On d-Folds whose Canonical Bundle is Not Numerically Effective, According to Mori and Kawamata (*).

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Sunto. – *In questo lavoro si studiano le varietà non singolari di dimensione d il cui divisore canonico non è numericamente effettivo e si estendono alcuni dei risultati ottenuti da Mori nel caso $d = 3$. Ciò viene ottenuto mediante un uso sistematico della teoria di Mori dei raggi estremali e di un forte teorema di Kawamata-Shokurov. Quest'ultimo risultato fornisce una varietà normale Y e un morfismo $\varphi: X \to Y$ che contrae un raggio estremale R e che dà la struttura di X. Se R è numericamente effettivo,* $\dim Y < \dim X$ *e φ è una (generica) fibrazione in varietà di Fano. Se R è non numericamente effettivo, φ è un morfismo birazionale e in questo caso lavoriamo nell'ipotesi che il luogo E dove φ non è isomorfismo sia un divisore in X. Se $d = 4$ diamo una descrizione abbastanza dettagliata di E.*

Introduction.

The aim of this paper is to give an attempt of classification of the complex nonsingular varieties of any dimension d, whose canonical bundle is not numerically effective. The method we use here is to extend some of the theory of the extremal rays stated by MORI in the case $d = 3$. As the title says, the ideas in this paper are mostly plagiarized from MORI [M1], [M2] and KAWAMATA [K1]. Indeed the extension to any dimension comes out from Mori's theory by using a recent result due to KAWAMATA and SHOKUROV (see 1.1). Through the paper we give for completeness almost all the proofs in detail, even those that differ from the proofs in the three-dimensional case mainly in technical matters.

Roughly speaking the Kawamata-Shokurov's result gives a normal projective variety Y and a morphism $\varphi = \mathrm{cont}_R: X \to Y$, contracting an extremal ray R, which gives the structure of X. We divide our analysis according to whether the extremal ray we consider is numerically effective or not. In the second case we work under the assumption that the dimension of the locus of X where φ is not an isomorphism is greater than or equal to $d - 1$. This condition is always satisfied if $d = 3$ but it is no longer true as the dimension increases (see [R1], 3.9).

In section 2 we consider the case when R is not numerically effective. Then Y is a $\mathbb{Q}$-factorial variety with only terminal singularities, φ is a birational morphism

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and there exists a unique irreducible exceptional divisor E which can be a Fano variety of index $\geqq 2$ or a (general) Fano fibering on $\varphi(E)$. Unlike the case $d = 3$, E may be not a $\mathbb{P}^1$-bundle when $\dim \varphi(E) = d - 2$.

In section 3 the extremal ray R is assumed to be numerically effective. Then Y is again $\mathbb{Q}$-factorial, $0 \leqq \dim Y < \dim X$, and φ is a (general) Fano fibering. In fact, unlike the case $d = 3$, the morphism φ may be not flat: indeed special fibres of big dimension can occur. In some cases we find birational models of X.

Clearly it turns out that in view of the lack of a classification of Fano varieties of any dimension the results above become less and less satisfactory as the dimension increases. The most explicit results are contained in section 4 which is devoted to the case $d = 4$. Especially when the extremal ray is not numerically effective and E contracts to a point via the morphism φ, a rather detailed description can be done. In such a case we find for E a three-dimensional Fano variety of index $\geqq 2$ with Gorenstein singularities and we work under the assumption that the degree of E is less than or equal to 72. To this purpose, some classical examples of singular Fano varieties whose degree actually reaches the upper bound 72, justify the assumption above which, in fact, can be reasonably considered as a conjecture. Indeed, $d = 4$ is a special case and when $d > 4$, up to a classification of (possibly singular) Fano varieties of any dimension, it seems we have not to expect much more precise results than those stated in sections 2, 3.

Definitions and some preliminaries are given in sections 0, 1.

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We would like to thank P. FRANCIA for many stimulating discussions on the subject. We are also indepted to M. REID for pointing out a fatal error in a previous proof.

After the paper was written down we knew that very similar results have been also obtained by T. ANDO [A].

0. – Notation, conventions and terminology.

Throughout the paper we work on the complex number field $\mathbb{C}$. By a *variety* (resp. d-fold) we shall mean an irreducible reduced projective scheme of dimension d (resp. nonsingular). For a variety V, we define

$$N_1(V) = (\{1\text{-cycles}\}/\sim) \otimes \mathbb{R}$$

where « $\sim$ » means numerical equivalence and a 1-cycle is an element of the free abelian group generated by all the irreducible reduced subvarieties of dimension 1;

$NE(V)$ = the convex cone in $N_1(V)$ generated by effective 1-cycles;

$\overline{NE}(V)$ = the closure of $NE(V)$ in $N_1(V)$ with respect to the real topology.

We shall express the intersection of cycles by the symbol « $\cdot$ » and the linear equivalence of divisors by « $\equiv$ ». The real vector spaces $N_1(V)$ and $N_{d-1}(V) = (\{\text{Cartier divisors}\}/\sim) \otimes \mathbb{R}$ are dual to each other via « $\cdot$ ». They are of finite dimension $\varrho(V)$, which is called the *Picard number* of V.

We shall use the following Kleiman's criterion for ampleness [K]:

(0.1) *A* Cartier divisor D on V is ample if and only if $(D\cdot Z) > 0$ for all

$$Z \in \overline{NE}(V) \cap \{Z \in N_1(V),\ \|Z\| = 1\}.$$

The *dualizing sheaf* of V will be denoted by ω_V. If V is normal, the *canonical divisor* K_V is defined to be a Weil divisor on V such that $\mathcal{O}_{\mathrm{Reg}(V)}(K_V) = \bigwedge^d \Omega_{\mathrm{Reg}(V)}$. We say that V has only *terminal* (resp. *canonical*) singularities if the following conditions are satisfied:

1) mK_V is a Cartier divisor for some integer m;

2) for a resolution of singularities $f\colon V' \to V$ we can write

$$mK_{V'} = f^*(mK_V) + \sum r_i E_i, \qquad r_i \in \mathbb{Z}$$

such that $r_i > 0$ (resp. $r_i \geqq 0$), where E_i are all the prime exceptional divisors of V'.

Moreover, V is called $\mathbb{Q}$-*factorial* if for every Weil divisor D on V, there exists an integer m such that mD is a Cartier divisor.

Let D be a divisor on V. We shall use the symbols

$$|D| \;= \text{the complete linear system associated to } D;$$
$$h^i(D) = \dim_{\mathbb{C}} H^i(V, \mathcal{O}_V(D)), \qquad i = 0, \dots, d;$$
$$\chi(D) = (-1)^i \sum_{i=0}^{d} h^i(D).$$

If D is numerically effective, the *numerical Kodaira dimension* is defined to be

$$\varkappa_{\mathrm{num}}(D) = \max\{s : D^s \nsim 0\}.$$

Then

$$\max\{0, \varkappa(D)\} \leqq \varkappa_{\mathrm{num}}(D) \leqq d$$

where $\varkappa(D)$ is the *Iitaka D-dimension.*

A part of the Mori's theory of extremal rays is to be used throughout the paper. As far as generalities about numerically effective cycles (*nef* cycles from now on), extremal rays, extremal rational curves, etc. are concerned, we shall refer to Mori's papers [M1], [M2]. Let us recall that $R = \mathbb{R}_+[Z]$ in $\overline{NE}(V)$, V nonsingular, is an *extremal ray* if

1) $(K_V \cdot Z) < 0$;

2) $Z_1, Z_2 \in \overline{NE}(V)$ satisfy $Z_1, Z_2 \in R$ when $Z_1 + Z_2 \in R$.

1. – Some preliminary results.

