geq 4$, $0\leq s\leq 2N$ and hence $A_{n,N}(s)$, $N\leq s\leq 2N$, $n\geq 3$, $N\geq 4$. Then we have to prove $A_{n,3}(g+3)$, $n\geq 4$, $0\leq g\leq 3$ but with the condition $q'\geq 2n-3$ (see IV.4.1 and VI.2): this will be implied by the statement $A'_{n,3}(g+3)$, $n\geq 4$, $0\leq g\leq 3$, of this section.

For the proofs we need two other inductive steps: $B_n(s)$ and $C_n(s, k)$. Then after to prove $A_{3,3}(g+3)$, $g=0$ or 2 and $H_{3,4}(g)$ for $g=1$ or 3, to get theorem 1. This will be done at the end of this section.

X.1. THE STATEMENT $B_n(s)$, PRELIMINARIES. – For any genus $g\geq 0$ and any integer $n\geq 1$ we have defined numbers $r(n,g)$, $q(n,g)$ by:

$$\binom{n+3}{3} = n\cdot r(n,g) - g + 1 + q(n,g); \qquad 0\leq q(n,g)\leq n-1 \ \ (\text{see [BE2]}) .$$

The $r(n,g)$ are the critical degrees for the postulation of curves of genus g in P^3 (at least for non special curves).

If $g=0$, writting more simply $r(n)$, $q(n)$, we have, according to the congruence of n mod. 6:

$$\begin{aligned}
r(6k+1) &= 6k^2+8k+3\,, & q(6k+1) &= 0\\
r(6k+2) &= 6k^2+10k+4\,, & q(6k+2) &= 3k+1\\
r(6k+3) &= 6k^2+12k+6\,, & q(6k+3) &= 2k+1\\
r(6k+4) &= 6k^2+14k+8\,, & q(6k+4) &= 3k+2\\
r(6k+5) &= 6k^2+16k+11\,, & q(6k+5) &= 0\\
r(6k+6) &= 6k^2+18k+13\,, & q(6k+6) &= 5k+5\,.
\end{aligned}$$

Since $\binom{n+3}{3} = nr(n)+1+q(n)$, we may write:

$$\binom{n+3}{3} = nr(n) - (1-c) + 1 + s \quad \text{with } c = 1 + q(n) - s\,,$$

or:

$$\binom{n+3}{3} = n(r(n)-1) - (1-c) + 1 + s \quad \text{with } c = n + q(n) + 1 - s\,.$$

These latter numbers are $h^0(\mathcal{O}_Y(n))$ if Y is the disjoint union of s points and of c disjoint bamboos such that the curve part of Y has degreee $r(n)$ (resp. $r(n)-1$). This is the motivation for the definition of $B(s)$ (see X.2, 4).

Note also that for the first part of this work (proof of theorem 1) we only need $B_n(0)$. But for theorem 2 we will need $B_n(s)$ for some $n > s > 0$. This explains why we let s vary in the definitions but only prove, in this section, the initial cases for $s = 0$.

Finally, before to start, we need one further definition.

X.1.1 DEFINITION. - Let $T = \bigcup_{i=1}^{r} T_i$ be the union of r disjoint bamboos. An s-uple of disjoint lines $(D_1, \ldots, D_s)$ is a b-connecting secant for T if there exist b connected components of T, for example $T_1, \ldots, T_b$ such that: $T_1 \cup \ldots \cup T_b \cup D_1 \cup \cup \ldots \cup D_s$ is a bamboo (hence is connected); $T \cup D_1 \cup \ldots \cup D_s$ is a union of disjoint bamboos.

X.2. $B_n(s)$ *for* n *odd.*

$B_{6k+1}(s)$, $0 \leqslant s \leqslant 2k-1$. For every $s+1$ points $P_0, P_1, \ldots, P_s$ in linear general position there exist (Y, Z, Q) where

(1) Y is the disjoint union of the P_i, $1 \leqslant i \leqslant s$, and of a curve Z which satisfies: $h^0(\boldsymbol{P}^3, \mathcal{I}_Y(6k+1)) = 0$

(2) Z has degree $6k^2 + 8k + 2$ and is the union of $6k + 2 - s$ disjoint bamboos; P_0 is contained in a final line of Z

(3) Q is a smooth quadric intersecting Y transversally. Moreover Q contains:

(*a*) $2k - s - 1$ pairs (D_i, D_i'), $1 \leqslant i \leqslant 2k - s - 1$, of 3-connecting secants for Z

(*b*) one $(2s+3)$-connecting secant (δ_j), $1 \leqslant j \leqslant 2s+2$ for Z. The lines D_i, D_i', δ_j are all in the same system of lines of Q.

REMARK. - It follows that $Z' := Z \cup (\bigcup D_i) \cup (\bigcup D_i') \cup (\bigcup \delta_j)$ is the union of $2k + 2 - s$ disjoint bamboos and has degree $6k^2 + 12k + 2$. We will use Z' to prove $B_{6k+3}(s)$.

$B_{6k+3}(s)$, $0 \leqslant s \leqslant 2k+1$. For every $s+1$ points $P_0, P_1, \ldots, P_s$ in l.g.p. there exist (Y, Z) where:

(1) Y is the disjoint union of $P_1, \ldots, P_s$ and of a curve Z and satisfies $h^0(\boldsymbol{P}^3, \mathcal{I}_Y(6k+3)) = 0$.

(2) Z has degree $6k^2 + 12k + 6$ and is the union of $2k + 2 - s$ disjoint bamboos; P_0 is contained in a final line of Z.

$B_{6k+5}(s)$, $0 \leqslant s \leqslant 6k+5$. For every $s+1$ points $P_0, \ldots, P_s$ in l.g.p. there exist (Y, Z) where:

(1) Y is the disjoint union of $P_1, \ldots, P_s$ and of a curve Z and satisfies $h\left(\boldsymbol{P}^3, \mathfrak{J}_Y(6k+5)\right) = 0$

(2) Z has degree $6k^2 + 16k + 10$ and is the union of $6k+6-s$ disjoint bamboos; P_0 is contained in a final line of Z.

