Wiener Estimates for a Class of Systems of Parabolic Variational Inequalities (*) (**).

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Summary. – *Wiener estimates at a point for parabolic diagonal systems of parabolic variational inequalities with obstacle are proved by a Green function method.*

0. – Introduction.

In the present papier we study the pointwise regularity of local bounded weak solutions of non linear diagonal systems of parabolic variational inequalities which are relative to the convex set

$$K = \{u \in (L^2(0, T; H^1(\boldsymbol{R}))^N, \, u \in K \text{ for q.e. } (x, t) \in E\}$$

where $K \subset R^N$ is a closed convex set and $E \subset \bar{E} \subset Q = \Omega \times (0, T)$ is a Borel set.

The continuity of u at an arbitrarly given point $z_0 = (x_0, t_0) \in \bar{E}$ is obtained by estimating the quantity

$$V(r) = (\operatorname{osc}_{Q(z_0,r)} u) + \Bigg(\int\limits_{Q(z_0,r)} |D_x u|^2 G^{z_0} \, dx \, dt\Bigg)^{\frac{1}{2}}$$

(where G^{z_0} is the Green function of the linear part of the parabolic operator with singularity at z_0) and this also gives an estimate of the modulus of continuity of u at z_0 in terms of the so called « Wiener integral ».

We recall that the analogous elliptic problem was solved in [8].

In the parabolic case the continuity of weak solutions of scalar obstacle problems has been studied by M. BIROLI and U. MOSCO in [3]. Here we use a refinement of the methods in [2, 3] and a suitable Poincaré's inequality to solve our problem.

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Finally we observe that the capacity used in the present paper, which is the same as the one used in [3], is weaker than the capacity used by W. P. Ziemer [11], and M. Biroli, U. Mosco [2], studying a qualitative Wiener criterion and Wiener estimates at the boundary for weak solutions of parabolic equations.

The present capacity is also weaker than the one used by R. Gariepy and W. P. Ziemer [7], to give a qualitative Wiener criterion for weak solutions of parabolic equations, but seems to be the adapted notion for problems of variational inequalities.

1. – Notations and preliminaries.

By Ω we denote a bounded open set in R^n, $n\geqslant 3$, and by $B(x, r)$, $x\in \boldsymbol{R}^n$, we denote the open ball

$$B(x, r) = \{y \in \boldsymbol{R}^n;\ |x-y| < r\}$$

For a given $T>0$ we put

$$Q = \Omega\times(0, T)$$

For every $z=(x, t)\in \boldsymbol{R}^{n+1}$ and $r>0$ we define

$$\begin{aligned} Q(z, r) &= B(x, r)x(t-r^2, t+r^2)\\ Q^-(z, r) &= B(x, r)x(t-r^2, t)\\ Q^-_\theta(z, r) &= B(x, r)\times(t-r^2, t-6\theta r^2) \end{aligned}$$

where $\theta\in(0, \frac{1}{6})$.

Let C be a compact subset of Q. We define the capacity of C relative to Q, by setting.

$$\operatorname{cap}_Q(C) = \inf\left\{\iint\limits_Q |D_x\varphi|^2\,dx\,dt;\ \varphi\in C_0^\infty(Q), \varphi\geqslant 1 \text{ on a neighbourhood of } C\right\}.$$

We have so defined a Choquet capacity [4], and we observe that every Borel set is capacitable [4.5].

We recall that if $B\subset\bar{B}\subset Q$ is a Borel set then

$$\operatorname{cap}_Q(B) = \int_0^T N - \operatorname{cap}_Q(B_t^{\circ})\,dt$$

where B_t denotes the section of B at the instant t and the N-$\operatorname{cap}_\Omega$ is the Newtonian elliptic capacity relative to Ω, [3]. By $H^{1,p}(\Omega)$, $1\leqslant p<+\infty$, we denote as usual the Sobolev space of all functions $w\in L^p(\Omega)$ with distributional derivatives $D_{x_i}w\in$

$\in L^p(\Omega)$, normed by

$$\|w\|_{H^{1,p}} = \left(\|w\|_{L^p}^p + \sum_{i=1}^{n} \|D_{x_i} w\|_{L^p}^p\right)^{1/p}$$

and by $H_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega)$.

We define also $H^1(\Omega) = H^{1,2}(\Omega)$, $H_0^1(\Omega) = H_0^{1,2}(\Omega)$.

By $L^2(0, T; H^1(\Omega))\big(L^2(0, T; H_0^1(\Omega))\big)$ we denote the space of all functions $w(x, t)$ such that $w(\cdot, t) \in H^1(\Omega)(H_0^1(\Omega))$ a.e. in $(0, T)$ and $t \to w(\cdot, t)$ is square-integrable in $(0, T)$ with values in $H^1(\Omega)(H_0^1(\Omega))$.

By $L^\infty(0, T; L^2(\Omega))$ we denote the space of all functions $w(x, t)$ on Q such that $w(\cdot, t) \in L^2(\Omega)$ for a.e. $t \in (0, T)$ and $\|w(\cdot, t)\|_{L^2(\Omega)}$ is (essentially) bounded on $(0, T)$.

