Existence of Symmetric Solutions for the Skyrme's Problem (*).

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Summary. – *We consider Skyrme's problem, a direct variational approach to the study of the structure of static configurations of mesons in a field of weak energy. In this paper we restrict ourselves to the consideration of two particular symmetry-conditions and prove the existence of minima for the corresponding energy among all the functions which satisfy those symmetry-conditions and have a fixed degree.*

1. – Introduction.

Let S^3 be the unit sphere of $\boldsymbol{R}^4$. Then for any function $\varphi\colon \boldsymbol{R}^3 \to S^3$ we define the functional:

(1) $$\mathcal{E}(\varphi) = \lambda \int_{\boldsymbol{R}^3} \left(K^2 |\nabla\varphi|^2 + |A(\varphi)|^2\right) dx$$

where λ and K^2 are positive physical constants and

$$|A(\varphi)|^2 = \sum_{\alpha,\beta=1}^{3} \left|\frac{\partial\varphi}{\partial x_\alpha} \wedge \frac{\partial\varphi}{\partial x_\beta}\right|^2 .$$

(We denote by $a \wedge b$ the alternating exterior product of any $a, b \in \boldsymbol{R}^4$.)

If we define the set

$$Y = \{\varphi \in C^1(\boldsymbol{R}^3, S^3)\colon \nabla\varphi, A(\varphi) \in L^2(\boldsymbol{R}^3, \boldsymbol{R}^4)\} ,$$

then the functional $\mathcal{E}$ is indeed well defined and finite on the following set:

$$X = \left\{\varphi\colon \boldsymbol{R}^3 \to S^3 \;\middle|\; \begin{array}{l} \nabla\varphi, A(\varphi) \in L^2(\boldsymbol{R}^3, \boldsymbol{R}^4) \text{ and } \exists\{\varphi_n\} \subset Y \quad \text{s.t.} \\ \nabla\varphi_n \to \nabla\varphi , \quad A(\varphi_n) \to A(\varphi) \text{ in } L^2(\boldsymbol{R}^3, \boldsymbol{R}^4) \end{array}\right\} .$$

We are interested in the existence of critical points of $\mathcal{E}$ in X. In view of Skyrme's work (see [S1, 2, 3]), it appears clearly that the elements of X which minimize the

(*) Entrata in Redazione il 15 dicembre 1985.

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energy $\mathcal{E}$ (in some sense to be defined below) are closely related to the stable configurations of a field of mesons. On the other hand, we observe that the quantity $1/2\pi^2\int_{\boldsymbol{R}^3} \det(\varphi, \nabla\varphi)\,dx$ is relevant in the study of those stable configurations of a field of mesons. In fact we notice that this quantity has also a geometrical meaning, since, for all φ in X, the quantity

$$d(\varphi) = \frac{1}{2\pi^2}\int_{\boldsymbol{R}^3} \det(\varphi, \nabla\varphi)\,dx \tag{2}$$

is the degree of $\varphi\circ E$, where E is a stereographic projection from S^3 to $\boldsymbol{R}^3$. Hence, $d(\varphi)$ is an integer number for all φ in X.

For all $k \in \boldsymbol{Z}$, we define the set

$$X_k = \{\varphi \in X : d(\varphi) = k\}\ . \tag{3}$$

Then we wish to solve the following family of minimization problems:

$$I_k = \inf\{\mathcal{E}(\varphi) : \varphi \in X_k\}\ . \tag{I_K^*}$$

This problem will be studied in [E2]. There we will restrict ourselves to a class of simpler problems which appear when making some symmetry assumption on the elements of the set where we minimize: In [S3] Skyrme introduced the class of functions $\varphi \in C(\boldsymbol{R}^3, S^3)$ such that:

$$\varphi(x) = \varphi_\omega(x) = \left(\frac{x}{|x|}\sin\omega(|x|),\ \cos\omega(|x|)\right), \tag{4}$$

where ω is a real function from $\boldsymbol{R}^+$ to $\boldsymbol{R}$ such that $\omega(0)$, $\omega(+\infty) = k\pi$ for some $k \in \boldsymbol{Z}$.