X.3. INDUCTION FOR THE ODD CASES.

X.3.1. $B_{6k-1}(s)$ *implies* $B_{6k+1}(s)$, $0 \leqslant s \leqslant 2k-1$, $k \geqslant 1$. Let (Y, Z) be given by $B_{6k-1}(s)$ for the P_i's. By definition Z has degree $6k^2 + 4k$ and is the union of $6k - s$ disjoint bamboos. Let Q be a smooth quadric not containing any irreducible component of Y. On Q we consider $4k+2$ disjoint lines (L_i) such that: if $1 \leqslant i \leqslant 4k$,. L_i is linked to a final line of a connected component of Z. If $i > 4k$: $L_i \cap Z = \emptyset$ It follows that $Z' := Z \cup \left(\bigcup^{4k+2} L_i\right)$ has degree $6k^2 + 8k + 2$ and is the union of $6k + 2 - s$ disjoint bamboos. Take $f \in H^0\left(\mathfrak{J}_{Y'}(6k+1)\right)$ where $Y' = Z' \cup \{P_1, \ldots, P_s\}$ Then $f' := f|Q$ vanishes on the $4k+2$ lines L_i and on the $12k^2 + 4k = (6k+2)(2k)$ points of $S = Z \cap \left(Q \setminus \left(\bigcup^{4k+2} L_i\right)\right)$.

Claim: every form of type $(2k-1, 6k-1)$ vanishing on S is identically zero.

Assuming the claim we get $f' = 0$. Thus $f = qg$ with $g \in H^0\left(\mathfrak{J}_Y(6k-1)\right)$. By $B_{6k-1}(s)$, $g = 0$ hence $f = 0$ as wanted. Finally the claim above is just a little variation of [BE, 2] lemma 5.2. This proves (1) and (2) of $B_{6k+1}(s)$.

Now it remains to show condition (3) of $B_{6k+1}(s)$. Let T_i, $1 \leqslant i \leqslant 6k - s$, be the connected components of Z. If length $(T_i) \geqslant 2$, T_i has two final lines, say: $T_i^f, R_i^{f'}$. If length $(T_i) = 1$, T_i intersects Q in two distinct points: $t_i^f, t_i^{f'}$. From the construction above we have, for example, that L_i is linked to T_i^f if $1 \leqslant i \leqslant 4k$. For $1 \leqslant i \leqslant 2k - s - 1$ let x_i be a point of $T_{4k+i}^f \cap Q$ $\left(\text{if } l(T_{4k+1}) = 1, x_i = t_{4k+i}^f\right)$ and x_i' be a point of $T_i^{f'} \cap Q$ $\left(x_i' = t_i^{f'} \text{ if } l(T_i) = 1\right)$. Let Δx_i (resp. $\Delta x_i'$) be the line on Q through x_i (resp. x_i') and intersecting L_1. Define: $y_i = \Delta x_i \cap L_i$ (resp. $y_i' = \Delta x_i' \cap L_{2k-s+i}$), $1 \leqslant i \leqslant 2k - s - 1$. Put $D_i := [x_i, y_i]$, $D_i' := [x_i', y_i']$.

Note that (D_i, D_i') connect $l_i \cup T_i$, L_{2k-s+i} and T_{4k+i}, $1 \leqslant i \leqslant 2k - s - 1$. They will be the $(2k - s - 1)$ 3-connecting secants.

Define $x_{2k-s}, y_{2k-s}, x_{2k-s}', y_{2k-s}'$ as $x_i, \ldots$ for $i \leqslant 2k - s - 1$. Furthermore for $0 \leqslant j \leqslant 2s - 1$ let α_j be a point of $T_{4k-2s+j}^{f'} \cap Q$ $[\alpha_j = t_{4k-2s+j}^{f'}$ if $l(T_{4k-2s+j}) = 1]$.

Let $\Delta\alpha_j$ be the line through α_j intersecting L_1 and put $z_{j+1} = \Delta\alpha_j \cap L_{4k-2s+j+1}$, $0 \leqslant j \leqslant 2s-1$. Finally set:

$$\delta_j = [\alpha_{j-1}, z_j], \quad 1 \leqslant j \leqslant 2s; \qquad \delta_{2s+1} = D_{2k-s}'; \qquad \delta_{2s+2} = D_{2k-s}.$$

Now we deform the lines L_i into lines L_i', $1 \leqslant i \leqslant 4k+2$, such that: L_i' is transversal to Q for

$$1 \leqslant i \leqslant 2k - s - 1 \quad y_i \in L_i', \quad y_i' \in L_{2k-s+i}'; \quad z_j \in L_{4k-2s+j}', \quad 1 \leqslant j \leqslant 2s;$$
$$y_{2k-s} \in L_{2k-s}', \quad y_{2k-s}' \in L_{4k-2s}'.$$

The resulting curve $\bar{Z} := Z \cup \big(\bigcup^{4k+2} L_i'\big)$ satisfies (1) and (2) of $B_{6k+1}(s)$. Now we claim that $\bar{Z}$ also satisfies (3) of $B_{6k+1}(s)$. Indeed (D_i, D_i'), $1 \leqslant i \leqslant 2k-1-s$, are the $(2k-1-s)$ 3-connecting secants and $(\delta_1, \ldots, \delta_{2s+2})$ is the $(2s+3)$-connecting secant to $\bar{Z}$. Note also that we may assume that P belongs to $T_{4k}^{f'}$: Hence

$$Z' = \bar{Z} \cup \big(\bigcup D_i\big) \cup \big(\bigcup D_i'\big) \cup \big(\bigcup \delta_j\big)$$

is the union of $2k - s + 2$ disjoint bamboos and P is contained in a final line of Z'. This finishes the proof of $B_{6k+1}(s)$.