Finally, given a space X, we denote $X^N = X \times \dots \times X$ N-times. For some given functions $a_{ij} \in L^\infty(Q)$, $i, j = 1, \dots, n$, satisfying the conditions

$$|a_{ij}(x, t)| \leqslant \Lambda \tag{1.1}$$

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\, \xi_i \xi_j \geqslant \lambda |\xi|^2 \qquad \text{for a.e. } (x, t) \in Q \tag{1.2}$$

where $\Lambda \geqslant \lambda > 0$, we consider the (formal) operator

$$P = D_t - \sum_{i,j=1}^{n} D_{x_i}\big(a_{ij}(x, t) D_{x_j}\big) .$$

Given $z = (x, t)$ we denote by $G^z = G(z, w)$ the Green function for the operator P in a large cylinder $Q_0 = \Omega_0 \times (-T_0, T_0)$ with homogeneous Cauchy-Dirichlet boundary conditions [1]. In [1] the following estimate on $G^z(w)$, $w = (y, s)$, has been proved

$$\gamma_1 \frac{1}{|t-s|^{n/2}} \exp\left(-\gamma_1' \frac{|x-y|^2}{|t-s|}\right) \leqslant G^z(w) \leqslant \gamma_2 \frac{1}{|t-s|^{n/2}} \exp\left(-\gamma_2' \frac{|x-y|^2}{|t-s|}\right)$$

for arbitrary $y \in \Omega$ and $s < t$, $G^z(w) = 0$ for $s > t$ and $\gamma_1, \gamma_1', \gamma_2, \gamma_2'$ are suitable positive constants which depend only on n, λ, Λ.

By G_ϱ^z, $\varrho > 0$, we denote the regularized Green function which is the (unique) solution in $L^2(-T_0, T_0; H_0^1(\Omega))$ of the problem

$$\int_{Q_0} G_\varrho^z D_t \varphi \, dx\, dt + \sum_{i,j=1}^{n} \int_{Q_0} a_{ij} D_{x_i} G_\varrho^z D_{x_j} \cdot \varphi \, dx \, dt = \fint_{Q(z,\varrho)} \varphi \, dx \, dt$$

for every $\varphi \in C_0^\infty(Q_0)$, with the «initial» condition $G_\varrho^z(x, T_0) = 0$ for a.e. $x \in \Omega_0$, where $\fint_Q v \, dx\, dt$ denotes the average of v on Q. By Nash-Moser theorem G_ϱ^z is Hölder continuous in Q_0 and as $\varrho \to 0+$ we have $G_\varrho^z(w) \to G^z(w)$ for every $w \neq z$ and uniformly on every compact set of $Q_0 - \{z\}$; moreover $G_\varrho^z \to G^z$ weakly in $H^1(Q_0 - \{z_0\})$ and in $L^1(-T_0, T_0; W^{1,1}(\Omega_0))$.

2. – Results.

We state now the problem we are interested in.

Let $H(x, t, u, p)\colon Q\times R^N\times R^{nN}\to R^N$ be a function such that H is measurable in (x, t) and continuous in (u, p) for a.e. $(x, t)\in Q$. We suppose that the function H satisfies the following condition

$$|H(x, t, u, p)|\leq a(M)|p|^2 + b(M) \tag{2.1}$$

for $(x, t)\in Q$, $|u|\leq M$ and $p\in \boldsymbol{R}^{Nn}$.

We fix now a Borel set $E\subset \bar{E}\subset Q$ and a closed convex $K\subset \boldsymbol{R}^N$ and we define

$$\mathcal{K} = \left\{u\in \left(L^2(0, T; H^1(\Omega))\right)^N\cap (L^\infty(Q))^N,\ u(x, t)\in K \text{ q.e. in } E\right\}$$

We look at a function u in $\mathcal{K}$ satisfying

$$\begin{aligned}\int_0^T\int_\Omega \Big\{D_t v\varphi(v-u) + \sum_{i,j=1} a_{i_j} D_{x_j} u D_{x_i}(\varphi(v-u)) &+ \\ + \tfrac{1}{2} D_t\varphi|v-u|^2 + H(\cdot,\cdot, u, D_x u)\varphi(v-u)\Big\}\,dx\,dt &\geq 0\end{aligned} \tag{2.2}$$

$$\forall v\in \mathcal{K}\cap \left(H^1(0, T; L^2(\Omega))\right)^N$$

$$\forall \varphi\in L^2(0, T; H_0^1(\Omega))\cap H^1(0, T; L^2(\Omega))\cap L^\infty(Q)\,,\qquad \varphi\geq 0\,,$$

$$\varphi(\cdot, o) = 0\,.$$

We say that u is *a bounded local weak solution* of our variational inequality.