Furthermore he considered the set

$$Z_1 = \{\varphi \text{ of the form (4)}: \omega \in C(\boldsymbol{R}, \boldsymbol{R}) \text{ and } \omega(0) = \pi, \omega(+\infty) = 0\}\ ,$$

and solved the problem:

$$I_1^* = \operatorname{Min}\{\mathcal{E}(\varphi_\omega) : \varphi_\omega \in Z_1\}\ . \tag{I_1^*}$$

We will see later that minimizing $\mathcal{E}$ on Z_1 is equivalent to minimizing it on the set of φ's of the form (4) with degree equal to 1.

In [K-L] we find a clearer proof of the above result and also of the fact that any minimum of $\mathcal{E}$ in the set Z_1 (resp. Z_k) is indeed a critical point of $\mathcal{E}$ in X or X_1 (resp. X_k).

Here we will show that problem (I_k^*) has a solution for all $k \in \boldsymbol{Z}$ (and not only for the particular case treated in [Sk 3] and [K-L]).

Moreover we will consider a class of intermediate problems between (I_k) and (I_k^*) as follows:

We consider the fonctions $\varphi \in C^1(\boldsymbol{R}^3, S^3)$ such that there exists $\omega\colon \boldsymbol{R}^3 \to \boldsymbol{R}$ with

$$\varphi(x) \equiv \varphi_\omega(x) \equiv \left(\frac{x}{|x|}\sin\omega(x),\ \cos\omega(x)\right). \tag{5}$$

Here we will not ask ω to be radially symmetric. Then we define the sets:

$$Y_k = \{\varphi \in X\colon \varphi \text{ is of the form (5) and } d(\varphi) = k\}\ .$$

Then we show that the problem:

$$\tilde{I}_k = \operatorname{Min}\{\mathcal{E}(\varphi)\colon \varphi \in Y_k\} \tag{$\tilde{I}_k$}$$

has a solution for all $k \in \boldsymbol{Z}$.

We note that the type of symmetry we assume on the functions φ is very important, since from the physical point of view it means that we fix a priori the shape of the mesons. This is why the less symmetry we assume, the more general result we obtain about the existence and shape of the stable configurations of mesons in a field of weak energy.

Let us also remark that from a mathematical point of view, problem (I_k) is very close to the problem of finding large solutions for harmonic maps in a bounded domain of $\boldsymbol{R}^2$ (see [B-C]). Furthermore, similar technical problems appear when trying to find solutions for the Yang-Mills equations (see [T1, 2], [U1, 2]). For more details about the relationship between all these problems, see [E2].

Finally we note that in [E1] we prove some inequalities of isoperimetric type in $\boldsymbol{R}^3$ which are crucial to prove the results about (I_k) that we give in [E2].

Notation

For any three vectors of $\boldsymbol{R}^4$, a, b, c, we will denote by $a\wedge b$ (resp. $a\wedge b\wedge c$) the alternating exterior product of a, b (resp. a, b and c), which is an element of $\Lambda^2(\boldsymbol{R}^4)$ (resp. $\Lambda^3(\boldsymbol{R}^4) \simeq \boldsymbol{R}^4$).

On the other hand, we will use the following notations for sets of $\boldsymbol{R}^N$:

$$B_R(y) = \{x \in \boldsymbol{R}^3\colon |y - x| < R\}\ ,$$
$$S_R(y) = \{x \in \boldsymbol{R}^3\colon |y - x| = R\}\ ,$$

and when $y = 0$, we will just write B_R, S_R. Moreover we denote by B^N, S^{N-1} the following:

$$B^N = \{x \in \mathbf{R}^N: |x| < 1\}$$

$$S^{N-1} = \{x \in \mathbf{R}^N: |x| = 1\}$$

and for any $A \subset \mathbf{R}^N$, we will write indistinctly, meas (A) and $|A|$.

Finally we will denote by C various positive constants.

2. – Main results.

THEOREM 1. – *For any $k \in \mathbf{Z}$, problem $(\tilde{I}_k)$ has a solution, i.e., there is $\omega_k \in Y_k$ such that $\mathcal{E}(\varphi_{\omega_k}) = \tilde{I}_k$.*

COROLLARY 2. – *For any $k \in \mathbf{Z}$, there is a $\omega_k^* \in Z_k$ such that $\mathcal{E}(\varphi_{\omega_k^*}) = I_k^*$.*

REMARK. – If we consider the case $k = 0$, we find that $I_0 = I_0^* = \tilde{I}_0 = 0$, and these infima are attained by all the constant functions from $\mathbf{R}^3$ to S^3.

