# On the Genus of a Hyperplane Section of a Geometrically Ruled Surface (*).

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**Summary.** – *In this paper we estimate the minimal genus of hyperplane sections of a geometrically ruled surface.*

## Introduction.

Let $D$ be a divisor on a geometrically ruled surface $\pi: X \to C$. If $C_0$ is a minimal section and $f$ is fiber on $X$ we can write $D \equiv aC_0 + bf$. For a fixed number $a$ we have studied two related problems:

I) What is the minimal $b$ (call it $b_a$) such that $D \equiv aC_0 + bf$ is very ample?

II) What is the minimal genus $\lambda_a$ of a very ample divisor $D$?

For $g = g(C) = 0$ (see [Ha, Corollary, V.2.18]) we have $b_a = ae + 1$ and $\lambda_a = (1/2)a(a-1)e$, where $e = - C_0 \cdot C_0$ is an invariant of $X$.

In this paper we obtain some answers for $g \geqslant 1$. In particular if $g = 1$ our answer (§ 6) is sharp i.e.

$$b_a = \begin{cases} ae + 3 & \text{if } e \geqslant 0 \text{ and any } a \text{ or } e = -1 \text{ and } a \leqslant 3 \\ 1 - (a/2) + \varepsilon(a) & \text{if } e = -1 \text{ and } a \geqslant 4 \end{cases}$$

where

$$\varepsilon(a) = \begin{cases} 1 & \text{if } a \text{ even} \\ (1/2) & \text{if } a \text{ odd} \end{cases}$$

and

$$\lambda_a = \begin{cases} (1/2)a(a-1)e + 3a - 2 & \text{if } e \geqslant 0 \text{ and any } a \text{ or } e = -1 \text{ and } a \leqslant 3 \\ (a-1)\varepsilon(a) + a & \text{if } e = -1 \text{ and } a \geqslant 4 \,. \end{cases}$$

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For $g \geqslant 2$ we found (§ 5) that if $e \geqslant 0$

$$ae + 1 \leqslant b_a \leqslant ae + 2g + 1$$

$$\big(a(a-1)/2\big)\, e + ag \leqslant \lambda_a \leqslant \big(a(a-1)/2\big)\, e + (3a-2)\, g$$

and if $e < 0$

$$(1/2)\, ae + \varepsilon(ae) \leqslant b_a \leqslant (1/2)\, ae + 2g + \varepsilon(ae)$$

$$ag + \big(\varepsilon(ae) - 1\big)(a-1) \leqslant \lambda_a \leqslant (3a-2)\, g + \big(\varepsilon(ae) - 1\big)(a-1)\,.$$

For the case $g = 2$ we can improve the above bounds (see § 7). In particular for $e \geqslant 0$, $b_a = ae + 5$ and $\lambda_a = (1/2)a(a-1)e + 6a - 4$.

Our results are very useful in the study of smooth, connected, projective, ruled surfaces with the genus of a hyperplane section less than or equal to seven. See [Li$_1$], [Li$_2$], [Bi-Li].

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## 0. – Background material.

The notation, throughout this paper, is essentially that used in [Ha].

(0.1) Let $X$ be an analytic space. We let $\mathcal{O}_X$ denote its structure sheaf and let $h^{i,0}(X) = \dim H^i(X, \mathcal{O}_X)$. If $X$ is a complex manifold, we let $K_X$ denote its canonical bundle.

(0.2) Let $X$ be a smooth connected projective surface. Let $D$ be an effective Cartier divisor on $X$. We denote by $L(D)$, the holomorphic line bundle associated to $D$. If $L$ is a holomorphic line bundle on $X$, $|L|$ denotes the linear system of Cartier divisors associated to $L$. Of course if $|L|$ is non-empty then $L(D) = L$ for $D \in |L|$. Let $E$ be a second holomorphic line bundle on $X$, then $L \cdot E$ denotes the evaluation of the cup product, $C_1(L) \wedge C_1(E)$ on $X$, where $C_1(L)$ and $C_1(E)$ are the Chern classes of $L$ and $E$ respectively. If $D \in |L|$ and $C \in |E|$, it is convenient to let $D \cdot C = D \cdot E = L \cdot C = L \cdot E$. We often let $g = g(L) = (1/2)(L \cdot L + K_X \cdot L + 2)$, which is called the *adjunction formula*. If there is a smooth $D \in |L|$, then

$$g = g(L) = h^{1,0}(D)\,.$$

(0.3) Let $L$ be a line bundle on a projective variety. We say $L$ is *spanned* if $\Gamma(L)$ is generated by its global sections. By [Ha, lemma 7.8] this is equivalent to saying that $\Gamma(L)$ is base-point-free. We say $L$ is *very ample* if $L$ is spanned and the map $\varphi: X \to \boldsymbol{P}^N_{\boldsymbol{C}}$ associated to $\Gamma(L)$ is an embedding. We say that $L$ is *ample* if some power of $L$ is very ample.