Let $\theta\in(o, \frac{1}{6})$ be given and $r>0$; we denote $E^\theta(r) = E(r) = E\cap Q_\theta^-(z_0, r)$ and

$$\delta_\theta(r) = \operatorname{cap}_{Q(z_0,2r)}(E\cap Q_\theta^-(z_0, r))\,(\operatorname{cap}_{Q(z_0,2r)} Q(z_0, r))^{-1}\,.$$

THEOREM 1. – *Let u be a bounded local weak solution of* (2.2) *satisfying*

$$2a(M)M < \lambda \tag{2.3}$$

where $M = \operatorname{Sup}_Q |u|$.

Let $z_0\in \bar{E}$ be fixed; there exists a constant θ_0 (dependent on $\lambda, \Lambda, M, a = a(M)$, $b = b(M)$) *such that* for $\theta\in(0, \theta_0)$ we *have*

$$V(r)\leq C\exp\left(-\beta\int_r^R \delta_\theta(\varrho)\frac{d\varrho}{\varrho}\right)(V(R) + C_1 R) + C_2 r$$

where

$$V(r) = \left(\int_{Q(z_0,r)} |D_x u|^2 G^{z_0}\, dx\, dt \right)^{\frac{1}{2}} + \operatorname{osc}_{Q(z_0,r)} u$$

and C is an absolute constant, β, C_1, C_2 are constants dependent on λ, Λ, M, a, b, θ.

From Th. 1 corollaries 1.2 follow easily:

COROLLARY 1. – *Let the conditions in Theoem 1 hold and suppose*

$$\int_r^1 \delta_\theta(\varrho) \frac{d\varrho}{\varrho} \to + \infty \quad \text{as } r \to 0$$

for a fixed $\theta \in (0, \theta_0)$.

Then u is continuous at z_0

COROLLARY 2. – *Let the conditions in Theorem 1 hold and suppose*

$$\liminf_{r\to 0} \frac{1}{|\lg r|} \int_r^1 \delta_\theta(\varrho) \frac{d\varrho}{\varrho} = c_0 > 0$$

for a fixed $\theta \in (0, \theta_0)$.

Then u is Hölder continuous at z_0.

In Sec. 3 we prove a Caccioppoli-De Giorgi inequality for bounded local weak solutions satisfying (2.3); this inequality is a preliminary to the proof of Th. 1.

In Sec. 4 we prove a Poincaré inequality for bounded local weak solutions relative to the capacity introduced in Sec. 0 and involving only the spatial derivatives.

In Sec. 5, using the Poincaré inequality and the integration lemma of [6], we finish the proof of Th. 1.

In Sec. 6 we give the easy proofs of Corollaries 1, 2.

We remark that the method used to prove Th. 1 is a refinement of the method used in [3] to study the continuity of local solutions of parabolic obstacle problems and that the elliptic case of our problem has been studied in [8].

3. – Estimates of Caccioppoli-De Giorgi type.

In this section we consider arbitrary bounded local solutions of our problem satisfying (2.2).

We prove the following

LEMMA 1. – *Let be a constant vector in K with $|d| \leqslant M$. Then for $\bar{z} \in Q(z_0, R/4)$,*

$R \leqslant R_0$, q.e. *the following estimate holds*

$$\int_{\bar{t}-3\theta R^2}^{\bar{t}} \int_{B(\bar{x},R/8)} |D_x u|^2 G^{\bar{z}}\, dx\, dt + |u-d|^2(\bar{z}) \leqslant$$
$$\leqslant CR^{-2} \int_{\bar{t}-5\theta R^2}^{\bar{t}} \int_{B(\bar{x},R/3)-B(\bar{x},R/16)} |u-d|^2 \cdot \left(G^{\bar{z}} + \sigma R^{-n/2} \cdot (G^{\bar{z}})^{\frac{1}{2}} + \sigma^{-1} R^{n/2} (G^{\bar{z}})^{\frac{3}{2}} + \sigma R^{-n}\right) dx\, dt +$$
$$+ CR^{-2}\theta^{-1} \int_{\bar{t}-5\theta R^2}^{t-3\theta R^2} \int_{B(\bar{x},R/4)} |u-d|^2 G^{\bar{z}}\, dx\, d + C(\theta) R^2$$

where $\sigma > o$ *is suitable,* $\theta \in (0, 1/16)$ *and the constant* C *depends only on* $(n, \lambda, \Lambda, a, b, M)$.

Let $\eta = \eta(x)$ and $\tau = \tau(t)$ be such that

$$n = 1 \qquad x \in B(\bar{x}, R/8)$$
$$\eta = 0 \qquad x \in B(\bar{x}, R/4)$$
$$0 \leqslant \eta \leqslant 1 \qquad x \in B(\bar{x}, R/4)$$
$$\eta \in C_0^\infty(\boldsymbol{R}^n) \qquad |D_x \eta| \leqslant CR^{-1}$$
$$\tau = 1 \qquad t \geqslant \bar{t} - 3\theta R^2$$
$$\tau = 0 \qquad t \leqslant \bar{t} - 5\theta R^2$$
$$\tau \in C_0^\infty(\boldsymbol{R}) \qquad |D_t \tau| \leqslant C\theta^{-1} R^{-2}.$$