PROOF OF THEOREM 1. – First we write the energy $\mathcal{E}(\varphi)$ for the φ_ω of the form (5) and we obtain:

$$\begin{aligned}(6)\qquad \tilde{\mathcal{E}}(\omega) = \lambda \int_{\mathbf{R}^3} |\nabla\omega|^2 \left(K^2 + \frac{8\sin^4\omega}{|x|^2} + \frac{4\sin^2\omega\cos^2\omega}{|x|^2}\right) dx \,+ \\ + \lambda \int_{\mathbf{R}^3} \frac{2\sin^2\omega}{|x|^2}\left(K^2 + \frac{2}{|x|^2}\sin^2\omega\right) dx \,+ \\ + \lambda \int_{\mathbf{R}^3} \frac{4\sin^2\omega}{|x|^4}\left(\sin^2\omega|x\wedge\nabla\omega|^2 + (\cos^2\omega)(x\cdot\nabla\omega)^2\right) dx\,.\end{aligned}$$

In the next section we will prove that minimizing $\mathcal{E}$ over Y_k is equivalent to minimizing $\mathcal{E}$ over the class of ω such that $\mathcal{E}(\omega) < +\infty$ and $\omega(0) = k\pi$, $\omega(+\infty) = 0$. Let us note that here these values of ω at 0 and at $+\infty$ are to be understood in a weak sense, as follows:

i) $\omega(+\infty) = 0$ if $\omega \in L^p(\mathbf{R}^3)$ for some $p \in (1, +\infty)$.

ii) $\omega(0) = \pi$ if $\displaystyle\lim_{r\to 0}\frac{1}{|B_r|}\int_{B_r}|\omega - \pi|^2\,dx = 0$.

Let us then consider a sequence $\{\omega_n\}_n$ in $\tilde{Y}_k$ such that:

$$(7)\qquad \tilde{\mathcal{E}}(\omega_k) \to \tilde{I}_K\,, \quad \omega_n(0) = k\pi\,, \quad \omega_n(+\infty) = 0\,.$$

Then there exists a positive constant C such that $\tilde{\mathcal{E}}(\omega_n) \leqq C$.

Moreover we may assume that $\|\omega_n\|_{L^\infty(\mathbf{R}^3)} \leqq k\pi$. Indeed, if this were not the case we would define continuous functions $\hat{\omega}_n$ as follows:

$$\hat{w}_n = \begin{cases} \min(\omega_n, k\pi) & \text{when } \omega_n \geqq 0 \\ 0 & \text{when } \omega_n \leqq 0 . \end{cases}$$

For this new sequence we have:

$$\hat{w}_n(0) = k\pi , \quad \hat{w}_n(+\infty) = 0 , \quad \|\hat{w}_n\|_{L^\infty(\mathbf{R}^3)} = k\pi \quad \text{and } \tilde{\mathcal{E}}(\hat{\omega}_n) \leqq \tilde{\mathcal{E}}(\omega_n) ,$$

and therefore $\tilde{\mathcal{E}}(\hat{\omega}_n)$ converges also towards $\tilde{I}_k$.

From (7) we infer the existence of ω in $L^6(\mathbf{R}^3)$ such that:

$$\begin{cases} \nabla\hat{\omega}_n \rightharpoonup \nabla\omega & \text{in } L^2(\mathbf{R}^3)\text{-weak} \\ \hat{\omega}_n \rightharpoonup \omega & \text{in } L^6(\mathbf{R}^3)\text{-weak} . \end{cases} \tag{8}$$

Moreover, since $\|\hat{\omega}_n\|_{L^\infty} \leqq k\pi$, we have also:

$$\begin{cases} \hat{\omega}_n \to \omega & \text{in } L^2_{\mathrm{loc}}(\mathbf{R}^3) \\ \hat{\omega}_n \to \omega & \text{a.e. in } \mathbf{R}^3 . \end{cases} \tag{9}$$

From (9) we obtain easily that:

$$\tilde{\mathcal{E}}(\omega) \leqq \varliminf_{n\to+\infty} \tilde{\mathcal{E}}(\hat{\omega}_n) = \tilde{I}_k \tag{10}$$