(0.4) Let $D$ be an effective divisor on a smooth connected, projective surface $X$, $D$ is $k$-connected if $D \cdot D > 0$ and for every decomposition $D = D_1 + D_2$ into effective divisors $D_1 \cdot D_2 \geqslant k$.

(0.5) (*Ruled surfaces*). Let $C$ be a smooth curve of genus $g$ and $\pi: X \to C$ a (geometrically) ruled surface. A section of $X$ is a map $\sigma: C \to X$ such that $\pi \circ \sigma = \mathrm{id}_C$. The image of $\sigma$ is a divisor $C_0$ which we will also call a section. Let $C_0 \subseteq X$ be a section, and let $f$ be a fiber, then $\mathrm{Pic}\, X \simeq \boldsymbol{Z} \oplus \pi^* \mathrm{Pic}\, C$, where $\boldsymbol{Z}$ is generated by $C_0$. Also $\mathrm{Num}\, X \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$ is generated by $C_0$, $f$ with $C_0 \cdot f = 1$ and $f \cdot f = 0$. For any ruled surface there exist a rank 2 vector bundle on $C$, $p: E \to C$ such that $\boldsymbol{P}(E) \simeq X$ and viceversa. We have $\boldsymbol{P}(E) \simeq \boldsymbol{P}(E')$ if and only if there is a line bundle $L$ such that $E' = E \otimes L$. Moreover it is always possible to write $X = \boldsymbol{P}(E)$ with $H^0(C, E) \neq 0$ and $H^0(C, E \otimes L) = 0$, for every line bundle with $\deg L < 0$. Such an $E$ is said to be normalized. It is not necessarily unique but $\deg E$ is uniquely determined and is an invariant of $X$. Let $\mathfrak{e}$ be the divisor on $C$ corresponding to $\Lambda^2 E$. Set $e = -\deg \mathfrak{e} = -\deg \Lambda^2 E$. We fix a section $C_0$ of $X$ with $\mathcal{L}(C_0) = \mathcal{O}_{\boldsymbol{P}(E)}(1)$. We have $C_0^2 = \deg \mathfrak{e} = -e$ and $C_0 \cdot f = 1$. If $\mathfrak{b}$ is any divisor on $C$, then we denote the divisor $\pi^* \mathfrak{b}$ by $\mathfrak{b} f$. Thus any element of $\mathrm{Pic}\, X$ can be written $aC_0 + \mathfrak{b} f$ with $a \in \boldsymbol{Z}$ and $\mathfrak{b} \in \mathrm{Pic}\, C$. Any element of $\mathrm{Num}\, X$ can be written $aC_0 + bf$ with $a, b \in \boldsymbol{Z}$. If $D_h \equiv a_h C_0 + b_h f$, $h = 1, 2$ we get

$$
(0.6) \qquad \begin{cases} D_1 \cdot D_2 = a_1 b_2 + a_2 b_1 - a_1 a_2 e \\ D_1^2 = 2a_1 b_1 - a_1^2 e\,. \end{cases}
$$

Moreover since

$$
(0.7) \qquad K_X \equiv -2C_0 + (2g - 2 - e) f
$$

we get

$$
K_X^2 = 8(1 - g)\,.
$$

If $D \equiv aC_0 + bf$ and setting $h^i(D) = \dim H^i(X, L(D))$, $i \geqslant 0$ then by the Riemann-Roch Theorem we have

$$
(0.8) \qquad h^0(D) - h^1(D) = (a + 1)\big(b - (ae/2) - g + 1\big)\,.
$$

Let $D \equiv aC_0 + bf$ be a divisor on $X$. Then $D$ is ample if and only if

$$
(0.9) \qquad a > 0 \quad \text{and} \quad b > \begin{cases} ae & \text{if } e \geqslant 0 \\ (1/2)ae & \text{if } e < 0\,. \end{cases}
$$