The main step to go on is the following contraction theorem due to KAWAMATA [K2] and SHOKUROV [S], and also proved in [K1], [R] in the case $d = 3$.

THEOREM 1.1 (Kawamata-Shokurov). – *Let V be a d-dimensional variety with canonical singularities. Let $D \in \operatorname{Pic}(V)$ be nef. Suppose that $aD - K_V$ is nef and $(aD - K_V)^d > 0$ for some integer $a \geqq 1$. Then $\varkappa(D) \geqq 0$ and $|mD|$ is base point free for $m \gg 0$.* ■

Let $\sigma \colon V \to Z$ be the morphism associated to $|mD|$. Then the Stein factorization of σ gives a normal projective variety Y and a morphism $\varphi \colon V \to Y$ with connected fibres, the contraction of $D^{\perp} \cap \overline{NE}(V)$, which satisfies the following conditions (here « $\perp$ » means the orthogonal)

(1.1.1) $\varphi_* \mathcal{O}_V = \mathcal{O}_Y$;

(1.1.2) for every curve C in V, $\dim \varphi(C) = 0$ if and only if $D \cdot C = 0$;

(1.1.3) the pair (Y, φ) is unique up to isomorphism;

(1.1.4) $D \in \varphi^* \operatorname{Pic}(Y)$.

As far as (1.1.4) is concerned, note that to prove Theorem 1.1 it is shown that $|p^t D|$, $|q^s D|$ are base point free for two primes $p > 1$, $q > 1$ and some positive integers t, s (cfr. [K2], 2.6 and also [K1], Thm. 2). Then $p^t D$, $q^s D \in \varphi^* \operatorname{Pic}(Y)$ so we get the result. ■

From now on, let X be a d-fold and suppose the canonical bundle K_X to be *not nef*. Then in view of Mori's theorem on cone ([M1], 1.4.2) the assumption on K_X is equivalent to saying that there exists an extremal ray R on X. We define:

$$R^{\perp} = \{D \in N_{d-1}(X),\ D \cdot R = 0\} \subset N_1(X)^*,$$

$$R^* = \{D \in N_{d-1}(X),\ D \cdot Z \geqq 0 \text{ for every } Z \in \overline{NE}(X) \text{ and } D \cdot C = 0 \text{ iff } [C] \in R\},$$

where « [ ] » denotes the class of a 1-cycle in $\overline{NE}(X)$.

The following Lemmas are proved by Mori in the case $d = 3$ (see [M2], 3.1, 3.2, 3.3, 3.4); up to obvious changes the same proof works in any dimension.

LEMMA 1.2. – *The set R^* is a non-empty open convex cone of $R^{\perp}$. Furthermore if H, D are divisors on X such that $H \in R^*$, $(D \cdot R) > 0$ then $nH + D$ is ample for $n \gg 0$.*

LEMMA 1.3. – *For every divisor H on X such that $H \in R^*$ one has $h^i(nH) = 0$, $i \geqq 1$, $n \gg 0$.*

LEMMA 1.4. – *Let H be a divisor on X such that $H \in R^*$. Then R is not nef if and only if $(H^d) > 0$.*

LEMMA 1.5. – *Let H be a divisor on X such that $H \in R^*$ and let B be an irreducible subvariety of X such that $H \cdot R \sim 0$ on X. Then for every irreducible curve C in B, $[C] \in R$.* ∎

We fix now an extremal ray R and a divisor $H \in R^*$. Since $(K_X \cdot R) < 0$ the divisor $nH - K_X$ is ample for $n \gg 0$ in view of Lemma 1.2. Then Theorem 1.1 applies to give a normal projective variety Y and a morphism $\varphi = \mathrm{cont}_R \colon X \to Y$ contracting R and with connected fibres which satisfies conditions (1.1.1), ..., (1.1.4). Indeed by construction φ contracts $H^\perp \cap \overline{NE}(X)$ and $H^\perp \cap \overline{NE}(X) = R$ since $H \in R^*$.

Let $A = \{x \in X;\ \varphi$ is not an isomorphism at $x\}$. We say that R is of type (a, b) if $a = \dim A$ and $b = \dim \varphi(A)$, where « dim » means the maximum of the dimensions of irreducible components. Of course $d \geqq a > b \geqq 0$. In the case of 3-folds we know from Mori's theory that it has to be $a \geqq d - 1 = 2$. This is no longer true in general when the dimension increases: an example of a 4-fold having an extremal ray of type (a, b) with $a < d - 1 = 3$ can be found in [B1] (example 3.9). Throughout the paper we work under the assumption that R is of type (a, b) with $a \geqq d - 1$; such a restriction is relevant only if R is not nef, because when R is nef it has to be $a = d$. If $a < d$ the contraction $\varphi \colon X \to Y$ is a birational morphism. In case $a < d - 1$ one only knows that Y has rational not $\mathbb{Q}$-Gorenstein singularities (see [K2], 5, (B)).

2. – The case when R is not nef.

Throughout this section the extremal ray R is assumed to be not nef. The following structure theorem is a consequence of Thm. 1.1 and it is essentially due to Kawamata (see [K1] for a proof in the three-dimensional singular case and [K2], section 5, (B) for the statement in any dimension).

THEOREM 2.1. – *Let X be a d-fold and let R be an extremal ray of type (a, b) with $a \geqq d - 1$. Suppose R to be not nef. Then there exist a normal projective variety Y and a birational morphism $\varphi = \mathrm{cont}_R \colon X \to Y$ such that*

(2.1.1) *for every integral curve C in X, $\dim \varphi(C) = 0$ iff $[C] \in R$;*

(2.1.2) *the sequence*

$$0 \to \mathrm{Pic}\,(Y) \xrightarrow{\varphi^*} \mathrm{Pic}\,(X) \xrightarrow{(\cdot R)} \mathbb{Z}$$

is exact and $\varrho(X) = \varrho(Y) + 1$;

(2.1.3) *$-K_X$ is φ-ample;*

(2.1.4) *there exists a unique irreducible reduced divisor E on X such that $(E \cdot R) < 0$, φ induces an isomorphism on $X \setminus E$ and $\dim \varphi(E) \leqq d - 2$.*

(2.1.5) *Y is locally $\mathbb{Q}$-factorial and has only terminal singularities.*

PROOF. – Let $\varphi = \mathrm{cont}_R\colon X \to Y$ be the contraction of R given by Theorem 1.1 and satisfying conditions (1.1.1), ..., (1.1.4). Lemmas 1.3, 1.4 say that $\chi(nH) = h^0(nH)$ goes to the infinity as n^d, that is $\varkappa(H) = d$. Then $a = d-1$ and φ is a birational morphism. Now (2.1.1) is clear while the ampleness of $nH - K_X$, $n \gg 0$, gives (2.1.3). To prove (2.1.2) consider the sequence

$$0 \to \mathrm{Pic}\,(Y) \xrightarrow{\varphi^*} \mathrm{Pic}\,(X) \xrightarrow{(\cdot\ell)} \mathbb{Z}$$

where $R = \mathbb{R}_+[\ell]$. The condition $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$ implies $\varphi_*\varphi^* = \mathrm{id}$, so φ^* is injective. To see the exactness in Pic (X) let L be an element of Pic (X) such that $(L\cdot\ell) = 0$. Then for $\alpha \gg 0$ the divisor $L + \alpha H$ satisfies the conditions

1) $L + \alpha H$ is nef;
2) $L + \alpha H - K_X$ is ample;
3) $(L + \alpha H)\cdot C = 0$ iff $H\cdot C = 0$, C curve in X.