X.3.2. $B_{6k+1}(s)$ *implies* $B_{6k+3}(s)$, $0 \leqslant s \leqslant 2k-1$, $k \geqslant 1$. Let (Y, Z, Q) be given by $B_{6k+1}(s)$ for the points $P_0, \ldots, P_s$. In Q we have the $(2k-1-s)$ 3-connecting secants to Z, (D_i, D_i'), and the $(2s+3)$-connecting secant (δ_j). This all together yields $4k$ lines and $Z' = Z \cup \big(\bigcup D_i \cup D_i'\big) \cup \big(\bigcup \delta_j\big)$ has $2k+2-s$ connected components. Also, by construction, two of these components are lines transversal to Q (cf. the lines L_{4k+1}', L_{4k+2}' of the previous proof). In Q we consider four further lines B_l, $1 \leqslant l \leqslant 4$, such that $\bar{Z} := Z' \cup \big(\bigcup^{4} B_l\big)$ has degree $6k^2 + 12k + 6$ and is the union of $2k + 2 - s$ disjoint bamboos. Now if $f \in H^0(\mathfrak{J}_Y(6k+3))$ where $Y = \bar{Z} \cup \{P_1, \ldots, P_s\}$ then $f' := f|Q$ vanishes on the $4k+4$ lines D_i, D_i', δ_j, B_l and on the

$$2(6k^2 + 8k + 2) - (8k + 4) = (6k + 4)2k$$

points of $Z \cap \Big(Q \setminus \big(\bigcup D_i \cup D_i' \cup \delta_j \cup B_l\big)\Big)$. As before using [BE2] § 6 we see that $f' = 0$, hence $f = 0$ as wanted.

X.3.3. $B_{6k+3}(s)$ *implies* $B_{6k+5}(s)$, $0 \leqslant s \leqslant 2k+1$, $k \geqslant 0$. Let (Y, Z) be given by $B_{6k+3}(s)$ for the P_i's. Let Q be a smooth quadric not containing any irreducible component of Y. In Q we consider $4k+4$ disjoint lines, L_i, such that: $L_i \cap Z = \emptyset$, $1 \leqslant i \leqslant 4k+4$. So $Z' = Z \cup \big(\bigcup L_i\big)$ has degree $6k^2 + 16k + 10$ and is the union of $6k + 6 - s$ disjoint bamboos. Arguing as in the previous cases we may assume that $Y' = Z' \cup \{P_1, \ldots, P_s\}$ satisfies $h^0(\mathfrak{J}_{Y'}(6k+5)) = 0$.

X.4. $B_n(s)$ *for* n *even.*

$B_{6k+2}(s)$, $0 \leqslant s \leqslant 2k+1$. For every $s+1$ points $P_0, P_1, \ldots, P_s$ in l.g.p. there exist (Y, Z) where:

(1) Y is the disjoint union of $P_1, \ldots, P_s$ and of a curve Z and satisfies $h^0(\boldsymbol{P}^3, \mathfrak{J}_Y(6k+2)) = 0$

(2) Z has degree $6k^2+10k+4$ and is the union of $3k+2-s$ disjoint bamboos; P_0 is contained in a final line of Z.

$B_{6k+4}(s)$, $0\leqslant s\leqslant 3k+2$. For every $s+1$ points $P_0, \ldots, P_s$ in l.g.p. there exist (Y, Z) where:

(1) Y is the disjoint union of $P_1, \ldots, P_s$ and of a curve Z and satisfies $h^0(\boldsymbol{P}^3, \mathfrak{I}_Y(6k+4))=0$

(2) Z has degree $6k^2+14k+8$ and is the union of $3k+3-s$ disjoint bamboos; P_0 is contained in a final line of Z.

$B_{6k+6}(s)$, $0\leqslant s\leqslant 4k+4$. For every $s+1$ points $P_0, \ldots, P_s$ in l.g.p. there exist (Y, Z, Q) where:

(1) Y is the disjoint union of $P_1, \ldots, P_s$ and of a curve Z and satisfies $h^0(\mathfrak{I}_Y(6k+6))=0$

(2) Z has degree $6k^2+18k+13$ and is the union of $5k+6-s$ disjoint bamboos; P_0 is contained in a final line of Z

(3) Q is a smooth quadric intersecting Y transversally. Furthermore Q contains $2k+1$ disjoint 2-connecting secants, D_i, $1\leqslant i\leqslant 2k+1$, to Z.

X.5. INDUCTION FOR THE EVEN CASES. – The proofs are similar as in the odd case so we will just sketch them.

X.5.1. $B_{6k+2}(s)$ *implies* $B_{6k+4}(s)$, $0\leqslant s\leqslant k-1$, $k\geqslant 1$. Let (Y, Z) be given by $B_{6k+2}(s)$ for the P'_is. Let Q be a smooth quadric surface containing no irreducible component of Y. Let T_i, $1\leqslant i\leqslant 3k+2-s$, be the connected components of Z. On Q we take $4k+4$ disjoint lines, L_i, in the following way: for $1\leqslant i\leqslant k+s+2$, L_i and $L_{k+s+2+i}$ intersect T_i so that $L_i \cup L_{k+s+2+i} \cup T_i$ is a bamboo. For $2k+2s+5\leqslant i\leqslant 4k+3$, L_i is linked to a final line of $T_{i-k-s-2}$; $L_{4k+4}\cap Z=\emptyset$. It follows that $Z'=Z\cup\big(\bigcup^{4k+4} L_i\big)$ has degree $6k^2+14k+8$ and is the union of $3k+3-s$ disjoint bamboos. As in the previous cases we may show that $h^0(\mathfrak{I}_{Y'}(6k+4))=0$ where $Y'=Z'\cup\{P_1, \ldots, P_s\}$.