## 1. - Vanishing theorems.

Let $\pi\colon X\to C$ be a ruled surface. Let $X\simeq \boldsymbol{P}(E)$, where $E$ is a normalized rank 2 vector bundle. Set $L(\mathfrak{e})=\Lambda^2E$. If $\deg\mathfrak{e}=-e$ then we write (numerically) $L(\mathfrak{e})=L(-e)$. We have

$$E=E^*\otimes\Lambda^2E=E^*\otimes L(\mathfrak{e})$$

where $E^*$ is dual to $E$. Let $S^aE$ be the $a$-symmetric product of $E$. Then we have

$$H^i(X,aC_0+bf)\simeq H^i(C,S^aE^*\otimes L(b))\,,\qquad i\geqslant 0 \tag{1.1}$$

$$S^aE\simeq S^aE^*\otimes(\Lambda^2E)^{\otimes a}=S^aE^*\otimes L(ae)\,. \tag{1.2}$$

Since $E$ is normalized we have

$$H^0(C,E\otimes L(b))\simeq H^0(X,C_0+bf)=0 \tag{1.3}$$

for any $b<0$. As before we set $h^i(D)=\dim H^i(X,L(D))$. Let $D\equiv aC_0+bf$ be a divisor on $X$. Then by the Kodaira Vanishing Theorem we have $h^1(D)=0$ if $D-K_X$ is ample. Therefore using (0.7) and (0.9) we get $h^1(D)=0$ if

$$a\geqslant -1\quad\text{and}\quad b>\begin{cases}(a+1)e+2g-2 & \text{if } e\geqslant 0\\ (1/2)ae+2g-2 & \text{if } e<0\,.\end{cases} \tag{1.4}$$

By Serre duality, (1.1) and (1.2) we get

$$\begin{cases}H^0(X,aC_0+bf)\simeq H^1(X,aC_0+(ae-b+2g-2)f)\\ H^1(X,aC_0+bf)\simeq H^0(X,aC_0+(ae-b+2g-2)f)\,.\end{cases} \tag{1.5}$$

THEOREM 1.1. - Let $D\equiv aC_0+bf$ be a divisor on $X$, with $a\geqslant 1$. We have $h^1(D)=0$ if

$$b>\begin{cases}ae+2g-2 & \text{if } a=1 \text{ and any } e \text{ or } a\geqslant 2 \text{ and } e\geqslant 0\\ (1/2)ae+2g-2 & \text{if } a\geqslant 2 \text{ and } e<0\end{cases} \tag{1.6}$$

and $h^1(D)>0$ if

$$b<(1/2)ae+g-1\,. \tag{1.7}$$

PROOF. - (1.7) follows from (0.8). Consider now (1.6). The case $e<0$ was already done in (1.4). We prove the case $e\geqslant 0$ by induction. By (1.3) and (1.5) we get $h^1(D)=0$ if $a=1$, $b>e+2g-2$ and any $e$.

Suppose (1.6) true for $a-1$. We have the short exact sequence

$$0 \to L(D - C_0) \to L(D) \to L(-ae + b) \to 0$$

since $L(D)|_{C_0} \cong L(-ae+b)$. Then

$$H^1(X, L(D - C_0)) \to H^1(X, L(D)) \to H^1(C, L(-ae + b)) \to 0 .$$

We have $b > ae + 2g - 2 \geqslant (a-1)e + 2g - 2$ since $e \geqslant 0$. Thus by induction $H^1(X, L(D - C_0)) = 0$. Moreover $-ae + b > 2g - 2$ implies $H^1(C, L(-ae+b)) = 0$. Hence $h^1(D) = 0$. □

THEOREM 1.2. – Let $D \equiv aC_0 + bf$ be a divisor on $X$, with $a \geqslant 1$. Then $h^0(D) = 0$ if

$$(1.8) \qquad b < \begin{cases} 0 & \text{if } a = 1 \text{ and any } e \text{ or } a \geqslant 2 \text{ and } e \geqslant 0 \\ (1/2)ae & \text{if } a \geqslant 2 \text{ and } e < 0 \end{cases}$$

and $h^0(D) > 0$ if

$$(1.8) \qquad b > (1/2)ae + g - 1 .$$

PROOF. – (1.9) follows from (0.8). Consider now (1.8). The case $a = 1$ is just (1.3). By (1.5) and (1.6) we get (1.8) in the case $a \geqslant 2$. □

## 2. – Very ample line bundles on ruled surfaces.

Let $D \equiv aC_0 + bf$ be a divisor on a ruled surface $\pi: X \to C$, with $a \geqslant 1$. We set $f_x = \pi^{-1}(x)$ for $x \in C$.

LEMMA 2.1. – If $h^1(D - f_x) = 0$ then $L(D)|_{f_x} = \mathcal{O}_{\mathbf{P}_1}(a)$.