Then in view of (1.1.2), (1.1.3), (1.1.4), Theorem 1.1 applies to $L + \alpha H$ to give $L + \alpha H = \varphi^* M$, $M \in \mathrm{Pic}\,(Y)$. Whence $L \in \varphi^*\,\mathrm{Pic}\,(Y)$, therefore

$$\ker\,(\cdot, \ell) \subset \varphi^*\,\mathrm{Pic}\,(Y)\,.$$

The converse is clear since

$$\varphi^* N\cdot\ell = N\cdot\varphi_*\ell = 0$$

for every $N \in \mathrm{Pic}\,(Y)$. Thus the sequence is exact, $\varrho(X) = \varrho(Y) + 1$ and (2.1.2) is proved.

Now, since R is not nef, there exists an irreducible reduced divisor E such that

$$(E\cdot R) < 0\,.$$

This means that $E\cdot C < 0$ for every curve C such that $[C] \in R$, hence E contains every such a curve. Then it follows that E is unique since $a = d - 1$. The fact that φ is isomorphism on $X\setminus E$ is clear since $nH - E$, $n \gg 0$, is ample by Lemma 1.2. Moreover the condition $\varrho(X) = \varrho(Y) + 1$ implies $\dim \varphi(E) \leq d - 2$ and (2.1.4) is proved.

To prove (2.1.5) let D be an arbitrary prime divisor on Y and let D' be the strict transform of D under φ. Write

$$-\alpha = (E\cdot R)\,, \qquad \beta = (D'\cdot R)\,.$$

Then $(\alpha D' + \beta E)\cdot R = 0$ so that (2.1.2) yields

$$\alpha D' + \beta E \equiv \varphi^* \Delta\,, \qquad \Delta \in \mathrm{Pic}\,(Y)\,.$$

Hence, for some rational function f on X,

$$\alpha D' + \beta E = \varphi^* \Delta + (f) .$$

Since φ is birational, we can also write $\alpha D = \Delta + (f)$, so αD is a Cartier divisor.

Finally, let m be a positive integer such that mK_Y is a Cartier divisor. Since φ is isomorphism outside of E we can write

$$mK_X \equiv \varphi^* mK_Y + aE$$

for some $a \in \mathbb{Z}$. The relations $(K_X \cdot R) < 0$, $(E \cdot R) < 0$, $(\varphi^* K_Y \cdot R) = 0$ imply a to be positive and (2.1.5) is proved. q.e.d.

REMARK (2.1)$'$. – As it is clear from the proof, we need the assumption « $a \geqq d - 1$ » only to get (2.1.4), (2.1.5).

QUESTION 2.2. – If $d = 3$, the contraction φ is the blowing-up of Y along $\varphi(E)$. Does the same in general hold true? [1]. ■

Now a problem arising here is to describe the exceptional divisor E of the contraction $\varphi: X \to Y$. We can go on by considering several cases according to numerical properties of the cycle $H \cdot E$. We need a preliminary result.

LEMMA 2.3. – *Let E be the exceptional divisor as in* 2.1 *and let a be a positive integer. Then*

$$H^i(\mathcal{O}_{aE}(nH + K_X)) = (0) , \quad i > 0, \ n \gg 0 .$$

Further, let L be a divisor on X such that $(L - K_X - \lambda E) \cdot R > 0$ for every integer $\lambda \geqq 0$. Then

$$H^i(\mathcal{O}_{aE}(nH + L)) = (0) , \quad i > 0, \ n \gg 0 .$$

PROOF. – Consider the standard exact sequence, $a \geqq 1$,

$$0 \to \mathcal{O}_X(nH + K_X - aE) \to \mathcal{O}_X(nH + K_X) \to \mathcal{O}_{aE}(nH + K_X) \to 0 .$$

For $n \gg 0$, $nH - aE$ is ample by Lemma 1.2 hence

$$H^i(\mathcal{O}_X(nH + K_X - aE)) = (0) , \quad i > 0, \ n \gg 0 .$$

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[1] This is certainly true if $\mathcal{O}_E(-E)$ is very ample on E and also in case when $\dim \varphi(E) = \dim E - 1 = d - 2$ and $\varphi_{|E}$ is equidimensional (see [A], 2.3).

By the Kawamata-Viehweg vanishing one has $H^i(\mathcal{O}_X(nH + K_X)) = (0)$, $i > 0$, since H is nef and $H^d > 0$ in view of Lemma 1.4. Then the first part of the statement follows.

The assumption on L implies that $nH + L - K_X - \lambda E$ is ample, $n \gg 0$, $\lambda \geqq 0$, again by Lemma 1.2. Therefore the Kodaira vanishing reads

$$H^i(\mathcal{O}_X(nH + L - \lambda E)) = (0), \quad i \geqq 1, \lambda \geqq 0, \ n \gg 0 .$$

Thus the exact sequence

$$0 \to \mathcal{O}_X(mH + L - aE) \to \mathcal{O}_X(mH + L) \to \mathcal{O}_{aE}(mH + L) \to 0$$

gives the result. q.e.d.

First, we consider the case when $H \cdot E \sim 0$. To this purpose let us give the following

DEFINITION. – Let V be an irreducible, reduced variety with invertible dualizing sheaf ω_V. We say that V is a *Fano variety of index* r if r is the maximal integer such that $\omega_V^{-1} \sim rL$, L ample sheaf on V.

PROPOSITION 2.4. – *Let* $\varphi\colon X \to Y$ *be as in Theorem* 2.1 *and assume* $H \cdot E \sim 0$. *Then* $\varphi(E)$ *is a point and* E *is a Fano variety of dimension* $d-1$, *index* $r > 1$ *and* $h^i(\mathcal{O}_E) = 0$, $i > 0$. *Furthermore*

$$\chi(\mathcal{O}_E(K_X)) = \chi(\mathcal{O}_E(E)) = 0 .$$

PROOF. – The assumption $H \cdot E \sim 0$ says that for every curve C in E, $[C] \in R$. This implies that the restriction map

$$N_1(X)^* \to N_1(E)^*$$

has image of dimension 1 (see Lemma 1.5). Then there exist an ample divisor L on E and intergers a, b such that

$$-aL \sim \mathcal{O}_E(K_X), \quad -bL \sim \mathcal{O}_E(E) .$$

Since $K_X \cdot R < 0$, $E \cdot R < 0$, then a, b are positive, so

$$\omega_E^{-1} \equiv -(K_X + E)_{|E} \sim (a + b)L$$

is ample and E is a Fano variety of index $r \geqq a + b \geqq 2$. To prove that $h^i(\mathcal{O}_E) = 0$, note that $\mathcal{O}_E(mH) \cong \mathcal{O}_E$ because $H \cdot E \sim 0$ and $h^0(\mathcal{O}_E(mH)) > 0$, $m \gg 0$, since $|mH|$, $m \gg 0$, is base point free by Theorem 1.1. Then Lemma 2.3 with $L \equiv 0$ gives the result.

Now consider the polynomial

$$P(n) = \chi(\mathcal{O}_E(nL)) .$$

By using Lemma 2.3 we find, for $n \gg 0$,

$$P(-a) = \chi(\mathcal{O}_E(K_X)) = \chi(\mathcal{O}_E(mH + K_X)) = h^0(\mathcal{O}_E(mH + K_X)) .$$

Hence $P(-a) = 0$ since $(-mH - K_X)_{|E} \equiv -K_{X|E} \sim aL$ is ample on E. The Serre duality says, for every integer n, that

$$P(n) = (-1)^{d-1} P(-a - b - n) .$$

Then, if $a \neq b$, $P(-a) = 0$ implies $P(-b) = 0$, so $\chi(\mathcal{O}_E(E)) = 0$. q.e.d.