By choosing $v = d$ and $\Phi = \eta^2 \tau^2 G_\varrho^{\bar{z}}$ with $\varrho < \theta R^2$ in the variational inequality, we have

$$\int_{Q(\bar{z},R)} |D_x u|^2 \eta^2 \tau^2 G_\varrho^{\bar{z}}\, dx\, dt + \frac{1}{|Q(\bar{z},\varrho)|} \int_{Q(\bar{z},\varrho)} |u-d|^2 \eta^2\, dx\, dt \leqslant \tag{3.1}$$
$$\leqslant C \left\{ \int_{t-5\theta R^2}^{\bar{t}+\varrho^2} \int_{B(\bar{x},R/4)} |u-d|^2 |D_x \eta| G_\varrho^{\bar{z}}\, dx\, dt + \int_{\bar{t}-5\theta R^2}^{\bar{t}+\varrho^2} \int_{B(\bar{x},R,4)} u-d|^2 \eta \tau^2 |D_x \eta|\, |D_x G_\varrho^{\bar{z}}|\, dx\, dt + \right.$$
$$\left. + 2bM \int_{\bar{t}-5\theta R^2}^{\bar{t}+\varrho^2} \int_{B(\bar{x},R/4)} \eta^2 \tau^2 G_\varrho^{\bar{z}}\, dx\, dt + \int_{t-5\theta R^2}^{\bar{t}+\varrho^2} \int_{B(\bar{x},R/4)} |u-d|^2 |D_t \tau|\, G_\varrho^{\bar{z}}\, dx\, dt \right\}$$

where by C we denote constants which depend only on $n, \lambda, \Lambda, a, b, M$ (actually in (3.1) C does not depend on b).

Passing to the limit as $\varrho \to 0$ and taking into account the estimates on $G^{\bar z}$, see [1], we have for q.e. $\bar z$

$$\int_{Q(\bar z, R)} |D_x u|^2 \eta^2 \tau^2 G^{\bar z} + |u-d|^2(\bar z) \leqslant C\Big\{R^{-2} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(\bar x, R/4) - B(\bar x, R/8)} |u-d|^2 G^{\bar z}\, dx\, dt + \tag{3.2}$$

$$+ R^3 + R^{-2}\theta^{-1} \int_{\bar t - 5\theta R^2}^{\bar t - 3\theta R^2} \int_{B(\bar x, R/4)} |u-d|^2 G^{\bar z}\, dx\, dt + \int_{\bar t - R}^{\bar t} \int_{B(\bar x, R/4)} |u-d|^2 \eta\tau |D_x \eta|^2 |D_x G^{\bar z}| dx\, dt\Big\}.$$

Let us consider the last term on the right hand side of (3.2); we obtain easily

$$\int_{\bar t - R^2}^{\bar t} \int_{B(\bar x, R/4)} |u-d|^2 \eta \tau^2 |D_x \eta| |D_x G^{\bar z}|\, dx\, dt \leqslant \tag{3.3}$$

$$\leqslant \sigma R^{-n/2} \int_{\bar t - R^2}^{\bar t} \int_{B(x, R/4) - B(x, R/8)} \eta^2 \tau |u-d|(|D_x G^{\bar z}|^2 (G^{\bar z})^{-\frac{3}{2}}\, dx\, dt +$$

$$+ C\sigma^{-1} R^{n/2-2} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(\bar x, R/4) - B(\bar x, ER/8)} 4\sigma^{-1} R^{n/2-2} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(\bar x, R/4) - B(\bar x, R/8)} |u-d|^2 (G^{\bar z})^{\frac{3}{2}}\, dx\, dt$$

where $\sigma > 0$ is to be choosen.

The first term can be estimated by the lemma 2 below and we obtain

$$\int_{\bar t - R^2}^{\bar t} \int_{B(\bar x, R/4) - B(\bar x, R/8)} \eta^2 \tau^2 |u-d|^2 |D_x G^{\bar z}| (G^{\bar z})^{-\frac{3}{2}}\, dx\, dt \leqslant \tag{3.4}$$

$$\leqslant C\Big[R^{-2} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(\bar x, R/3) - B(\bar x, R/16)} |u-d|^2 (G^{\bar z})^{\frac{1}{2}}\, dx\, dt + R^{n/2} \int_{\bar t - R^2}^{\bar t} \int_{B(\bar x, R)} |D_x u|^2 \eta^2 \tau^2 G^{\bar z}\, dx\, dt +$$

$$+ R^{-(n/2+2)} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(x, R/3) - B(x, R/16)} |u-d|^2\, dx\, dt + R^{n/2+2}\Big]$$

from (3.2) (3.3) and (3.4) choosing $\sigma > 0$ suitable we have

$$\int_{\bar t - R^2}^{\bar t} \int_{B(\bar x, R)} |D_x u|^2 \eta^2 \tau^2 G^{\bar z}\, dx\, dt + |u-d|^2(\bar z) \leqslant \tag{3.5}$$

$$\leqslant C\Big[R^{-2} \int_{\bar t - 5\theta R^2}^{\bar t} \int_{B(\bar x, R/3) - B(\bar x, R/16)} |u-d|^2 (G^{\bar z} + \sigma R^{-n/2} (G^{\bar z})^{\frac{1}{2}} +$$

$$+ \sigma^{-1} R^{n/2} (G^{\bar z})^{\frac{3}{2}} + \sigma R^{-n})\, dx\, dt + R^{-2}\theta^{-1} \int_{\bar t - 5\theta R^2}^{\bar t - 3\theta R^2} \int_{B(\bar x, R/4)} |u-d|^2 G^{\bar z}\, dx\, dt + R^2\Big]$$

which is the result of lemma 1.