X.5.2. $B_{6k+4}(s)$ *implies* $B_{6k+6}(s)$, $0\leqslant s\leqslant k+1$, $k\geqslant 0$. Let (Y, Z) be given by $B_{6k+4}(s)$ for the P_i's. Let Q be a smooth quadric containing no irreducible component of Y. On Q we consider $4k+5$ disjoint lines, L_i, as follows: if $1\leqslant i\leqslant 2k+3$, L_i is linked to a final line of a connected component of Z. If $i\geqslant 2k+3$, $L_i\cap Z=\emptyset$. We may assume that $Z'=Z\cup\big(\bigcup L_i\big)$ satisfies, with $\{P_1, \ldots, P_s\}$, (1) and (2) of $B_{6k+6}(s)$. For condition (3) let x_i be a point of L_i, $1\leqslant i\leqslant 2k+1$, and let Δx_i be the line on Q through x_i and intersecting L_1. Set $y_i=\Delta x_i\cap L_{i+2k+1}$. We deform the lines L_i, $1\leqslant i\leqslant 4k+5$, into lines L'_i such that: L'_i is transversal to Q; for $1\leqslant i\leqslant 2k+1$, L'_i (resp. L'_{2k+1+i}) passes through x_i (resp. y_i); $Y:=Z\cup\big(\bigcup L'_i\big)\cup\{P_1, \ldots, P_s\}$ satisfies $B_{6k+6}(s)$.

X.5.3. *$B_{6k+6}(s)$ implies $B_{6k+8}(s)$, $0 \leqslant s \leqslant k-1$, $k \geqslant 1$.* Let (Y, Z, Q) be given by $B_{6k+6}(s)$ for $P_0, P_1, \ldots, P_s$. On Q we have also the $2k+1$ disjoint 2-connecting secants, D_i, to z. Let $(T_i)_{1 \leqslant i \leqslant 5k+6-s}$ be the connected components of Z. If $l(T_i) \geqslant 2$ let T_i^f, $T_i^{f'}$ be the two final lines of T_i. If $l(T_i) = 1$ then let $T_i \cap Q = \{t_i^f, t_i^{f'}\}$. We may assume that T_i^f, T_{i+2k+1}^f intersect D_i (or t_i^f (resp. t_{i+2k+1}^f) belongs to D_i, if $l(T_i) = 1$ (resp. $l(T_{i+2k+1}) = 1$) $1 \leqslant i \leqslant 2k+1$. Then we take $2k+1$ further disjoint lines on Q, D_i', such that: $D_i' \cap D_i = \emptyset$, $T_{i+2k+1}^{f'}$ intersects D_i' (or $t_{i+2k+1}^{f'} \in D_i'$). Finally we take, always in the same system of lines, five other lines, $\bar{D}_i$, such that $\bar{D}_i$ is linked to a final line of T_{4k+2+i}, $1 \leqslant i \leqslant 5$. Then $Z' = Z \cup \left(\bigcup^{2k+1} D_i \cup D_i'\right) \cup \left(\bigcup^{5} \bar{D}_j\right)$ has degree $6k^2 + 22k + 20$ and is the union of $2k+5-s$ disjoint bamboos. Arguing as before $Y_i = Z' \cup \{P_1, \ldots, P_s\}$ may be taken to satisfy $B_{6k+8}(s)$.

Now we turn to the special case $s = 0$ and prove the missing initial cases:

X.6. PROPOSITION. – For $n \geqslant 1$, $B_n(0)$ is true.

PROOF. – As two skew lines are never contained in a plane $B_1(0)$ is clear. For $B_2(0)$ we take the disjoint union of a line and of a bamboo of length three. Then, using a plane instead of a quadric, we prove $B_2(0)$ implies $B_3(0)$. In this way the general union of two disjoint bamboos of length 3 satisfies $B_3(0)$. Now for odd cases the induction can start.

Using again a plane we show $B_3(0)$ implies $B_4(0)$. The curve satisfying $B_4(0)$ is the disjoint union of T_1, T_2 two bamboos of length 3 and of T_3, a degenerate conic. We show $B_4(0)$ implies $B_6(0)$ adding 5 skew lines L_i in a quadric Q such that $T_i \cup L_i$, $1 \leqslant i \leqslant 2$, are disjoint bamboos and $L_i \cap (T_1 \cup T_2 \cup T_3) = \emptyset$, $i > 2$. With the usual procedure this yields a curve satisfying $B_6(0)$ and having three 2-connecting secants. With a slight modification of the general proof we can show $B_6(0)$ implies $B_8(0)$ and the induction starts also in the even case.

Now we introduce the second inductive statement of this section.

X.7. $C_n(s, k)$, $0 \leqslant s \leqslant n+k-1$, $n \geqslant 1$, $k \geqslant 0$. – For every $s+1$ points $P_0, P_1, \ldots, P_s$ in l.p.g. there exists a triple (Z, D, S) where:

(1) Z is the disjoint union of $n+k-s$ bamboos, $\deg(Z) = a(n, 3, k)$ (see below) and P_0 is contained in a final line of Z

(2) D is a line with $Z \cup D$ disjoint union of $n+k-s$ bamboos and P_0 is contained in a final line of $Z \cup D$

(3) $S \subset D$ is a set of $b(n, 3, k) - n$ points

(4) $h^0(\mathcal{J}_{Z'}(n)) = 0$ where $Z' = Z \cup \{P_1, \ldots, P_s\}$.

Recall that $a(n, 3, k)$, $b(n, 3, k)$ are defined by:

$$n \cdot a(n, 3, k) + k + b(n, 3, k) = \binom{n+3}{3}, \qquad n \leqslant b(n, 3, k) \leqslant 2n-1 .$$

In this section we are interested in the case $s = k$, $0 \leqslant k \leqslant 3$, $n \geqslant 3$. If $0 \leqslant k \leqslant 3$, $n \geqslant 3$ then $a(n, 3, k) \geqslant b(n, 3, k) \geqslant n$ except if $n = 3$, $k = 3$. In this case: $a(3, 3, 3) = 4$, $b(3, 3, 3) = 5$.