REMARK 2.5. – Let E, L be as in Proposition 2.4 and let us consider the particular case when E is normal with isolated singularities. Then a generalization of a result of Kobayashi-Ochiai (see [F-S], (0.6.1)) shows that Pic $(E) \cong H^2(E, \mathbb{Z})$ is torsion free and the index of E is $r \leq d$. If $r = d, (E, \mathcal{O}_E(L)) \cong (\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(1))$ while if $r = d - 1$, then E is isomorphic to a hyperquadric in $\mathbb{P}^d$ and L is linearly equivalent to a hyperplane section.

In the general case, looking over the proof of 2.4 and using Lemma 2.3 one sees that the polynomial, of degree $d - 1$, $\chi(\mathcal{O}_E(nL))$ vanishes for $n = -1, \ldots, -a$, so that it has to be $a \leq d - 1$. ■

Now, assume $H \cdot E \nsim 0$ and let $s \in \{1, \ldots, d - 2\}$ be the maximal integer such that $H^s \cdot E \nsim 0$. If $d = 3$ we know that $\varphi: X \to Y$ is the blowing-up along $\varphi(E)$ and $\varphi_{|E}: E \to \varphi(E)$ is a $\mathbb{P}^1$-bundle. The situation becomes more and more complicated as the dimension increases and one has to analyse several cases according to the value of s. The following example shows that $\varphi_{|E}: E \to \varphi(E)$ can be not a $\mathbb{P}^1$-bundle when $s = d - 2$.

EXAMPLE 2.6. – Let Y be the ordinary double point of a 4-fold (given in a 5-dimensional space by $xy - zt + w^2 = 0$). Let $\varphi: X \to Y$ be the blow-up of the plane $x = z = w = 0$; an easy calculation in coordinates shows that X is nonsingular, that the exceptional locus of φ is a $\mathbb{P}^1$-bundle outside the origin, but the fibre $\varphi^{-1}(O)$ is 2-dimensional. ■

A rough description of E is given by the following

PROPOSITION 2.7. – *Let $\varphi: X \to Y$ be as in Theorem 2.1. Assume $H \cdot E \nsim 0$ and let $s \in \{1, \ldots, d - 2\}$ be the maximal integer such that $H^s \cdot E \nsim 0$. Then* $\dim \varphi(E) = s$ *and every irreducible, reduced fibre of $\varphi_{|E}$ is a Fano variety of index $r > 1$.*

PROOF. – The restriction $\varphi_{|E}$ is the morphism associated to the complete linear system $|mH_{|E}|$, $m \gg 0$, in view of Lemma 2.3. Since $|mH_{|E}|$ is base point free for $m \gg 0$, $\varkappa_{\mathrm{num}}(H_{|E}) = \varkappa(H_{|E})$, so that $\dim \varphi(E) = s$.

Now the proof runs as in Proposition 2.4. Let F be an irreducible, reduced fibre of $\varphi_{|E}$. Then $H \cdot F \sim 0$, so for every irreducible curve C in F, $[C] \in R$ and the image of the restriction map

$$N_1(X)^* \to N_1(F)^*$$

has dimension 1 (see Lemma 1.5). Therefore there exist an ample divisor L on F and integers a, b such that

$$-aL \sim \mathcal{O}_F(K_X)\,, \qquad -bL \sim \mathcal{O}_F(E)$$

where a, b are positive because $K_X \cdot R < 0$, $E \cdot R < 0$. Hence

$$\omega_F^{-1} \equiv \omega_{E|F}^{-1} \equiv -(K_X + E)_{|F} \sim (a+b)L$$

is ample on F and F is a Fano variety of index $r \geqq a + b \geqq 2$. q.e.d.

3. – The case when R is nef.

Throughout this section the extremal ray R is assumed to be nef. The structure Theorem 3.2 is again a direct consequence of Theorem 1.1 (see also [K2], section 5, (B)). We need the following

LEMMA 3.1. – *Under the assumptions as above, let $H \in R^*$. Then*

(3.1.1) $\qquad \varkappa(H) = d - 1 \quad$ *if* $H^{d-1} \cdot - K_X > 0\,;$

(3.1.2) $\qquad H^{d-1} = 0$ *in* $N_1(X) \quad$ *if* $H^{d-1} \cdot - K_X \leqq 0\,.$

PROOF. – In view of Lemmas 1.3, 1.4, the Riemann-Roch theorem gives for $n \gg 0$, λ positive integer

$$h^0(nH) = \lambda(H^{d-1} \cdot - K_X)n^{d-1} + (\text{terms of degree} \leqq d-2)\,,$$

which proves (3.1.1). Since H is nef one has $H^{d-1} \in \overline{NE}(X)$ and $nH - K_X$ is ample, $n \gg 0$, by Lemma 1.2. Then $(H^{d-1} \cdot nH - K_X) = 0$ because $H^d = 0$ and the Kleiman's ampleness criterion implies that $H^{d-1} \sim 0$. q.e.d.

THEOREM 3.2. – *Let X be a d-fold and let R be an extremal ray. Suppose R to be nef. Then there exist a normal projective variety Y and a morphism* $\varphi = \mathrm{cont}_R \colon X \to Y$

such that $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$ *and for an arbitrary irreducible curve* C *in* X, $[C] \in R$ *if and only if* $\dim \varphi(C) = 0$. *Moreover* $-K_X$ *is* φ*-ample,* $\varrho(X) = \varrho(Y) + 1$, Y *is* $\mathbb{Q}$*-factorial and we have*

(3.2.1) $\dim Y = d - 1$, *for every general fibre* F, $F \cong \mathbb{P}^1$, $(F \cdot K_X) = -2$, $[F] \in R$;

(3.2.2) $\dim Y = d - s$, $s = 2, \ldots, d - 1$, $\varphi\colon X \to Y$ *is a Fano fibering, that is every general fibre is a Fano variety. Furthermore when* $s = d - 1$, *then* Y *is a nonsingular curve and every fibre is irreducible, reduced of dimension* $d - 1$.

(3.2.3) $\dim Y = 0$ *and* X *is a Fano* d*-fold.*

PROOF. – The first part of the proof runs as in Theorem 2.1, and the fact that Y is $\mathbb{Q}$-factorial is proved in [K2], 5.2. The morphism $\varphi\colon X \to Y$ is the Stein factorization of the morphism associated to $|mH|$, $m \gg 0$, and $H = \varphi^* D$, $D \in \operatorname{Pic}(Y)$. Let F be a general fibre of φ.

Assume $H^{d-1} \cdot - K_X > 0$. Then $\varkappa(H) = d - 1$ by Lemma 3.1, so that $\dim Y = d - 1$. Hence $\omega_F = K_{X|F}$ and $[F] \in R$, therefore

$$\deg \omega_F = K_X \cdot F = K_X \cdot R < 0\,.$$

Hence $F \cong \mathbb{P}^1$, $\deg \omega_F = (F \cdot K_X) = -2$ and (3.2.1) is proved.