Let now ω be such that

$$\omega = \tilde{\omega}\eta$$

where $\tilde{\omega} \in C_0^\infty((B(\bar{x}, R))$

$$\tilde{\omega} = o \text{ in } B(\bar{x}, R/16) \text{ and for } x \notin B(\bar{x}, (5/16)R)$$

$$\tilde{\omega} = 1 \text{ in } B(\bar{x}, R/4) - B(\bar{x}, R/8)$$

$$0 \leqslant \tilde{\omega} \leqslant 1 .$$

We now complete the proof of the lemma 1 by proving:

Lemma 2. – *Let u be a bounded weak solution of our problem satisfying* (2.2). *Then the following estimate holds*

$$\int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},R)} \omega^2\tau^2|u-d|^2|D_xG^{\bar{z}}|^{-\frac{3}{2}}\,dx\,dt \leqslant C\Big[R^{-2}\int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},5R/16)-B(\bar{x},R/16)} \tau|u-d|^2(G^{\bar{z}})^{\frac{1}{2}}\,dx\,dt + \\ + R^{n/2}\int_{\bar{t}-R^2}^{\bar{t}} \int_{B(\bar{x},R)} |D_xu|^2G^{\bar{z}}\eta^2\tau^2\,dx\,dt + R^{-n/2-2}\int_{\bar{t}-R^2}^{\bar{t}} \int_{(B\,x,(5/16)R)-B(\bar{x},R/16)} |u-d|^2\,\tau^2\,dx\,dt + R^{n2+2}\Big] .$$

From the definition of the regularized Green function we have

$$\int_{t-R^2}^{t-\chi} \langle D_tG_\varrho^{\bar{z}} + \sum_{i,j=1}^N D_{x_j}(a_{ij}\,D_{x_i}G_\varrho^{\bar{z}}), \omega^2\tau^2(\delta G_\varrho^{\bar{z}})^{\frac{1}{2}}|u-d|^2\rangle\,dt = 0 \qquad ({}_\delta G_\varrho^{\bar{z}} = G_\varrho^{\bar{z}} + \delta,\ \delta > 0) .$$

Then since u is a local solution of our variational inequality we obtain, after some computations.

$$\text{(3.6)}\qquad \int_{\bar{t}-R^2}^{\bar{t}-\chi} \int_{B(\bar{x},R)} ({}_\delta G_\varrho^{\bar{z}})^{\frac{1}{2}}D_t\tau\,\tau\omega^2\,|u-d|^2\,dx\,dt - \\ -\tfrac{1}{2}\sum_{i,j=1}^n \int_{\bar{t}-R^2}^{\bar{t}-\chi} \int_{B(z,R)} a_{ij}D_{x_i}G_\varrho^{\bar{z}}({}_\delta G_\varrho^{\bar{z}})\,D_{x_j}G_\varrho^{\bar{z}}\omega^2\tau^2|u-d|^2\,dx\,dt + \\ + \sum_{i,j=1}^n \int_{\bar{t}-R^2}^{\bar{t}-\chi} \int_{B(\bar{x},R)} 2a_{ij}D_{x_i}G_\varrho^{\bar{z}}D_{x_j}\omega\,\omega\tau^2\,|u-d|^2(G_\varrho^{\bar{z}})^{-\frac{1}{2}}\,dx\,dt - \\ - \sum_{i,j=1}^n \int_{\bar{t}-R^2}^{\bar{t}-\chi} \int_{B(\bar{x},R)} 2a_{ij}({}_\delta G_\varrho^{\bar{z}})^{\frac{1}{2}}D_{x_i}\omega D_{x_j}(u-d)\,\omega(u-d)\,\tau^2\,dx\,dt + \\ + b\int_{\bar{t}-R^2}^{\bar{t}-\chi} \int_{B(x,R)} |u-d|\omega^2\tau^2\,({}_\delta G_\varrho^{\bar{z}})^{\frac{1}{2}}\,dx\,dt \geq 0 .$$

From (3.6) by the same methods as in Appendix of [2] and taking into account the estimates on $G^{\bar{z}}$ given in [1] we obtain the result.