X.8. LEMMA. - For $0 \leqslant k \leqslant 3$ and $n \geqslant 3$, $B_{n-2}(0)$ implies $C_n(k, k)$.

PROOF. - Let (Y, Z) be given by $B_{n-2}(0)$ for the point P_0: Let Q be a smooth quadric containing $P_1, \ldots, P_k$ $(k > 0)$ but not P_0. Let (u, v) be respectively the degree and number of connected components of Z. In Q consider $x := a(n, 3, k) - u$ disjoint lines $(L_1, \ldots, L_x)$ in such a way that $T = Z \cup (\bigcup^{k} L_i)$ is the union of n disjoint bamboos. According to X.8.1 below, this is possible. Now let D be a line on Q not intersecting T and take $S \subset D$ with $\#(S) = b(n, 3, k) - n$. We have to show that every form of type (n, n) vanishing on $(T \cap Q) \cup S \cup \{P_1, \ldots, P_k\}$ is identically zero (then the lemma will follow because the residual scheme to Q satisfies $B_{n-2}(0)$). Again this follows from [BE, 2] § 6, especially lemma 6.2. Note that the points $P_1, \ldots, P_k$ cause no trouble since $k \leqslant 3$ and we may assume that Q does not contain any line $[P_i, P_j]$.

X.8.1. SUB-LEMMA. - With the notations of the proof of X.8 we have: (i) $x > 0$, (ii) $v \leqslant n$, (iii) $n \geqslant x \geqslant n - v$.

PROOF. - (i) By definition we have:

$$\binom{n+3}{3} = n \cdot a(n, 3, k) + k + b(n, 3, k), \qquad n \leqslant b(n, 3, k) \leqslant 2n - 1$$

$$\binom{k+1}{3} = (n-2)u + v.$$

Hence

$$(n+1)^2 = nx + 2u + k + b(n, 3, k) - v \ (*).$$

If $x \leqslant 0$ then $(n+1)^2 \leqslant 2u + 2n + 2$. Since $u \leqslant r(n-2)$ we get $(n+1)^2 \leqslant 2r(n-2) + 2n + 2$ which is impossible if $n \geqslant 3$ (use X.1).

(ii) This follows from the definition of $B_{n-2}(0)$.

(iii) If $n < x$ then $\binom{n+3}{3} > n(n+u) + k + b(n, 3, k)$ this latter being greater than $n^2 + n + nu$. Since $u \geqslant r(n-2) - 1$, using the definition of $r(n-2)$ we get $\binom{n+3}{3} > \binom{n+1}{3} + n^2 - n + 2 + 2r(n-2)$, i.e. $3n - 1 > 2r(n-2)$. If $n \geqslant 7$ this is impossible. For $n \leqslant 6$ the lemma is checked by direct computations.

If $x < n - v$, using $(*)$ we get: $(n+1)^2 < n(n-v) + 2u + k + b(n, 3, k) - v$, i.e.: $v(n+1) + (n+1)^2 < n^2 + 2u + k + b(n, 3, k) \leqslant n^2 + 2u + 2n + 2$. Finally:

$v(n+1) < 2r(n-2)+1$, which is impossible if $n \geq 3$, except if $n \equiv 5 \pmod 6$ and $k = 3$. In this latter case a direct computation shows $x = n - v$ (this is due to the fact that $b(n, 3, 3) = 2n - 2$ if $n \equiv 5 \pmod 6$).

X.9. PROPOSITION. – For $n \geq 2$, $C_n(3, 3)$ is true.

PROOF. – According to X.6, X.8 it remains to show that $C_2(3, 3)$ holds. But this means that for every P_0, P_1, P_2, P_3 spanning $\boldsymbol{P}^3$ we can find two lines D, R, with $P_0 \in D$ such that $D \cup R \cup \{P_1, P_2, P_3\}$ is contained in exactly one quadric. Just take a smooth quadric containing all the P_i's and two skew lines on it.

X.10. PROPOSITION. – For $n \geq 2$, $C_n(0, 0)$ is true.

PROOF. – According to X.6, X.8 it remains to show that $C_2(0, 0)$ holds. For this take the disjoint union, X, of a line and of a bamboo of degree 3. It is easy to check that X is not contained in a quadric surface.

X.11. THE STATEMENT $A'_{n,3}(s)$, $n \geq 4$. – The statement is similar to $\bar{A}_{n,3}(s)$. The only difference is that instead of condition (c) in $\bar{A}_{n,3}(s)$ (see VII) we require:

(c') $q' \geq \min(r, n) + n - 3$ for $n \geq 4$.

X.12. LEMMA. – For $0 \leq g \leq 3$ and $n \geq 4$: $C_{n-2}(3, 3)$ implies $A'_{n,3}(g+3)$.

PROOF. – Let $P_0, \ldots, P_{g+3}$ be the points considered. By $C_{n-2}(3, 3)$ for $P_0, \ldots, P_3$ we are given (Z, D, S). Note that Z has $n-2$ connected components. Let Q be a smooth quadric containing $P_4, \ldots, P_{g+3}$ but not P_i for $i \leq 3$. First assume $r \geq a(n-2, 3, 3)$, hence $r \geq n$. In this case we add in Q the union T of $r - a(n-2, 3, 3)$ disjoint lines in such a way that $Z \cup T$ is the union of n disjoint bamboos. This is possible since: $2(n-2) \geq r - a(n-2, 3, 3) - 2$ (see below). Outside Q we have the $b(n-2, 3, 3) - n + 2$ points of S and in Q we add the q'' collinear points and further $q' - b(n-2, 3, 3) - 2$ points in general position (note that $q' - b(n-2, 3, 3) - 2 \geq 0$ because of the new assumption on q'). We claim that the union, X, of Z, T, S, $\{P_1, \ldots, P_{g+3}\}$, the q'' collinear points and the $q' - b(n-2, 3, 3) - 2$ points satisfies $A'_{n,3}(g+3)$. Indeed $\operatorname{Res}_Q(X)$ satisfies $C_{n-2}(3, 3)$ and we may assume (by [BE, 2] § 6) that any form of type (n, n) vanishing on $X \cap Q$ is identically zero. Finally we can always reduce to the case $r \geq a(n-2, 3, 3) + 2$ since $n[a(n-2, 3, 3) + 2] + n + (n-3) + 6 + (n-1) \leq \binom{n+3}{3}$. Indeed if not then

$$(+) \qquad na + 5q + 2 > \binom{n+3}{3}$$

But from the definition: $(n-2)a \leq \binom{n+1}{3} - n - 1$. Combining with $(+)$ $2a > n^2 - 2n$. Using again the definition of $a = a(n-2, 3, 3)$ we get: $0 > 2n^2(n-6) + 19n + 6$ which is false if $n \geq 3$.