Suppose now $H^{d-1} \cdot - K_X \leqq 0$. Then $H^{d-1} \sim 0$ by Lemma 3.1, so $\varkappa(H) \leqq \varkappa_{\text{num}}(H) \leqq d - 2$, that is $\dim Y \leqq d - 2$. Since $H \in \varphi^* \operatorname{Pic}(Y)$, the restriction $H_{|F}$ is numerically trivial. Then the ampleness of $nH - K_X$, $n \gg 0$, implies the ampleness of

$$\omega_F^{-1} \equiv -K_{X|F} \sim (nH - K_X)_{|F}\,.$$

When $\dim Y = 1$, the condition $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$ implies that Y is a nonsingular curve, so φ is a flat morphism (see [H], III, 9.7) and all fibres F are of dimension $d - 1$. Let G be an irreducible component of any fibre F of φ. Since φ contracts R and R is nef one sees that $(G \cdot R) = 0$ and $(G \cdot C) = 0$ for every irreducible curve $C \subset F$. This means that $G = F_{\text{red}}$. Otherwise, write $F = G_1 + G_2 + \Delta$, G_1, G_2, Δ effective divisors, G_1, G_2 irreducible. The fibre F is connected so there exists a point $p \in G_1 \cap G_2$ if $G_1 \neq G_2$. Let L be a very ample divisor, $L \ni p$, $L \not\supset G_1 \cap G_2$ (if $\dim G_1 \cap G_2 \geqq 1$) and look at the 1-cycle $\gamma = L_{|G_2}^{d-2}$ on G_2. Thus $[\gamma] \in R$ and $(G_1 \cdot \gamma) = 0$, contradiction; so F is irreducible.

To prove that F is also reduced write $F = aG$, $G = F_{\text{red}}$, a positive integer. Since $F^2 \sim 0$ then $\mathcal{O}_G(G) \sim \mathcal{O}_G$, so

$$\chi(\mathcal{O}_F) = \chi(\mathcal{O}_{aG}) = \chi(\mathcal{O}_G) + \chi(\mathcal{O}_G(-G)) + \ldots + \chi(\mathcal{O}_G(-a+1)G) = a\chi(\mathcal{O}_G)\,.$$

The flatness of φ implies that $\chi(\mathcal{O}_F) = 1$, therefore $a = 1$ and (3.2.2) is proved.

Finally, when $\dim Y = 0$, we find $\varrho(X) = 1$, then the condition $(K_X \cdot R) < 0$ shows that $-K_X$ is ample. q.e.d.

Let X be as in cases (3.2.1) or (3.2.2) with $s = 2$ of Theorem 3.2. Then X can be described as follows, up to birationality, by using a result by Sarkisov together with the Enriques-Iskovskih classification of the minimal rational surfaces.

SUPPLEMENT 3.3. *Let $\varphi: X \to Y$ be as in cases* (3.2.1) *or* (3.2.2) *with $s = 2$ of Theorem* 3.2. *Then X is birationally equivalent to one of the following types of d-folds $\tilde{X}$:*

a) $\tilde{X} = Y \times \mathbb{P}^2$;

b) $\tilde{X} \to \tilde{Y}$ *is a conic bundle, $\tilde{Y}$ birationally equivalent either to Y or to a conic bundle on Y;*

c) there exists a morphism $\tilde{\varphi}: \tilde{X} \to Y$ such that the generic fibre (on $K = \mathbb{C}(Y)$) is

c_1) a quadric $Q = \mathbb{P}^1_K \times \mathbb{P}^1_K \subset \mathbb{P}^3_K$ with Pic $(Q) \simeq \mathbb{Z}$ *generated by $\mathcal{O}_Q(1)$;*

c_2) a Del Pezzo surface S with Pic $(S) \simeq \mathbb{Z}$ *generated by the anticanonical sheaf ω_S^{-1} and $\omega_S^2 = 1, \ldots, 6$.*

PROOF. - Let X be as in (3.2.1). Then X belongs to class b) of the statement in view of [Sa], 1.13. Therefore we can assume X to be as in (3.2.2) with $s = 2$. Let $K = \mathbb{C}(Y)$ and denote by $F_\eta = X \otimes_Y \operatorname{Spec} K$ the generic fibre of $\varphi: X \to Y$. Since the general fibre of φ is a rational surface over $\mathbb{C}$, one has $\varkappa(F_\eta) < 0$, $q(F_\eta) = 0$ (cf. [I], § 10.3, [H], III, 9.9) so that F_η is a rational surface over K. Let S be a minimal model of F_η. Hence S belongs to one of the following families of surfaces ([Is], Thm. 1):

i) $S = \mathbb{P}^2_K$;

ii) $S = \mathbb{P}^1_K \times \mathbb{P}^1_K$ is a quadric in $\mathbb{P}^3_K$, having Pic $(S) \simeq \mathbb{Z}$ generated by a hyperplane section;

iii) S is a Del Pezzo surface with Pic $(S) \simeq \mathbb{Z}$ generated by the anticanonical sheaf ω_S^{-1};

iv) there exists a morphism $S \to B$ such that the generic fibre and the base curve B are nonsingular of genus 0.

It is not difficult to prove that there exist a d-fold $\tilde{X}$ and a birational map $\tilde{X} \dashrightarrow X$ such that the composition $\tilde{X} \to Y$ is a morphism whose generic fibre is isomorphic to the minimal model S of F_η. Moreover the function field $K(S)$ of S is isomorphic to the function field $\mathbb{C}(X)$ of X (see [EGA], I, 3.4.6). Then case i) yields $\mathbb{C}(X) \simeq$ $\simeq K(\mathbb{P}^2_K) \simeq \mathbb{C}(Y \times \mathbb{P}^2)$, hence X is birationally equivalent to $Y \times \mathbb{P}^2$ and we get a), while ii) gives c_1) of the statement. Suppose now case iii) holds. Therefore we get c_2), after proving that $1 \leq \omega_S^2 \leq 6$. This follows by [M1], 3.5.2 where it is proved that the case $\omega_S^2 = 7$ does not occur, while if $\omega_S^2 = 8, 9$ the Picard group Pic (S) is not generated by the anticanonical sheaf. Finally in case iv) one has an embedding of the function fields $K(B) \hookrightarrow \mathbb{C}(X)$ corresponding to the surjective morphism $S \to B$. On the other hand $K(B)$ is nothing but the function field of a conic bundle $V \to W$, W birationally equivalent to Y, whose generic fibre is isomorphic to B. Then the inclusion $K(B) \hookrightarrow \mathbb{C}(X)$ gives a rational map $X - \to V$ whose generic fibre is isomorphic to a conic. Therefore, by using again [Sa], 1.13, X belongs to class *b*) of the statement. q.e.d.

Following an idea due to P. IONESCU, we get a slight improvement of (3.2.2) (compare with example 3.6 below).

SUPPLEMENT 3.4. – *Let* $\varphi\colon X \to Y$ *be as in* (3.2.2) *and suppose* $\dim Y = 2$. *Then every fibre of* φ *has dimension* $d-2$.

PROOF. – Let $y \in Y$ be a closed point and denote by F the fibre over y. Then $\omega_F^{-1} \sim D_{|F}$, where $D = nH - K_X$ is ample for $n \gg 0$ (see the proof of (3.2.2)). Consider the embedding of X given by the complete linear system $|mD|$, $m \gg 0$. Let $\tilde{Y}$ be the nonsingular surface obtained by intersecting $d-2$ general members of $|mD|$ and denote by $\tilde{\varphi}$ the restriction of φ to $\tilde{Y}$. Assume $\dim F = d-1$. Then we see that $\tilde{\varphi}$ contracts the curve $\gamma = \tilde{Y} \cap F$ to a point and $F_{|F} \sim 0$ since $(F\cdot R) = 0$. Therefore we find

$$0 > (\gamma^2)_{\tilde{Y}} = m^{d-2}(D^{d-2}\cdot F^2)_X = m^{d-2}(D_{|F}^{d-2}\cdot F_{|F})_F = 0\,,$$

contradiction. Then $\dim F = d-2$ and we are done. q.e.d.