From the result of lemma 1 taking the supremum for $\bar{z} \in Q\ (z_0, \theta^{\frac{1}{2}} R)$ and using again the estimates on $G^{\bar{z}}$ we have by a suitable choice of σ:

LEMMA 3. – *Let the conditions of lemma* 1 *hold, there exists a constants* $\theta_0 > 0$ *such that* for $\theta \in (0, \theta_0)$ *we have*

$$\int_{{}^0t-\theta R^2}^{t_0} \int_{B(x_0,\theta^{\frac{1}{2}}R)} |D_x u|^2 G^{z_0}\,dx\,dt + \operatorname{Sup}_{Q(z_0,\theta^{\frac{1}{2}}R)} |u-d|^2 \leqslant$$
$$\leqslant C \exp(-C\theta^{-1})\,\theta^{-3n/4} R^{-(n+2)} \int_{Q(z_0,R)} |u-d|^2\,dx\,dt +$$
$$+ C\theta^{-(n/2+1)} R^{-(n+2)} \int_{t_0-6\theta R^2}^{t_0-2\theta R^2} \int_{B(x_0,(3/8)R)} |u-d|^2\,dx\,dt + C(\theta) R^2 .$$

We have

$$\operatorname*{Sup}_{Q(z_0,\theta^{\frac{1}{2}}R)} |u-d|^2 \geqslant \tfrac{1}{4} (\operatorname{osc}_{Q(z_0,\theta^{\frac{1}{2}}R)} u)^2 . \tag{3.7}$$

We now choose d such that

$$\operatorname*{Sup}_{Q(z_0,R)} |u-d|^2 \leqslant C (\operatorname{osc}_{Q(z_0,R)} u)^2 . \tag{3.8}$$

From (3.7) (3.8) and the lemma 3 we obtain.

LEMMA 4. – *Let the conditions of lemma* 1 *hold and let d be such that* (3.8) *holds. Then for* θ_0 *as in lemma* 3 *and* $\theta \in (0, \theta_0)$ *we have*

$$\int_{t_0-\theta R^2}^{t_0} \int_{B(x_0,\theta^{\frac{1}{2}}R)} |D_x u|^2 G^{z_0}\,dx\,dt + (\operatorname{osc}_{Q(z_0,\theta^{\frac{1}{2}}R)} u)^2 \leqslant C \exp(-C\theta^{-1})\,\theta^{-3n/4} (\operatorname{osc}_{Q(z_0,R)} u)^2 +$$
$$+ C\theta^{-(n/2+1)} R^{-(n+2)} \int_{t_0-6\theta R^2}^{t_0-2\theta R^2} \int_{B(x_0,(3/8)R)} |u-d|^2\,dx\,dt + C(\theta) R^2 .$$

4. – A Poincaré inequality.

Let z_0 be such that cap $(E(R)) \neq 0$ and $\bar{u} \in R^N$ be defined by the minimization problem

$$\inf \left\{ \int_{t_0-6\theta R^2}^{t_0-\theta R^2} \int_{B(z_0,R/2)} |u-c|^2\,dx\,dt;\ c \in K,\ |c| \leqslant M \right\} . \tag{4.1}$$

REMARK. – Since cap $(E(R) \neq 0$ there is a constant vector $c \in K$

$$|c| \leqslant M \quad \text{such that} \quad |u-c| \leqslant \omega = \operatorname{osc}_{Q(z_0,R)} u .$$

Due to the definition of $\bar{u}$.

$$\inf_{Q(z_0,R)} |u-\bar{u}| \leqslant \omega$$

and this implies

$$\operatorname{Sup}_{Q(z_0,R)} |u-\bar{u}| \leqslant 3\omega\ .$$

We denote in the following by $E_t(R)$ the section of $E(R)$ at the instant t.

From the variational inequality we obtain

$$(4.2)\quad \tfrac{1}{2}\left\|\big(u(s)-\bar{u}(t)\big)\zeta\right\|^2_{L^2(\Omega)} + \lambda\int_t^s\int_\Omega |D_x u|^2\zeta^2\,dx\,d\eta \leqslant \tfrac{1}{2}\left\|\big(u(t)-\bar{u}(t)\big)\zeta\right\|^2_{L^2(\Omega)} - \\ - 2\Lambda\int_t^s\int_\Omega (D_x u)(D_x)\zeta\,\big(u-\bar{u}(t)\big)\zeta\,dx\,d\eta + \int_t^1\int_\Omega \big[a|D_x u|^2\big(u-\bar{u}(t)\big)\zeta^2 + b\left|\big(u(t)-\bar{u}(t)\big)\right|\zeta^2\big]dx\,d\eta$$

Here $\bar{u}(t) = \int_{E_t(R)} u(x,t)\,d\mu(x)$ where μ is a unit measure on $E_t(R)$ such that

$$(4.2)\qquad \int_{B(x_0,R)} |u(x,t)-\bar{u}(t)|^2\,dx \leqslant \frac{C}{N-\operatorname{cap}\big(E_t(R)\big)}\int_{B(x_0,R)} |D_x u|^2(x,t)\,dx\ .$$

The existence of such a measure μ has been proved in [9] (see also [8]).