X.12.1. With notations of the proof of X.12; we have:

$$(*)\qquad 2(n-2)\geqslant r-a(n-2,3,3)-2$$

First for $n\geqslant 4$ we have $r\leqslant a(n,3,0)$ (because $q''+q'+g+3\geqslant 2n-1$). On the other hand, from the definitions, we get: $a(n-2,3,3)\geqslant a(n-2,3,0)-1$ for $n\geqslant 4$. Moreover:

$$(**)\qquad a(n,3,0)-a(n-2,3,0)+1\leqslant (n+1)\,,\qquad n\geqslant 4$$

and (*) follows. To prove (**) note that the definition of $a(n,3,3)=a$, $a(n-2,3,3)=a'$ yields $(n+1)^2=n(a-a')+2a'+b-b'$ and

$$a'\geqslant\left[\binom{n+1}{3}-2n+5\right]/(n-2)$$

from which (**) follows.

In a similar way we have:

X.13. LEMMA. – For $n\geqslant 4$, $C_{n-2}(0,0)$ implies $A'_{n,3}(0)$. Hence $\bar{A}_{n,3}(0)$ is true for $n\geqslant 4$.

X.14. LEMMA. – $\bar{A}_{3,3}(0)$ holds.

PROOF. – One easily computes the possible values of r, q', q''. For example $5\geqslant r\geqslant 1$. If $r\geqslant 3$ take a smooth quadric Q. Let L, L' be two disjoint lines intersecting Q transversally. Then in Q consider a bamboo, T, with $\deg(T)=r-2$; a set, S, of q'' collinear points and a set, P, of $q'-3$ general points. Then $X=L\cup L'\cup T\cup S\cup P$ satisfies $h^0(\mathfrak{I}_X(3))=0$. Indeed if $f\in H^0(\mathfrak{I}_X(3))$ then $f|Q\equiv 0$ and moreover $L\cup L'$ is not contained in a plane. If $1\leqslant r\leqslant 2$, do as above but with one line, L, outside Q and $\deg(T)=r-1$.

Almost the same proof shows:

X.15. LEMMA. – If $g=0$ or $g=2$ then $A_{3,3}(g+3)$ is true.

The following lemma concludes the proof of theorem 1.

X.16. LEMMA. – For $g=1$ or $g=3$, $H_{3,4}(g)$ is true.

PROOF. – Let C be an elliptic quintic in $\boldsymbol{P}^4$ and H be a general hyperplane. Set $H\cap C=\{P_1,\dots,P_5\}$. Let L_i be a general line in H through P_i, $1\leqslant i\leqslant 5$. Also let D be a general line (not contained in H) but intersecting L_1. Finally let $S=\{x,y\}$ be a set of two distinct points such that $[x,y]$ meets D. We may assume $\mathfrak{L}:=L_1\cup\dots\cup L_5$ to be in general position in $H\simeq\boldsymbol{P}^3$.

Hence $h^0(\mathfrak{J}_{\mathfrak{L}}(3)) = 0$ (see [HH1]). Hence a cubic containing $X = C \cup \mathfrak{L} \cup D \cup S$ has to split into the union of H and of an hyperquadric containing $D \cup S$. But we may assume that $h^0(\boldsymbol{P}^4, \mathfrak{J}_{D \cup S}(2)) = 0$ (for instance let H' be an hyperplane containing $D \cup S$, then $D \cup S \cup (C \cap H')$ is not contained in a quadric of H' and C is non degenerated). If $g = 3$, C is a curve of degree 7 and we modify a little bit the above construction: in H we take four lines $L_1, \dots, L_4$ through $P_1, \dots, P_4$ and one further point P.

Outside H we take D intersecting L_1. As the union in $\boldsymbol{P}^3$ of four general lines and four general points is not contained in a cubic surface, we conclude as above.

A) The second theorem.

These last sections are devoted to the proof of:

THEOREM 2. – Fix N, g with $N \geqslant 4$, $g \geqslant 0$. There exists an integer $d(g, N)$ such that for every $d \geqslant d(g, N)$, for every smooth curve X of genus g and every L in $\mathrm{Pic}^d(X)$, the general projection of $\varphi_L(X)$ into $\boldsymbol{P}^N$ has maximal rank.

From the proof we could obtain inductively an explicit upper bound for $d(g, N)$, $N \geqslant 4$. But the construction is highly inefficient and therefore the bound is useless. Nevertheless theorem 2 seems interesting because it shows that asymptotically in a fixed $\boldsymbol{P}^N$ the postulation of a general embedding of every curve does not depend on the geometry of the curve. The proof is, as usual, by induction on N but we write with details only the «starting» case $N = 4$. Then the induction from $N - 1$ to N, $N \geqslant 5$, is similar and even simpler.

B) Structure of the proof for $N = 4$.