PROOF. – Since $h^1(D - f_x) = 0$ we have

$$(2.1) \qquad 0 \to H^0(X, L(D - f_x)) \to H^0(X, L(D)) \xrightarrow{\beta} H^0(X, L(D))|_{f_x} \to 0 .$$

If $D'|_{f_x} \not\equiv 0$ for some $D' \in |L(D)|$ we would have $L(D)|_{f_x} \cong \mathcal{O}_{\mathbf{P}_1}(a)$. But $D'|_{f_x} \equiv 0$ for every $D' \in |L(D)|$ implies $\beta = 0$, hence $h^0(D - f_x) = h^0(D)$. On the other hand from (0.8) we get $h^0(D) = h^0(D - f_x) + (a+1) + h^1(D)$ which implies $h^0(D) > h^0(D - f_x)$ since $a \geqslant 1$ and $h^1(D) \geqslant 0$. Therefore $L(D)|_{f_x} = \mathcal{O}_{\mathbf{P}_1}(a)$. □

PROPOSITION 2.2. – If $h^1(D - f_x) = 0$ then $L(D)$ is spanned.

PROOF. - Since $h^1(D - f_x) = 0$ for every $x$, by lemma 2.1 we have

$$0 \to L(D - f_x) \to L(D) \to \mathcal{O}_{\boldsymbol{P}_1}(a) \to 0$$

since $\mathcal{O}_{\boldsymbol{P}_1}(a)$ is very ample for $a \geqslant 1$ we get that $L(D)$ is spanned. □

PROPOSITION 2.3. - If $h^1(D - f_x) = 0$ and $h^1(D - 2f_x) = 0$ then $L(D)$ is very ample.

PROOF. - We have to prove that $|L(D)|$ separates points and tangent vectors.

*Case* 1. - $P$ and $Q$ (or $P$ and $t$) not in the same fiber. Let $f_P$ and $f_Q$ be the fibers which $P$ and $Q$ are on respectively. Since $h^1(D - f_P - f_Q) = 0$ and $L(D - f_P)|_{f_Q} = \mathcal{O}_{\boldsymbol{P}_1}(a)$ we have

$$0 \to H^0(X, L(D - f_P - f_Q)) \to H^0(X, L(D - f_P)) \to H^0(\boldsymbol{P}_1, \mathcal{O}_{\boldsymbol{P}_1}(a)) \to 0 .$$

So we can find $D' \simeq D - f_P$ such that $Q \notin D'|_{f_Q}$ i.e. $Q \notin D'$. Hence $Q \notin D' + f_P \simeq D$ but $P \in D' + f_P$. In the case ($P$ and $t$) we do the same considering $P \equiv Q$. Then we get $P \in D' + f_P$ but $2P \notin D' + f_P$ so $t$ is not a tangent vector to $D' + f_P$ at $P$.

*Case* 2. - $P$ and $Q$ (or $P$ and $t$) are both in the same fiber $f_x$ for some $x \in C$. From (2.1) we can find $D' \simeq D$ such that $P \in D'|_{f_x}$ but $Q \notin D'|_{f_x}$ ($P \in D'|_{f_x}$ but $2P \notin D'|_{f_x}$). Hence $P \in D'$ and $Q \notin D'$ (or $P \in D'$ but $t$ is not tangent to $D'$ at $P$).

COROLLARY 2.4. - $D$ is spanned if

$$b > \begin{cases} ae + 2g - 1 & \text{if } a = 1 \text{ and any } e \text{ or } a \geqslant 2 \text{ and } e \geqslant 0 \\ (1/2)ae + 2g - 1 & \text{if } a \geqslant 2 \text{ and } e < 0 \end{cases}$$

and $D$ is very ample if

$$b > \begin{cases} ae + 2g & \text{if } a = 1 \text{ and any } e \text{ or } a \geqslant 2 \text{ and } e \geqslant 0 \\ (1/2)ae + 2g & \text{if } a \geqslant 2 \text{ and } e < 0 . \end{cases}$$

## 3. - On the 3-connectedness of a divisor on a ruled surface.

Let $D \equiv aC_0 + bf$ be a divisor on a ruled surface. If $D = D_1 + D_2$ we have $D_1 \equiv xC_0 + yf = (a - \tilde{x})C_0 + (b - \tilde{y})f$ and $D_2 \equiv (a - x)C_0 + (b - y)f = \tilde{x}C_0 + \tilde{y}f$. Assume $D^2 = a(2b - ae) > 0$, i.e. $a > 0$ and $b > (1/2)ae$. In order to prove that $D$ is 3-connected we have to prove that for any decomposition $D = D_1 + D_2$ with $D_i \not\simeq 0$ and $h^0(D_i) \geqslant 1$ we get that $D_1 \cdot D_2 \geqslant 3$.