REMARK 3.5. – Let X be a Fano d-fold as in (3.2.3). Then $\operatorname{Pic}(X) \cong H^2(X, \mathbb{Z})$ is torsion free (see Remark 2.5), so $K_X \equiv -rL$ for some (unique) ample divisor L on X, where $r = 1, \ldots, d+1$ is the index of X. If $r = d+1$, d, then (X, L) is isomorphic to $(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$, $(Q, \mathcal{O}_Q(1))$, Q hyperquadric in $\mathbb{P}^{d+1}$, respectively. If $r = d-1$, the pairs (X, L) are also called *Del Pezzo manifolds*: it has to be $L^d = 1, \ldots, 8$ and they are completely classified by Fujita in several papers (see [F1], § 2, [F2], §§ 5, 6, [F3]). ■

In the three-dimensional case the contraction φ in Theorem 3.2 is a flat morphism and X is a conic bundle in case (3.2.1) as proved by Mori. This is no longer true when $d \geqq 4$, as the following example shows.

EXAMPLE 3.6. – Let L_1, L_2, L_3 be general hyperplanes in $\mathbb{P}^n$, $n \geqq 3$, and let $\mathbb{P}^2$ be the base of the net of hyperplanes $\sum \lambda_i L_i$. Now let $X \subset \mathbb{P}^2 \times \mathbb{P}^n$ be the incidence correspondence $\{(p, L);\ p \in L\}$; then the projection $X \to Y = \mathbb{P}^n$ is a contraction, is a $\mathbb{P}^1$-bundle outside the axis, but has fibre $\mathbb{P}^2$ over every point of the axis.

4. – The case of 4-folds.

From now on we shall suppose $d = 4$. This section is essentially devoted to study the particular case when the extremal ray R is not nef and $H\cdot E \sim 0$ as in Proposition 2.4. Then $\varphi\colon X \to Y$ is a birational morphism contracting E to a point, E is a Fano variety of dimension 3 and index $r > 1$, there exist positive integers a, b and an ample divisor $L = \mathscr{L}_{|E}$ on E, $\mathscr{L} \in \operatorname{Pic}(X)$, such that

$$-aL \sim \mathcal{O}_E(K_X), \qquad -bL \sim \mathcal{O}_E(E).$$

To say more on E we have to study the polynomial, of degree 3,

$$P(n) = \chi(\mathcal{O}_E(nL)).$$

Let us state some preliminaries. From Proposition 2.4 we know that

$$P(0) = 1, \qquad P(-a) = 0. \tag{4.1.1}$$

Further, the divisor $\mathscr{L}$ verifies the condition $(n\mathscr{L} - K_X - \lambda E)\cdot R > 0$, $\lambda \geqq 0$, for every $n > -a$. Then, since $H_{|E} \equiv 0$ (see the proof of 2.4), we get $h^i(nL) = 0$, for any $n > -a$, $i > 0$, in view of Lemma 2.3. It follows:

(4.1.2) $P(1) = h^0(L)$;

(4.1.2)′ $P(n) = 0$ for every $n = -1, \ldots, -a$, hence in particular $a \leqq 3$;

(4.1.2)″ $P(n) = 0$ for every $n = 1 - a - b, \ldots, -b$ by the Serre duality.

Following Fuijta, we call *Δ-genus* the integer

$$\Delta(L) = 3 + L^3 - h^0(L).$$

The base locus $Bs|L|$ of $|L|$ satisfies the inequality ([F1], 1.9)

$$\dim Bs|L| < \Delta(L). \tag{4.1.3}$$

Let $\sigma = \sigma_{|L|}\colon E \dashrightarrow E' \subset \mathbb{P}^N$ be the rational map associated to the ample divisor L. When $\deg \sigma \neq \infty$ one has

$$L^3/\deg \sigma \geqq \deg E' \geqq \operatorname{codim}_{\mathbb{P}^N} E' + 1 \tag{4.1.4}$$

and all the cases when the equality holds are classically classified (see f.e. [Mu], 2.7).

Finally let us assume the following extra condition

$$1 \leqq (\omega_E^{-1})^3 \leqq 72. \tag{4.1.5}$$

The theory of Fano 3-folds says that (4.1.5) holds true whenever E is nonsingular (indeed in this case the upper bound is known to be 64).

PROPOSITION 4.1. – *Let $\varphi: X \to Y$ be as in Theorem 2.1, let E be the exceptional divisor and assume $H \cdot E \sim 0$. With the notation as above, the 3-ple $(E, \omega_E, \mathcal{O}_E(E))$ can be described as in the table below (where ω_E, $\mathcal{O}_E(E)$ are given up to linear equivalence modulo torsion).*

L	$\Delta(L)$	$h^0(L)$	L^3	Structure of $\sigma: E \xrightarrow{\lvert L\rvert} \mathbb{P}^N$	$\mathcal{O}_E(E)$	ω_E
very ample	0	4	1	$\sigma: E \xrightarrow{\sim} \mathbb{P}^3$	$-bL$ $b = 1, 2, 3$	$-4L$
	1	3		$\sigma: E \to \mathbb{P}^2$, $Bs\lvert L\rvert$ a simple point	$-L$	$-2L$
very ample	0	5	2	$\sigma: E \xrightarrow{\sim} V_3^2$ a normal quadric in $\mathbb{P}^4$	$-bL$ $b = 1, 2$	$-3L$
	1	4		$Bs\lvert L\rvert = \emptyset$ and $\sigma: E \to \mathbb{P}^3$ a 2-sheeted covering of $\mathbb{P}^3$, or $\deg \sigma = \infty$ and E singular at $Bs\lvert L\rvert$	$-L$	$-2L$
very ample	1	5	3	$\sigma: E \xrightarrow{\sim} V_3^3$ a cubic in $\mathbb{P}^4$	$-L$	$-2L$
				$\deg \sigma = \infty$, E singular at $Bs\lvert L\rvert$		
very ample	1	6	4	$\sigma: E \xrightarrow{\sim} V_3^4$ a complete intersection of 2 quadrics in $\mathbb{P}^5$	$-L$	$-2L$
				$\deg \sigma = \infty$, E singular at $Bs\lvert L\rvert$		
very ample	1	$L^3 + 2$	$5 \leqslant L^3 \leqslant 9$	$\sigma: E \xrightarrow{\sim} V_3^N \subset \mathbb{P}^{N+1}$, $N = L^3$	$-L$	$-2L$
				$\deg \sigma = \infty$, E singular at $Bs\lvert L\rvert$		

PROOF. – First, let us assume $a \neq b$. The Serre duality gives, for every $n \in \mathbb{Z}$,

$$P(n) = - P(- a - b - n) .$$

Hence $P(- a) = 0$ implies $P(- b) = 0$. Let x be the third root of $P(n)$. Then clearly either $x = - a$, $x = - b$ or $x = (- a - b)/2$. Since $P(0) = 1$ we find for $P(n)$ the following forms

$$P(n) = (1/a^2 b)(n + a)^2(n + b) , \quad P(n) = (1/ab^2)(n + a)(n + b)^2 ,$$
$$P(n) = (2/a^2 b + ab^2)(n + a)(n + b)\left(n + \frac{a + b}{2}\right).$$

The leading coefficient of $P(n)$ is known to be $L^3/6$, so we have respectively

$$L^3(a^2b) = 6\,,\qquad L^3(ab^2) = 6\,,\qquad L^3(a^2b + ab^2) = 12\,.$$

Recalling that $h^0(L) = P(1)$ (see (4.1.2)) we find, case by case, the following possibilities,

$$L^3 = 1\,,\quad a = 1\,,\quad b = 6 \quad \text{or} \quad a = 6\,,\quad b = 1\,,\quad h^0(L) \notin \mathbb{Z},\ \text{contradiction}\,;$$

$$L^3 = 2\,,\quad a = 1\,,\quad b = 3 \quad \text{or} \quad a = 3\,,\quad b = 1\,,\quad h^0(L) \notin \mathbb{Z},\ \text{contradiction}\,;$$

$$L^3 = 3\,,\quad a = 1\,,\quad b = 2 \quad \text{or} \quad a = 2\,,\quad b = 1\,,\quad h^0(L) = 6\,;$$

$$L^3 = 1\,,\quad a = 1\,,\quad b = 3 \quad \text{or} \quad a = 3\,,\quad b = 1\,,\quad h^0(L) = 4\,;$$

$$L^3 = 2\,,\quad a = 1\,,\quad b = 2 \quad \text{or} \quad a = 2\,,\quad b = 1\,,\quad h^0(L) = 5\,.$$

In the last three cases one has $\Delta(L) = 3 + L^3 - h^0(L) = 0$, then E is normal and L is very ample as proved in [F1], 4.8. Note that case $L^3 = 3$ above does not occur: indeed in this case $\omega_E \sim -3L$ which contradicts the sectional genus formula (4.1.7) below.