Indeed, from (9) we know that $\sin\hat{\omega}_n$ (resp. $\cos\hat{\omega}_n$) converges a.e. to $\sin\omega$ (resp. $\cos\omega$) as n goes to $+\infty$. Moreover, from (9) and (10) we infer also that:

$$\nabla\hat{\omega}_n\left(K^2 + \frac{8\sin^4\hat{\omega}_n}{|x|^2} + \frac{4\sin^2\hat{\omega}_n\cos^2\hat{\omega}_n}{|x|^2}\right)$$

converges weakly to

$$\nabla\omega\left(K^2 + \frac{8\sin^4\omega}{|x|^2} + \frac{4\sin^2\omega\cos^2\omega}{|x|^2}\right)$$

in $L^2(\mathbf{R}^3)$ as n goes to $+\infty$.

Therefore, by Fatou's lemma we obtain (10).

We have now to prove that ω is in Y_k. First we note that $\omega \in L^6(\mathbf{R}^3)$ and so, $\omega(+\infty) = 0$. Then it remains only to prove that $\omega(0) = k\pi$ in the weak sense defined above. Assume that this were not the case. Then there would exist a set $A \subset \mathbf{R}^4$ of positive measure with:

i) $\|\omega - k\pi\|_{L^\infty(A\cap B)} \geqq \alpha > 0$, for all $r > 0$, small.

ii) $|A \cap S_{r_m}| \geqq \delta |S_{r_m}|$, where $\delta > 0$ and $\lim_{m\to+\infty} r_m = 0$.

But then we observe that (5), together with (i)-(ii), implies the existence of a set $B \subset \boldsymbol{R}^4$ such that $|B \cap S^3| > 0$ and $B \subset \varphi_{\omega_n}(B_{r_m})$ for n and m large. And we show next that this is impossible. Indeed, by using Cauchy-Schwarz's inequality and the fact that for any three vectors of $\boldsymbol{R}^4$, a, b, c, we have: $|a\wedge b\wedge c| \leqq |a\wedge b|^{\frac{1}{2}}|a\wedge c|^{\frac{1}{2}}|b\wedge c|^{\frac{1}{2}}$, we obtain for all $\varphi \in X$, for all $K \subset \boldsymbol{R}^3$:

$$\text{(11)} \qquad \text{area } (\varphi(K)) = \int_K \left|\frac{\partial\varphi}{\partial x_1}\wedge\frac{\partial\varphi}{\partial x_2}\wedge\frac{\partial\varphi}{\partial x_3}\right| dx \leqq |K|^{\frac{1}{4}}\left(\int_K |A(\varphi)|^2\right)^{\frac{3}{4}} \leqq \mathcal{E}(\varphi)^{\frac{3}{4}}|K|^{\frac{1}{4}}$$

and so, area $(\varphi_{\omega_n}(B_{r_m})) \leqq Cr_m^{\frac{3}{4}}$ for all n, m which is contradictory with the existence of the set B.

PROOF OF COROLLARY 2. – If $\{\omega_n\}$ is a sequence of radially symmetric fonctions in Z_k such that $\mathcal{E}(\varphi_{\omega_n}) = \tilde{\mathcal{E}}(\omega_n) \xrightarrow[n\to+\infty]{} I_k^*$, we have in particular that:

$$\text{(12)} \qquad \tilde{\mathcal{E}}(\omega_n) \leqq C \quad \text{ and } \omega_n \in Y_k \quad \text{ for all } n\,.$$

Proceeding as in the proof of Theorem 1, from (12) we infer the existence of ω^* such that $\tilde{\mathcal{E}}(\omega^*) \leqq I_k^*$ and $\omega^*(0) = k\pi$, $\omega^*(+\infty) = 0$. Therefore we know that ω_n and ω^* are continuous for all n and $\omega_n \xrightarrow[n\to+\infty]{} \omega$ everywhere in $\boldsymbol{R}^3$. Therefore $\omega^* \in Z_k$ and the proof is complete.

3. – Complementary results.

In this section we will prove some results about the degree of the fonctions φ which are of the form (4) or (5). We will also make some comments about the relationship between I_k, $\tilde{I}_k$ and I_k^*.