Lemma 3.1. – Assume that $h^0(xC_0 + yf) \geqslant 1$, then *a*) $x \geqslant 0$

$$b) \quad y \geqslant \begin{cases} (1/2)xe & \text{if } x \geqslant 2 \text{ and } e < 0 \\ 0 & \text{if } x = 0, 1 \text{ and } e < 0 \text{ or } e \geqslant 0 \text{ and any } x . \end{cases}$$

Proof. – *a*) If $x < 0$ then $h^0(D_1) = 0$. It is enough to prove it for $x = -1$, since $h^0(D_1 + C_0) = 0$ implies $h^0(D_1) = 0$ for $x \leqslant -2$. We have

$$0 \to L(D_1) \to L(yf) \to L(yf)|_{C_0} \simeq L(y) \to 0 .$$

Since $h^0(yf) = h^0(y)$ and $H^0(X, L(yf)) \to H^0(C, L(y))$ is surjective, we have $h^0(D_1) = 0$.

*b*) If $x = 0$, then $h^0(D_1) = 0$ if $y < 0$. Therefore if $x = 0$ and $h^0(D_1) > 0$ it follows that $y \geqslant 0$. If $x \geqslant 1$ then by (1.8) we get part *b*). □

Proposition 3.2. – Assume that $e < 0$ and $a \geqslant 3$. Then $D$ is 3-connected if

$$b > \begin{cases} 0 & \text{if } e = -1 \text{ and } a = 3 \\ (1/2)ae + 1 & \text{otherwise} . \end{cases}$$

Proof. – By lemma 3.1 $x$ and $\tilde{x}$ are non-negative.

*Case* 1. – Assume $x = 0$ (or $\tilde{x} = 0$) from

$$D_1 \cdot D_2 = \begin{cases} y(a - 2x) + x(-(a - x)e + b) \\ \tilde{y}(a - 2\tilde{x}) + \tilde{x}(-(a - \tilde{x})e + b) \end{cases} \tag{3.1}$$

we get $D_1 \cdot D_2 = ya$ (or $= \tilde{y}a$). Since $a > 2$ and $y > 0$ ($\tilde{y} > 0$) we obtain $D_1 \cdot D_2 > 2$.

*Case* 2. – Assume $x = 1$ (or $\tilde{x} = 1$). From $a - 2x > 0$ and $y \geqslant 0$ and (3.1) we have $D_1 \cdot D_2 \geqslant b - (a - 1)e$. If $(a, e) = (3, -1)$ then $b > 0$ so $-(a - 1)e + b > 2$, hence $D_1 \cdot D_2 > 2$. If $(a, e) \neq (3, -1)$ then $b > (1/2)ae + 1$ so

$$D_1 \cdot D_2 \geqslant b - (a - 1)e > 1 + (1/2)ae - (a - 1)e = 1 - (a - 2)(e/2) \geqslant 2 ,$$

then $D_1 \cdot D_2 > 2$.

*Case* 3. – $2 \leqslant x \leqslant a - 2$ and $a - 2x \geqslant 0$ (or $2 \leqslant \tilde{x} \leqslant a - 2$ and $-(a - 2x) = a - 2\tilde{x} \geqslant 0$) we treat only the part $a - 2x \geqslant 0$. The other part is similar. In this case $y \geqslant xe/2$ so $D_1 \cdot D_2 \geqslant (xe/2)(a - 2x) + x(-(a - x)e + b) = x(b - (1/2)ae)$. If $(a, e) = (3, -1)$ then $b > 0$ and we have $D_1 \cdot D_2 > 3$. If $(a, e) \neq (3, -1)$ then $D_1 \cdot D_2 \geqslant x(b - (ae/2)) \geqslant$ $\geqslant 2(b - (ae/2)) > 2$. So $D_1 \cdot D_2 > 2$. □

## 4. – Very ampleness by Bombieri's method.

We would like to find new conditions for $L$ to be very ample. In order to do this we shall use the following theorem.

THEOREM 4.1. – Let $L$ be a line bundle over a surface $X$. We put $L_0 = L \otimes K_X^{-1}$. If i) $h^0(L_0) \geqslant 7$; ii) $L_0 \cdot L_0 \geqslant 10$; iii) $L_0$ is 3-connected, then $L$ is very ample.