Now, assume $a = b$ and let $x \neq -a$ be a root of $P(n)$. Then the Serre duality says $P(x) = -P(-2a - x) = 0$ and $-2a - x \neq x$, $-2a - x \neq -a$. Therefore, since $P(0) = 1$, we get

$$P(n) = (-1/2a^2x + ax^2)(n + a)(n - x)(n + 2a + x)$$

and

$$L^3(2a^2x + ax^2) = -6 \tag{4.1.6}$$

because the leading coefficient is $L^3/6$. In view of (4.1.2)$'$, (4.1.2)$''$, case $a = b = 3$ is excluded while $a = b = 2$ gives $x = -3, -1$, hence

$$P(n) = \tfrac{1}{6}(n + 2)(n + 3)(n + 1)$$

and $h^0(L) = P(1) = 4$. Again $\Delta(L) = 0$, so E is normal and L is very ample. Thus the cases in the table when $\Delta(L) = 0$ are proved. In the remaining case $a = b = 1$ we find from (4.1.2), (4.1.6) that

$$h^0(L) = P(1) = L^3 + 2\,.$$

Hence

$$\Delta(L) = 1\,.$$

If $L^3 = 1$ then $h^0(L) = 3$, $\sigma: E - \to \mathbb{P}^2$, $Bs|L|$ is a simple point on E. To go on, consider the *sectional genus* $g(L)$ defined by the equality ([F2], § 1)

$$2g(L) - 2 = (\omega_E + 2L)\cdot L^2 . \tag{4.1.7}$$

Now $\omega_E \sim -2L$, so $g(L) = \Delta(L) = 1$. When E is nonsingular at each point of $Bs|L|$ then Theorem 3.6 in [F2] applies to say that $Bs|L| = \emptyset$ if $L^3 = 2$, so $\deg \sigma = 2$ in this case by (4.1.4), and L is very ample when $L^3 \geqq 3$. In particular E is a cubic in $\mathbb{P}^4$ or a complete intersection of two quadrics in $\mathbb{P}^5$ when $L^3 = 3$ or 4 respectively, as proved again in [F2], 5.4, 5.6. While if E is singular at some base point, then it has to be $\deg \sigma = \infty$ by combining [F2], 3.3, 3.6. Assumption (4.1.5) gives $L^3 \leqq 9$ and this completes the proof. q.e.d.

SUPPLEMENT (4.1)′. – Consider the case $N = L^3 \geqq 5$, $\deg \sigma < \infty$ in 4.1. Then $\sigma_{|L|}: E \xrightarrow{\sim} V_3^N \subset \mathbb{P}^{N+1}$ and let us assume in addition that the restriction mapping

$$H^0(\mathbb{P}^{N+1}, \mathcal{O}(1)) \to H^0(E, L)$$

is bijective. Therefore, since $\Delta(L) = 1$ and L is very ample, from the results stated in [F3] we can deduce for E the possibilities listed in the table below. Note in particular that if $L^3 = 7$, 8 and E is singular then E is necessarily not normal, while if $L^3 = 9$, E has to be singular and not normal. When E is nonsingular we find the Fano 3-folds of index two and degree $N = 5, \dots, 8$. Further, every connected component of the singular locus of E is a linear space (of dimension $\leqq 2$) and when E is not normal then its singular locus is connected. Here $\gamma: \tilde{E} \to E$ denotes the normalization of E. We say that $(\tilde{E}, \gamma^*L)$ is a *rational scroll* if $(\tilde{E}, \gamma^*L) = (\mathbb{P}(\mathscr{E}), \mathcal{O}(1))$ for some ample vector bundle $\mathscr{E}$ on $\mathbb{P}^1$.

L^3	Structure of E ($\gamma: \tilde{E} \to E$ the normalization of E)
5	— E nonsingular, a linear section of the Grassmann variety parametrizing lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker coordinates; — E not normal, $(\tilde{E}, \gamma^*L)$ a rational scroll; — E singular with normal, rational singularities.
6	— E nonsingular, a hyperplane section of a Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, or a Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$; — E not normal, $(\tilde{E}, \gamma^*L)$ a rational scroll; — E singular with normal, rational singularities.
7	— E nonsingular, a blowing-up of $\mathbb{P}^3$ at a point; E not normal, $(\tilde{E}, \gamma^*L)$ a rational scroll.
8	— E nonsingular, the Veronese image of $\mathbb{P}^3$; — E not normal, $(\tilde{E}, \gamma^*L)$ a rational scroll.
9	— E not normal, $(\tilde{E}, \gamma^*L)$ a rational scroll.

Note that in the present case it has to be $(\omega_E^{-1})^3 \leqq 64$ whenever E is normal. ■

As far as the case R nef is concerned, we have the following

PROPOSITION 4.2. – *Let $\varphi: X \to Y$ be as in (3.2.1) and let f be a 1-dimensional fibre of φ. Write $C = f_{\mathrm{red}}$. Then every irreducible component A of C is isomorphic to $\mathbb{P}^1$; moreover if C_1, C_2 are distinct irreducible components of C such that $C_1 \cap C_2 \neq \emptyset$, then C_1 and C_2 intersect at only one point and transversally. Furthermore*

$$(K_X \cdot A) = -2 - p_1 - p_2 - p_3$$

where $\mathcal{N}_{A/X} = \mathcal{O}_A(p_1) \oplus \mathcal{O}_A(p_2) \oplus \mathcal{O}_A(p_3)$ *and* $(p_1, p_2, p_3) = (0, 0, 0)$, $(0, 0, -1)$, $(1, 0, -1)$, $(1, 0, -2)$ or $(1, -1, -1)$ (²).

PROOF. – Let C' be an arbitrary closed subscheme of X such that $C'_{\mathrm{red}} \subset C$. Then we claim that

$$\chi(\mathcal{O}_{C'}) > 0 . \tag{4.2.1}$$

Choose D_1, D_2, D_3 irreducible divisors on Y which intersect properly at $y = \varphi(f)$ and let E_1, E_2, E_3 be irreducible divisors on X such that

$$\operatorname{Supp} E_i = \varphi^{-1}(D_i) , \quad i = 1, 2, 3 .$$

Clearly one has

$$E_i \neq E_j , \quad i \neq j , \ i, j = 1, 2, 3 ; \quad (E_i \cdot R) = 0 ; \quad C \subset E_1 \cap E_2 \cap E_3 .$$