We define statements D_t, $H'_{t,4}(g)$ (see C). We show that if n is big enough then D_n holds and that there is a chain of implications:

$$D_{t-g} \Rightarrow D_{t-g+1} \Rightarrow \dots \Rightarrow D_{t-1} \Rightarrow H'_{t,4}(g) \tag{D, E}$$

Thus $H'_{t,4}(g)$ holds for $t \geqslant t_0$. This implies theorem 2 (see F). Thus the proof is essentially reduced to the proof of D_t, $t \geqslant n$. The starting point, D_n, is proved using a curve in $\boldsymbol{P}^3$ and a specific construction (D). Then to prove D_t, $t \geqslant n$, we use: $D_{t-1} + A'_{t,3}(g+3) \Rightarrow D_t$. To prove $A'_{t,3}(s)$ with $t \gg s$, we use the chain of implications: $B_{t-2}(s) \Rightarrow C_t(s, s) \Rightarrow A'_{t+2}(s)$ (these statements are defined in section X). Finally it remains to prove $B_l(s)$, $l \gg s$. For this we use (E.1) the constructions of X.2, ..., X.6 and the fact (see X.6) that $B_n(0)$, $n \geqslant 1$, is true.

C) **The statements** D_t, $H'_{t,4}(g)$.

From now on we fix a smooth curve, X, of genus g.

C.1. D_t:

There exist (Y, W, Y_i, h) such that:

(1) $h\colon W \to X$ is an isomorphism

(2) $Y = W \cup \left(\bigcup Y_i\right)$, $p_a(Y) = g$, $\deg(Y) = r(t, 4, g)$ and Y has $q(t, 4, g) + 1$ connected components.

(3) The Y_i's are disjoint bamboos; every Y_i intersects W at most at one point and quasi-transversally. At least t of the Y_i's intersect W.

(4) $h^0\big(\boldsymbol{P}^4, \mathfrak{I}_Y(t)\big) = 0$.

C.2. $H'_{t,4}$:

There exist (Y, W, D, S, h) with:

(1) $h\colon W \to X$ is an isomorphism

(2) D is a line, $S \subset D$, $\#(S) = q(t, 4, g)$

(3) Y and $Y \cup D$ are connected and union of W and t disjoint bamboos

(4) $\deg(Y) = r(t, 4, g)$ and $h^0\big(\boldsymbol{P}^4, \mathfrak{I}_{Y \cup S}(t)\big) = 0$.

D) **Proof of** D_n **for some** n **large enough.**

First of all let C be the image of X through an embedding of degree $g + 3$ in $\boldsymbol{P}^3$ with $\mathcal{O}_C(1)$ not special. In the lemma below we consider $\boldsymbol{P}^3$ as an hyperplane, H, in $\boldsymbol{P}^4$.

We define integers $r(t), q(t)$ by the relations:

$$(t+1)\,r(t) + q(t) = \binom{t+3}{3}, \qquad 0 \leqslant q(t) \leqslant t\,.$$

We have

$$r(t) = (t+2)(t+3)/6\,, \qquad q(t) = 0 \qquad \text{if } t \equiv 0, 1 \pmod 3$$

and

$$r(t) = (t+1)(t+4)/6\,, \qquad q(t) = (t+1)/3 \qquad \text{if } t \equiv 2 \pmod 3 \ \big(\text{see [HH1]}\big)\,.$$

Finally let m be the least integer $\geqslant 2g + 6$ with $m \equiv 0 \pmod 6$.

D.1. LEMMA. – For every $n \geqslant m$ with $n - m$ even there exist (Y, Z, D, U, S) where:

(1) $Y = C \cup Z \cup U \cup S$

(2) Z is the union of $r(n)$-$r(m)$ disjoint lines

(3) D is a line, $S \subset D$, $\#(S) = q(n)$, U is a set of

$$\binom{n+3}{3} - (n+1)\big(r(n) - r(m)\big) - n(g+3) - 1 + g - q(n) =$$
$$= \binom{m+3}{3} - m(g+3) - 1 + g + \big(r(m) - g - 3\big((n - m) \text{ points}$$

(4) $h^0\big(\boldsymbol{P}^3, \mathfrak{J}_{Y,H}(n)\big) = 0$.

PROOF. – By Castelnuovo theorem $h^1\big(\mathfrak{J}_{C,H}(m)\big) = 0$. Thus there exists the union, V, of $x := \binom{m+3}{3} - m(g+3) - 1 + g$ points with $r_{C \cup V,H}(m)$ bijective. Consider a general quadric Q. In Q add $r(m+2) - r(m)$ disjoint lines and $2\big(r(m) - g - 3\big)$ points in such a way that the union Y has $r_{Y,H}(m+2)$ bijective. Repeating this process we get the lemma.

The condition $m \geqslant 2g + 6$ ensures us that $C \cap Q$ is in general position for forms of type (x, y) on Q, $y \geqslant m + 2$, since we may assume that Q does not contain any secant line to C [BE2].

Now we can construct curves in $\boldsymbol{P}^4$ isomorphic to X and with maximal rank.

D.2. LEMMA. – For some n large enough (n depending only on g) D_n is true.

PROOF. – Since asymptotically on n we have $a(n, 4, 0) \sim r(n, 4, g) \sim n^3/24$, we can find a large integer n, $n \geqslant m$, $n \equiv m \pmod 2$ (n depending only on g) such that: $r(n, 4, g) > a(n-1, 4, 0) + g + 3 + 3n$ and such that $r(n-1, 4, g) > 10n + (n - m)(2x + 1)$ say. By $P_{n-1,4}(0)$ (see VII) there exists in $\boldsymbol{P}^4$ a reduced curve T union of $q(n-1, 4, 0)$ disjoint bamboos, with $h^0\big(\boldsymbol{P}^4, \mathfrak{J}_T(n-1)\big) = 0$. We may assume that T intersects C only at one point. We can find in H the union B of $r(n, 4, g) - a(n-1, 4, 0) - g - 3$ disjoint lines, say $B = B_1 \cup B_2$, such that $\deg(B_2) = q(n, 4, g)$, $C \cup T \cup B_1$ is connected, of arithmetic genus g, with only ordinary double points and has at least n lines intersecting only another irreducible component. Also we assume $B_2 \cap (C \cup T \cup B_1) = \emptyset$. Let $Y = C \cup T \cup B$. The points of $T \cap H$ not on C are general points in H. Thus by lemma D.1, $C \cup B \cup (T \cap B)$ is not contained in any degree n surface of H and $r_Y(n)$ is bijective. By the choice of T, Y has maximal rank. We can deform $C \cup T \cup B_1$ to $Z \cup A$ where Z is isomorphic to X, A is the union of n disjoint lines each intersecting Z only at one point.