PROOF. – See [VdV].

Theorem 4.1 has been proved using a method of Bombieri. See also [Be], [Bo], [So$_1$] and [So$_2$].

We will apply Theorem 4.1 for $L = L(D)$ where $D \equiv aC_0 + bf$ is a divisor over a ruled surface $X$. Then $L_0 = L(D_0)$ where

$$D_0 = D - K_X \equiv a_0 C_0 + b_0 f = (a+2)\, C_0 + (b - 2g + 2 + e) f\,.$$

We are interested in the case $e < 0$ and $a \geqslant 2$. By (0.8) we have

$$h^0(D_0) - h^1(D_0) = (a_0 + 1)\big(b_0 + 1 - g - (a_0 e/2)\big)\,.$$

Since $h^1(D_0) \geqslant 0$ we have $h^0(D_0) \geqslant 7$ if

$$b > 7/(a+3) + (ae/2) + 3g - 3\,. \tag{4.1}$$

By (0.6) we have $L_0 \cdot L_0 = 2a_0\big(b_0 - (1/2)a_0 e\big) = 2(a+2)\big(b - 2g + 2 - (ae/2)\big)$. Therefore $L_0 \cdot L_0 \geqslant 10$ when

$$b > 5/(a+2) + (ae/2) + 2g - 2\,. \tag{4.2}$$

Moreover by Proposition 3.2 we have $L_0$ is 3-connected if

$$b > (ae/2) + 2g - 1\,. \tag{4.3}$$

We set

$$K_1 = 7/(a+3) + (ae/2) + 3g - 3\,, \qquad K_2 = 5/(a+2) + (ae/2) + 2g - 2\,,$$
$$K_3 = (ae/2) + 2g - 1 + \varepsilon(ae)\,, \qquad K_4 = K_3 + 1\,,$$

where

$$\varepsilon(n) = \begin{cases} 1 & \text{if } n \text{ even} \\ 1/2 & \text{if } n \text{ odd}\,. \end{cases}$$

Using Theorem 4.1 we have $D$ is very ample if $b \geqslant K_0 = \operatorname{Max}\{K_1, K_2, K_3\}$. We have $K_1 \geqslant K_2$. For $g = 1$ we have $K_0 = K_3$. For $g = 2$

$$K_0 = \begin{cases} K_1 & \text{if } \begin{cases} a = 2, 3, 5, 7, 9 \text{ and } e = -1 \\ a = 2, 3 \qquad\quad \text{ and } e = -2 \end{cases} \\ K_3 & \text{otherwise}. \end{cases}$$

For $g \geqslant 3$ then $K_0 = K_1$. Therefore by Theorem 4.1 and Corollary 2.4, in the case $e < 0$ and $a \geqslant 2$, $D$ is very ample when

$$b \geqslant \operatorname{Min}\{K_0, K_4\} = K$$

and

$$K = K_0 = K_3 \quad \text{if } g = 1 \tag{4.4}$$

$$K = \begin{cases} K_1 = K_4 & \text{if } \begin{cases} a = 2, 3, 5, 7, 9 \text{ and } e = -1 \\ a = 2, 3 \qquad\quad \text{ and } e = -2 \end{cases} \\ K_3 & \text{otherwise}. \end{cases} \qquad \text{if } g = 2 \tag{4.5}$$

For $g \geqslant 3$ we have $K = K_4$.

## 5. – The genus of a very ample divisor on a ruled surface.

Let $D = aC_0 + \mathfrak{b}f$ be a very ample divisor on a ruled surface $X$. Let $b = \deg \mathfrak{b}$ and $\gamma = g(D)$. Then by the Adjunction formula we have $2\gamma - 2 = D\cdot(D + K_X)$ where $K_X = -2C_0 + (K_C + \mathfrak{e})f$. Therefore

$$(D + K_X)D = 2(a-1)(b - 1 - (1/2)ae) + 2ag - 2$$

and hence

$$\gamma = (a-1)(b - 1 - (1/2)ae) + ag .$$

We set $\lambda_a = \lambda_a(C, X)$ and $b_a = b_a(C, X)$ which are respectively the minimum genus and the minimum $b$ of a very ample divisor $D \equiv aC_0 + bf$ on a ruled surface $X$ over the curve $C$. We have

$$\lambda_a = (a-1)(b_a - 1 - (1/2)ae) + ag . \tag{5.1}$$

So finding $\lambda_a$ is equivalent to finding $b_a$. The next step is finding an estimate for $b_a$ (or $\lambda_a$). We are interested in the case $a \geqslant 2$. Since if $yD$ is very ample it is ample. Hence

$$b_a > \begin{cases} ae & \text{if } e \geqslant 0 \\ (1/2)ae & \text{if } e < 0 \end{cases}$$

and by corollary 2.4 we have

$$b_a \leqslant \begin{cases} ae + 2g + 1 & \text{if } e \geqslant 0 \\ (1/2)ae + 2g + \varepsilon(ae) & \text{if } e < 0 . \end{cases}$$