PROPOSITION 3. – *Let φ be a function of X of the form* (4). *Then $d(\varphi) = k$ if and only if $\omega(0) - \omega(+\infty) = k\pi$.*

PROOF. – First we observe that it is enough to prove the result for φ in Y. Then, as we proved in [E1], if φ is in Y, then there exists an $e \in S^3$ such that

$$\int_{\boldsymbol{R}^3} |\varphi(x) - e|^6\, dx < +\infty\,.$$

Hence, by (4), there must exist $l, m \in \boldsymbol{Z}$ s.t. $\omega(0) = l\pi$, $\omega(+\infty) = m\pi$. Let us then prove that $l - m = k$.

Let E be the following stereographic projection:

$$E\colon S^3 \to \boldsymbol{R}^3$$
$$(y_1, \dots, y_4) \to \left(\frac{y_1}{1-y_4}, \frac{y_2}{1-y_4}, \frac{y_3}{1-y_4}\right).$$

Then we see that for any φ in Y,

$$\tilde{\varphi}(y) \equiv (\varphi \circ E)(y) \equiv \left(\frac{y'}{s} \sin \omega(r),\ \cos \omega(r)\right)$$

where

$$y' = (y_1, y_2, y_3)\,, \qquad s = \sqrt{1 - y_4^2} \qquad \text{and} \qquad r^2 = \sum_{i=1}^{3} \frac{y_i^2}{(1 - y_4)^2} = \frac{1 + y_4}{1 - y_4}\,.$$

We know that $d(\varphi) =$ degree of $\tilde{\varphi}$. Let us compute the jacobian matrix of $\tilde{\varphi}$ at any point y of S^3. We obtain:

$$D\tilde{\varphi}(y) = \frac{\sin \omega(r)}{s} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{y_1}{s} (\cos \omega) r' \omega' + (\sin \omega) \frac{y_1 y_4}{s^3} \\ 0 & 0 & 0 & \frac{y_2}{s} (\cos \omega) r' \omega' + (\sin \omega) \frac{y_2 y_4}{s^3} \\ 0 & 0 & 0 & \frac{y_3}{s} (\cos \omega) r' \omega' + (\sin \omega) \frac{y_3 y_4}{s^3} \\ 0 & 0 & 0 & -(\sin \omega)\, \omega' r' \end{pmatrix}$$

where

$$\omega' = \frac{d}{dr} \qquad \text{and} \qquad r' = \frac{dr}{dy_4} = \frac{1}{r(1 - y_4)^2}\,.$$

So, the jacobian determinant of $\tilde{\varphi}$ at y is:

$$J(y) = \det D\tilde{\varphi}(y) = -(\sin \omega)^4 \frac{\omega' r'}{s^3} = -\frac{\omega'(\sin^4 \omega)}{s^3 r (1 - y_4)^2}$$

and since $\operatorname{sign} J(y) = -\operatorname{sign}(\omega'(r))$, the proposition follows easily.

REMARK 4. – Let ω be a continuous function from $\boldsymbol{R}^+$ to $\boldsymbol{R}$ such that $\omega(0) = l\pi$ and $\omega(+\infty) = m\pi$, $l, m \in \boldsymbol{Z}$. Then, if we define $\hat{\omega}(r) \equiv \omega(r) + h\pi$, $h \in \boldsymbol{Z}$, we observe that this transformation in the image of ω conserves the degree and the energy $\tilde{\mathcal{E}}$ as well. Thus, translations of $h\pi$ ($h \in \boldsymbol{Z}$) in the image of ω leave the minimizing problem (I_k^*) invariant. From Proposition 3 it follows then immediatly that minimizing $\mathcal{E}$ over the functions $\varphi \in X$ of the form (4) that have degree equal to k is equivalent to minimizing $\tilde{\mathcal{E}}$ over the functions ω which satisfy $\omega(0) = k\pi$, $\omega(+\infty) = 0$.

Let us now prove a result which corresponds to Proposition 3 when φ is only of the form (5).

PROPOSITION 5. – *Let ω be a function from $\boldsymbol{R}^3$ into $\boldsymbol{R}$. Then if $\varphi \in X$ is of the form* (5), *$d(\varphi) = k$ if and only if $\omega(0) - \omega(+\infty) = k\pi$.*

PROOF. – As already said, this is enough to prove the result for $\omega \in C^1(\boldsymbol{R}^3, \boldsymbol{R})$. Moreover one may still prove the existence of $l, m \in \boldsymbol{Z}$ s.t. $\omega(0) = l\pi$ and $\omega(+\infty) = m\pi$. Then the proposition will be proved as soon as we show that $d(\varphi) = k$ if and only if $l - m = k$.