The function ζ is such that

$$\zeta \in C_0^\infty\big(B(x_0,R)\big)\qquad 0 \leqslant \zeta \leqslant 1$$
$$\zeta = 1 \qquad \text{in } B(x_0,R/2)\ .$$

From (4.2) we have for $t \in [t_0 - R^2, t_0 - 6\theta R^2]$.

$$(4.4)\qquad \operatorname{Sup}_{s\in(t,t_0-\theta R^2)} \|u(s)-\bar{u}(t)\|^2_{L^2(B(x_0,R/2))} \leqslant \\ \leqslant C\left[\|u(t)-\bar{u}(t)\|^2_{L^2(B(x_0,R))} + \int_t^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dx\,d\eta + R^{n+2}\right].$$

Now we apply the elliptic Poincaré inequality (4.3) to the first term on the right side of (4.4) and we obtain

$$(4.5)\qquad \operatorname{Sup}_{s\in(t_0-6\theta R^2,t_0-\theta R^2)} \|u(s)-\bar{u}(t)\|^2_{L^2(B(x_0,R/2))} \leqslant \\ \leqslant C\left[\frac{R^n}{N-\operatorname{cap}\big(E_t(R)\big)}\int_{B(x_0,R)} |D_x u|^2(cx,t)\,dx + \int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)} |D_x u|^2\,dx\,d\eta + R^{n+2}\right].$$

In (4.5) and in the following of the section we denote

$$N-\operatorname{cap}\left(E_t(R)\right)=N-\operatorname{cap}_{B(x_0,2R)}\left(E_t(R)\right), \qquad \operatorname{cap}\left(E(R)\right)=\operatorname{cap}_{Q(z_0,2R)}\left(E(R)\right).$$

We observe that $\bar{u}(t)\in K$, $|\bar{u}(t)|\leqslant M$, then

$$(4.6)\qquad \int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,R/2)}|u-\bar{u}|^2\,dx\,d\eta\leqslant\int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,R/2)}|u-\bar{u}(t)|^2\,dx\,d\eta+C\theta\omega R^{n+2}.$$

From (4.5) and (4.6) we have

$$(4.7)\qquad \int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,R/2)}|u-\bar{u}|^2\,dx\,d\eta\leqslant C\left[\frac{R^{n+2}}{N-\operatorname{cap}\left(E_t(R)\right)}\int_{B(x_0,R)}|D_x u|^2(x,t)\,dx+\right.$$
$$\left.+R^2\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)}|D_x u|^2\,dx\,d\eta+R^{n+4}+C\theta\omega R^{n+2}\right].$$

We integrate for $t\in(t_0-R^2,\,t_0-6\theta R^2)$; then

$$(4.8)\qquad \int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,R/2)}|u-\bar{u}|^2\,dx\,d\eta\leqslant$$
$$\leqslant C\left[\frac{R^{n+2}}{\operatorname{cap}\left(E(R)\right)}\int_{t_0-R^2}^{{}_0t-\theta R^2}\int_{B(x_0,R)}|D_x u|^2\,dx\,d\eta+R^{n+4}+C\theta\omega R^{n+2}\right].$$

We have so proved the following

PROPOSITION 1. – *Let* $\operatorname{cap}\left(E(R)\right)\neq 0$ *and* $\bar{u}$ *defined as in* (4.1). *Then we have*

$$\int_{t_0-6\theta R^2}^{t_0-\theta R^2}\int_{B(x_0,R/2)}|u-\bar{u}|^2\,dx\,d\eta\leqslant C\left[\frac{R^2}{\delta_\theta(R)}\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)}|D_x u|^2\,dx\,d\eta+R^{n+4}+C\theta\omega R^{n+2}\right].$$

5. – Proof of Theorem 1.

From the lemma 4 and the Poincarθ inequality we obtain choosing $d=\bar{u}$ (this is possible by the Remark of Sec. 4)

$$(5.1)\qquad \int_{t_0-\theta R^2}^{t_0}\int_{B(x_0,\theta^{\frac{1}{2}}R)}|D_x u|^2 G^{z_0}\,dx\,dt+\left(\operatorname{osc}_{Q(zz_0\,\theta^{\frac{1}{2}}R)}u\right)^2\leqslant$$
$$\leqslant C\left(\exp\left(-C\theta^{-1}\right)\theta^{-3n/4}\right)\left(\operatorname{osc}_{Q(z_0,R)}u\right)^2+$$
$$+C\theta^{-(n/2+1)}\frac{1}{R^n\,\delta_\theta(R)}\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)}|D_x u|^2\,dx\,dt+C(\theta)R^2.$$