Therefore if $e \geqslant 0$

$$ae + 1 \leqslant b_a \leqslant ae + 2g + 1 \tag{5.2}$$

and

$$\big(a(a-1)/2\big)e + ag \leqslant \lambda_a \leqslant a(a-1)/2 + (3a-2)g \tag{5.3}$$

if $e < 0$

$$(1/2)ae + \varepsilon(ae) \leqslant b_a \leqslant (1/2)ae + 2g + \varepsilon(ae) \tag{5.4}$$

and

$$ag + \big(\varepsilon(ae) - 1\big)(a-1) \leqslant \lambda_a \leqslant (3a-2)g + \big(\varepsilon(ae) - 1\big)(a-1) . \tag{5.5}$$

If $g = 0$ then $e \geqslant 0$ and $b_a = ae + 1$ hence

$$\lambda_a = (1/2)a(a-1)e \tag{5.6}$$

In the case $g = 1$ or $2$ we can improve the lower bound. By the short exact sequence

$$0 \to L(D - C_0) \to L(D) \to L(D)|_{C_0} \simeq L(a\mathfrak{e} + \mathfrak{b}) \to 0$$

we get that $L(D)$ very ample implies $L(a\mathfrak{e} + \mathfrak{b})$ very ample.

In the case $g = 1$ or $2$, $L(a\mathfrak{e} + \mathfrak{b})$ is very ample if and only if $b > ae + 2g$. If $e \geqslant 0$ we have $ae + 2g + 1 \geqslant ae + 1$ and

$$b_a = ae + 2g + 1 , \tag{5.7}$$

$$\lambda_a = (1/2)a(a-1)e + (3a-2)g . \tag{5.8}$$

If $e < 0$ we have

$$(1/2)ae \begin{cases} \geqslant ae + 2g & \text{if } a \geqslant -4g/e \\ < ae + 2g & \text{if } a < -4g/e . \end{cases}$$

So

$$b_a \geqslant \begin{cases} ae + 2g + 1 & \text{if } a < -4g/e \\ (1/2)ae + \varepsilon(ae) & \text{if } a \geqslant -4g/e . \end{cases} \tag{5.9}$$

## 6. - The case $g = 1$.

If $e \geqslant 0$ we have (5.7) and (5.8). It only remains to study the case $e = -1$. By (5.4), (5.9) and (4.4) we have for $a \geqslant 2$

$$1 - (a/2) + \varepsilon(a) \geqslant b^3 \geqslant \begin{cases} -a + 3 & \text{if } a \leqslant 3 \\ -(a/2) + \varepsilon(a) & \text{if } a \geqslant 4 . \end{cases} \tag{6.1}$$

We already know $b_1 = 2$. From (6.1) we have $b_2 = 1$ and $b_3 = 0$. If $a \geqslant 4$ then $b_a$ is either $-(a/2) + \varepsilon(a)$ or $1 - (a/2) + \varepsilon(a)$. We set $D_a \equiv aC_0 + \big(-(a/2) + \varepsilon(a)\big) f$.

THEOREM 6.1. - $D_a$ is not very ample.
In order to prove Theorem 6.1 we need the following.

LEMMA 6.2. - Let $X$ be a ruled surface over $C$. Assume $e = -1$. Then there is $P \in C$ such that $h^0(2C_0 - Pf) \geqslant 1$.