As in the proof of Proposition 3 we have only to check the jacobian determinant of $\tilde{\varphi} \equiv \varphi \circ E$. In this case we obtain:

$$J_{\tilde{\varphi}}(y) = -\frac{\sin^4 \omega\big(y'/(1-y_4)\big)}{\big((1-y_4)(1-y_1^2)\big)^{\frac{3}{2}}}(y' \cdot \nabla\omega)\,, \qquad \text{where } y' = (y_1, y_2, y_3)\,.$$

Thus the sign of det $\big(D(\varphi \circ E)(y)\big)$ coincides with the sign of $-(y' \cdot \nabla\omega)$. Therefore, when $\omega(0) = l\pi$ and $\omega(+\infty) = m\pi$, we see that the number of inverse images of all the points of S^3 is $(l - m)$, this of course when one takes into account the orientation of $\tilde{\varphi}$ at those inverse images. ∎

Let us finish this paper with some remarks about the differences between the different minimization problems we have considered.

We begin with a proposition which tells us how behaves I_k^* with respect to k.

PROPOSITION 6. – *For any* $k, l \in \boldsymbol{Z} - \{0\}$, $k \neq l$, *we have*:

$$I_k^* > I_l^* + I_{k-l}^* \tag{13}$$

PROOF. – Take for simplicity the case $k > l \geq 1$. Then let $\varphi_\omega \in Z_k$ be such that $\tilde{\mathcal{E}}(\omega) = I_k^*$. As we already know, we can take ω such that $\omega(0) = k\pi$, $\omega(+\infty) = 0$.

Let $\bar{r}$ be the minimum of the $r > 0$ such that $\omega(r) = l\pi$. Then we define two new functions, ω_1 and ω_2, as follows:

$$\omega_1(r) = \begin{cases} \omega(r)\,, & r \leqq \bar{r} \\ \omega(\bar{r})\,, & r \geqq \bar{r} \end{cases}$$

$$\omega_2(r) = \begin{cases} \omega(\bar{r})\,, & r \leqq \bar{r} \\ \omega(r)\,, & r \geqq \bar{r}\,. \end{cases}$$

It is immediate that these two functions satisfy:

$$\tilde{\mathcal{E}}(\omega_1) \geqq I_{k-l}^*\,, \quad \tilde{\mathcal{E}}(\omega_2) \geqq I_l^* \quad \text{and} \quad d(\varphi_{\omega_1}) = k - l\,, \quad d(\varphi_{\omega_2}) = l\,.$$

Then, if $I_k^* \leqq I_{k-l}^* + I_l^*$, $(\omega_1 - l\pi)$ would be a minimum of I_{k-l}^* with compact support. And this is impossible, since the corresponding Euler equation for ω is an O.D.E., with regular coefficients in any interval $(\delta, +\infty)$, $\delta > 0$, and therefore no compactly supported solution may exist. ∎

REMARK 7. – The above result shows that the different problems considered in this paper are not just the same. Indeed, as we prove in [E2], for all $k, l \in \boldsymbol{Z}$,

$I_k \leqq I_l + I_{k-l}$, $I_k \leqq |k| I_1$. But the opposite happens with the symmetric problems I_k^*. Hence at least for all $k \neq 0, \pm 1$, $I_k^* \neq I_k$, and in fact, $I_k < I_k^*$ for $|k| > 1$.

Let us finally remark that the fact that the subadditivity condition:

$$f(k) \leqq f(l) + f(k-l), \quad l \in \boldsymbol{Z} - \{k, 0\}$$

is valid for $f(k) = I_k$ but not for I_k^* is not surprising at all. Indeed, as P.L.L. shows in [L1, 2] the subadditivity inequality takes place in a large class of minimization problems in $\boldsymbol{R}^N$ which are invariant under the group of translations of $\boldsymbol{R}^N$. Moreover as soon as this invariance is broken, the subadditivity inequality does not hold any more. And this is the case for the problem I_k^*.

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