By the estimates on the Green function [1], we obtain

$$\int_{t_0-\theta R^2}^{t_0}\int_{B(x_0,\theta^{\frac12}R)}|D_x u|^2 G^{z_0}\,dx\,dt+(\operatorname{osc}_{Q(z_0,\theta^{\frac12}R)}u)^2\leqslant \tag{5.2}$$

$$\leqslant CK_1(\theta)(\operatorname{osc}_{Q(z_0,R)}u)^2+\frac{1}{CK_2(\theta)\delta_\theta(R)}\int_{t_0-R^2}^{t_0-\theta R^2}\int_{B(x_0,R)}|D_x u|^2 G^{z}{}_{0}\,dx\,dt+C(\theta)R^2$$

where

$$K_1(\theta)=\exp(-C\theta^{-1})\theta^{-3n/4}$$

$$K_2(\theta)=\exp(-C\theta^{-1})\theta\,.$$

LEMMA 5. *There exists* $0_1>0$, *depending only on* n, λ, Λ, a, b, M, *such that for* $\theta\in(0,\theta_1)$ *if*

$$V^2(\theta^{\frac12}R)\geqslant 2C(\theta)R^2\,, \tag{5.3}$$

then we have

$$V^2(\theta^{\frac12}R)\leqslant\frac{1+K_3(\theta)\delta_\theta(R)}{1}V^2(R)$$

where $V(R)$ *is defined as in Sec.* 0 *and* $K_3(\theta)=C\theta\exp(-C\theta^{-1})$.

Let

$$\Phi(r)=(\operatorname{osc}_{Q(z_0,r)}u)^2$$

$$\Psi(r)=\int_{Q(z_0,r)}|D_x u|^2G^{z_0}\,dx\,dt\,.$$

Let (5.3) hold. From (5.2) we dotain

$$\big(1+C_1K_2(\theta)\delta_\theta(R)\big)\big(\Phi(\theta^{\frac12}R)+\Psi(\theta^{\frac12}R)\big)\leqslant\big(1+C_2K_1(\theta)K_2(\theta)\delta_\theta(R)\big)\Phi(R)+\Psi(R)\,.$$

Then

$$\Phi(\theta^{\frac12}R)+\Psi(\theta^{\frac12}R)\leqslant\frac{1+C_2K_1(\theta)K_2(\theta)\delta_\theta(R)}{1+C_1K_2(\theta)\delta_\theta(R)}\Phi(R)+\frac{1}{1+C_1K_2(\theta)\delta_\theta(R)}\Psi(R)\,.$$

We observe that

$$\frac{1+C\sigma x}{1+Cx}\leqslant\frac{1}{1+C/2^u}\qquad 0\leqslant x\leqslant 1$$

for $0<C<1$ and $0<\sigma<\frac13$, then if we choose θ_1 such that $0<C_1K_2(\theta_1)<1$ and $0<(C_2/C_1)K_1(\theta_1)<\frac13$ we have the result. From lemma 5 we have

$$V^2(\theta^{\frac12}R)\leqslant\frac{1}{1+K_3(\theta)\delta_\theta(R)}V^2(R)+2C(\theta)R^2$$

then for $\theta \in (0, \theta_2)$, $\theta_2 > 0$ suitable, there exists a constant $C_3(\theta)$ such that, if we define

$$W(r) = V^2(r) + C_3(\theta) r^2 ,$$

we have

$$W(r) \leqslant \frac{1}{1 + K_3(\theta)\,\delta_\theta(R)}\, W(R) \qquad (r \leqslant \theta^{\frac{1}{2}} R) . \tag{5.4}$$

Form (5.4) and the integration lemma given in [6, 10] (where the continuity of the solution of an elliptic variational inequality with wiener obstacles is studied) we obtain

$$W(r) \leqslant C \exp\left(-2\beta \int_r^R \delta_\theta(\varrho)\,\frac{d\varrho}{\varrho}\right) W(R) \tag{5.5}$$

where

$$\beta = -\frac{K_4(\theta)}{|\lg \theta|}, \qquad K_4(\theta) = \frac{K_3(\theta)}{1 + K_3(\theta)} \qquad (\text{then } \beta \to 0 \text{ as } \theta \to 0) .$$

From (5.5) we obtain easily.

$$V(r) \leqslant C \exp\left(-\beta \int_r^R \delta_\theta(\varrho)\,\frac{d\varrho}{\varrho}\right) (V(R) + C_4 R) + C_5 r$$

where C_4 and C_5 depends on θ.

The result of Th. 1 is so proved.

6. – Corollaries 1 and 2.

The result of corollary 1 follows from the estimate in Th. 1 for $R = R_0$, R_0 suitable and fixed, and $r \to 0$.

We now give a proof of corollary 2; for $r \leq \bar{R}$, $\bar{R}$ suitable, we have

$$2^{-1} c_0 \leqslant |\lg r| \int_r^1 \delta_\theta(\varrho)\,\frac{d\varrho}{\varrho} \leqslant 1$$

then

$$\int_r^{R_0} \delta_\theta(\varrho)\,\frac{d\varrho}{\varrho} \leqslant 2^{-1} c_0\, |\lg r| - |\lg R_0| .$$

For $R = R_0$ suitable and fixed, we obtain (from the estimate of Th. 1)

$$V(r) \leq C r^{2^{-1} c_0 \beta}\, R_0^{-\beta} (V(R_0) + C_1 R_0) + C_2 r . \tag{6.1}$$

From (6.1) the Hölder continuity of u at x_0 can be easily deduced.

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