PROOF. - We put $D \equiv 2C_0 - f$. By (0.8) we have $h^0(D) = h^1(D)$ and $h^0(2C_0) - h^1(2C_0) = 3$. By (1.6) $h^1(2C_0) = 0$, so $h^0(2C_0) = 3$. Now $h^0(2C_0) = h^0(S^2E)$ so there is a section $\sigma$ in $S^2E$ which has some zero, otherwise $S^2E$ would be trivial which implies $\Lambda^3 S^2 E = L(3e)$ is trivial which is a contradiction. Then by (1.8) we have $D[\sigma] = (P)$, i.e. only one point, and $h^0(2C_0 - Pf) \geqslant 1$. □

PROOF OF THEOREM 6.1. - Suppose $D_a$ very ample. We set $D_0 = 2C_0 - Pf$. We have $D_a \cdot D_0 = 2\varepsilon(a)$, i.e. $D_a \cdot D_0 = 1$ if $a$ is odd and $D_a \cdot D_0 = 2$ if $a$ is even. In both cases $D_Q$ is a smooth rational curve (since $D_0$ is irriducible) with respect to the embedding provided by $|D_a|$. But $\pi|_{D_0} : D_0 \to C$ is a 2:1 map over an elliptic curve, which is a contradiction. □

THEOREM 6.3. - Let $D \equiv aC_0 + bf$ be divisor on a ruled surface $X$ over an elliptic curve $C$. Assume that $a \geqslant 1$. Then $D$ is very ample if and only if

$$b > \begin{cases} ae + 2 & \text{if } e \geqslant 0 \text{ and any } a \text{ or } e = -1 \text{ and } a \leqslant 3 \\ 1 - (a/2) & \text{if } e = -1 \text{ and } a \geqslant 4 . \end{cases} \tag{6.2}$$

COROLLARY 6.4. – Let $D$ be as above. Then

$$(6.3)\qquad b_a=\begin{cases} ae+3 & \text{if } e\geqslant 0 \text{ and any } a \text{ or } e=-1 \text{ and } a\leqslant 3\\ 1-(a/2)+\varepsilon(a) & \text{if } e=-1 \text{ and } a\geqslant 4\end{cases}$$

$$(6.4)\qquad \lambda_a=\begin{cases} (1/2)a(a-1)e+3a-2 & \text{if } e\geqslant 0 \text{ and any } a \text{ or } e=-1 \text{ and } a\leqslant 3\\ (a-1)\varepsilon(a)+a & \text{if } e=-1 \text{ and } a\geqslant 4\,.\end{cases}$$

## 7. – The case $g=2$.

Let $X$ be a ruled surface over a curve $C$ with $g=g(C)=2$. Let $D\equiv aC_0+bf$ be a divisor over $X$ with $a\geqslant 2$. As for the case $g=1$, if $e\geqslant 0$ we have

$$(7.1)\qquad b_a=ae+5 \quad (\text{actually it holds also for } a=1 \text{ and } e<0)\,.$$

When $e<0$ we have two cases $e=-1$ and $e=-2$. At first we consider the cases $e=-1$. From (5.4), (5.9) and (4.5) we have

$$(7.2)\qquad \begin{cases} b_a\leqslant\begin{cases} -a+5 & \text{if } a\leqslant 7\\ -(a/2)+\varepsilon(a) & \text{if } a\geqslant 8\end{cases}\\ b_a\geqslant\begin{cases} -(a/2)+4+\varepsilon(a) & \text{if } a=2,3,5,7,9\\ -(a/2)+3+\varepsilon(a) & \text{otherwise}\end{cases}\end{cases}$$

Therefore

$$(7.3)\qquad \begin{cases} \lambda_a\geqslant\begin{cases} 6a-4-(1/2)a(a-1) & \text{if } a\leqslant 7\\ (a-1)(\varepsilon(a)-1)+2a & \text{if } a\geqslant 8\end{cases}\\ \lambda_a\leqslant\begin{cases} 6a-4+(a-1)(\varepsilon(a)-1) & \text{if } a=2,3,5,7,9\\ 5a-3+(a-1)(\varepsilon(a)-1) & \text{otherwise}\,.\end{cases}\end{cases}$$

Now we consider the case $e=-2$. From (5.4), (5.9) and (4.5) we have

$$(7.4)\qquad \begin{cases} b_a\geqslant\begin{cases} -2a+5 & \text{if } a\leqslant 3\\ -a+1 & \text{if } a\geqslant 4\end{cases}\\ b_a\leqslant\begin{cases} -a+5 & \text{if } a\leqslant 3\\ -a+4 & \text{if } a\geqslant 4\,.\end{cases}\end{cases}$$

Therefore

$$
(7.5)\qquad \begin{cases} \lambda_a \geqslant \begin{cases} (a-1)(4-a)+2a & \text{if } a\leqslant 3 \\ 2a & \text{if } a\geqslant 4 \end{cases} \\ \\ \lambda_a \leqslant \begin{cases} 6a-4 & \text{if } a\leqslant 3 \\ 5a-3 & \text{if } a\geqslant 4\,. \end{cases} \end{cases}
$$

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