E) **Proof of** D_t, $t \geqslant n$.

We have just proved D_n. From now on n is fixed. We will prove below that D_{t-1} and $A'_{t,3}(g+3)$ imply D_t, $t \geqslant n+1$. But first we have to prove $A'_{t\,3}(s)$ when t is much bigger than s. This is a consequence of the following lemmas:

E.1. LEMMA. - We have:

$$B_{6k+1}(s) \Rightarrow B_{6k+3}(s) \Rightarrow B_{6k+5}(s+1) \Rightarrow B_{6k+7}(s+1) \quad \text{if } s \leqslant k$$

$$B_{6k+2}(s) \Rightarrow B_{6k+4}(s) \Rightarrow B_{6k+6}(s+1) \Rightarrow B_{6k+8}(s+1) \quad \text{if } s \leqslant k\,.$$

PROOF. - Most of the lemma has been proved in section X. Slight modifications of the constructions of X.2, ..., X.6 prove the implications above when s increases to $s+1$.

With the same type constructions as in section X we have:

E.2. LEMMA. - $B_{t-2}(s)$ implies $C_t(s, s)$ if $6s < t-2$ and $C_{t-2}(s, s)$ implies $A'_{t,3}(s)$ when t is big and s small with respect to t.

Using X.6 and the lemmas above we have proved $A'_{t,3}(g+3)$ for $t > K(g)$, $K(g)$ a constant depending on g. We may assume $n > K(g)$ (n is defined in D.2).

E.3. LEMMA. - For $t \geqslant n+1$, D_{t-1} and $A'_{t,3}(g+3)$ imply D_t. Also D_{t-1} implies $H'_{t,4}(g)$.

PROOF. - The proof is similar to the one of IV.4. However we have to be a little careful. Let (Y, W, Y_i) be given by D_{t-1}. Let $P \in W$ be a point not contained in any Y_i. Suppose that to prove D_t we add to Y some bamboo intersecting W only at P. For this we have to control the postulation of $H \cap W$, H a general hyperplane through P. By theorem 0 we can degenerate W in $\mathrm{Pr}_d\left(\mathcal{O}_W(1), \boldsymbol{P}^4\right)$ to $W' = M \cup N$ where

(*a*) $M \xrightarrow[\approx]{f} W$, $\deg(M) = g+4$ and $L = \mathcal{O}_M(1)$ is non special

(*b*) N is the union of $d-g-4$ disjoint lines R_i, each R_i intersecting M only at one point, P_i, quasi-transversally;

$$f^*\mathcal{O}_{\widetilde{W}}(1) = L(P_1 + \ldots + P_{d-g-4})\,.$$

We may even assume that $p' = f^{-1}(p)$ is a smooth point of W'. Thus we can apply $A'_{t,3}(g+3)$ to W' taking a general hyperplane H through p'.

F) **End of the proof.**

Take L in $\mathrm{Pic}^d(X)$ and suppose $r(t, 4, g) < d \leqslant r(t+1, 4, g)$. If d is large enough then D_{t-g} holds. We have proved $D_{t-g+1}, \ldots, D_{t-1}, H'_{t,4}$ for some W, in g steps. At each steps we have linked one more bamboo to W through a point we can fix before. Then we add $d - r(t, 4, g)$ lines, one of them being the line D in the definition of $H'_{t,4}(g)$, to the curve given by $H'_{t,4}(g)$. The lines are added in such a way that we obtain a connected curve T, union of W and of t disjoint bamboos, Y_i, each intersecting W at one point, P_i. Set $a_i = \deg(Y_i)$ and let $L_i \subset Y_i$ be the line intersecting W. By construction $t-g$ of the L_i were added in the construction of a curve satisfying D_{t-g}; say L_i for $1 \leqslant i \leqslant t-g$. On the other hand L_i, $i > t-g$, where added during the chain of implications:

$$D_{t-g} \Rightarrow D_{t-g+1} \Rightarrow \ldots \Rightarrow D_{t-1} \Rightarrow H'_{t,4}(g)\,. \tag{*}$$

Set $M = \mathcal{O}_W(1)$. Unfortunately $M\Big(\sum_{i=1}^{t} a_i P_i\Big)$ in general need not to be isomorphic to L. But by Abel's theorem there exist g points, $P'_1, \ldots, P'_g$ such that:

$$M\Big(\sum_{i=1}^{t-g} a_i P_i + \sum_{i=t-g+1}^{t} a_i P'_i\Big) \simeq L\,.$$

If in the chain of implications (*) we can take $P_i = P'_i$, $i > t-g$, we are done, if we are sure that $P'_i \neq P_j$.

This can be achieved moving the points and using semi-continuity. Now by theorem 0 and semi-continuity a general element of $\mathrm{Pr}_d(L, \boldsymbol{P}^4)$ has $r(t)$ injective. In the same way, using $H'_{t+1,4}(g)$ we can prove that a general element X of $\mathrm{Pr}_d(L, \boldsymbol{P}^4)$ has $r_X(t+1)$ surjective (and hence $r_X(k)$ surjective for $k \geqslant t+1$, see [M] p. 99). Since $\mathrm{Pr}_d(L, \boldsymbol{P}^4)$ is irreducible, this proves theorem 2 for $N = 4$.

The full proof of theorem 2 now follows by induction from $N-1$ to N, $N \geqslant 5$, in a similar way.

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