Indeed, by construction, E_1, E_2, E_3 are irreducible divisors containing C, $\dim E_1 \cap \cap E_2 \cap E_3 = 1$ and $(E_i \cdot f) = (E_i \cdot R) = 0$ since $[f] \in R$. For every positive integer a, the divisor

$$mH - \varepsilon_1 E_1 - \varepsilon_2 E_2 - \varepsilon_3 E_3 - K_X , \quad \varepsilon_i = 0, \ a$$

is ample for $m \gg 0$ because $(-\varepsilon_1 E_1 - \varepsilon_2 E_2 - \varepsilon_3 E_3 - K_X) \cdot R < 0$. Hence

$$H^i(\mathcal{O}_X(mH - \varepsilon_1 E_1 - \varepsilon_2 E_2 - \varepsilon_3 E_3)) = (0) , \quad i > 0, \ m \gg 0, \ \varepsilon_i = 0, \ a .$$

Then in view of Lemma 1.3, the exact sequence

$$0 \to \mathcal{O}_X(-a(E_1 + E_2 + E_3)) \to \mathcal{O}_X(-aE_1) \oplus \mathcal{O}_X(-aE_2) \oplus \mathcal{O}_X(-aE_3) \to \\ \to \mathcal{O}_X \to \mathcal{O}_{aE_1 \cap aE_2 \cap aE_3} \to 0$$

(²) In [A] p. 351 is proved that $2p_1 + p_2 + p_3 \leqq 0$ so case $(1, 0, -1)$ can be excluded.

gives

$$H^1(\mathcal{O}_{aE_1\cap aE_2\cap aE_3}(mH)) = (0)\,, \qquad m \gg 0\,.$$

Since $E_1 \cap E_2 \cap E_3 \supset C \supset C'_{\text{red}}$ one has, for some positive integer a,

$$aE_1 \cap aE_2 \cap aE_3 \supset C'\,.$$

Then there exists a surjection

$$\mathcal{O}_{aE_1\cap aE_2\cap aE_3}(mH) \to \mathcal{O}_{C'}(mH) \to 0\,,$$

which gives

$$H^1(\mathcal{O}_{C'}(mH)) = (0)\,, \qquad m \gg 0\,.$$

Now, $\dim \varphi(C'_{\text{red}}) = 0$ so that $H\cdot C'_{\text{red}} = H\cdot C' = 0$. Therefore $\mathcal{O}_{C'}(mH) \sim \mathcal{O}_{C'}$ and

$$\chi(\mathcal{O}_{C'}) = \chi(\mathcal{O}_{C'}(mH)) = h^0(\mathcal{O}_{C'}(mH))\,.$$

Indeed in view of Theorem 1.1, mH is effective and $|mH|$ is base point free for $m \gg 0$. Then $\chi(\mathcal{O}_{C'}) = h^0(\mathcal{O}_{C'}(mH)) > 0$, $m \gg 0$ and (4.2.1) is proved. Up to obvious changes, the proof of the second part of the statement runs as in [M2], 5.6 and we omit it. Thus, let $A \cong \mathbb{P}^1$ be an irreducible component of C. If $\mathcal{N}_{A/X} = \mathcal{O}_A(p_1) \oplus \mathcal{O}_A(p_2) \oplus \oplus\, \mathcal{O}_A(p_3)$ the adjunction formula reads

$$-2 = \deg \omega_A = (K_X\cdot A) + p_1 + p_2 + p_3$$

whence

$$p_1 + p_2 + p_3 \geqq -1\,. \tag{4.2.2}$$

We can assume

$$p_1 \geqq p_2 \geqq p_3 \tag{4.2.3}$$

so (4.2.2) implies that

$$p_1 \geqq 0\,. \tag{4.2.4}$$

Now, let I be the sheaf of ideals defining A in X and write

$$I/I^2 = \mathcal{O}_A(a) \oplus \mathcal{O}_A(b) \oplus \mathcal{O}_A(c)$$

where $a \geqq b \geqq c$, $a = -p_3$, $b = -p_2$, $c = -p_1$. Since A is locally complete intersection in X of codimension 3, we can assume I to be generated by a regular sequence (a, b, c) in a neighbourhood of an arbitrary point. Let J be the ideal defined by

$$I^2 \subset J \subset I\,, \qquad J/I^2 = \mathcal{O}_A(a) \oplus \mathcal{O}_A(b)\,.$$

Then

$$J = (a, b) \text{ mod. } I^2 = (a, b, c^2)\,.$$

Look at the exact sequences

$$0 \to I^3 \cap J^2/I^4 \to I^3/I^4 \to I^3/I^3 \cap J^2 \to 0\,,$$
$$0 \to I^3 + J^2/I^3 \to I^2/I^3 \to I^2/I^3 + J^2 \to 0\,.$$

One can sees that

(4.2.5) $$I^3/I^3 \cap J^2 = \mathcal{O}_A(3c)\,,$$

(4.2.6) $$I^2/I^3 + J^2 = \mathcal{O}_A(2c) \oplus \mathcal{O}_A(a + c) \oplus \mathcal{O}_A(b + c)\,,$$

(4.2.7) $$\chi(\mathcal{O}/J^2) = \chi(\mathcal{O}_A) + \chi(I/I^2) + \chi(I^2/I^3 + J^2) + \chi(I^3/J^2 \cap I^3)\,.$$

Then (4.2.2), (4.2.6), (4.2.7) give

$$\chi(\mathcal{O}/J^2) = 8 + 6c + 2(a + b + c) \leqq 10 + 6c\,,$$

so in view of (4.2.1) we find $c \geqq -1$ that is

(4.2.8) $$p_1 \leqq 1\,.$$

Relations (4.2.3), (4.2.4), (4.2.8) imply that the only 3-ples (p_1, p_2, p_3) can occur are those listed in the table below

p_1	p_2	p_3
0	0	0
0	0	-1
1	1	1
1	1	0
1	1	-1
1	1	-2
1	1	-3
1	0	0
1	0	-1
1	0	-2
1	-1	-1

A direct computation shows that

$$\chi(\mathcal{O}/I^3) = \chi(\mathcal{O}_A) + \chi(I/I^2) + \chi(S^2(I/I^2)) = 10 + 5(a+b+c)\,,$$
$$\chi(\mathcal{O}/I^4) = \chi(\mathcal{O}_A) + \chi(I/I^2) + \chi(S^2(I/I^2)) + \chi(S^3(I/I^2)) = 20 + 15(a+b+c)\,,$$
$$\chi(\mathcal{O}/J^3) = \chi(\mathcal{O}_A) + \chi(I/I^2) + \chi(S^2(I/I^2)) + \chi(I^4/J^3 \cap I^4) + \\ + \chi(I^3/J^3 + I^4) = 19 + 10a + 10b + 25c\,.$$

By (4.2.1), the first two equalities exclude the 3-ples $(p_1, p_2, p_3) = (1, 1, 1)$, $(1, 1, 0)$ respectively, while the third implies that $(p_1, p_2, p_3) = (1, 1, -1)$, $(1, 0, 0)$ cannot occur.

Now let J' be the ideal defined by

$$I^2 \subset J' \subset I\,, \qquad J'/I^2 = \mathcal{O}_A(a)\,.$$

Then

$$J' = (a) \bmod. I^2 = (a, b^2, c^2, bc)$$

and we find

$$\chi(\mathcal{O}/J'^2) = \chi(\mathcal{O}_A) + \chi(I/I^2) + \chi(I^2/J'^2 + I^3) + \chi(I^3/J'^2 \cap I^3) = 13 + 3a + 11b + 11c\,.$$

Again (4.2.1) shows that the cases $(p_1, p_2, p_3) = (1, 1, -2)$, $(1, 1, -3)$ do not occur. This completes the proof. q.e.d.

Added in proof. – In the paper of the same author: *Contractions of non numerically effective extremal rays in dimension* 4, Proceedings of Algebraic Geometry Conference, Humboldt University, Berlin (1985), Teubner-Texte zur Math., Band 92 (1986), some improvements of the results contained in § 4 are given. In particular conjectured condition (4.1.5) is proved.

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